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Inference for the Two Parameter Reduced Kies Distribution under Progressive Type-II Censoring

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Abstract: In this paper, we obtained several recurrence relations for the single and product moments under progressively Type-II right censored order statistics and then use these results to compute the means and variances of two parameter reduced Kies distribution. Besides, these moments are then utilized to derived best linear unbiased estimators of the scale and location parameters of two parameter reduced Kies distribution. The parameters of the two parameter reduced Kies distribution are estimated under progressive type-II censoring scheme. The model parameters are estimated using the maximum likelihood estimation method. Further, we explore the asymptotic confidence intervals for the model parameters. Monte Carlo simulations are performed to compare between the proposed estimation methods under progressive type-II censoring scheme. Based on our study, we can conclude that maximum likelihood estimators is decreasing with respect to an increase of the schemes and comparing the three censoring schemes, it is clear that the mean sum of squares, confidence interval lengths are smaller for scheme 1 than schemes 2 and 3.

Keywords: progressive type-II censoring; moments; recurrence relations; reduced Kies distribution; maximum likelihood estimation; best linear unbiased estimator

1. Introduction

The one parameter reduced Kies (RK) distribution was introduced by Kumar and Dharmaja [1] for modeling data and a generalization of Kies distribution. The two parameter RK distribution is a flexible model which provides left-skewed, symmetrical, right-skewed, and reversed-J shaped densities (see Figure 1). Its hazard rate function (HRF) can provide decreasing, increasing, upside-down bathtub, bathtub, and reversed-J shaped hazard rates (see Figure 2). It is noted that the bathtub and modified bathtub hazard rates are very important in the reliability engineering context. John et al. [2] investigated modified-bathtub hazard rate shape is widely used in industrial and medical applications. For example, thermal stress screening is an assembly-level electronics manufacturing process that evolved from the burn-in processes used in NASA and DoD programs. While burn-in subjects the product to expected field extremes to expose infant mortalities (patent failures), thermal stress screening briefly exposes a product to fast temperature rate-of-change and out-of-spec temperatures to trigger failures that would otherwise occur during the useful life of the product. Also Xie and Lai [3], Lai et al. [4], Chakherloo et al. [5] and Al abbasi et al. [6] pointed out bathtub hazard rate shape is widely used in reliability engineering. The motivation for using this distribution here is that it has many applications in several areas of life such as accelerated life testing, survival analysis, reliability, biology, material science, engineering, physics, chemistry, economics, business administration, meteorology, hydrology, medicine, psychology and pharmacy. For a detailed account in this regard see Murthy et al. [7] or Rinne [8] and references therein. RK distribution can be viewed as a functional form of the Weibull distribution with shape parameter λ and it can be useful for modeling data sets with increasing and bathtub shaped hazard rate functions. Simple probability distributions generally do not exhibit bathtub-shaped failure rate, including Weibull, gamma, and log-normal. In most cases, bathtub shaped hazard functions have at least two parameters, whereas reduced Kies distribution has two parameter which exhibit both increasing and bathtub shaped hazard rate. In Engineering and Medical situations, Kumar and Dharmaja [1] observed that RK distribution is a better model compared to the Weibull as well as its extended models such as beta Weibull distribution, beta generalised Weibull distribution etc. in terms of hazard function is decreasing, increasing and bathtub shaped where Weibull models are inappropriate. Kumar and Dharmaja [1] studied the estimation of the parameters by using maximum likelihood estimation method of the reduced Kies distribution. Kumar and Dharmaja [9] considered the estimation of the Kies parameters under maximum likelihood estimation method. Dey et al. [10] studied the estimation of the reduced Kies parameter under progressive type-II censoring. They compared the performance of these estimators, for small and large samples, using extensive simulations. The only paper we were able to find on progressive type-II censoring of the one parameter RK distribution is Dey et al. [10]. This paper gives recurrence relations for single moments and product moments of progressive type-II censoring order statistics based on one parameter RK distribution. It did not consider two parameter RK distribution.

Let $Y_1, Y_2, \ldots, Y_n$ be a random variable come from a two parameter RK distribution, then its non negative probability density function (pdf) and cumulative distribution function (cdf) are given as follows:

$$f(y;\lambda,\mu) = \lambda(y-\mu)^{\lambda-1}[1-(y-\mu)]^{-\lambda-1}\, e^{-\left(\frac{y-\mu}{1-(y-\mu)}\right)^{\lambda}},\ y>\mu,\ \lambda>0 \tag{1}$$

and

$$F(y;\lambda,\mu) = 1-e^{-\left(\frac{y-\mu}{1-(y-\mu)}\right)^{\lambda}},\ y>\mu>0,\ \lambda>0 \tag{2}$$

Here λ is a shape parameter and μ is a location parameter. From Equations (1) and (2), we obtain

$$f(y) = \lambda \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{\mu^b\,(\lambda+1)_a}{a!}\, y^{\lambda+a-1-b}\, \bar{F}(y), \tag{3}$$

where $(\xi)_c = \xi(\xi+1), \cdots, (\xi+c-1)$ denotes the ascending factorial.

In Figures 1 and 2, various graphs of the pdf and the hazard rate function for the two parameter RK distribution for different parameters values. These plots show that the pdf is uni-modal, positively skewed and approximately symmetric. The plots in Figure 2 indicate that the hazard rate function for the two parameter RK distribution is very flexible. It can have increasing (IFR), decreasing (DFR), upside down bathtub (UBT) or bathtub (BT) failure rate functions.

Let the experimenter decides to carry out the life-test for a pre-fixed length of time, say T. Then the data arising from such a time-constrained life-test would be of the form $Y_{1:s} \leq \cdots \leq Y_{r:s}$ with the remaining $s-s$ lifetimes being more than T; here, r is random $(0 \leq r \leq s)$ and has a binomial distribution with parameters $(s, F(T))$. This situation is referred to as Type-I censoring. Suppose the experimenter decides to carry out the life-test until the time of the r^{th} failure, then the data arising from such a life-test would be of the form $Y_{1:s} \leq \cdots \leq Y_{r:s}$ with the remaining $s-r$ lifetimes being more than $Y_{r:s}$. This situation is referred to as Type-II censoring see Balakrishnan [11].

In life testing experiments, it is common to come across incomplete or censored data. This happens particularly when the experimenter does not observe the failure times of all units placed on the life test and this may be intentional or unintentional or may be due to time constraints or owing to the structure of a technical system. Obviously, in such a situation, the probabilistic structure of the resulting incomplete data affects the censoring mechanism and therefore suitable inferential procedures become necessary. In literature, there are various censoring schemes which include right, left and interval censoring, single or multiple censoring and type-I or type-II censoring. However, classical Type-I and Type-II censoring schemes are not flexibile as they do not allow removal of units at point other than the terminal point of the experiment. A mixture of type-I and type-II schemes is known as the hybrid censoring scheme. For this reason, we consider here a more general censoring scheme called progressive type-II censoring scheme.

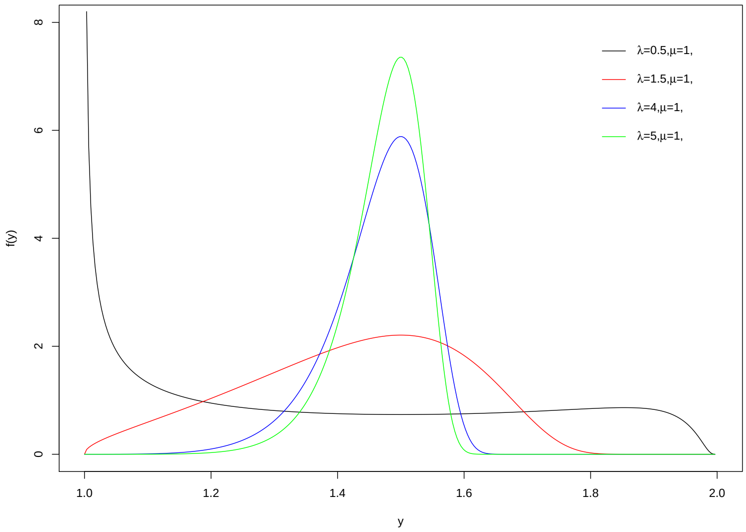

Figure 1. The pdfs of two parameter RK distribution for various parameter values.

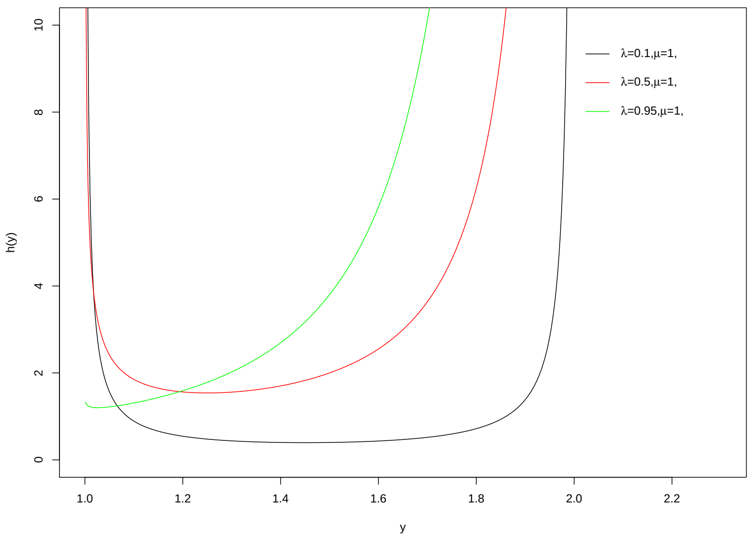

Figure 2. The hazard rate functions of two parameter RK distribution for various parameter values.

