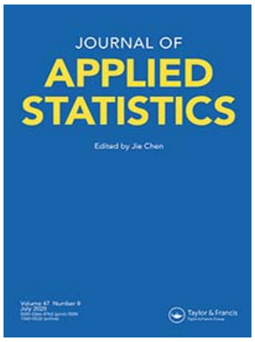

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Statistical inference based on generalized Lindley record values

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ABSTRACT

This paper addresses the problems of frequentist and Bayesian estimation for the unknown parameters of generalized Lindley distribution based on lower record values. We first derive the exact explicit expressions for the single and product moments of lower record values, and then use these results to compute the means, variances and covariance between two lower record values. We next obtain the maximum likelihood estimators and associated asymptotic confidence intervals. Furthermore, we obtain Bayes estimators under the assumption of gamma priors on both the shape and the scale parameters of the generalized Lindley distribution, and associated the highest posterior density interval estimates. The Bayesian estimation is studied with respect to both symmetric (squared error) and asymmetric (linear-exponential (LINEX)) loss functions. Finally, we compute Bayesian predictive estimates and predictive interval estimates for the future record values. To illustrate the findings, one real data set is analyzed, and Monte Carlo simulations are performed to compare the performances of the proposed methods of estimation and prediction.

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1. Introduction

Record values are of great significance in many statistical applications, statistical modeling, and inference involving data pertaining to hydrology, sports, industry, weather forecast, seismology, athletics, economics, life-testing studies and so on. For example, record values of the fastest time taken to recite the periodic table of the elements or shortest duration of tennis matches or fastest indoor marathon, etc. In practice, several attempts are performed but a record makes or breaks only when the attempt is a success, and usually the data on all of the attempts made to break a record around the world do not available. But the available data in the form of records have attracted the attentions of several researchers. In fact the frequency of weather conditions first motivated Chandler [8] to study record values for independent and identically distributed sequences of random variables. Since then several scientific findings and books have been published on record-breaking data and their distributional properties, and among them one may refer to Glick [15], Ahsanullah [1], Gulati

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and Padgett [14], Arnold et al. [3], and literature cited therein for the developments and applications using different probability distributions till the last three decades.

In practice, various phenomena can be modeled through different statistical distributions, and the properties of the phenomenon can be studied through the statistical properties of the considered particular distribution. For example, mean, variance and moments of a considered distribution can provide enough information of modeled phenomena. In this regard a significant amount of research has been done for several distributions to study record values. For an instance Balakrishnan and Ahsanullah [5] established some recurrence relations satisfied by single and product moments for the generalized Pareto distribution based on upper record values. Furthermore, single and product moments based on record observations are obtained by Balakrishnan and Chan [6] for normal distribution, Raqab [25] for generalized exponential distribution, MirMostafaee *et al.* [16] for NH distribution, and recently Devendra *et al.* [19] for exponentiated moment exponential distribution. In statistical theory, problems of estimating unknown parameters of a considered distribution under classical and Bayesian approaches are of fundamental importance as many statistical inferences can be made through the estimates of parameters based on the observed record values. Also, problem of prediction may help the practitioners in the field of reliability and statistical analysis to infer future record observations. In the existing literature, a lot of work has also been done in this direction for different statistical distributions based on record values. For instance, Ahmadi and Doostparast [2] considered problems of Bayesian estimation and prediction for some lifetime distributions such as Exponential, Weibull, Pareto and Burr type XII, Dey et al. [11] and [10] respectively for generalized exponential and generalized inverted exponential distributions, Doostparast *et al.* [13] and Singh *et al.* [32] for lognormal distribution, and Yoon *et al.* [35] for exponentiated Pareto distribution. The objective of this paper is to compute single and product moments, and estimation and prediction problems under both classical and Bayesian approaches for generalized Lindley (GL) distribution based on lower record values.

GL distribution was introduced by Nadarajah et al. [23] as an interesting lifetime model alternative to the gamma, the lognormal and the Weibull distributions. The main advantage of this distribution is that it can accommodate lifetime data having monotonically decreasing, monotonically increasing and bathtub shaped hazard rate functions which occurs in many practical problems, see Bebbington et al. [7] and references cited therein. The probability density function (PDF) and the cumulative distribution function (CDF) of a two-parameter $GL(\alpha,\beta)$ distribution are respectively given by

$$f(x;\alpha,\beta) = \frac{\alpha\beta^2(1+x)}{(1+\beta)}\left[1-\frac{1+\beta+\beta x}{(1+\beta)}\mathrm{e}^{-\beta x}\right]^{\alpha-1}\mathrm{e}^{-\beta x};\quad x>0,\ \alpha,\beta>0, \tag{1}$$

$$F(x;\alpha,\beta) = \left[1-\frac{1+\beta+\beta x}{(1+\beta)}\mathrm{e}^{-\beta x}\right]^{\alpha};\quad x>0,\ \alpha,\ \beta>0, \tag{2}$$

