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Bézier-Summation-Integral-Type Operators That Include Pólya–Eggenberger Distribution

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Abstract: We define the summation-integral-type operators involving the ideas of Pólya–Eggenberger distribution and Bézier basis functions, and study some of their basic approximation properties. In addition, by means of the Ditzian–Totik modulus of smoothness, we study a direct theorem as well as a quantitative Voronovskaja-type theorem for our newly constructed operators. Moreover, we investigate the approximation of functions with derivatives of bounded variation (DBV) of the aforesaid operators.

Keywords: Stancu operators; Pólya–Eggenberger distribution; Bézier curves; rate of convergence

MSC: 41A36; 41A25; 26A15

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1. Introduction and Preliminaries

In 1968, Stancu opened up new vistas for researchers working in the field of approximation theory by constructing positive linear operators involving the idea of the Pólya–Eggenberger distribution [1] as follows. The operator $P_n^{[\tau]}$ (τ is a non-negative parameter) acting from $C[0,1]$ (the space of continuous functions on $[0,1]$) to itself is defined by

$$P_r^{[\tau]}(\zeta;u)=\sum_{i=0}^{r}p_{r,i}^{[\tau]}(u)\zeta\left(\frac{i}{r}\right)\qquad(u\in[0,1])\tag{1}$$

for $r\in\mathbb{N}$ ($\mathbb{N}:=$ the set of natural numbers) and any function $\zeta\in C\,[0,1]$, where

$$p_{r,i}^{[\tau]}(u)=\binom{r}{i}\frac{u^{[i,-\tau]}(1-u)^{[r-i,-\tau]}}{1^{[r,-\tau]}}.\tag{2}$$

In the expression (2), the rth factorial power of u with increment k is given by

$$u^{[r,k]}=u(u-k)\ldots(u-(r-1)k),\quad u^{[0,k]}=1.$$

Equivalently, one writes the above expressions as

$$P_r^{[\tau]}(\zeta;u)=\sum_{i=0}^{r}\binom{r}{i}\frac{\prod_{\lambda=0}^{i-1}(u+\lambda\tau)\prod_{\mu=0}^{r-i-1}(1-u+\mu\tau)}{(1+\tau)(1+2\tau)\cdots(1+(r-1)\tau)}\zeta\left(\frac{i}{r}\right).\tag{3}$$

For $\tau = 0$, we obtain

$$P_r^{[0]}(\zeta;u) = \sum_{i=0}^{r} p_{r,i}^{[0]}(u)\zeta\left(\frac{i}{r}\right), \quad p_{r,i}^{[0]}(u) = \binom{r}{i}\frac{u^{[i,0]}(1-u)^{[r-i,0]}}{1^{[r,0]}}$$

which coincides with Bernstein operators [2]. For $\tau = \frac{1}{r}$, (1) has been discussed in [3], given by

$$\begin{aligned} P_r^{[\frac{1}{r}]}(\zeta;u) &= \sum_{i=0}^{r} p_{r,i}^{[\frac{1}{r}]}(u)\zeta\left(\frac{i}{r}\right) \\ &= \frac{2(r!)}{(2r)!}\sum_{i=0}^{r}\binom{r}{i}\prod_{\lambda=0}^{i-1}(ru+\lambda)\prod_{\mu=0}^{r-i-1}(r-ru+\mu)\zeta\left(\frac{i}{r}\right). \end{aligned} \tag{4}$$

Later, Miclăuş [4,5] presented an interesting work on these operators using the idea of convex functions and divided differences including Voronovskaja-type theorem while the classical Voronovskaja theorem [6] is stated as follows.

Theorem 1. *Let $\zeta(u)$ be bounded on $[0,1]$. Then, for any $u \in [0,1]$ at which $\zeta''(u)$ exists, one has*

$$\lim_{r\to\infty}\left[2r\left(P_r^{[0]}(\zeta;u) - \zeta(u)\right)\right] = u(1-u)\zeta''(u), \tag{5}$$

and the equality (5) holds uniformly on $[0,1]$ if $\zeta'' \in C[0,1]$.

Some other work in this sense was discussed by Gupta et al. [7–9], Agrawal et al. [10,11], Razi [12], Wang et al. [13], Finta [14,15], Deo et al. [16], Abel et al. [17], and Kajla et al. [18].

In what follows, $L_B[0,1]$ and Π_n, respectively, will be used for the class of bounded Lebesgue integrable functions on $[0,1]$ and the polynomials of degree at most n in $\mathbb{N}$. For $\varrho > 0$, the operators $\mathscr{B}_{n,\varrho} : L_B[0,1] \to \Pi_n$ (see [19]) are defined by

$$\mathscr{B}_{r,\varrho}(\zeta;u) = \sum_{i=0}^{r} p_{r,i}(u)F_{r,i}^{\varrho}(\zeta) \qquad (\varrho > 0),$$

where

$$p_{r,i}(u) = p_{r,i}^{[0]}(u), \qquad \beta(u_1,u_2) = \int_0^1 s^{u_1-1}(1-s)^{u_2-1}ds \quad (u_1,u_2 > 0),$$

and

$$F_{r,i}^{\varrho}(\zeta) = \left(\int_0^1 \frac{s^{i\varrho-1}(1-s)^{(r-i)\varrho-1}}{\beta(i\varrho,(r-i)\varrho)}\zeta(s)ds\right),$$

or,

$$\mathscr{B}_{r,\varrho}(\zeta;u) = (1-x)^r\zeta(0) + x^r\zeta(1) + \sum_{i=1}^{r-1} p_{r,i}(u)\left(\int_0^1 \frac{s^{i\varrho-1}(1-s)^{(r-i)\varrho-1}}{\beta(i\varrho,(r-i)\varrho)}\zeta(s)ds\right). \tag{6}$$

Gonska and Păltănea [20] discussed the recursion formula and simultaneous approximation of derivatives for the operators (6). In addition, they established that $\mathscr{B}_{r,\varrho}(\zeta;u)$ presents a link between $P_r^{[0]}(\zeta;u)$ and its genuine Durrmeyer form.