If the failure times are based continuous cdf $F(y)$ with pdf $f(y)$, the joint pdf of the progressively censored failure times $Y_{1:r:s}, Y_{2:r:s}, \cdots, Y_{r:r:s}$, is given by Balakrishnan and Aggarwala [12].

$$f_{Y_{1:r:s},Y_{2:r:s},\cdots,Y_{r:r:s}}(y_1, y_2, \cdots, y_r) = \Delta(s, r-1) \prod_{i=1}^{r} f(y_i)[1 - F(y_i)]^{T_i} \\ , -\infty < y_1 < y_2 < \cdots < y_r < \infty, \quad (4)$$

where

$$\Delta(s, r-1) = s(s - T_1 - 1) \cdots (s - T_1 - T_2 - \cdots - T_{r-1} - r + 1), \quad (5)$$

with $\Delta(s,0) = s$. Here T is the progressive censoring scheme, $T_1, T_2, \ldots, T_r$ are numbers which are prefixed, s is the number of units we put on the life testing experiment and r is the predetermined number of failures at which experiment will be terminated.

Let $y_1, y_2, \cdots, y_s$ be a random sample of size s from the two parameter RK distribution with pdf and cdf given in (1) and (2) respectively. The corresponding progressive Type-II right censored order statistics with censoring scheme $(T_1, T_2, \cdots T_r)$, $r \leq s$ will be

$$Y_{1:r:s}^{(T_1,T_2,\ldots,T_r)}, Y_{2:r:s}^{(T_1,T_2,\ldots,T_r)}, \ldots, Y_{r:r:s}^{(T_1,T_2,\ldots,T_r)}.$$

Let us define the single moments of the progressive Type-II right censored order statistics

$$\begin{aligned} \alpha_{i:r:s}^{(T_1,T_2,\ldots,T_r)^{(k)}} &= E\left[Y_{i:r:s}^{(T_1,T_2,\ldots,T_r)^{(k)}}\right] \\ &= \Delta(s,r-1)\int\int\cdots\int_{0<y_1<y_2<\cdots<y_r<\infty} y_i^k f(y_1) \\ &\times [1-F(y_1)]^{T_1} f(y_2)[1-F(y_2)]^{T_2} f(y_3)[1-F(y_3)]^{T_3}\ldots f(y_r) \\ &\times [1-F(y_r)]^{T_r} dy_1 dy_2 dy_3 \ldots dy_r, \end{aligned} \tag{6}$$

In the last few decades, researchers have focused their attention to recurrence relation for moments of progressive type-II censoring. Many researchers considered moments of progressive type-II censoring in their studies. For example, Aggarwala and Balakrishnan [13] studied censored order statistics of a exponential and truncated exponential distribution. Balakrishnan et al. [14] discussed the inference under progressive type-II censoring of extreme value distribution. Fernandez [15] discussed the information of estimate the parameter of exponential distribution. With regard to progressive type-II censoring order statistics, readers may refer to the works of Cohen [16] discussed in progressively censored samples in life testing experiments. Viveros and Balakrishnan [17] obtained the interval estimation of life characteristics under progressively censored data. Balakrishnan and Aggarwala [18] discussed in details the progressive Censoring including theory, method and applications. Mahmoud et al. [19] studied the parameters estimation of linear exponential distribution under Progressively censored data. Sultan et al. [20] discussed the moments and estimation of parameters of the half logistic distribution based on progressively censored data, Balakrishnan et al. [21] obtained relations for moments of progressively censored order statistics from logistic distribution. Balakrishnan and Saleh [22] discussed relations for single and product moments of progressively Type-II censored order statistics from a generalized half logistic distribution. Dey et al. [23] discussed the estimation of parameters of Rayleigh distribution under progressively Type-II censored data. Kumar et al. [24] obtained the moments of extended exponential distribution under order statistics. Malik and Kumar [25] studied moments of progressively type-II Right censored order statistics from Erlang-truncated exponential distribution. Hu and Gui [26] discussed Bayesian and Non-Bayesian inference for the generalized Pareto distribution based on Progressive Type II Censored Sample. Malik and Kumar [27] obtained the moments of exponential-Weibull distribution based on progressively censored data. Singh and Khan [28] discussed the moments of progressively type-II right censored order statistics from additive Weibull distribution. Kumar et al. [29] studied the moments and estimation of parameters of extended exponential distribution based on progressive type-II right censored order statistics and Kumar et al. [30] considered estimation of the location and scale parameters of generalized Pareto distribution based on progressively type-II censored order statistics.

The key role of this article is two fold: first, we derive recurrence relations for the single and product moments of the RK distribution based on progressive type-II right censored order statistics. The so-obtained relationships enable us to compute all these moments for all sample sizes and all possible censoring schemes, using some mathematical softwares (Mathematica, Maple), second, we discuss the maximum likelihood estimators (MLEs) and BLUEs of the scale and location-scale parameters and compare them on the basis of bias and mean squared errors.

The rest of the paper is organized as follows. Relations for single moments is presented in Section 2. The relations for double (product) moments are given in Section 3. Parameter estimation along with approximate confidence intervals are computed in Section 4. In Section 5, the potentiality of the estimation approaches is assessed via simulation results. Finally, some remarks are offered in Section 7.

2. Relations for Single Moments

Here, we obtain some relations for the moments of progressive type-II right censored order statistics from the two parameters reduced Kies distribution.

Theorem 1. *For* $2 \leq r \leq s$ *and* $k \geq 0$,

$$\begin{aligned}
\alpha_{1:r:s}^{(T_1,T_2,\cdots,T_r)^{(k)}} &= \lambda \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{\mu^b(\lambda+1)_a}{a!(k+\lambda+a-b)} \\
&\times \left[(s-T_1-1)\alpha_{1:r-1:s}^{(T_1+1+T_2,\cdots,T_r)^{(k+\lambda+a-b)}} + (1+T_1)\alpha_{1:r:s}^{(T_1,T_2,\cdots,T_r)^{(k+\lambda+a-b)}} \right]. \qquad (7)
\end{aligned}$$

Proof. We have, from Equations (3) and (6)

$$\begin{aligned}
\alpha_{1:r:s}^{(T_1,T_2,\cdots,T_r)^{(k)}} &= \Delta(s,r-1) \int\int\cdots\int_{0<y_1<y_2<\cdots<y_r<\infty} \\
&\times \Psi(y_2)f(y_2)[1-F(y_2)]^{T_2} f(y_3)[1-F(y_3)]^{T_3}\cdots f(y_r) \\
&\times [1-F(y_r)]^{T_r} dy_2 dy_3 \cdots dy_r, \qquad (8)
\end{aligned}$$

where

$$\begin{aligned}
\Psi(y_2) &= \int_0^{y_2} y_1^k f(y_1)[1-F(y_1)]^{T_1} dy_1 \\
&= \int_0^{y_2} y_1^k \left\{ \lambda \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{\mu^b(\lambda+1)_a\, y^{\lambda+a-1-b}}{a!} [1-F(y_1)] \right\} [1-F(y_1)]^{T_1} dy_1 \\
&= \lambda \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{\mu^b(\lambda+1)_a}{a!} \int_0^{y_2} y_1^{k+\lambda+a-b-1}[1-F(y_1)]^{T_1+1} dy_1. \qquad (9)
\end{aligned}$$

Integrating (9) by parts, we obtain

$$\begin{aligned}
&= \lambda \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{\mu^b(\lambda+1)_a}{a!(k+\lambda+a-b)} \Big[[1-F(y_2)]^{T_1+1} y_2^{k+\lambda+a-b} \\
&+ (T_1+1) \int_0^{y_2} y_1^{k+\lambda+a-b}[1-F(y_1)]^{T_1} f(y_1) dy_1 \Big]. \qquad (10)
\end{aligned}$$

Using Equations (6) and (10) the Equation (8) can be rewritten as

$$\begin{aligned}
\alpha_{1:r:s}^{(T_1,T_2,\cdots,T_r)^{(k)}} &= \lambda \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{\mu^b(\lambda+1)_a}{a!(k+\lambda+a-b)} \Big[\int\int\cdots\int y_2^{k+\lambda+a-b}(1-F(y_2))^{T_1+1} \\
&\times f(y_2)(1-F(y_2))^{T_2}\cdots f(y_r)(1-F(y_r))^{T_r} + (1+T_1)\alpha_{1:r:s}^{(T_1,T_2,\ldots,T_r)^{(k+\lambda+a-b)}} \Big] \\
&= \lambda \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{\mu^b(\lambda+1)_a}{a!(k+\lambda+a-b)} \\
&\times \left[(s-T_1-1)\alpha_{1:r-1:s}^{(T_1+1+R_2,\ldots,T_r)^{(k+\lambda+a-b)}} + (1+T_1)\alpha_{1:r:s}^{(T_1,T_2,\ldots,T_r)^{(k+\lambda+a-b)}} \right],
\end{aligned}$$

hence the result. □

Theorem 2. *For $r = 1, s = 1, 2, \ldots$ and $k \geq 0$,*

$$\alpha_{1:1:s}^{(s-1)^{(k)}} = s\lambda \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+s-1}{b} \frac{\mu^b (\lambda+1)_a}{a!(k+\lambda+a-b)} \alpha_{1:1:s}^{(s-1)^{(k+\lambda+a-b)}}. \tag{11}$$