where α and β are respectively the shape and the scale parameters. In fact, this distribution is an extended model of the one-parameter Lindley distribution (LD), and correspond to shape parameter $\alpha=1$, it reduces to LD. In the existing literature, Singh et al. [28] considered GL distribution under the progressive type-II censoring scheme which allows the removals of the live units from a life-test with Beta-binomial probability law during the execution of the experiment. Authors obtained maximum likelihood estimates, Bayesian estimates and Bayesian predictive estimates, and further discussed the behavior of the expected total test time. Later, Singh et al. [29] discussed maximum likelihood estimates and the problem of estimation and prediction from Bayesian viewpoint based on complete sample. In Bayesian paradigm, authors considered non-informative and independent gamma informative priors under squared error and general entropy loss functions. Recently, Singh *et al.* [30] further extended their work when data are observed under type-I hybrid censoring. We observed that the existing work under Bayesian approach suggest independent gamma priors. Furthermore, we observed that problems of estimation and prediction under both classical and Bayesian approaches for GL distribution based on lower record values have not been considered. With this motivation, we first derive the single and product moments of lower record values and then use these results to compute the means, variances and covariances between two record values in Section 3. Furthermore, we consider the problems of estimation based on classical and Bayesian approaches respectively in Sections 4 and 5. The problem of Bayesian prediction is discussed in Section 6. Real data analysis and simulation study are presented in Section 7. Finally, the paper ends with a conclusion in Section 8.

2. Preliminaries

This section presents some preliminaries for better understanding of lower record values. Suppose that $X_1, X_2, \ldots$ is a sequence of continuous random variables following a PDF $f(x;\theta)$ and a CDF $F(x;\theta)$, where θ represents a vector of unknown parameters. Then an observation X_j will be a lower record if its value is smaller than all of its previous observations, that is, if $X_j < X_i$ for every $i < j$. Now let us denote $\boldsymbol{r} = (r_1, r_2, \ldots, r_m)$ as m number of lower record values observed from a distribution having PDF and CDF respectively given by $f(x;\theta)$ and $F(x;\theta)$. Notice that if we consider the first observation as a lower record such that $r_1 = X_1$ then the next record will be $r_2 = X_j$ such that $X_j < r_1, j = 2, 3, \ldots$, and likewise the i-th record $r_i = X_j$ such that $X_j < r_{i-1}$, $i = 3, \ldots, m$ and $j = i, i+1, \ldots$. In fact observing record values in such a way is an example of inverse sampling, however if record values are observed from $X_1, X_2, \ldots, X_n$ observations then it is called random sampling. Therefore, under inverse sampling m is a pre-specified number but under random sampling n is a pre-specified number and m turns out to be a random number. The PDF for the lower record values $R_m = r, m = 1, 2, \ldots$.. can be written as [3]

$$f_{R_m}(r;\theta) = \frac{1}{(m-1)!}[-\log(F(r;\theta))]^{m-1} f(r;\theta), \quad r > 0, \; m = 1, 2, \ldots. \tag{3}$$

Furthermore, the joint PDF for $R_m = r$ and $R_n = s$ such that $r > s, m = 1, 2, \ldots, m < n$ is given by

$$\begin{aligned} f_{R_m,R_n}(r,s;\theta) = {} & \frac{1}{(m-1)!(n-m-1)!}[-\log(F(r;\theta))]^{m-1} \\ & \times [-\log(F(s;\theta)) + \log(F(r;\theta))]^{n-m-1} \frac{f(r;\theta)}{F(r;\theta)} f(s;\theta). \end{aligned} \tag{4}$$

The associated likelihood function of θ given the observed lower record values $\boldsymbol{r} = (r_1, r_2, \ldots, r_m)$ can be written as

$$L(\theta \mid \boldsymbol{r}) = f(r_m; \theta) \prod_{i=1}^{m-1} \frac{f(r_i; \theta)}{F(r_i; \theta)}, \quad r > 0,\ m = 1, 2, \ldots. \tag{5}$$

3. Moments for lower record values

The objective of this section is to obtain the single and product moments for lower record values when the lifetime data follows $GL(\alpha, \beta)$ distribution. We assume that $\boldsymbol{r} = (r_1, r_2, \ldots, r_m)$ are the m number of lower record values observed from $GL(\alpha, \beta)$ distribution. Then PDF for the lower record values can be written from (3) using PDF and CDF respectively as defined in (1) and (2). Subsequently the g-th moment ($g = 1, 2, \ldots,$) for lower record value r_m, ($m = 1, 2, \ldots,$) can be derived as

$$\begin{aligned}
\mu_m^{(g)} = E(R_m^g) &= \int_0^\infty r^g f_{R_m}(r; \alpha, \beta)\, \mathrm{d}r,\ g = 1, 2, \ldots, m = 1, 2, \ldots, \\
&= \frac{\alpha^m \beta^2}{(1+\beta)(m-1)!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{i+j+m-1} (-1)^i \binom{\alpha-1}{i} \binom{i+j+m-1}{k} \\
&\quad \times \phi_j(m-1) \left(\frac{\beta}{1+\beta}\right)^k \frac{\Gamma(g+k+1)}{[\beta(i+j+1)]^{g+k+1}} \left(1 + \frac{g+k+1}{\beta(i+j+m)}\right).
\end{aligned} \tag{6}$$