Kajla and Miclăuş [21] introduced the Stancu-type modification of Durrmeyer operators $\mathscr{B}_{n,\varrho}^{[\tau]} : L_B[0,1] \to \Pi_n$, defined by

$$\mathscr{B}_{r,\varrho}^{[\tau]}(\zeta;u) = \sum_{i=0}^{r} p_{r,i}^{[\tau]}(u)F_{r,i}^{\varrho}(\zeta)$$

$$= \frac{(1-u)^{[r,-\tau]}\zeta(0)}{1^{[r,-\tau]}} + \frac{u^{[r,-\tau]}\zeta(1)}{1^{[r,-\tau]}} + \sum_{i=1}^{r-1} p_{r,i}^{[\tau]}(u)\left(\int_0^1 \frac{s^{i\varrho-1}(1-s)^{(r-i)\varrho-1}}{\beta(i\varrho,(r-i)\varrho)}\zeta(s)ds\right) \quad (7)$$

for $\varrho > 0$.

Bézier curves [22] are widely used in computer-aided geometric design, as well as in other areas of computer science. To obtain a better understanding of positive linear operators, many efforts are devoted to studying operators involving Bézier basis functions; namely, Bernstein-type [23,24], Păltănea-type involving Appell and Gould–Hopper polynomials [25,26], Meyer–König and Zeller [27], Baskakov [28], Srivastava–Gupta [29,30], Chlodowsky [31], Kantorovich [32], Bleimann–Butzer–Hahn [33], Durrmeyer [34,35], and Bleimann–Butzer–Hahn–Kantorovich [36]. From a computational point of view, it is important to remember that linear positive operators converge with inherently slow convergence rates (see the seminal book by Korovkin [37]) due to the Voronoskaja-type saturation results, but the good news from an applicative viewpoint is that many of these operators (especially those of the Bernstein-type) admit asymptotic expansion with respect to the parameter r, when the function is smooth enough (see [38–41]). We remember that such expansions are used to construct extrapolation algorithms that converge very quickly to the given smooth function, so overcoming the most important limitation of linear positive operators.

For more details and further investigations regarding the study of Bézier curves and related operators, we refer to [42–49].

2. Bézier-Summation-Integral-Type Operators and Auxiliary Results

Let $\varrho > 0$ and $r \in \mathbb{N}$. For $\theta \geq 1$ and $\zeta \in L_B[0,1]$, we present the operators

$$\mathscr{B}_{r,\varrho,\theta}^{[\tau]}(\zeta;u) = \sum_{i=0}^{r} Q_{r,i,\theta}^{[\tau]}(u)F_{r,i}^{\varrho}(\zeta) \qquad (u \in [0,1]), \quad (8)$$

where

$$Q_{r,i,\theta}^{[\tau]}(u) = [H_{r,i}(u)]^{\theta} - [H_{r,i+1}(u)]^{\theta} \quad (9)$$

and

$$H_{r,i}(u) = \sum_{j=i}^{r} p_{r,j}^{[\tau]}(u) \quad \text{if} \quad i \leq r,$$

and $H_{r,i}(u) = 0$ otherwise.

Alternatively, we rewrite (8) as

$$\mathscr{B}_{r,\varrho,\theta}^{[\tau]}(\zeta;u) = \int_0^1 \mathscr{K}_{r,\varrho,\theta}^{[\tau]}(u,s)\zeta(s)ds \qquad (u \in [0,1]), \quad (10)$$

where

$$\mathscr{K}_{r,\varrho,\theta}^{[\tau]}(u,s) = \sum_{i=1}^{r-1} Q_{r,i,\theta}^{[\tau]}(u)F_{r,i}^{\varrho}(\zeta) + Q_{r,0,\theta}^{[\tau]}(u)\delta(s) + Q_{r,r,\theta}^{[\tau]}(u)\delta(1-s).$$

In this case, $\delta(w)$ is the Dirac-delta function.

Lemma 1 ([21]). *For the operators $\mathscr{B}_{r,\varrho}^{[\tau]}(\zeta;u)$, we have*

$$\mathscr{B}_{r,\varrho}^{[\tau]}(e_0;u) = 1; \qquad \mathscr{B}_{r,\varrho}^{[\tau]}(e_1;u) = u;$$

$$\mathscr{B}_{r,\varrho}^{[\tau]}(e_2;u) = \frac{u^2\varrho(r-1)}{(1+\tau)(1+r\varrho)} + \frac{u(1+\tau+\varrho+r\tau\varrho)}{(1+\tau)(1+r\varrho)};$$

$$\begin{aligned}\mathscr{B}^{[\tau]}_{r,\varrho}(e_3;u) &= \frac{(r-1)(r-2)u^3\varrho^2}{(1+\tau)(1+2\tau)(1+r\varrho)(2+r\varrho)} + \frac{3(r-1)(1+\varrho+\tau(2+r\varrho))u^2\varrho}{(1+\tau)(1+2\tau)(1+r\varrho)(2+r\varrho)}\\ &+ \frac{((1+\varrho)(2+\varrho)+3\tau(1+\varrho)(2+r\varrho)+2\tau^2(1+r\varrho)(2+r\varrho))u}{(1+\tau)(1+2\tau)(1+r\varrho)(2+r\varrho)};\end{aligned}$$

$$\begin{aligned}\mathscr{B}^{[\tau]}_{r,\varrho}(e_4;u) &= \frac{(r-1)(r-2)(r-3)u^4\varrho^3}{(1+\tau)(1+2\tau)(1+3\tau)(1+r\varrho)(2+r\varrho)(3+r\varrho)}\\ &+\frac{6(r-1)(r-2)(1+\varrho+\tau(3+r\varrho))u^3\varrho^2}{(1+\tau)(1+2\tau)(1+3\tau)(1+r\varrho)(2+r\varrho)(3+r\varrho)}\\ &\frac{(r-1)u^2\varrho}{(1+\tau)(1+2\tau)(1+3\tau)(1+r\varrho)(2+r\varrho)(3+r\varrho)}\Big\{11(1+\tau(5+6\tau))\\ &+18(1+3\tau)(1+r\tau)\varrho+(1+r\tau)(7+(11r-1)\tau)\varrho^2\Big\}\\ &+\frac{u\varrho^3}{(1+\tau)(1+2\tau)(1+3\tau)(1+r\varrho)(2+r\varrho)(3+r\varrho)}\Big\{6(1+\tau)\\ &\times(1+2\tau)(1+3\tau)+11(1+r\tau)(1+\tau(5+6\tau))\varrho\\ &+6(1+3\tau)(1+r\tau)(1+2r\tau)\varrho^2+(1+r\tau)(1+\tau(6r(1+r\tau)-1))\Big\}.\end{aligned}$$

Let $M^{[\tau]}_{r,\varrho,r_1}(u) := \mathscr{B}^{[\tau]}_{r,\varrho}((e_1-u)^{r_1};u)$, where $r \geq 1$, $r_1 \geq 0$ and $u \in [0,1]$.