Proof. Similar to the proof of Theorem 1. □

Theorem 3. *For $2 \leq i \leq r-1$, $r \leq s$ and $k \geq 0$,*

$$\begin{aligned} \alpha_{i:r:s}^{(T_1,T_2,\cdots,T_r)^{(k)}} &= \lambda \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{\mu^b (\lambda+1)_a}{a!(k+\lambda+a-b)} \Big[(s - T_1 - T_2 - \ldots - T_i - i) \\ &\times \alpha_{i:r-1:s}^{(T_1,T_2,\ldots,T_{i-1},T_i+T_{i+1}+1,T_{i+2},\cdots,T_r)^{(k+\lambda+a-b)}} + (1+T_i)\alpha_{i:r:s}^{(T_1,T_2,\cdots,T_r)^{(k+\lambda+a-b)}} \\ &- (s - T_1 - T_2 - \ldots - T_{i-1} - i + 1) \\ &\times \alpha_{i-1:r-1:s}^{(T_1,T_2,\ldots,T_{i-2},T_{i-1}+T_i+1,T_{i+1},\ldots,T_r)^{(k+\lambda+a-b)}} \Big]. \end{aligned} \tag{12}$$

Proof. Similar to the proof of Theorem 1. □

Theorem 4. *For $2 \leq r \leq s$, and $k \geq 0$,*

$$\begin{aligned} \alpha_{r:r:s}^{(T_1,T_2,\ldots,T_r)^{(k)}} &= \lambda\,(1+T_r) \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{\mu^b (\lambda+1)_a}{a!(k+\lambda+a-b)} \Big[\alpha_{r:r:s}^{(T_1,T_2,\ldots,T_r)^{(k+\lambda+a-b)}} \\ &- \alpha_{r-1:r-1:s}^{(T_1,T_2,\ldots,T_{r-2},T_{r-1}+T_r+1,T_{i+1},\cdots,T_r)^{(k+\lambda+a-b)}} \Big]. \end{aligned} \tag{13}$$

Proof. Similar to the proof of Theorem 1. □

Special cases For $T_1 = T_2 = \cdots = T_r = 0$ this implies that $r = s$ then the progressive censored order statistics reduced to the order statistics $Y_{1:s}, Y_{2:s}, \cdots, Y_{s:s}$, then

1. For $k \geq 0$, then Equation (7), we obtain

$$\begin{aligned} \alpha_{1:s}^{(k)} &= \lambda \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{\mu^b (\lambda+1)_a}{a!(k+\lambda+a-b)} \\ &\times \Big[\alpha_{1:s}^{k} + (s-1)\alpha_{1:s-1:s}^{(1,0,0,\ldots,0)^{(k+\lambda+a-b)}} \Big]. \end{aligned} \tag{14}$$

2. For $k \geq 0$, then Equation (12), we obtain

$$\begin{aligned} \alpha_{i:s}^{(k)} &= \lambda \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{\mu^b (\lambda+1)_a}{a!(k+\lambda+a-b)} \\ &\times \Big[\alpha_{i:s}^{(k+\lambda+a-b)} + (s-i)\alpha_{i:s}^{(k+\lambda+a)} - (s-i+1)\alpha_{i-1:s}^{(k+\lambda+a-b)} \Big]. \end{aligned} \tag{15}$$

3. Relations for Product Moments

Here, we present the relations for product moments of the progressive type-II right censored order statistics from the two parameters reduced Kies distribution. The (i, j) th product moment of the progressive type-II right censored order statistics can be written as

$$
\begin{aligned}
\alpha_{i,j:r:s}^{(T_1,T_2,\ldots,T_r)} &= E\left[y_{i:r:s}^{(T_1,T_2,\ldots,T_r)} y_{j:r:s}^{(T_1,T_2,\ldots,T_r)}\right] \\
&= \Delta(s,\, r-1) \int\int \ldots \int_{0<y_1<y_2<\cdots<y_r<\infty} y_i y_j f(y_1)[1-F(y_1)]^{T_1} f(y_2) \\
&\times \; [1-F(y_2)]^{T_2} \ldots f(y_r)[1-F(y_r)]^{T_r} dy_1 dy_2 dy_3 \ldots dy_r.
\end{aligned} \tag{16}
$$

Theorem 5. *For $1 \le i < j \le r-1$ and $r \le s$,*

$$
\begin{aligned}
\alpha_{i:r:s}^{(T_1,T_2,\ldots,T_r)} &= \lambda(T_j+1) \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{\mu^b (\lambda+1)_a}{a!(\lambda+a-b)} \Bigg[\alpha_{i,j:r:s}^{(T_1,T_2,\ldots,T_r)^{(1,\,\lambda+a-b)}} \\
&+ \; (s-T_1-1-\cdots-T_j-j)\alpha_{i,j:r-1:s}^{(T_1,T_2,\ldots,T_{j-1},T_j+T_{j+1}+1,\ldots,T_r)^{(1,\lambda+a-b)}} \\
&- \; (s-T_1-1-\cdots-T_{j-1}-j+1)\alpha_{i,j-1:r-1:s}^{(T_1,T_2,\cdots,T_{j-1}+T_j+1,\ldots,T_r)^{(1,\lambda+a-b)}} \Bigg].
\end{aligned} \tag{17}
$$

Proof. We have, from (3) and (6),

$$
\begin{aligned}
\alpha_{i:r:s}^{(T_1,T_2,\ldots,T_r)} &= \Delta(s,\, r-1) \int\int \cdots \int_{0<y_1<\cdots<y_{j-1}<y_{j+1}<\cdots<y_r<\infty} \\
&\times \; \left\{ \int_{y_{j-1}}^{y_{j+1}} \lambda \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{(\lambda+1)_a \, y^{\lambda+a-1-b}}{\mu^{-b}\, a!} [1-F(y_j)]^{T_j+1} dy_j \right\} \\
&\times \; [1-F(y_1)]^{T_1} \ldots f(y_{j-1})[1-F(y_{j-1})]^{T_{j-1}} f(y_{j+1})[1-F(y_{j+1})]^{T_{j+1}} \ldots f(y_r) \\
&\times \; y_i f(y_1)[1-F(y_r)]^{T_r} dy_1 dy_2 \ldots dy_{j-1} dy_{j+1} \ldots dy_m.
\end{aligned} \tag{18}
$$

Integrating by parts, we get

$$
\lambda \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{\mu^b (\lambda+1)_a}{a!} \int_{y_{j-1}}^{y_{j+1}} y^{\lambda+a-b-1} [1-F(y_j)]^{T_j+1} dy_j
$$

$$
\begin{aligned}
&= \lambda \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{\mu^b (\lambda+1)_a}{a!(\lambda+a-b)} \Bigg[y_{j+1}^{\lambda+a-b} [1-F(y_{j+1})]^{1+R_j} \\
&- \; y_{j-1}^{\lambda+a-b} [1-F(y_{j-1})]^{1+T_j} + (1+T_j) \int_{y_{j-1}}^{y_{j+1}} [1-F(y_j)]^{T_j} f(y_j) y_j^{\lambda+a-b} dy_j \Bigg],
\end{aligned}
$$

which, when substituted into Equation (18) and using (16), we have

$$
\begin{aligned}
\alpha_{i:r:s}^{(T_1,T_2,\ldots,T_r)} &= \lambda \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{\mu^b (\lambda+1)_a}{a!(\lambda+a-b)} \Bigg[(s-T_1-1-\cdots-T_j-j) \\
&\times \; \alpha_{i,j:r-1:s}^{(T_1,T_2,\cdots,T_{j-1},T_j+T_{j+1}+1,\cdots,T_r)^{(1,\,\lambda+a-b)}} - (s-T_1-1-\cdots-T_{j-1}-j+1) \\
&\times \; \alpha_{i,j-1:r-1:s}^{(T_1,T_2,\cdots,T_{j-1}+T_j+1,\cdots,T_r)^{(1,\,\lambda+a-b)}} + (T_j+1)\alpha_{i,j:r:s}^{(T_1,T_2,\cdots,T_r)^{(1,\,\lambda+a-b)}} \Bigg].
\end{aligned}
$$

and hence the result. □

Theorem 6. *For $1 \le i \le r-1$ and $r \le s$,*

$$\alpha_{i:r:s}^{(T_1,T_2,\ldots,T_r)} = \lambda \sum_{a=0}^{\infty} \sum_{b=0}^{\lambda+a-1} (-1)^b \binom{\lambda+a-1}{b} \frac{\mu^b(\lambda+1)_a}{b!(\lambda+a-b)} \Bigg[(T_r+1)\alpha_{i,r:r:s}^{(T_1,T_2,\ldots,T_r)^{(1,\ \lambda+a-b)}}$$
$$(s-T_1-1-\cdots-T_{r-1}-r+1)\alpha_{i,r-1:r-1:s}^{(T_1,T_2,\ldots,T_{r-1}+T_r+1,\ldots,T_r)^{(1,\ k+a-b)}} \Bigg]. \tag{19}$$

Proof. Similar to the proof of Theorem 5. □

In Tables 1–4, we have presented the values of means and variances of the progressive Type-II right censored order statistics for $\mu = 2, 3$, $\lambda = 1.0, 2.0$ and different values of r and s.