To compute the final above expression, the expansions of, say, $(1-t)^j$ and $(-\ln(1-t))^j = (\sum_{p=1}^{\infty} (t^p/p))^j = \sum_{p=0}^{\infty} a_p(j) t^{j+p}$ such that $|t| < 1$ are used, and the complete gamma function is considered, see Shawky and Bakoban [27]. Furthermore, from (4) using PDF and CDF as respectively defined in (1) and (2), covariance between the lower records $R_m = r$ and $R_n = s$ such that $m < n$, is given by

$$\begin{aligned}
\mu_{m,n}^{(g,h)} = E(R_m^g R_n^h) &= \int_0^\infty \int_0^\infty R_m^g R_n^h f_{R_m,R_n}(r, s; \alpha, \beta)\, \mathrm{d}r\, \mathrm{d}s,\ m = 1, 2, \ldots, m, \text{ and } m < n \\
&= \frac{\beta^4 \alpha^n}{(1+\beta)^2 (m-1)!(n-m-1)!} \sum_{i=0}^{n-m-1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{n-i+j+k-2} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \sum_{w=0}^{u+v+i} \\
&\quad \times (-1)^{n-m-1-i+u} \binom{n-m-1}{i} \binom{n-i+j+k-2}{l} \binom{\alpha-1}{u} \binom{u+v+i}{w} \phi_v(i) \\
&\quad \times \phi_k(n-i-2) \left(\frac{\beta}{1+\beta}\right)^{l+w} (g+l+1)! \sum_{t=0}^{g+l+1} \frac{[\beta(n-i+j+k-1)]^{t-(g+l+1)}}{t!} \\
&\quad \times [1 + (g+l+1)(\beta(n-i+j+k-1))] \left[\frac{\Gamma(l+t+w+1)}{[\beta(n-i+j+k+u+v+i)]^{l+t+w+1}} \right. \\
&\quad \left. + \frac{\Gamma(l+t+w+2)}{[\beta(n-i+j+k+u+v+i)]^{l+t+w+2}}\right], \quad r, s = 1, 2, \ldots.
\end{aligned} \tag{7}$$

Table 1. Means of lower record values r_m for $GL(\alpha, 2.0)$ distribution.

m	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1.0$	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2.0$
1	0.57932	0.77435	0.93333	1.06702	1.18220	1.28330	1.37333
2	0.22531	0.36650	0.49333	0.60590	0.70642	0.79697	0.87916
3	0.12016	0.23087	0.33797	0.43671	0.52725	0.61041	0.68700
4	0.07276	0.16391	0.25773	0.34658	0.42973	0.50734	0.57967
5	0.04676	0.12417	0.20853	0.28988	0.36725	0.44046	0.50939
6	0.03084	0.09781	0.17522	0.25064	0.32330	0.39289	0.45900
7	0.02043	0.07892	0.15114	0.22173	0.29044	0.35697	0.42069
8	0.01339	0.06458	0.13291	0.19949	0.26481	0.32870	0.39034
9	0.00859	0.05318	0.11862	0.18182	0.24416	0.30575	0.36556
10	0.00533	0.04378	0.10712	0.16741	0.22712	0.28666	0.34485

Now the covariance between the R_m and the R_n records is given by $\sigma_{m,n} = \mu_{m,n} - \mu_m \mu_n$, here $\mu_m = \mu_m^{(1)}$. Furthermore, for $n = m$, the variance is given by $\sigma_m^2 = \mu_m^{(2)} - (\mu_m)^2$. Therefore, means, variances and covariances for the lower records can be computed using the expressions given by (6) and (7). Next, we consider the parameter values of $GL(\alpha, \beta)$ distributions as $\alpha = 0.5(0.25)2.0$ and $\beta = 2$, and report the means in Table 1 and variances and covariances for the first 10 record values in Table 2. Tabulated values suggest that the mean of the record values decrease when more number of records are taken into account. However, for a fixed value of the scale parameter β, mean do increase with a higher value of the shape parameter α. Furthermore, the values of variances do decrease with a large number of records, and increase with a higher value of the shape parameter α. A similar type of behavior is observed for the covariances as well. We mention that the reported observations also hold true for other values of scale parameter β.