Lemma 2 ([21])**.** *We have*

$$M^{[\tau]}_{r,\varrho,1}(u) = 0; \qquad M^{[\tau]}_{r,\varrho,2}(u) = \frac{(1+\tau+\varrho+r\tau\varrho)u(1-u)}{(1+\tau)(1+r\varrho)};$$

$$\begin{aligned}M^{[\tau]}_{r,\varrho,4}(u) &= \frac{1}{(1+\tau)(1+2\tau)(1+3\tau)(1+r\varrho)(2+r\varrho)(3+r\varrho)}\Big\{3u^4\Big(2\tau(1+\varrho)\\ &\times((r-4)\varrho-6)(3+r\varrho)+\tau^2((r-12)\varrho-11)(2+r\varrho)(3+r\varrho)\\ &-6\tau^3(1+r\varrho)(2+r\varrho)(3+r\varrho)+(1+\varrho)(-6+\varrho(r-6+(r-2)\varrho))\Big)\\ &+6u^3\Big(6(1+\tau)(1+2\tau)(1+3\tau)+(1+\tau(5+6\tau))(12+r(11\tau-1))\varrho\\ &+2(1+3\tau)(1+r\tau)(4+r(6\tau-1))\varrho^2+(1+r\tau)\\ &\times(2+r(6\tau-1)(1+r\tau))\varrho^3\Big)-u^2\Big(24(1+\tau)(1+2\tau)(1+3\tau)\\ &+(1+\tau(5+6\tau))(47+r(44\tau-3))\varrho+6(1+3\tau)(1+r\tau)(5+r(8\tau-1))\varrho^2\\ &+(1+r\tau)(7-\tau+3r(8\tau-1)(1+r\tau))\varrho^3\Big)+u\Big(6(1+\tau)(1+2\tau)(1+3\tau)\\ &+11(1+r\tau)(1+\tau(5+6\tau))\varrho+6(1+3\tau)(1+r\tau)(1+2r\tau)\varrho^2\\ &+(1+r\tau)(1+\tau(-1+6r(1+r\tau)))\varrho^3\Big)\Big\}.\end{aligned}$$

Lemma 3 ([21])**.** *For any $r \in \mathbb{N}$, we can write*

$$M^{[\tau]}_{r,\varrho,2}(u) = \mathscr{B}^{[\tau]}_{r,\varrho}((e_1-u)^2;u) \leq \frac{\mathcal{C}^{[\tau]}_{\varrho}\ u(1-u)}{1+r\varrho},$$

$$M^{[\tau]}_{r,\varrho,4}(u) = \mathscr{B}^{[\tau]}_{r,\varrho}((e_1-u)^4;u) \leq \frac{\mathcal{D}^{[\tau]}_{\varrho}\ u(1-u)}{(1+r\varrho)^2},$$

where $\mathcal{C}_{\varrho}^{[\tau]}, \mathcal{D}_{\varrho}^{[\tau]}$ are positive constant depending on ϱ and τ.

Lemma 4 ([21])**.** *If $\tau \to 0$ as $r \to \infty$ and $\lim_{r\to\infty} r\tau = c \in \mathbb{R}$ (the set of real numbers), then*

$$\lim_{r\to\infty} r \cdot M_{r,\varrho,1}^{[\tau]}(u) = 0,$$

$$\lim_{r\to\infty} r \cdot M_{r,\varrho,2}^{[\tau]}(u) = \frac{(1+\varrho+c\varrho)u(1-u)}{\varrho},$$

$$\begin{aligned}\lim_{r\to\infty} r^2 \cdot M_{r,\varrho,4}^{[\tau]}(u) &= \frac{u^4(3+3\varrho(2+\varrho+2c(1+\varrho+c\varrho)))}{\varrho^2} - \frac{6u^3(1+\varrho+c\varrho)^2}{\varrho^2} \\ &\quad + \frac{3u^2(1+\varrho)(1+\varrho+2c\varrho)}{\varrho^2}.\end{aligned}$$

Remark 1. *We have*

$$\begin{aligned}\mathscr{B}_{r,\varrho,\theta}^{[\tau]}(e_0;u) &= \sum_{i=0}^{r} Q_{r,i,\theta}^{[\tau]}(u) = [H_{r,0}(u)]^\theta \\ &= \left[\sum_{j=0}^{r} p_{r,j}^{[\tau]}(u)\right]^\theta.\end{aligned}$$

Since

$$\sum_{j=0}^{r} p_{r,j}^{[\tau]}(u) = 1,$$

so $\mathscr{B}_{r,\varrho,\theta}^{[\tau]}(e_0;u) = 1$.

Lemma 5. *Let $\zeta \in C[0,1]$. Then,*

$$\|\mathscr{B}_{r,\varrho,\theta}^{[\tau]}(\zeta)\| \leq \theta\|\zeta\|$$

for $x \in [0,1]$.