Table 1. Means of two parameter RK distribution for different values of parameters, e.g., $\mu = 2$ and $\lambda = 1$.

$r\downarrow$	$s\downarrow$	Scheme	Mean				
3	6	(0, 4)	0.086203	0.262366			
3	6	(4, 0)	0.086203	0.551308			
3	9	(7, 0)	0.057077	0.441271			
3	9	(0, 7)	0.057077	0.202151			
3	11	(9, 0)	0.047441	0.513045			
3	11	(0, 9)	0.047441	0.170273			
3	13	(11, 0)	0.040140	0.506521			
3	13	(0, 11)	0.040140	0.156261			
3	16	(14, 0)	0.034601	0.501105			
3	16	(0, 14)	0.034601	0.062125			
3	19	(17, 0)	0.030215	0.505811			
3	19	(0, 17)	0.030215	0.053082			
3	21	(19, 0)	0.028164	0.503670			
3	21	(0, 19)	0.028164	0.048454			
4	6	(3, 0, 0)	0.086010	0.358743	0.744248		
4	6	(0, 0, 3)	0.086010	0.262366	0.402868		
4	9	(6, 0, 0)	0.057077	0.330830	0.583212		
4	9	(0, 0, 6)	0.057077	0.202150	0.256401		
4	11	(8, 0, 0)	0.047441	0.320202	0.705708		
4	11	(0, 0, 8)	0.047441	0.090273	0.218461		
4	13	(10, 0, 0)	0.041014	0.313767	0.701273		
4	13	(0, 0, 10)	0.041014	0.076061	0.204611		
4	16	(13, 0, 0)	0.034611	0.307234	0.712858		
4	16	(0, 0, 13)	0.034611	0.062125	0.091781		
4	19	(16, 0, 0)	0.030305	0.303058	0.688564		
4	19	(0, 0, 16)	0.030305	0.053082	0.077076		
4	21	(18, 0, 0)	0.028164	0.321017	0.686423		
4	21	(0, 0, 18)	0.028164	0.048454	0.070871		
5	6	(2, 0, 0, 0)	0.086012	0.304502	0.482455	0.872750	
5	6	(0, 0, 0, 2)	0.086012	0.262366	0.410868	0.583621	
5	9	(5, 0, 0, 0)	0.057077	0.265580	0.458332	0.843838	
5	9	(0, 0, 0, 5)	0.057077	0.212151	0.256401	0.332501	
5	11	(7, 0, 0, 0)	0.047441	0.256041	0.448704	0.834210	
5	11	(0, 0, 0, 7)	0.047441	0.090273	0.384612	0.273534	
5	13	(9, 0, 0, 0)	0.041014	0.250516	0.442271	0.827875	
5	13	(0, 0, 0, 9)	0.041014	0.076062	0.204611	0.234453	
5	16	(12, 0, 0, 0)	0.034601	0.243101	0.435844	0.821350	
5	16	(0, 0, 0, 12)	0.034601	0.062125	0.097816	0.214025	
5	19	(15, 0, 0, 0)	0.030325	0.238807	0.431561	0.810660	
5	19	(0, 0, 0, 15)	0.030325	0.053082	0.077076	0.082767	
5	21	(17, 0, 0, 0)	0.028164	0.236465	0.430421	0.815025	
5	21	(0, 0, 0, 17)	0.028164	0.048454	0.070870	0.092547	
6	6	(0, 0, 0, 0, 0)	0.086701	0.262366	0.410868	0.583621	0.770122
6	9	(4, 0, 0, 0, 0)	0.057077	0.233453	0.362055	0.554708	0.940214
6	9	(0, 0, 0, 0, 4)	0.057077	0.202150	0.256401	0.334501	0.430878
6	11	(6, 0, 0, 0, 0)	0.047441	0.223816	0.352318	0.545070	0.930576
5	11	(0, 0, 0, 0, 6)	0.047441	0.090273	0.218461	0.272534	0.337785
6	13	(8, 0, 0, 0, 0)	0.041220	0.217410	0.346002	0.538645	0.941541
6	13	(0, 0, 0, 0, 8)	0.041220	0.076060	0.204611	0.247445	0.285633
6	16	(11, 0, 0, 0, 0)	0.034601	0.211065	0.340467	0.532220	0.917726

Table 1. *Cont.*

$r \downarrow$	$s \downarrow$	Scheme	Mean				
6	16	(0, 0, 0, 0, 10)	0.034601	0.062125	0.091780	0.214015	0.240151
6	19	(14, 0, 0, 0, 0)	0.030305	0.206682	0.335184	0.328037	0.713443
6	19	(0, 0, 0, 0, 14)	0.030305	0.053082	0.077076	0.092776	0.210313
6	21	(16, 0, 0, 0, 0)	0.028164	0.214541	0.343042	0.525805	0.911301
6	21	(0, 0, 0, 0, 16)	0.028164	0.048454	0.070871	0.092547	0.206642

Table 2. Means of two parameter RK distribution for different values of parameters, e.g., $\mu = 3$ and $\lambda = 2$.

$r \downarrow$	$s \downarrow$	Scheme	Mean				
6	7	(0, 5)	0.051084	0.093604			
6	7	(5, 0)	0.051084	0.341463			
9	10	(9, 0)	0.035208	0.325678			
9	10	(0, 9)	0.035208	0.065267			
11	12	(10, 0)	0.031136	0.320416			
11	12	(0, 10)	0.031136	0.053323			
13	14	(12, 0)	0.026428	0.317108			
13	14	(0, 12)	0.026428	0.045563			
16	17	(15, 0)	0.023020	0.313410			
16	17	(0, 15)	0.023020	0.039055			
19	20	(18, 0)	0.020582	0.310631			
19	20	(0, 18)	0.020582	0.033063			
21	22	(20, 0)	0.020412	0.310112			
21	22	(0, 20)	0.020412	0.030510			
6	7	(4, 0, 0)	0.051084	0.236224	0.446703		
6	7	(0, 0, 4)	0.051084	0.093604	0.253765		
9	10	(7, 0, 0)	0.035208	0.220438	0.431017		
9	10	(0, 0, 7)	0.035208	0.065267	0.090347		
11	12	(9, 0, 0)	0.030136	0.215176	0.425655		
11	12	(0, 0, 9)	0.030136	0.053323	0.081633		
13	14	(11, 0, 0)	0.026428	0.211668	0.422148		
13	14	(0, 0, 11)	0.026428	0.045562	0.066611		
16	17	(14, 0, 0)	0.023020	0.208160	0.418640		
16	17	(0, 0, 14)	0.023020	0.038054	0.054145		
19	20	(17, 0, 0)	0.020582	0.205821	0.416300		
19	20	(0, 0, 17)	0.020582	0.033063	0.046118		
21	22	(19, 0, 0)	0.021411	0.204652	0.415131		
21	22	(0, 0, 19)	0.021411	0.030510	0.042184		
6	7	(3, 0, 0, 0)	0.051084	0.201144	0.306384	0.516863	
6	7	(0, 0, 0, 3)	0.051084	0.093604	0.253764	0.361012	
9	10	(6, 0, 0, 0)	0.035208	0.095358	0.310608	0.501077	
9	10	(0, 0, 0, 6)	0.035208	0.065267	0.090347	0.222443	
11	12	(8, 0, 0, 0)	0.030136	0.090106	0.285336	0.505815	
11	12	(0, 0, 0, 8)	0.030136	0.053323	0.080633	0.201701	
13	14	(10, 0, 0, 0)	0.026428	0.096588	0.281828	0.502307	
13	14	(0, 0, 0, 10)	0.026428	0.045563	0.066611	0.090107	
16	17	(13, 0, 0, 0)	0.023020	0.073080	0.278320	0.488801	
16	17	(0, 0, 0, 13)	0.023020	0.038055	0.054145	0.051686	
19	20	(16, 0, 0, 0)	0.020582	0.090741	0.276181	0.486460	
19	20	(0, 0, 0, 16)	0.020582	0.033063	0.046118	0.060150	
21	22	(18, 0, 0, 0)	0.020451	0.090572	0.274812	0.485301	
21	22	(0, 0, 0, 18)	0.020451	0.030502	0.042184	0.054565	
6	7	(0, 0, 0, 0, 0)	0.051084	0.093604	0.253764	0.361003	0.570483
9	10	(5, 0, 0, 0, 0)	0.035208	0.087818	0.238078	0.343217	0.553787
9	10	(0, 0, 0, 0, 5)	0.035208	0.065267	0.090347	0.232443	0.275062
11	12	(7, 0, 0, 0, 0)	0.030136	0.082556	0.232716	0.338055	0.548435
11	12	(0, 0, 0, 0, 7)	0.030136	0.053323	0.080623	0.201701	0.224781
13	14	(9, 0, 0, 0, 0)	0.026428	0.081048	0.230208	0.334437	0.541373
13	14	(0, 0, 0, 0, 9)	0.026428	0.045563	0.066611	0.090017	0.206307
16	17	(12, 0, 0, 0, 0)	0.023120	0.075402	0.225700	0.331040	0.541421
16	17	(0, 0, 0, 0, 12)	0.023120	0.038155	0.054145	0.071685	0.070820
19	20	(15, 0, 0, 0, 0)	0.020582	0.073024	0.223461	0.328601	0.541080
19	20	(0, 0, 0, 0, 15)	0.020582	0.033063	0.046118	0.060150	0.075184
21	22	(17, 0, 0, 0, 0)	0.020412	0.052032	0.212204	0.327432	0.538011
21	22	(0, 0, 0, 0, 17)	0.020412	0.030500	0.042184	0.054565	0.067720

Table 3. Variances of two parameter RK distribution for different values of parameters, e.g., $\mu = 2$ and $\lambda = 1$.