4. Maximum likelihood estimation

The objective of this section is to derive maximum likelihood estimators (MLEs) for the unknown parameters of GL distribution based on lower record values. Suppose that $\boldsymbol{r} = (r_1, r_2, \ldots, r_m)$ be the m number of lower record values observed from $GL(\alpha, \beta)$ distribution. Then the associated likelihood function of (α, β) given the lower record values $\boldsymbol{r} = (r_1, r_2, \ldots, r_m)$ is given by (5) using PDF and CDF as respectively defined in (1) and (2). Now on equating partial derivative of the log-likelihood function, say, $l = \ln L(\alpha, \beta \mid \boldsymbol{r})$ with respect to α to zero, we get

$$\alpha = -m/\ln A(\beta, r_m). \tag{8}$$

Here $A(\beta, x) = 1 - (1 + ((\beta)/(1+\beta))x)e^{-\beta x}$. Notice that $\lim_{\beta \to 0} A(\beta, x) \to 0$ and $\lim_{\beta \to \infty} A(\beta, x) \to 1$. The limits also hold true for the values of x, and therefore $\ln A(\beta, x)$ always become negative which gives the positive unique estimate for α. Also the second derivative of the log-likelihood function $\partial^2 l/\partial \alpha^2 = -m/\alpha^2 < 0$ corresponding to any value of β ensures the maximization of the associated log-likelihood equation with respect to α. Now if $\alpha^{(k+1)} = -m/\ln A(\beta^{(k)}, x)$ is the estimate of α at $(k+1)$th stage, then the estimate of β at the $(k+1)$th stage can be obtained on solving the following equation:

$$\frac{2m}{\beta} - \frac{m}{1+\beta} - \sum_{i=1}^{m} r_i - \sum_{i=1}^{m} \frac{A_\beta(\beta, r_i)}{A(\beta, r_i)} + \alpha^{(k+1)} \frac{A_\beta(\beta, r_m)}{A(\beta, r_m)} = 0. \tag{9}$$

Table 2. Variances and covariances of lower record values for $GL(\alpha, 2.0)$ distribution.

m	n	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1.0$	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2.0$
1	1	0.38906	0.52138	0.62980	0.72134	0.80048	0.87014	0.93231
	2	0.14833	0.24227	0.32728	0.40319	0.47131	0.53290	0.58899
	3	0.07828	0.15110	0.22212	0.28808	0.34890	0.40502	0.45691
	4	0.04710	0.10660	0.16836	0.22731	0.28283	0.33491	0.38366
	5	0.03015	0.08040	0.13563	0.18933	0.24076	0.28967	0.33594
	6	0.01983	0.06313	0.11359	0.16318	0.21130	0.25764	0.30186
	7	0.01311	0.05082	0.09773	0.14400	0.18936	0.23354	0.27605
	8	0.00858	0.04151	0.08576	0.12929	0.17230	0.21463	0.25566
	9	0.00550	0.03414	0.07641	0.11763	0.15860	0.19931	0.23905
	10	0.00341	0.02807	0.06891	0.10815	0.14732	0.18660	0.22519
2	2	0.14010	0.22893	0.30939	0.38128	0.44582	0.50422	0.55742
	3	0.07385	0.14261	0.20975	0.27214	0.32971	0.38287	0.43204
	4	0.04441	0.10054	0.15887	0.21459	0.26711	0.31640	0.36256
	5	0.02842	0.07580	0.12792	0.17866	0.22727	0.27355	0.31734
	6	0.01868	0.05950	0.10710	0.15393	0.19940	0.24322	0.28506
	7	0.01235	0.04788	0.09212	0.13580	0.17865	0.22041	0.26061
	8	0.00808	0.03910	0.08082	0.12190	0.16252	0.20252	0.24131
	9	0.00518	0.03215	0.07200	0.11089	0.14957	0.18803	0.22559
	10	0.00321	0.02644	0.06492	0.10193	0.13891	0.17602	0.21249
3	3	0.06987	0.13498	0.19861	0.25778	0.31242	0.36288	0.40958
	4	0.04199	0.09510	0.15034	0.20315	0.25295	0.29972	0.34354
	5	0.02686	0.07166	0.12100	0.16906	0.21514	0.25903	0.30057
	6	0.01765	0.05624	0.10127	0.14561	0.18869	0.23024	0.26992
	7	0.01167	0.04525	0.08709	0.12843	0.16902	0.20860	0.24672
	8	0.00764	0.03695	0.07639	0.11526	0.15373	0.19163	0.22840
	9	0.00489	0.03037	0.06804	0.10483	0.14145	0.17789	0.21349
	10	0.00303	0.02497	0.06134	0.09635	0.13135	0.16650	0.20106
4	4	0.03980	0.09019	0.14262	0.19280	0.24014	0.28462	0.32631
	5	0.02545	0.06794	0.11475	0.16038	0.20417	0.24589	0.28540
	6	0.01672	0.05330	0.09601	0.13810	0.17902	0.21850	0.25623
	7	0.01105	0.04287	0.08254	0.12178	0.16032	0.19792	0.23415
	8	0.00723	0.03500	0.07240	0.10927	0.14579	0.18178	0.21673
	9	0.00463	0.02877	0.06447	0.09937	0.13413	0.16873	0.20255
	10	0.00287	0.02365	0.05811	0.09132	0.12454	0.15791	0.19073
5	5	0.02418	0.06456	0.10908	0.15251	0.19420	0.23395	0.27160
	6	0.01588	0.05063	0.09124	0.13129	0.17024	0.20784	0.24379
	7	0.01049	0.04072	0.07843	0.11575	0.15242	0.18822	0.22274
	8	0.00687	0.03324	0.06878	0.10384	0.13859	0.17285	0.20614
	9	0.00440	0.02732	0.06124	0.09442	0.12749	0.16042	0.19262
	10	0.00273	0.02246	0.05520	0.08676	0.11836	0.15011	0.18136
6	6	0.01512	0.04821	0.08691	0.12508	0.16224	0.19812	0.23243
	7	0.00999	0.03877	0.07469	0.11025	0.14523	0.17939	0.21233
	8	0.00654	0.03164	0.06548	0.09890	0.13203	0.16472	0.19647
	9	0.00418	0.02601	0.05830	0.08991	0.12144	0.15285	0.18357
	10	0.00259	0.02137	0.05254	0.08262	0.11273	0.14301	0.17282
7	7	0.00953	0.03699	0.07127	0.10524	0.13866	0.17131	0.20281
	8	0.00623	0.03018	0.06248	0.09439	0.12604	0.15727	0.18764
	9	0.00399	0.02480	0.05562	0.08580	0.11591	0.14593	0.17530
	10	0.00247	0.02039	0.05012	0.07883	0.10759	0.13652	0.16502
8	8	0.00596	0.02885	0.05973	0.09025	0.12054	0.15044	0.17952
	9	0.00381	0.02370	0.05317	0.08203	0.11085	0.13958	0.16770
	10	0.00236	0.01948	0.04790	0.07536	0.10288	0.13057	0.15785
9	9	0.00365	0.02269	0.05091	0.07857	0.10618	0.13373	0.16070
	10	0.00226	0.01865	0.04587	0.07217	0.09854	0.12509	0.15125
10	10	0.00217	0.01788	0.04399	0.06923	0.09454	0.12004	0.14516