Proof. It follows from the definition of $Q_{r,i,\theta}^{[\tau]}(u)$ and using the inequality

$$|a^\theta - b^\theta| \leq \theta|a-b| \qquad (0 \leq a,b \leq 1, \theta \geq 1)$$

that

$$\begin{aligned}0 &< [H_{r,i}(u)]^\theta - [H_{r,i+1}(u)]^\theta \leq \theta(H_{r,i}(u) - H_{r,i+1}(u)) \\ &\leq \theta p_{r,i}^{[\tau]}(u).\end{aligned}$$

Using (8) and ([21], Proposition 1), we may write

$$\|\mathscr{B}_{n,\varrho,\theta}^{[\tau]}(\zeta)\| \leq \theta\|\mathscr{B}_{n,\varrho}^{[\tau]}(\zeta)\| \leq \theta\|\zeta\|.$$

□

3. Direct and Quantitative Voronovskaja-Type Results

Recall as in [50] that $\vartheta(u) = \sqrt{u(1-u)}$ and $\zeta \in C[0,1]$. We denote by

$$\omega_\vartheta(\zeta,s) = \sup_{0<h\leq s}\left\{\left|\zeta\left(u+\frac{h\vartheta(u)}{2}\right) - \zeta\left(u-\frac{h\vartheta(u)}{2}\right)\right|, u \pm \frac{h\vartheta(u)}{2} \in [0,1]\right\},$$

the Ditzian–Totik first-order modulus of smoothness of ζ.

The K-functional is given as

$$\overline{K}_{\vartheta}(\zeta,s)=\inf_{f\in W_{\vartheta}[0,1]}\{\|\zeta-f\|+s\|\vartheta f'\|\}\qquad (s>0),$$

where

$$W_{\vartheta}[0,1]=\{f: f\in AC_{loc}[0,1],\ \|\vartheta f'\|<\infty\}.$$

and $f\in AC_{loc}[0,1]$ means that f is absolutely continuous on every interval $[a,b]\subset(0,1)$. From Theorem 3.1.2 of [50], we have $\overline{K}_{\vartheta}(\zeta,s)\sim\omega_{\vartheta}(\zeta,s)$, so there is a constant $C>0$ such that

$$C^{-1}\omega_{\vartheta}(\zeta,s)\le\overline{K}_{\vartheta}(\zeta,s)\le C\omega_{\vartheta}(\zeta,s). \tag{11}$$

We are now ready to study the direct approximation theorem for $\mathscr{B}^{[\tau]}_{r,\varrho,\theta}$.

Theorem 2. *Consider $\zeta\in C[0,1]$ with $\vartheta(u)=\sqrt{u(1-u)}$. Then*

$$|\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(\zeta;u)-\zeta(u)|\ \le\ C\omega_{\vartheta}\left(\zeta,\sqrt{\frac{C^{[\tau]}_{\varrho}}{1+r\varrho}}\right)\qquad(\forall u\in[0,1)),$$

where $C^{[\tau]}_{\varrho}>0$ is a constant depending on ϱ and τ.

Proof. Using the relation $f(s)=f(u)+\displaystyle\int_u^s f'(w)dw$, we can write

$$\left|\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(f;u)-f(u)\right| = \left|\mathscr{B}^{[\tau]}_{r,\varrho,\theta}\left(\int_u^s f'(w)dw;u\right)\right|. \tag{12}$$

For any $u,s\in(0,1)$, we have

$$\left|\int_u^s f'(w)dw\right|\le||\vartheta f'||\left|\int_u^s\frac{1}{\vartheta(w)}dw\right|. \tag{13}$$

Therefore,

$$\begin{aligned}
\left|\int_u^s\frac{1}{\vartheta(w)}dw\right| &= \left|\int_u^s\frac{1}{\sqrt{w(1-w)}}dw\right|\\
&\le \left|\int_u^s\left(\frac{1}{\sqrt{w}}+\frac{1}{\sqrt{1-w}}\right)dw\right|\\
&\le 2\left(\,|\sqrt{s}-\sqrt{u}\,|+|\,\sqrt{1-s}-\sqrt{1-u}\,|\,\right)\\
&= 2|s-u|\left(\frac{1}{\sqrt{s}+\sqrt{u}}+\frac{1}{\sqrt{1-s}+\sqrt{1-u}}\right)\\
&< 2|s-u|\left(\frac{1}{\sqrt{u}}+\frac{1}{\sqrt{1-u}}\right)\\
&< \frac{2\sqrt{2}\,|s-u|}{\vartheta(u)}.
\end{aligned}\tag{14}$$

Taking (12)–(14) and applying the Cauchy–Bunyakovsky–Schwarz inequality, we obtain

$$|\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(f;u)-f(u)|\ <\ 2\sqrt{2}\|\vartheta f'\|\vartheta^{-1}(u)\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(|s-u|;u)$$

$$\leq 2\sqrt{2}\|\vartheta f'\|\vartheta^{-1}(u)\left(\mathscr{B}_{r,\varrho,\theta}^{[\tau]}((s-u)^2;u)\right)^{1/2}.$$

It follows from Lemma 3 that

$$|\mathscr{B}_{r,\varrho,\theta}^{[\tau]}(f;u)-f(u)|<C\sqrt{\frac{\mathcal{C}_{\varrho}^{[\tau]}}{1+r\varrho}}\|\vartheta f'\|. \tag{15}$$

Using [21] (Proposition 3.1) and (15), we obtain

$$\begin{aligned}|\mathscr{B}_{r,\varrho,\theta}^{[\tau]}(\zeta;u)-\zeta| &\leq |\mathscr{B}_{r,\varrho,\theta}^{[\tau]}(\zeta-f;u)|+|\zeta-f|+|\mathscr{B}_{r,\varrho,\theta}^{[\tau]}(f;u)-f(u)| \\ &\leq C\left(\|\zeta-f\|+\sqrt{\frac{\mathcal{C}_{\varrho}^{[\tau]}}{1+r\varrho}}\|\vartheta f'\|\right). \end{aligned} \tag{16}$$

Letting $\inf_{f\in W_\vartheta}$ in (16) gives

$$|\mathscr{B}_{n,\varrho,\theta}^{[\tau]}(\zeta;x)-\zeta(x)|\leq C\overline{K}_{\vartheta}\left(\zeta;\frac{\mathcal{C}_{\varrho}^{[\tau]}}{1+n\varrho}\right). \tag{17}$$

Using $\overline{K_\vartheta}(\zeta,s)\sim\omega_\vartheta(\zeta,s)$, we obtain the required inequality. □