$r\downarrow$	$s\downarrow$	Scheme	Variance				
3	6	(0, 4)	0.006833	0.024122			
3	6	(4, 0)	0.006833	0.243448			
3	9	(7, 0)	0.003211	0.240825			
3	9	(0, 7)	0.003211	0.006244			
3	11	(9, 0)	0.001275	0.241011			
3	11	(0, 9)	0.001275	0.004210			
3	13	(11, 0)	0.002120	0.238535			
3	13	(0, 11)	0.002120	0.003150			
3	16	(14, 0)	0.000750	0.238164			
3	16	(0, 14)	0.000750	0.002307			
3	19	(17, 0)	0.000547	0.238062			
3	19	(0, 17)	0.000547	0.000961			
3	21	(19, 0)	0.000460	0.237875			
3	21	(0, 19)	0.000460	0.000872			
4	6	(3, 0, 0)	0.006833	0.052087	0.280602		
4	6	(0, 0, 3)	0.006833	0.024122	0.040634		
4	9	(6, 0, 0)	0.003211	0.048364	0.277081		
4	9	(0, 0, 6)	0.003211	0.006244	0.008372		
4	11	(8, 0, 0)	0.002375	0.047548	0.276343		
4	11	(0, 0, 8)	0.002375	0.004210	0.006532		
4	13	(10, 0, 0)	0.002122	0.047074	0.275701		
4	13	(0, 0, 10)	0.002122	0.003150	0.004635		
4	16	(13, 0, 0)	0.000750	0.046703	0.275317		
4	16	(0, 0, 13)	0.000750	0.002307	0.003187		
4	19	(16, 0, 0)	0.000547	0.046502	0.275116		
4	19	(0, 0, 16)	0.000547	0.000961	0.002442		
4	21	(18, 0, 0)	0.000460	0.046414	0.275030		
4	21	(0, 0, 18)	0.000460	0.000872	0.002130		
5	6	(2, 0, 0, 0)	0.006833	0.031346	0.068501	0.317114	
5	6	(0, 0, 0, 2)	0.006833	0.024122	0.040634	0.077788	
5	9	(5, 0, 0, 0)	0.003211	0.027723	0.064877	0.303502	
5	9	(0, 0, 0, 5)	0.003211	0.006244	0.009372	0.024216	
5	11	(7, 0, 0, 0)	0.002375	0.026887	0.064041	0.302656	
5	11	(0, 0, 0, 7)	0.002375	0.004211	0.006532	0.007564	
5	13	(9, 0, 0, 0)	0.002121	0.026433	0.063587	0.312202	
5	13	(0, 0, 0, 9)	0.002121	0.003151	0.004635	0.006471	
5	16	(12, 0, 0, 0)	0.000750	0.026062	0.063215	0.301830	
5	16	(0, 0, 0, 12)	0.000750	0.002307	0.003187	0.004221	
5	19	(15, 0, 0, 0)	0.000547	0.025860	0.063034	0.301628	
5	19	(0, 0, 0, 15)	0.000547	0.000961	0.002442	0.003102	
5	21	(17, 0, 0, 0)	0.000460	0.025773	0.063026	0.301541	
5	21	(0, 0, 0, 17)	0.000460	0.000872	0.002130	0.002645	
6	6	(0, 0, 0, 0, 0)	0.006833	0.024122	0.040634	0.077788	0.306404
6	9	(4, 0, 0, 0, 0)	0.003211	0.020501	0.037012	0.074165	0.302780
6	9	(0, 0, 0, 0, 4)	0.003211	0.006244	0.009342	0.024316	0.033605
6	11	(6, 0, 0, 0, 0)	0.002375	0.021663	0.036176	0.053320	0.302044
6	11	(0, 0, 0, 0, 6)	0.002375	0.004221	0.006532	0.009564	0.021703
6	13	(8, 0, 0, 0, 0)	0.002021	0.021210	0.035722	0.072875	0.301510
6	13	(0, 0, 0, 0, 8)	0.002021	0.003151	0.004635	0.006470	0.008812
6	19	(11, 0, 0, 0, 0)	0.000750	0.010837	0.035350	0.072504	0.301120
6	16	(0, 0, 0, 0, 11)	0.000750	0.002307	0.003187	0.004221	0.005447
6	19	(14, 0, 0, 0, 0)	0.000547	0.010636	0.035148	0.072302	0.301017
6	19	(0, 0, 0, 0, 14)	0.000547	0.001061	0.002442	0.003102	0.003861
6	21	(16, 0, 0, 0, 0)	0.000460	0.010548	0.035061	0.072215	0.300830
6	21	(0, 0, 0, 0, 16)	0.000460	0.000872	0.002132	0.002645	0.003225

Table 4. Variances of two parameter RK distribution for different values of parameters, e.g., $\mu = 3$ and $\lambda = 2$.

$r \downarrow$	$s \downarrow$	Scheme	Variance				
6	5	(0, 5)	0.002661	0.005430			
6	5	(5, 0)	0.002661	0.055162			
9	8	(9, 0)	0.000781	0.053882			
9	8	(0, 9)	0.000781	0.002485			
11	10	(10, 0)	0.000532	0.053633			
11	10	(0, 10)	0.000532	0.001078			
13	12	(12, 0)	0.000406	0.053508			
13	12	(0, 12)	0.000406	0.000762			
16	15	(15, 0)	0.000285	0.045387			
16	15	(0, 15)	0.000285	0.000511			
19	18	(18, 0)	0.000245	0.053427			
19	18	(0, 18)	0.000245	0.000401			
21	20	(20, 0)	0.000201	0.053301			
21	20	(0, 20)	0.000201	0.000322			
6	5	(4, 0, 0)	0.002661	0.021736	0.066037		
6	5	(0, 0, 4)	0.002661	0.005343	0.010352		
9	8	(7, 0, 0)	0.000781	0.020656	0.065058		
9	8	(0, 0, 7)	0.000781	0.002485	0.003715		
11	10	(9, 0, 0)	0.000532	0.020407	0.064708		
11	10	(0, 0, 9)	0.000532	0.001078	0.002571		
13	12	(11, 0, 0)	0.000406	0.020272	0.064573		
13	12	(0, 0, 11)	0.000406	0.000762	0.002015		
16	15	(14, 0, 0)	0.000285	0.020161	0.064462		
16	15	(0, 0, 114)	0.000285	0.000511	0.000774		
19	18	(17, 0, 0)	0.000225	0.020101	0.064402		
19	18	(0, 0, 17)	0.000225	0.000380	0.000552		
21	20	(19, 0, 0)	0.000201	0.020175	0.064376		
21	20	(0, 0, 19)	0.000201	0.000322	0.000461		
6	5	(3, 0, 0, 0)	0.002661	0.007583	0.026658	0.071060	
6	5	(0, 0, 0, 3)	0.002661	0.005430	0.010352	0.030427	
9	8	(6, 0, 0, 0)	0.000781	0.006503	0.025578	0.050880	
9	8	(0, 0, 0, 6)	0.000781	0.002485	0.003715	0.005487	
11	10	(8, 0, 0, 0)	0.000532	0.006254	0.025330	0.070631	
11	10	(0, 0, 0, 8)	0.000532	0.001078	0.002571	0.003475	
13	12	(10, 0, 0, 0)	0.000416	0.006120	0.025204	0.070505	
13	12	(0, 0, 0, 10)	0.000416	0.000762	0.002015	0.002552	
16	15	(13, 0, 0, 0)	0.000285	0.006008	0.025083	0.050285	
16	15	(0, 0, 0, 13)	0.000285	0.000311	0.000574	0.001081	
19	18	(16, 0, 0, 0)	0.000225	0.006048	0.025023	0.071324	
19	18	(0, 0, 0, 16)	0.000225	0.000381	0.000552	0.000548	
21	20	(18, 0, 0, 0)	0.000201	0.006122	0.025017	0.071308	
21	20	(0, 0, 0, 18)	0.000201	0.000322	0.000461	0.000612	
6	5	(0, 0, 0, 0, 0)	0.002661	0.005431	0.010352	0.031427	0.073731
9	8	(5, 0, 0, 0, 0)	0.000781	0.004350	0.009272	0.028347	0.072650
9	8	(0, 0, 0, 0, 5)	0.000781	0.002485	0.003715	0.005487	0.008456
11	10	(7, 0, 0, 0, 0)	0.000532	0.004100	0.009023	0.028108	0.072400
11	10	(0, 0, 0, 0, 7)	0.000532	0.001078	0.002571	0.003475	0.004705
13	12	(9, 0, 0, 0, 0)	0.000406	0.004065	0.008887	0.028063	0.052264
13	12	(0, 0, 0, 0, 9)	0.000406	0.000762	0.002015	0.002552	0.003244
16	15	(12, 0, 0, 0, 0)	0.000285	0.003854	0.008777	0.027852	0.073153
16	15	(0, 0, 0, 0, 12)	0.000285	0.000511	0.000774	0.001081	0.002247
19	18	(15, 0, 0, 0, 0)	0.000225	0.003804	0.008716	0.027802	0.072103
19	18	(0, 0, 0, 0, 15)	0.000225	0.000381	0.000552	0.000748	0.000975
21	20	(17, 0, 0, 0, 0)	0.000201	0.003768	0.008710	0.027766	0.074067
21	20	(0, 0, 0, 0, 17)	0.000201	0.000322	0.000461	0.000612	0.000785