Here $A_\beta(\beta, x)$ is the partial derivative of $A(\beta, x)$, and is given by $A_\beta(\beta, x) = xe^{-\beta x}[(1 + (\beta/(1+\beta))x) - (1/((1+\beta)^2))]$. Notice that the possible solution to the above equation can be seen graphically, and further the negative values of the second derivative given in

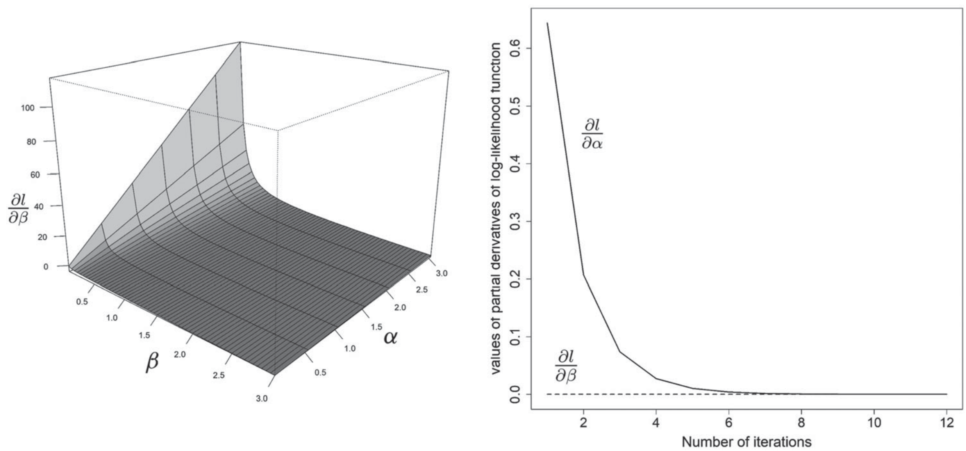

Figure 1. Plot for $(\partial l/\partial\beta)$ and convergence of partial derivatives of log-likelihood function.