Theorem 3. *Consider $\zeta\in C^2[0,1]$. Then, there hold:*

$$\left|r\left(\mathscr{B}_{r,\varrho,\theta}^{[\tau]}(\zeta;u)-\zeta(u)-\zeta'(u)\mathscr{B}_{r,\varrho,\theta}^{[\tau]}(s-u;u)-\frac{1}{2}\zeta''(u)\mathscr{B}_{r,\varrho,\theta}^{[\tau]}((s-u)^2;u)\right)\right| \leq C\omega_\vartheta\left(\zeta''(u),\vartheta(u)r^{-1/2}\right) \tag{18}$$

and

$$\left|r\left(\mathscr{B}_{r,\varrho,\theta}^{[\tau]}(\zeta;u)-\zeta(u)-\zeta'(u)\mathscr{B}_{r,\varrho,\theta}^{[\tau]}(s-u;u)-\frac{1}{2}\zeta''(u)\mathscr{B}_{r,\varrho,\theta}^{[\tau]}((s-u)^2;u)\right)\right| \leq C\vartheta(u)\omega_\vartheta\left(\zeta''(u),r^{-1/2}\right). \tag{19}$$

Proof. By Taylor's expansion, we write

$$\zeta(s)-\zeta(u)=(s-u)\zeta'(u)+\int_u^s(s-w)\zeta''(w)dw \qquad (\zeta\in C^2[0,1];u,s\in[0,1]).$$

Thus,

$$\begin{aligned}\zeta(s)-\zeta(u)-(s-u)\zeta'(u)-\frac{1}{2}(s-u)^2\zeta''(u) &= \int_u^s(s-w)\zeta''(w)dw \\ &\quad -\int_u^s(s-u)\zeta''(u)dw.\end{aligned} \tag{20}$$

We obtain by operating $\mathscr{B}_{r,\varrho,\theta}^{[\tau]}(\cdot;u)$ on (20) that

$$\left|\left(\mathscr{B}_{r,\varrho,\theta}^{[\tau]}(\zeta;u)-\zeta(u)-\zeta'(u)\mathscr{B}_{r,\varrho,\theta}^{[\tau]}(s-u;u)-\frac{1}{2}\zeta''(u)\mathscr{B}_{r,\varrho,\theta}^{[\tau]}((s-u)^2;u)\right)\right| \leq \mathscr{B}_{r,\varrho,\theta}^{[\tau]}\left(\left|\int_u^s|s-w|\,|\zeta''(w)-\zeta''(u)|dw\right|;u\right). \tag{21}$$

Therefore, $f \in W_{\vartheta}[0,1]$, we have

$$\left|\int_u^s |s-w|\,|\zeta''(w)-\zeta''(x)|dw\right| \leq 2||\zeta''-f||(s-u)^2 + 2\|\vartheta f'\|\vartheta^{-1}(u)|s-u|^3. \tag{22}$$

It follows from (21), (22), Lemma 3, and the Cauchy–Bunyakovsky–Schwarz inequality that

$$\begin{aligned}
&\left|\left(\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(\zeta;u)-\zeta(u)-\zeta'(u)\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(s-u;u)-\frac{1}{2}\zeta''(u)\mathscr{B}^{[\tau]}_{r,\varrho,\theta}((s-u)^2;u)\right)\right| \\
&\leq 2\|\zeta''-f\|\mathscr{B}^{[\tau]}_{r,\varrho,\theta}((s-u)^2;u)+2\|\vartheta f'\|\vartheta^{-1}(u)\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(|s-u|^3;u) \\
&\leq 2\theta\|\zeta''-f\|\frac{C^{[\tau]}_{\varrho}}{1+r\varrho}\vartheta^2(u) \\
&\quad +2\theta\|\vartheta f'\|\vartheta^{-1}(u)\left(\mathscr{B}^{[\tau]}_{r,\varrho}((s-u)^2;u)\right)^{1/2}\left(\mathscr{B}^{[\tau]}_{r,\varrho}((s-u)^4;u)\right)^{1/2} \\
&\leq 2\theta\|\zeta''-f\|\frac{C^{[\tau]}_{\varrho}}{1+r\varrho}\vartheta^2(u)+2\theta\frac{C}{(1+r\varrho)}\sqrt{\frac{C^{[\tau]}_{\varrho}}{1+r\varrho}}\vartheta(u)\|\vartheta f'\| \\
&\leq C\left(\frac{C^{[\tau]}_{\varrho}}{1+r\varrho}\vartheta^2(u)\|\zeta''-f\|+\frac{1}{(1+r\varrho)}\sqrt{\frac{C^{[\tau]}_{\varrho}}{1+r\varrho}}\vartheta(u)\|\vartheta f'\|\right) \\
&\leq \frac{C}{r}\left(\vartheta^2(u)\|\zeta''-f\|+r^{-1/2}\vartheta(u)\|\vartheta f'\|\right).
\end{aligned}$$

Since $\vartheta^2(u) \leq \vartheta(u) \leq 1, u \in [0,1]$, we have

$$\begin{aligned}
&\left|\left(\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(\zeta;u)-\zeta(u)-\zeta'(u)\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(s-u;u)-\frac{1}{2}\zeta''(u)\mathscr{B}^{[\tau]}_{r,\varrho,\theta}((s-u)^2;u)\right)\right| \\
&\qquad\leq \frac{C}{r}\left(\|\zeta''-f\|+r^{-1/2}\vartheta(u)\|\vartheta f'\|\right).
\end{aligned}$$

In addition,

$$\begin{aligned}
&\left|\left(\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(\zeta;u)-\zeta(u)-\zeta'(u)\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(s-u;u)-\frac{1}{2}\zeta''(u)\mathscr{B}^{[\tau]}_{r,\varrho,\theta}((s-u)^2;u)\right)\right| \\
&\qquad\leq \frac{C}{r}\vartheta(u)\left(\|\zeta''-f\|+r^{-1/2}\|\vartheta f'\|\right).
\end{aligned} \tag{23}$$