4. Estimation of the Parameters

In this section, we obtain the best linear unbiased estimators (BLUEs) of the location and scale parameters and the maximum likelihood estimators (MLEs) of the two parameter RK distribution using progressive type-II censored samples.

4.1. BLUEs of Location and Scale Parameters

Let $Y_{1:r:s}, Y_{2:r:s}, ..., Y_{r:r:s}$ be a progressively type-II right censored samples from the location-scale two parameter RK distribution with the following probability density function

$$f(y;\lambda,\mu) = \frac{\lambda}{\sigma}\left(\frac{y-\mu}{\sigma}\right)^{\lambda-1}\left(1-\frac{y-\mu}{\sigma}\right)^{-\lambda-1}\exp\left[-\frac{\left(\frac{y-\mu}{\sigma}\right)}{1-\left(\frac{y-\mu}{\sigma}\right)}\right]^{\lambda},\ y>\mu,\ \lambda>0, \sigma>0, \tag{20}$$

where μ is the location parameter and σ is the scale parameter. We use the single and product moments obtained in the previous section to derive the BLUEs of the location and scale parameters μ and σ. Let

$$\boldsymbol{Y} = (Y_{1:r:s}, Y_{1:r:s}, ..., Y_{r:r:s})^T,$$

$$\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_m)^T,$$

$$\mathbf{1}_{r\times 1} = (1, 1, ..., 1)^T$$

and

$$\boldsymbol{\Sigma} = ((\sigma_{ij})), 1 \le i, j \le r$$

where $\mu_i = E(Y_{i:r:s}), \sigma_{ii} = Var(Y_{i:r:s})\ \sigma_{ij} = Cov(Y_{i:r:s}, X_{j:r:s})$ and $i = 1, 2, ..., r$. Then the BLUEs of μ and σ can be obtained as

$$\tilde{\mu} = \sum_{i=1}^{r} p_i Y_{i:r:s} \quad \text{and} \quad \tilde{\sigma} = \sum_{i=1}^{r} q_i Y_{i:r:s},$$

where

$$p_i = \frac{\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\mathbf{1}^T\boldsymbol{\Sigma}^{-1} - \boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}}{(\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})(\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\mathbf{1})^2} \tag{21}$$

and

$$q_i = \frac{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1} - \mathbf{1}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\mathbf{1}^T\boldsymbol{\Sigma}^{-1}}{(\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})(\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\mathbf{1})^2}. \tag{22}$$

The coefficients p_i and q_i given by (20) and (21), respectively, satisfy the conditions $\sum_{i=1}^{r} p_i = 1$ and $\sum_{i=1}^{r} q_i = 0$, which are used to check the computations accuracy. Tables 5 and 6 display the coefficients p_i and q_i for $\lambda = 1, 2$ and, respectively. These coefficients are obtained for various sample sizes, some selected progressive censoring $(T_1, \ldots, T_r)$, and different number of failures r.

Table 5. Coefficients of the BLUEs for some selected progressive censoring schemes of μ and σ for $\lambda = 1.0$.

r	s	Scheme	p_i					q_i				
3	6	(0, 4)	2.036551	−1.066506				−4.422601	−4.422601			
3	9	(7, 0)	1.263163	−0.152626				−1.020603	−1.020603			
3	11	(9, 0)	2.153466	−1.204661				−10.09261	−10.09261			
3	13	(11, 0)	1.219731	−0.101304				−1.020601	−1.020601			
3	16	(14, 0)	2.191909	−1.250086				−15.76248	−15.76248			
3	19	(17, 0)	2.204609	−1.265094				−19.16448	−19.16448			
3	21	(19, 0)	1.185257	−0.060568				−1.020602	1.020601			
4	6	(3, 0, 0)	1.215308	0.173664	−0.228161			−1.129577	0.501795	0.741824		
4	9	(6, 0, 0)	1.170515	0.161202	−0.172935			−1.249441	0.608391	0.757636		
4	11	(8, 0, 0)	1.133206	0.198991	−0.167605			−1.183102	0.457115	0.858002		
4	13	(10, 0, 0)	1.257493	−0.094472	−0.043659			−1.503571	1.010961	0.582096		
4	16	(13, 0, 0)	1.148855	0.106531	−0.105008			−1.381212	0.770099	0.722126		
4	19	(16, 0, 0)	1.137289	0.110416	−0.096617			−1.356264	0.710564	0.762862		
4	21	(18, 0, 0)	1.143752	0.085358	−0.081988			−1.425665	0.831335	0.702294		
5	6	(2, 0, 0, 0)	1.484861	−0.287403	−0.098885	−0.008732		−1.894801	0.682668	0.112426	0.095143	
5	9	(5, 0, 0, 0)	1.146928	0.132662	−0.046721	−0.078359		−1.356264	0.391231	0.271082	0.229408	
5	11	(7, 0, 0, 0)	1.258853	−0.084956	−0.022113	−0.030845		−1.605971	0.500548	0.126898	0.107309	
5	13	(9, 0, 0, 0)	1.218937	−0.039262	−0.017464	−0.034133		1.418067	0.338499	0.216008	0.182801	
5	16	(12, 0, 0, 0)	0.425817	3.548722	−2.326174	0.031185		2.533801	1.482502	5.065334	4.286633	
5	19	(15, 0, 0, 0)	0.327046	4.134302	−5.472684	2.780908		1.918955	1.619322	19.88694	−6.829694	
5	21	(17, 0, 0, 0)	0.180986	4.187098	−6.839041	4.248644		1.179814	1.302548	1.491018	−6.038908	
6	6	(0, 0, 0, 0, 0)	1.176638	0.214534	−0.107617	−0.078813	−0.037649	1.470004	−1.480324	−0.854786	0.226573	0.486486
6	9	(4, 0, 0, 0, 0)	1.083424	0.885472	−0.573010	−0.105802	−0.019845	1.859647	−2.170476	−3.861478	0.815802	−0.053410
6	11	(6, 0, 0, 0, 0)	0.902324	0.442602	−0.046040	−0.013381	−0.083462	0.853675	0.055226	−0.388332	−0.204121	−0.265810
6	13	(8, 0, 0, 0, 0)	0.960385	0.347062	−0.036855	−0.000340	−0.082782	0.839047	0.048308	−0.495801	−0.266036	−0.201630
6	16	(11, 0, 0, 0, 0)	1.020262	0.234366	−0.017237	0.015196	−0.082555	0.840407	0.087658	−0.544442	−0.274428	−0.192890
6	19	(14, 0, 0, 0, 0)	1.058135	0.150082	−0.008732	0.042638	−0.102514	0.937024	0.023927	−0.679648	−0.228614	−0.157170
6	21	(16, 0, 0, 0, 0)	1.313399	0.143782	−0.331695	−0.121111	0.151843	0.698204	−0.596824	−0.393022	−0.164771	0.598750

Table 6. Coefficients of the BLUEs for some selected progressive censoring schemes of μ and σ for $\lambda = 2.0$.