the appendix can assure the maximization of the associated log-likelihood equation. In real data analysis and simulation studies described later in Section 7, we consider *nleqslv* package in R-programming language for the solution and maximization to above equation. Next, the iterative procedure given by Equations (8) and (9) can be terminated once the convergence, say $|\alpha^{(k+1)} - \alpha^{(k)}| + |\beta^{(k+1)} - \beta^{(k)}| \le \epsilon$ for some given $\epsilon > 0$, is achieved. From now onwards, we will denote the maximum likelihood estimates of α and β respectively as $\hat{\alpha}$ and $\hat{\beta}$. For illustration purpose, we consider the generated lower record values given in Section 7 under real data analysis. Now Equation (8) assures the unique estimate of α, and further for the given value of α, the estimate of β can be obtained from Equation (9). In Figure 1, we first present the graph for $\partial l/\partial\beta$ given by Equation (9) for a range of α and β in $(0, 3]$. It can be seen that for every value of α, the estimate of β can be easily observed. Next, for all the estimates of α and β obtained respectively from iterative procedure given by Equations (8) and (9), we plot both derivatives with respect to number of iterations. It can be seen that as the number of iteration increases both the derivatives converge to zero. Notice that alternatively as discussed in the work of Pak and Dey [24], for the given set of parameters and associated generated record values, both equations $\partial l/\partial\alpha$ and $\partial l/\beta$ can be plotted, and the unique intersection can ensure solution to the log-likelihood equations exists and is unique. Now the observed information matrix can be written as

$$I(\hat{\alpha}, \hat{\beta}) = -\begin{bmatrix} l_{20} & l_{11} \\ l_{11} & l_{02} \end{bmatrix}_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})}, \tag{10}$$

where $l_{ij} = \partial^2 l/\partial\alpha^i\partial\beta^j$, $i, j = 0, 1, 2$ such that $i + j = 2$, and the associated expressions are reported in Appendix. Now the asymptotic normal distribution for the MLEs can be derived in the usual way. Under some mild regularity conditions, the MLEs $(\hat{\alpha}, \hat{\beta})$ is approximately bivariate normal with mean (α, β) and covariance matrix which is the inverse of Fisher's information matrix. In practice, a simpler and equally valid procedure is to use the approximation $(\hat{\alpha}, \hat{\beta}) \sim N((\alpha, \beta), I^{-1}(\hat{\alpha}, \hat{\beta}))$, where $I(\hat{\alpha}, \hat{\beta})$ is the observed information matrix defined in (10). Therefore the two-sided $100(1 - \gamma)\%$,

$0 < \gamma < 1$, asymptotic confidence intervals for α and β can be respectively obtained as $\hat{\alpha} \pm \Phi^{-1}((\gamma/2))\sqrt{Var(\hat{\alpha})}$ and $\hat{\beta} \pm \Phi^{-1}(\gamma/2)\sqrt{Var(\hat{\beta})}$.

5. Bayesian estimation

In this section, we consider Bayesian estimation for the parameters of the GL distribution based on lower record values. In Bayesian estimation, selection of prior distribution and loss function play important roles. In the existing literature various prior distributions have been proposed for the unknown parameters of a particular distribution of interest. For example, in their work Singh et al. [28–30] considered independent gamma prior for the parameters of GL distribution, Kundu and Gupta [21] also considered independent gamma priors for the parameters of Weibull distribution, Ahmadi and Doostparast [2] proposed a bivariate prior, and Singh *et al.* [32] and Singh and Tripathi [31] considered a conditional prior for lognormal distribution. However, Arnold and Press [4] mentioned that there is clearly no way in which one can say that one prior is better than other. In the premises of the above arguments, we consider gamma priors on the shape and the scale parameters of the GL distribution such that

$$\pi(\alpha \mid c, d) = \frac{d^c}{\Gamma(c)}\alpha^{c-1}\mathrm{e}^{-\mathrm{d}\alpha}, \quad c > 0,\ d > 0,$$
$$\pi(\beta \mid a, b\alpha) = \frac{(b\alpha)^a}{\Gamma(a)}\beta^{a-1}\mathrm{e}^{-b\alpha\beta}, \quad a > 0,\ b > 0.$$

Here $\Gamma(.)$ represents the gamma function, and a, b, c and d are the hyper-parameters. Now the prior density is $\pi(\alpha,\beta) = \pi(\beta \mid a, b\alpha)\pi(\alpha \mid c, d)$, and the non-informative prior $\pi(\alpha,\beta) = (1/(\alpha\beta))$ corresponds to the zero approaching hyper-parameter values in the considered prior. Notice that posterior distribution using the Bayes theorem under consideration of prior $\pi(\alpha,\beta)$ for (α,β) can be obtained as

$$\pi(\alpha,\beta \mid \boldsymbol{r}) = \frac{\pi(\alpha,\beta)L(\alpha,\beta \mid \boldsymbol{r})}{\int_0^\infty \int_0^\infty \pi(\alpha,\beta)L(\alpha,\beta \mid \boldsymbol{r})\,\mathrm{d}\alpha\,\mathrm{d}\beta}.$$

In the existing literature squared error loss (SEL) function has been extensively considered, under which the Bayes estimator of, say $g(\alpha,\beta)$ is given by

$$\hat{g}(\alpha,\beta) \propto E\big(g(\alpha,\beta) \mid \boldsymbol{r}\big) = \int_0^\infty \int_0^\infty g(\alpha,\beta)\pi(\alpha,\beta \mid \boldsymbol{r})\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{11}$$