Taking $\inf_{f\in W_{\vartheta}[0,1]}$ in (23), we obtain

$$\begin{aligned}
&\left|r\left(\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(\zeta;u)-\zeta(u)-\zeta'(u)\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(s-u;u)-\frac{1}{2}\zeta''(u)\mathscr{B}^{[\tau]}_{r,\varrho,\theta}((s-u)^2;u)\right)\right| \\
&\qquad\leq \begin{cases} C\overline{K}_{\vartheta}\left(\zeta''(u),\vartheta(u)r^{-1/2}\right) \\ C\vartheta(u)\overline{K}_{\vartheta}\left(\zeta''(u),r^{-1/2}\right). \end{cases}
\end{aligned}$$

Applying $\overline{K_{\vartheta}}(\zeta,s) \sim \omega_{\vartheta}(\zeta,s)$, the theorem is proved. □

4. Rate of Convergence

We shall use the symbol $DBV(0,1)$ to denote the class of all absolutely continuous functions ζ on $[0,1]$ and having a derivative $\zeta^{'}$ on $(0,1)$, which is equivalent to a function of bounded variation (BV) on $[0,1]$. For $\zeta \in DBV(0,1)$, we have

$$\zeta(x) = \int_0^x f(s)ds + \zeta(0),$$

where f is a function of BV on $[0,1]$.

Lemma 6. *Let $u \in (0,1]$. Then, for sufficiently large r and $\theta \geq 1$, we obtain*

(i) $\eta^{[\tau]}_{r,\varrho,\theta}(u,y) = \int_0^y \mathscr{K}^{[\tau]}_{r,\varrho,\theta}(u,s)ds \leq \frac{\theta}{(u-y)^2}\frac{\mathcal{C}^{[\tau]}_{\varrho}}{1+n\varrho}\vartheta^2(u)\,,\ 0 \leq y < u,$

(ii) $1-\eta^{[\tau]}_{r,\varrho,\theta}(u,z) = \int_z^1 \mathscr{K}^{[\tau]}_{r,\varrho,\theta}(u,s)ds \leq \frac{\theta}{(z-u)^2}\frac{\mathcal{C}^{[\tau]}_{\varrho}}{1+r\varrho}\vartheta^2(u),\ u < z < 1.$

Theorem 4. *Let $\zeta \in DBV(0,1), \theta \geq 1$ and also let $\bigvee_a^b(\zeta'_u)$ be the total variation of ζ'_u on $[a,b] \subset [0,1]$. Then*

$$\begin{aligned}\left|\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(\zeta;u)-\zeta(u)\right| \leq\ & \frac{1}{\theta+1}\left|\zeta'(u+)+\theta\zeta'(u-)\right|\sqrt{\frac{\theta\mathcal{C}^{[\tau]}_{\varrho}}{1+r\varrho}}\,\vartheta(u)\\ &+\sqrt{\frac{\theta\mathcal{C}^{[\tau]}_{\varrho}}{1+r\varrho}}\,\vartheta(u)\frac{\theta}{\theta+1}\left|\zeta'(u+)-\zeta'(u-)\right|\\ &+\theta\frac{\mathcal{C}^{[\tau]}_{\varrho}(1-u)}{(1+r\varrho)}\sum_{i=1}^{[\sqrt{r}\,]}\bigvee_{u-(u/i)}^{u}(\zeta'_u)+\frac{u}{\sqrt{r}}\bigvee_{u-(u/\sqrt{r})}^{u}(\zeta'_u)\\ &+\theta\frac{\mathcal{C}^{[\tau]}_{\varrho}u}{(1+r\varrho)}\sum_{i=1}^{[\sqrt{r}\,]}\bigvee_{u}^{u+((1-u)/i)}(\zeta'_u)+\frac{1-u}{\sqrt{r}}\bigvee_{u}^{u+((1-u)/\sqrt{r})}(\zeta'_u)\end{aligned}$$

for every $u \in (0,1)$ and for sufficiently large r, where $\mathcal{C}^{[\tau]}_{\varrho} > 0$ and the auxiliary function ζ'_u is defined by

$$\zeta'_u(s) = \begin{cases} \zeta'(s)-\zeta'(u-), & 0 \leq s < u \\ 0, & s = u \\ \zeta'(s)-\zeta'(u+) & u < s \leq 1. \end{cases} \tag{24}$$

Proof. Using the fact that $\int_0^1 \mathscr{K}^{[\tau]}_{r,\varrho,\theta}(u,s)ds = \mathscr{B}^{[\tau]}_{r,\varrho,\theta}(e_0;u) = 1$, we obtain

$$\begin{aligned}\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(\zeta;u)-\zeta(u) &= \int_0^1[\zeta(s)-\zeta(u)]\mathscr{K}^{[\tau]}_{r,\varrho,\theta}(u,s)ds\\ &= \int_0^1\left(\int_u^s\zeta'(w)dw\right)\mathscr{K}^{[\tau]}_{r,\varrho,\theta}(u,s)ds.\end{aligned} \tag{25}$$

Using (24), we may write

$$\begin{aligned}\zeta'(s) =\ & \frac{1}{\theta+1}\left(\zeta'(u+)+\theta\zeta'(u-)\right)+\zeta'_u(s)\\ &+\frac{1}{2}\left(\zeta'(u+)-\zeta'(u-)\right)\left(sgn(s-u)+\frac{\theta-1}{\theta+1}\right)\end{aligned}$$

$$+\delta_u(s)\left(\zeta'(u)-\frac{1}{2}\left(\zeta'(u+)+\zeta'(u-)\right)\right), \tag{26}$$

where

$$\delta_u(s)=\begin{cases} 1\ ,\ u=s \\ 0\ ,\ u\neq s. \end{cases}$$

It is clear that

$$\int_0^1 \mathscr{K}^{[\tau]}_{r,\varrho,\theta}(u,s)\int_u^s\left[\zeta'(u)-\frac{1}{2}\left(\zeta'(u+)+\zeta'(u-)\right)\right]\delta_u(s)duds=0.$$