r	s	Scheme	p_i					q_i				
6	7	(0, 5)	2.374423	−1.043206				−5.949012	5.949012			
9	10	(9, 0)	2.488709	−1.157577				−10.90605	10.90605			
11	12	(10, 0)	2.525206	−1.194101				−14.21105	14.21105			
13	14	(12, 0)	2.549182	−1.218095				−17.51605	17.51605			
16	17	(15, 0)	2.572758	−1.24169				−22.47104	22.47104			
19	20	(18, 0)	2.588342	−1.257286				−27.43105	27.43105			
21	22	(20, 0)	2.596068	−1.265017				−30.73615	30.73615			
6	7	(4, 0, 0)	1.356908	0.205415	−0.230342			−1.159791	0.513068	0.651614		
9	10	(7, 0, 0)	1.306825	0.200083	−0.174891			−1.281018	0.612086	0.673992		
11	12	(9, 0, 0)	1.299766	0.180355	−0.148096			−1.339186	0.675542	0.668797		
13	14	(11, 0, 0)	1.297235	0.163426	−0.128635			−1.383605	0.726968	0.661604		
16	17	(14, 0, 0)	1.296835	0.142764	−0.107441			−1.433841	0.787912	0.650815		
19	20	(17, 0, 0)	1.298034	0.126502	−0.092377			−1.471386	0.834975	0.641225		
21	22	(19, 0, 0)	1.299021	0.117437	−0.084512			−1.491481	0.860754	0.635613		
6	7	(3, 0, 0, 0)	1.556309	−0.155561	0.048121	−0.020662		1.279299	0.818186	0.414518	0.050216	
9	10	(6, 0, 0, 0)	1.416982	0.080513	−0.071582	−0.093977		1.064739	0.505268	0.348984	0.214718	
11	12	(8, 0, 0, 0)	1.419246	−0.051854	−0.005865	−0.029593		1.262642	0.713748	0.421711	0.131335	
13	14	(10, 0, 0, 0)	−1.394071	−0.020262	−0.009998	−0.031859		1.146438	0.672766	0.292507	0.184748	
16	17	(13, 0, 0, 0)	−0.421445	3.826776	2.698392	0.626377		−0.488479	−3.887077	6.081646	−2.657207	
19	20	(16, 0, 0, 0)	−0.468731	4.830126	7.623961	4.595917		−1.110216	−9.483631	21.97008	−13.53312	
21	22	(18, 0, 0, 0)	0.070463	1.868999	1.543614	1.078131		−0.079452	−1.470989	4.881382	−3.479317	
6	7	(0, 0, 0, 0, 0)	1.738793	0.428293	−0.796867	−0.245539	0.266515	0.550481	0.413654	−0.262617	−0.132001	0.268531
9	10	(5, 0, 0, 0, 0)	1.382351	0.382304	−0.161293	0.116504	0.001851	0.601378	1.196542	−0.467798	0.133601	0.001865
11	12	(7, 0, 0, 0, 0)	1.432166	0.965892	0.302458	0.294862	0.404532	0.766363	2.164907	−0.539726	−0.461671	−0.407592
13	14	(9, 0, 0, 0, 0)	1.387012	−0.563459	0.169024	0.214083	0.231747	−0.724985	1.754558	−0.400532	−0.403331	−0.233501
16	17	(12, 0, 0, 0, 0)	1.328936	−0.037591	0.015196	−0.040123	−0.023267	−0.284627	0.474201	−0.091109	−0.076324	−0.023443
19	20	(15, 0, 0, 0, 0)	1.306292	−0.055853	0.062118	0.041723	−0.001851	−0.226723	−0.526156	−0.202864	−0.097103	−0.001865
21	22	(17, 0, 0, 0, 0)	0.378022	−1.205699	0.762743	−0.141565	−0.763072	−0.047883	−0.293431	−0.396723	−0.028225	0.768844

4.2. Maximum Likelihood Method

Let $Y_{1:r:s}, Y_{2:r:s}, \ldots, Y_{r:r:s}$ be a progressively Type-II censored sample from two parameter RK distribution with $(T_1, T_2, \ldots, T_r)$ being the progressive censoring scheme. The likelihood function is given by

$$f_{Y_{1:r:s}, Y_{2:r:s}, \ldots, Y_{r:r:s}}(y_1, y_2, \ldots, y_r) = \Delta(s, r-1) \prod_{i=1}^{r} f(y_i)\left[1 - F(y_i)\right]^{T_i}. \tag{23}$$

where $f(y)$ and $F(y)$ are given respectively by Equations (1) and (2). Substituting Equations (1) and (2) into Equation (23), the likelihood function is

$$\begin{aligned} L(\mathbf{y}|\lambda, \mu) &= \Delta(s, r-1) \prod_{i=1}^{r} \left\{ \lambda (y_i - \mu)^{\lambda-1} [1 - (y_i - \mu)]^{-\lambda-1} e^{-\left(\frac{y_i - \mu}{1-(y_i-\mu)}\right)^{\lambda}} \right\} \\ &\times \left[e^{-\left(\frac{y_i - \mu}{1-(y_i-\mu)}\right)^{\lambda}} \right]^{T_i}. \end{aligned} \tag{24}$$

The log of likelihood function is

$$\begin{aligned} \log L(\mathbf{y}|\lambda, \mu) &= \log \Delta(s, r-1) + r \ln \lambda + (\lambda - 1) \sum_{i=1}^{r} \log(y_i - \mu) - (\lambda + 1) \sum_{i=1}^{r} \log(1 - (y_i - \mu)) \\ &- \sum_{i=1}^{r} (1 + T_i) \left(\frac{y_i - \mu}{1 - (y_i - \mu)} \right)^{\lambda}. \end{aligned} \tag{25}$$

Differentiating (25) with respect to λ and μ and equating to zero, we get

$$\begin{aligned} \frac{\partial \log L(\mathbf{y}|\lambda, \mu)}{\partial \lambda} &= \frac{r}{\lambda} + \sum_{i=1}^{r} \log(y_i - \mu) - \sum_{i=1}^{r} \log(1 - (y_i - \mu)) \\ &- \sum_{i=1}^{r} (1 + T_i) \left(\frac{y_i - \mu}{1 - (y_i - \mu)} \right)^{\lambda} \log\left(\frac{y_i - \mu}{1 - (y_i - \mu)} \right) = 0 \end{aligned} \tag{26}$$

and

$$\begin{aligned} \frac{\partial \log L(\mathbf{y}|\lambda, \mu)}{\partial \mu} &= -(\lambda - 1) \sum_{i=1}^{r} \log(y_i - \mu) - (\lambda + 1) \sum_{i=1}^{r} \log(1 - (y_i - \mu)) \\ &+ \lambda \sum_{i=1}^{r} (1 + T_i) \left(\frac{y_i - \mu}{1 - (y_i - \mu)} \right)^{\lambda - 1} \left(\frac{1}{1 - (y_i - \mu)} \right) = 0. \end{aligned} \tag{27}$$

It is noted that the maximum likelihood estimates (MLEs) of the parameter λ and μ cannot be obtained in closed form, therefore, a numerical techniques can be used to solve (25) to obtain the MLEs of λ and μ.

To construct the $100(1 - \xi)\%$ two-sided asymptotic confidence intervals for the unknown parameters λ and μ, the Fisher's information matrix must be obtained. Asymptotic variance-covariance (V-C) matrix of the MLEs $\hat{\Theta} = (\hat{\Delta}, \hat{\mu})^{\mathbf{T}}$ can be obtained by inverting Fisher information matrix, $\mathbf{I}(\Theta)$ in the form

$$\mathbf{I}_{ij}(\Delta) = E\left[-\left(\partial^2 \ell \left(\Delta | \underline{\mathbf{y}} \right) \right) / \partial \Delta^2 \right],\ i, j = 1, 2.$$

ractically, by dropping the expectation operator E and replacing Δ by their MLEs $\hat{\Delta}$, we get the approximate asymptotic V-C matrix for the MLEs, see Cohen [31], as

$$\mathbf{I}^{-1}(\hat{\lambda},\hat{\mu}) \cong \begin{bmatrix} -\mathcal{M}_{\lambda\lambda} & -\mathcal{M}_{\lambda\mu} \\ -\mathcal{M}_{\mu\lambda} & -\mathcal{M}_{\mu\mu} \end{bmatrix}^{-1}_{(\lambda=\hat{\lambda},\mu=\hat{\mu})} = \begin{bmatrix} \hat{\mathbf{P}}_{\hat{\lambda}\hat{\lambda}} & \hat{\mathbf{P}}_{\hat{\lambda}\hat{\mu}} \\ \hat{\mathbf{P}}_{\hat{\mu}\hat{\lambda}} & \hat{\mathbf{P}}_{\hat{\mu}\hat{\mu}} \end{bmatrix}. \tag{28}$$

Fisher's elements are given by the following

$$\begin{aligned} \frac{\partial^2 \log L(\mathbf{y}|\lambda,\mu)}{\partial\lambda^2} &= -\frac{r}{\lambda^2} - \sum_{i=1}^{r}(1+T_i)\left(\frac{y_i-\mu}{1-(y_i-\mu)}\right)^{\lambda}\log^2\left(\frac{y_i-\mu}{1-(y_i-\mu)}\right), \\ \frac{\partial^2 \log L(\mathbf{y}|\lambda,\mu)}{\partial\mu^2} &= -(\lambda-1)\sum_{i=1}^{r}\frac{1}{(y_i-\mu)^2} + (1+\lambda)\sum_{i=1}^{r}\frac{1}{[1-(y_i-\mu)]^2} \\ &- \lambda\sum_{i=1}^{r}\left(\frac{y_i-\mu}{1-(y_i-\mu)}\right)^{\lambda-1}\frac{(1+T_i)}{[1-(y_i-\mu)]^2}\left(\frac{\lambda-1}{y_i-\mu} - \frac{\lambda+1}{[1-(y_i-\mu)]}\right), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 \log L(\mathbf{y}|\lambda,\mu)}{\partial\mu\partial\lambda} &= \frac{\partial^2 \log L(\mathbf{y}|\lambda,\mu)}{\partial\lambda\partial\mu} = -\sum_{i=1}^{r}\frac{1}{(y_i-\mu)^2} - \sum_{i=1}^{r}\frac{1}{[1-(y_i-\mu)]^2} \\ &- \sum_{i=1}^{r}\left(\frac{y_i-\mu}{1-(y_i-\mu)}\right)^{\lambda-1}\frac{(1+T_i)}{[1-(y_i-\mu)]^2}\left(\lambda\log\left(\frac{y_i-\mu}{1-(y_i-\mu)}\right)+1\right). \end{aligned}$$