However, the SEL is a symmetric loss function in which under and over estimations have same weight. But in many practical situations under/over-estimation may be more serious than the over/under-estimation, and in such situations asymmetric loss function can be used. Thus we next consider LINEX loss function defined by $\delta_{LL}(g(\alpha,\beta), \hat{g}(\alpha,\beta)) = \mathrm{e}^{\hat{g}(\alpha,\beta)-g(\alpha,\beta)} - \nu(\hat{g}(\alpha,\beta) - g(\alpha,\beta)) - 1$, here $\nu \neq 0$ is a shape parameter and further suggests seriousness of over/under-estimation according to $\nu > 0/\nu < 0$. The Bayes estimator of a function $g(\alpha, \beta)$ under LINEX loss function is given by

$$\hat{g}_{LINEX}(\alpha, \beta) = -\frac{1}{\nu} \ln \left[E(e^{-\nu g(\alpha,\beta)} \mid \boldsymbol{x})\right]$$
$$= -\frac{1}{\nu} \ln \left[\int_0^\infty \int_0^\infty e^{-\nu g(\alpha,\beta)} \pi(\alpha, \beta \mid \boldsymbol{x})\, d\alpha\, d\beta\right]. \quad (12)$$

We observe that Bayes estimators under the consideration of squared error and LINEX loss functions respectively given by (11) and (12) can not be simplified into a closed form expression. So by making use of some approximation methods, we can derive explicit expressions for these estimators. In the existing literature, Lindley's method [22] has been extensively taken into account for such situations. However, this method requires third derivatives of the log-likelihood function. Instead, we consider another approximation method proposed by Tierney and Kadane (TK) [33], in which derivatives only up to second order are required to compute the desired Bayes estimates.

5.1. TK's method

This section deals with the use of TK's [33] method to approximate the Bayes estimates. Suppose our objective is to estimate the expression $E(g(\theta) \mid \boldsymbol{x})$ using the TK method. Then, we first consider the following functions:

$$\delta(\alpha, \beta) = \frac{1}{n} \ln[L(\alpha, \beta)\pi(\alpha, \beta)] \quad \text{and} \quad \delta_g^*(\alpha, \beta) = \frac{1}{n} \ln[L(\alpha, \beta)\pi(\alpha, \beta)g(\theta)]. \quad (13)$$

Now suppose that values $(\hat{\alpha}_\delta, \hat{\beta}_\delta)$ and $(\hat{\alpha}_{\delta_g^*}, \hat{\beta}_{\delta_g^*})$ respectively maximize the functions $\delta(\alpha, \beta)$ and $\delta_g^*(\alpha, \beta)$. Then the approximation using the method of TK suggests that

$$E\big(g(\theta) \mid \boldsymbol{x}\big) = \sqrt{\frac{|\Sigma_g^*|}{|\Sigma_\delta|}} \exp\left[n\big(\delta_g^*(\hat{\alpha}_{\delta_g^*}, \hat{\beta}_{\delta_g^*}) - \delta(\hat{\alpha}_\delta, \hat{\beta}_\delta)\big)\right], \quad (14)$$

where $|\Sigma_\delta|$ and $|\Sigma_g^*|$ are the negatives of inverse Hessian matrices of $\delta(\alpha, \beta)$ and $\delta_g^*(\alpha, \beta)$ respectively obtained at $(\hat{\alpha}_\delta, \hat{\beta}_\delta)$ and $(\hat{\alpha}_{\delta_g^*}, \hat{\beta}_{\delta_g^*})$. We first demonstrate the method to obtain $(\hat{\alpha}_\delta, \hat{\beta}_\delta)$ and $(\hat{\alpha}_{\delta_g^*}, \hat{\beta}_{\delta_g^*})$. Suppose that our purpose is to obtain the Bayes estimate of α under the SEL and the LINEX loss functions. Then accordingly we have $g(\alpha, \beta) = \alpha$, and subsequently we need to approximate function $E(\alpha \mid \boldsymbol{r})$ and $E(e^{-\nu\alpha} \mid \boldsymbol{r})$ respectively for SEL and LINEX loss functions in expressions given by (11) and (12). Now on equating partial derivatives of $\delta(\alpha, \beta)$ and $\delta_g^*(\alpha, \beta)$ ($g(\theta) = \alpha$ under SEL and $g(\theta) = e^{-\nu\alpha}$ under LINEX loss) with respect to α to zero, we get estimate of α in the following form:

$$\alpha = -(m + C_\alpha + K_1)/\left[\ln A(\beta, r_m) - (d + b\beta) - K_2\right], \quad (15)$$

where $C_\alpha = (a + c - 1)$. Furthermore,

$$K_1 = \begin{cases} 0 & \text{for } \delta(\alpha, \beta) \text{ expression} \\ 1 & \text{for } \delta_g^*(\alpha, \beta) \text{ under SEL} \\ 0 & \text{for } \delta_g^*(\alpha, \beta) \text{ under LINEX} \end{cases} \quad \text{and} \quad K_2 = \begin{cases} 0 & \text{for } \delta(\alpha, \beta) \text{ expression,} \\ 0 & \text{for } \delta_g^*(\alpha, \beta) \text{ under SEL,} \\ \nu & \text{for } \delta_g^*(\alpha, \beta) \text{ under LINEX.} \end{cases}$$