From (10) and straightforward computations, we have

$$\begin{aligned} E_1 &= \int_0^1\left(\int_u^s\frac{1}{\theta+1}\left(\zeta'(u+)+\theta\zeta'(u-)\right)dw\right)\mathscr{K}^{[\tau]}_{r,\varrho,\theta}(u,s)ds \\ &= \frac{1}{\theta+1}\left|\zeta'(u+)+\theta\zeta'(u-)\right|\int_0^1|s-u|\mathscr{K}^{[\tau]}_{r,\varrho,\theta}(u,s)ds \\ &\leq \frac{1}{\theta+1}\left|\zeta'(u+)+\theta\zeta'(u-)\right|\left(\mathscr{B}^{[\tau]}_{r,\varrho,\theta}((s-u)^2;u)\right)^{1/2} \end{aligned}$$

and

$$\begin{aligned} E_2 &= \int_0^1\left(\int_u^s\frac{1}{2}\left(\zeta'(u+)-\zeta'(u-)\right)\left(sgn(w-u)+\frac{\theta-1}{\theta+1}\right)dw\right)\mathscr{K}^{[\tau]}_{r,\varrho,\theta}(u,s)ds \\ &= \frac{1}{2}(\zeta'(u+)-\zeta'(u-))\left[-\int_0^u\left(\int_s^u\left(sgn(w-u)+\frac{\theta-1}{\theta+1}\right)dw\right)\mathscr{K}^{[\tau]}_{r,\varrho,\theta}(u,s)ds\right. \\ &\quad \left.+\int_u^1\left(\int_u^s\left(sgn(w-u)+\frac{\theta-1}{\theta+1}\right)dw\right)\mathscr{K}^{[\tau]}_{r,\varrho,\theta}(u,s)ds\right] \\ &\leq \frac{\theta}{\theta+1}(\zeta'(u+)-\zeta'(u-))\int_0^1|s-u|\mathscr{K}^{[\tau]}_{r,\varrho,\theta}(u,s)ds \\ &= \frac{\theta}{\theta+1}(\zeta'(u+)-\zeta'(u-))\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(|s-u|;u) \\ &\leq \frac{\theta}{\theta+1}(\zeta'(u+)-\zeta'(u-))\left(\mathscr{B}^{[\tau]}_{r,\varrho,\theta}((s-u)^2;u)\right)^{1/2}. \end{aligned}$$

By using Lemma 3 and considering (25) and (26), we obtain

$$\begin{aligned} \left|\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(\zeta;u)-\zeta(u)\right| \leq & \left|H^{[\tau]}_{r,\varrho,\theta}(\zeta'_u,u)+L^{[\tau]}_{r,\varrho,\theta}(\zeta'_u,u)\right| \\ & +\tfrac{1}{\theta+1}|\zeta'(u+)+\theta\zeta'(u-)|\sqrt{\frac{\theta C^{[\tau]}_{\varrho}}{1+r\varrho}}\,\vartheta(u) \\ & +\tfrac{\theta}{\theta+1}|\zeta'(u+)-\zeta'(u-)|\sqrt{\frac{\theta C^{[\tau]}_{\varrho}}{1+r\varrho}}\,\vartheta(u), \end{aligned} \tag{27}$$

where

$$H^{[\tau]}_{r,\varrho,\theta}(\zeta'_u,u)=\int_0^u\left(\int_u^s\zeta'_u(w)dw\right)\mathscr{K}^{[\tau]}_{r,\varrho,\theta}(u,s)ds$$

and

$$L^{[\tau]}_{r,\varrho,\theta}(\zeta'_u,u)=\int_u^1\left(\int_u^s\zeta'_u(w)dw\right)\mathscr{K}^{[\tau]}_{r,\varrho,\theta}(u,s)ds.$$

We now estimate the terms $H^{[\tau]}_{r,\varrho,\theta}(\zeta'_u, u), L^{[\tau]}_{r,\varrho,\theta}(\zeta'_u, u)$. Since

$$\int_a^b d_s \eta^{[\tau]}_{r,\varrho,\theta}(u,s) \leq 1 \qquad (\forall [a,b] \subseteq [0,1]),$$

applying basic properties of integration and Lemma 6 with $y = u - (u/\sqrt{r})$, we obtain

$$\begin{aligned}
\left| H^{[\tau]}_{r,\varrho,\theta}(\zeta'_u, u) \right| &= \left| \int_0^u \left(\int_u^s \zeta'_u(w) dw \right) d_s \eta^{[\tau]}_{r,\varrho,\theta}(u,s) \right| \\
&= \left| \int_0^u \eta^{[\tau]}_{r,\varrho,\theta}(u,s) \zeta'_u(s) ds \right| \\
&\leq \left(\int_0^y + \int_y^u \right) |\zeta'_u(s)| \left| \eta^{[\tau]}_{r,\varrho,\theta}(u,s) \right| ds \\
&\leq \theta \frac{\mathcal{C}^{[\tau]}_\varrho}{1 + r\varrho} \vartheta^2(u) \int_0^y \bigvee_s^u (\zeta'_u)(u-s)^{-2} ds + \int_y^u \bigvee_s^u (\zeta'_u) ds \\
&\leq \theta \frac{\mathcal{C}^{[\tau]}_\varrho}{1 + r\varrho} \vartheta^2(u) \int_0^y \bigvee_s^u (\zeta'_u)(u-s)^{-2} ds + \frac{u}{\sqrt{r}} \bigvee_{u-(u/\sqrt{r})}^u (\zeta'_u).
\end{aligned}$$

Setting $w = u/(u-s)$, we obtain

$$\begin{aligned}
\theta \frac{\mathcal{C}^{[\tau]}_\varrho}{1 + r\varrho} \vartheta^2(u) \int_0^{u-(u/\sqrt{r})} (u-s)^{-2} \bigvee_s^u (\zeta'_u) ds &= \theta \frac{\mathcal{C}^{[\tau]}_\varrho (1-u)}{1 + r\varrho} \int_1^{\sqrt{r}} \bigvee_{u-(u/w)}^u (\zeta'_u) dw \\
&\leq \theta \frac{\mathcal{C}^{[\tau]}_\varrho (1-u)}{1 + r\varrho} \sum_{i=1}^{[\sqrt{r}]} \int_i^{i+1} \bigvee_{u-(u/w)}^u (\zeta'_u) dw \\
&\leq \theta \frac{\mathcal{C}^{[\tau]}_\varrho (1-u)}{1 + r\varrho} \sum_{i=1}^{[\sqrt{r}]} \bigvee_{u-(u/i)}^u (\zeta'_u).
\end{aligned}$$