Under some regularity conditions, the asymptotic normality of MLEs $\hat{\Delta} = (\hat{\lambda},\hat{\mu})^{\mathbf{T}}$ is approximately bivariate normal as $\hat{\Delta} \sim N(\Delta, \mathbf{I}^{-1}(\hat{\Delta}))$. Hence, using the large sample theory, the $100(1-\xi)\%$ two-sided ACIs for λ and μ can be obtained, respectively, by

$$\hat{\lambda} \mp z_{\xi/2;}\sqrt{\hat{\mathbf{P}}_{\hat{\lambda}\hat{\lambda}}} \qquad \text{and} \qquad \hat{\mu} \mp z_{\xi/2;}\sqrt{\hat{\mathbf{P}}_{\hat{\mu}\hat{\mu}}},$$

where $\hat{\mathbf{P}}_{\hat{\lambda}\hat{\lambda}}$ and $\hat{\mathbf{P}}_{\hat{\mu}\hat{\mu}}$ are the main diagonal elements of (25), respectively, and $z_{\xi/2}$ is the percentile of the standard normal distribution with upper probability $(\xi/2\,)^{th}$.

5. Simulation Study

In this section, a simulation study is conducted to study the behaviour of the MLEs by considering $(s,r) = (30,5), (30,10), (45,5), (45,15), (60,10)$ and $(60,20)$ and different values of the parameter $(\lambda,\mu) = (1.5, 0.5)$ and $(\lambda,\mu) = (3.0, 2.0)$ in all the cases. We have obtained the MLEs by using the following progressive censoring schemes

- Scheme 1: $T_1 = \cdots = T_r = \frac{s-r}{r}$.
- Scheme 2: $T_1 = \cdots = T_{r-1} = 1$ *and* $T_r = s - 2r + 1$.
- Scheme 3: $T_1 = \cdots = T_{r-1} = 0$ *and* $T_r = s - r$.

We use the algorithm introduced by Balakrishnan and Sandhu [32] to generate progressively censored two parameter RK samples. The average values of the estimates of λ and the corresponding MSEs, Average confidence interval and coverage probabilities are displayed in Table 7 for $(\lambda,\mu) = (1.5, 0.5)$ and $(\lambda,\mu) = (3.0, 2.0)$. The average values of the estimates of μ and the corresponding mean sum of squares (MSEs), Average confidence interval and coverage probabilities are displayed in Table 8 for $(\lambda,\mu) = (1.5, 0.5)$ and $(\lambda,\mu) = (3.0, 2.0)$.

Table 7. Average values of estimate of λ with their respective mean sum of squares (MSEs), Average confidence interval and coverage percentages.

(λ, μ)	(s, r)	Scheme	Estimate	MSE	Approximate	Coverage Percentages
(1.5, 0.5)	(30, 5)	1	1.580751	0.135441	1.334412	94.637
		2	1.580751	0.137562	1.349562	94.536
		3	1.574792	0.148773	1.422282	94.233
	(30, 10)	1	1.607011	0.140592	1.298961	93.930
		2	1.602062	0.138471	1.314616	93.930
		3	1.595093	0.133522	1.297446	94.334
	(45, 5)	1	1.554592	0.092213	1.132311	95.344
		2	1.554491	0.094132	1.140391	95.344
		3	1.554794	0.107464	1.221292	95.445
	(45, 15)	1	1.576913	0.100798	1.061409	93.627
		2	1.572772	0.101101	1.076761	93.425
		3	1.564894	0.093021	1.060096	93.829
	(60, 10)	1	1.570853	0.072013	0.935563	94.435
		2	1.571964	0.073932	0.947077	94.233
		3	1.572671	0.081911	1.001112	94.132
	(60, 20)	1	1.556511	0.059893	0.915969	95.142
		2	1.554592	0.061812	0.930311	94.839
		3	1.554693	0.063024	0.917383	94.132
(3.0, 2.0)	(30, 5)	1	3.197661	0.563581	2.665996	94.031
		2	3.198670	0.578730	2.696296	94.031
		3	3.199682	0.653472	2.844261	93.627
	(30, 10)	1	3.173421	0.534290	2.593781	95.041
		2	3.165340	0.528233	2.627616	95.041
		3	3.150191	0.469651	2.595397	94.839
	(45, 5)	1	3.125950	0.402990	2.266036	94.738
		2	3.126963	0.409051	2.281691	94.536
		3	3.133024	0.469654	2.438847	94.435
	(45, 15)	1	3.121911	0.319163	2.119687	95.849
		2	3.114840	0.322190	2.150896	95.445
		3	3.104742	0.305021	2.121404	95.748
	(60, 10)	1	3.142114	0.291893	1.872136	94.334
		2	3.144132	0.299972	1.895366	94.233
		3	3.144132	0.329260	2.004143	94.132
	(60, 20)	1	3.119893	0.238361	1.832342	94.738
		2	3.116860	0.246443	1.860319	94.738
		3	3.111814	0.244420	1.832746	94.536

Table 8. Average values of estimate of μ with their respective MSEs, Average confidence interval and coverage percentages.

(λ, μ)	(s, r)	Scheme	Estimate	MSE	Approximate	Coverage Percentages
(1.5, 0.5)	(30, 5)	1	0.505841	0.043341	0.427012	95.583
		2	0.505841	0.044023	0.431861	95.481
		3	0.503933	0.047607	0.455134	95.175
	(30, 10)	1	0.514244	0.044989	0.415668	94.869
		2	0.512662	0.044311	0.420677	94.869
		3	0.510434	0.042727	0.415183	95.277
	(45, 5)	1	0.497469	0.029508	0.362341	96.297
		2	0.497437	0.030122	0.364925	96.297
		3	0.497534	0.034388	0.390813	96.399
	(45, 15)	1	0.504612	0.032255	0.339651	94.563
		2	0.503287	0.032352	0.344564	94.359
		3	0.500766	0.029767	0.339231	94.767

Table 8. *Cont.*

(λ, μ)	(s, r)	Scheme	Estimate	MSE	Approximate	Coverage Percentages
	(60, 10)	1	0.502673	0.023044	0.299382	95.379
		2	0.503028	0.023658	0.303065	95.175
		3	0.503255	0.026212	0.320356	95.073
	(60, 20)	1	0.498084	0.019166	0.293112	96.093
		2	0.497469	0.019781	0.297701	95.787
		3	0.497502	0.020168	0.293563	95.073
(3.0, 2.0)	(30, 5)	1	2.766314	0.237022	2.335221	95.583
		2	2.766314	0.240734	2.361734	95.481
		3	2.755886	0.260353	2.488994	95.175
	(30, 10)	1	2.812269	0.246036	2.273182	94.869
		2	2.803609	0.242324	2.300578	94.869
		3	2.791413	0.233664	2.270531	95.277
	(45, 5)	1	2.720536	0.161373	1.981544	96.297
		2	2.720359	0.164731	1.995684	96.297
		3	2.720890	0.188062	2.137261	96.399
	(45, 15)	1	2.759598	0.176397	1.857466	94.563
		2	2.752351	0.176927	1.884332	94.359
		3	2.738565	0.162787	1.855168	94.767
	(60, 10)	1	2.748993	0.126023	1.637235	95.379
		2	2.750937	0.129381	1.657385	95.175
		3	2.752174	0.143344	1.751946	95.073
	(60, 20)	1	2.723894	0.104813	1.602946	96.093
		2	2.720536	0.108171	1.628044	95.787
		3	2.720713	0.110292	1.605420	95.073

6. Discussion

This study examined the recurrence relations for single and product moments of progressively type-II censored samples from two parameter RK distribution. We have presented the values of means and variances of the progressive Type-II right censored order statistics for $\mu = 2, 3$, $\lambda = 1.0, 2.0$ and different values of r and s. We observe that the means and variances are decreasing with respect to r, s, μ and λ. From our study it is to be noted that the MLEs is decreasing with respect to increase the Schemes. For fixed s, when the number of observed failure r increases, the MSEs and the confidence interval lengths decreases in all cases. Comparing the three censoring schemes, it is clear that the MSEs, confidence interval lengths are smaller for Scheme 1 than Schemes 2 and 3.

7. Conclusions

Based on our study, we can conclude that the the MLEs is decreasing with respect to increase the Schemes. For fixed s, when the number of observed failure r increases, the MSEs and the confidence interval lengths decreases in all cases. Comparing the three censoring schemes, it is clear that the MSEs, confidence interval lengths are smaller for Scheme 1 than Schemes 2 and 3. A future work may be to derive estimation procedures for the two parameter RK distribution based on order statistics, generalized order statistics and dual generalized order statistics. Another future work may be to characterize the two parameter RK distribution based on order statistics, generalized order statistics and dual generalized order statistics.

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