Hence,

$$\left| H^{[\tau]}_{r,\varrho,\theta}(\zeta'_u, u) \right| \leq \theta \frac{\mathcal{C}^{[\tau]}_\varrho (1-u)}{1 + r\varrho} \sum_{i=1}^{[\sqrt{r}]} \bigvee_{u-(u/i)}^u (\zeta'_u) + \frac{u}{\sqrt{r}} \bigvee_{u-(u/\sqrt{r})}^u (\zeta'_u). \tag{28}$$

Applying the basic property of integration and using Lemma 6 with $z = u + ((1-u)/\sqrt{r})$, we obtain

$$\begin{aligned}
|L^{[\tau]}_{r,\varrho,\theta}(\zeta'_u, u)| &= \left| \int_u^1 \left(\int_u^s \zeta'_u(w) dw \right) \mathscr{K}^{[\tau]}_{r,\varrho,\theta}(u,s) ds \right| \\
&= \left| \int_u^z \left(\int_u^s \zeta'_u(w) dw \right) d_s (1 - \eta^{[\tau]}_{r,\varrho,\theta}(u,s)) \right. \\
&\quad \left. + \int_z^1 \left(\int_u^s \zeta'_u(w) dw \right) d_s (1 - \eta^{[\tau]}_{r,\varrho,\theta}(u,s)) \right| \\
&= \left| \left[\left(\int_u^t f'_u(w) dw \right) (1 - \eta^{[\tau]}_{r,\varrho,\theta}(u,s)) \right]_u^z - \int_u^z f'_u(s)(1 - \eta^{[\tau]}_{r,\varrho,\theta}(u,s)) ds \right. \\
&\quad \left. + \int_z^1 \left(\int_u^s \zeta'_u(w) dw \right) d_s (1 - \eta^{[\tau]}_{r,\varrho,\theta}(u,s)) \right| \\
&= \left| \int_u^z f'_u(w) dw (1 - \eta^{[\tau]}_{r,\varrho,\theta}(u,z)) - \int_u^z f'_u(s)(1 - \eta^{[\tau]}_{r,\varrho,\theta}(u,s)) ds \right.
\end{aligned}$$

$$
\begin{aligned}
&+\left[\int_u^t f_u'(w)dw(1-\eta^{[\tau]}_{r,\varrho,\theta}(u,s))\right]_z^1 - \int_z^1 \zeta_u'(s)(1-\eta^{[\tau]}_{r,\varrho,\theta}(u,s))ds\Bigg| \\
= &\left|\int_u^z f_u'(s)(1-\eta^{[\tau]}_{r,\varrho,\theta}(u,s))ds + \int_z^1 \zeta_u'(s)(1-\eta^{[\tau]}_{r,\varrho,\theta}(u,s))ds\right| \\
< &\ \theta\frac{\mathcal{C}^{[\tau]}_{\varrho}}{1+r\varrho}\vartheta^2(u)\int_z^1 \bigvee_u^s(\zeta_u')(s-u)^{-2}ds + \int_u^z\bigvee_u^s(\zeta_u')ds \\
\leq &\ \theta\frac{\mathcal{C}^{[\tau]}_{\varrho}}{1+r\varrho}\vartheta^2(u)\int_{u+((1-u)/\sqrt{r})}^1 \bigvee_u^s(\zeta_u')(s-u)^{-2}ds + \frac{(1-u)}{\sqrt{r}}\bigvee_u^{u+((1-u)/\sqrt{r})}(\zeta_u').
\end{aligned}
$$

Setting $w=(1-u)/(s-u)$, we obtain

$$
\begin{aligned}
\theta\frac{\mathcal{C}^{[\tau]}_{\varrho}}{1+r\varrho}\vartheta^2(u)\int_{u+((1-u)/\sqrt{r})}^1 \bigvee_u^s(\zeta_u')(s-u)^{-2}ds &= \theta\frac{\mathcal{C}^{[\tau]}_{\varrho}u}{(1+r\varrho)}\int_1^{\sqrt{r}}\bigvee_u^{u+((1-u)/w)}(\zeta_u')dw \\
&< \theta\frac{\mathcal{C}^{[\tau]}_{\varrho}u}{(1+r\varrho)}\sum_{i=1}^{[\sqrt{r}]}\int_i^{i+1}\bigvee_u^{u+((1-u)/w)}(\zeta_u')dw \\
&\leq \theta\frac{\mathcal{C}^{[\tau]}_{\varrho}u}{(1+n\varrho)}\sum_{i=1}^{[\sqrt{r}]}\bigvee_u^{u+((1-u)/i)}(\zeta_u').
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
|L^{[\tau]}_{r,\varrho,\theta}(\zeta_u',u)| &\leq \theta\frac{\mathcal{C}^{[\tau]}_{\varrho}u}{(1+r\varrho)}\sum_{i=1}^{[\sqrt{r}]}\bigvee_u^{u+((1-u)/i)}(\zeta_u') \\
&\quad + \frac{1-u}{\sqrt{r}}\bigvee_u^{u+((1-u)/\sqrt{r})}(\zeta_u').
\end{aligned} \tag{29}
$$

Combining (27)–(29), we obtain the estimate. The theorem is proved. □

5. Conclusions

In our investigations, we first constructed summation-integral-type operators that include a Pólya–Eggenberger distribution and Bézier basis functions by

$$
\mathscr{B}^{[\tau]}_{r,\varrho,\theta}(\zeta;u)=\sum_{i=0}^{r}\Big([H_{r,i}(u)]^{\theta}-[H_{r,i+1}(u)]^{\theta}\Big)F^{\varrho}_{r,i}(\zeta). \tag{30}
$$

We then discussed an interesting direct theorem and a quantitative Voronovskaja-type theorem by taking into account the Ditzian–Totik modulus of smoothness for summation-integral-type operators (30). In the last section, we discussed the approximation of functions involving the idea of DBV of our operators. In this paper, we presented the modification of the operators (7) by taking into account the idea of Bézier basis functions so the constructed operators (30) are stronger than (7), as are our results. It is worth mentioning that one can study the asymptotic expansion of sequences of the aforementioned summation-integral-type operators in the multivariate case.

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