Now if $\alpha^{(k+1)}$ is the estimate of α at $(k+1)$th stage given by (15) then the respective estimate of β at the $(k+1)$th stage can be obtained on solving the following equation:

$$\frac{(2m+C_\beta)}{\beta} - \frac{m}{1+\beta} - \sum_{i=1}^{m} r_i - \sum_{i=1}^{m} \frac{A_\beta(\beta, r_i)}{A(\beta, r_i)} + \alpha^{(k+1)} \left(\frac{A_\beta(\beta, r_m)}{A(\beta, r_m)} - b \right) = 0, \quad (16)$$

where $C_\beta = (a-1)/\beta$. Consequently $(\hat{\alpha}_\delta, \hat{\beta}_\delta)$ and $(\hat{\alpha}_{\delta^*_\alpha}, \hat{\beta}_{\delta^*_\alpha})$ under SEL and LL can be obtained from iterative procedure given by (15) and (16) using the respective values of K_1 and K_2. Finally, Bayes estimate of α under SEL is given by (14) with

$$|\Sigma_\delta| = \left[\frac{\partial^2 \delta}{\partial \alpha^2} \times \frac{\partial^2 \delta}{\partial \beta^2} - \frac{\partial^2 \delta}{\partial \alpha \partial \beta} \times \frac{\partial^2 \delta}{\partial \beta \partial \alpha} \right]^{-1}_{\alpha=\hat{\alpha}_\delta, \beta=\hat{\beta}_\delta},$$

where

$$\frac{\partial^2 \delta}{\partial \alpha^2} = \frac{1}{n}\left[l_{20} - \frac{a+c-1}{\alpha^2} \right], \quad \frac{\partial^2 \delta}{\partial \beta^2} = \frac{1}{n}\left[l_{02} - \frac{a-1}{\beta^2} \right],$$

and

$$\frac{\partial^2 \delta}{\partial \beta \partial \alpha} = \frac{1}{n}[l_{11} - b].$$

Furthermore, under SEL function

$$|\Sigma^*_\alpha| = \left[\frac{\partial^2 \delta^*_\alpha}{\partial \alpha^2} \times \frac{\partial^2 \delta^*_\alpha}{\partial \beta^2} - \frac{\partial^2 \delta^*_\alpha}{\partial \alpha \partial \beta} \times \frac{\partial^2 \delta^*_\alpha}{\partial \beta \partial \alpha} \right]^{-1}_{\alpha=\hat{\alpha}_{\delta^*_\alpha}, \beta=\hat{\beta}_{\delta^*_\alpha}},$$

where

$$\frac{\partial^2 \delta^*_\alpha}{\partial \alpha^2} = \frac{1}{n}\left[l_{20} - \frac{a+c}{\alpha^2} \right], \quad \frac{\partial^2 \delta^*_\alpha}{\partial \beta^2} = \frac{\partial^2 \delta}{\partial \beta^2},$$

and

$$\frac{\partial^2 \delta^*_\alpha}{\partial \beta \partial \alpha} = \frac{\partial^2 \delta}{\partial \beta \partial \alpha}.$$

However, under LINEX loss function, the expressions are given by

$$\frac{\partial^2 \delta^*_\alpha}{\partial \alpha^2} = \frac{\partial^2 \delta}{\partial \alpha^2}, \quad \frac{\partial^2 \delta^*_\alpha}{\partial \beta^2} = \frac{\partial^2 \delta}{\partial \beta^2},$$

and

$$\frac{\partial^2 \delta^*_\alpha}{\partial \beta \partial \alpha} = \frac{\partial^2 \delta}{\partial \alpha \partial \beta}.$$

Furthermore, for Bayes estimate of β, we have $g = \beta$ under SEL and $g = e^{-\nu\beta}$ under LINEX loss. Therefore the estimate of α is given by

$$\alpha = -(m + C_\alpha) / \left[\ln A(\beta, r_m) - (d + b\beta) \right]. \quad (17)$$

Furthermore, estimate of β at the $(k+1)$th stage can be obtained on solving the following equation:

$$\frac{(2m+C_\beta+K_1)}{\beta}-\frac{m}{1+\beta}-\sum_{i=1}^{m} r_i-\sum_{i=1}^{m}\frac{A_\beta(\beta,r_i)}{A(\beta,r_i)} + \alpha^{(k+1)}\left(\frac{A_\beta(\beta,r_m)}{A(\beta,r_m)}-b\right)-K_2=0. \tag{18}$$

The iterative procedure given by (17) and (18) provides estimates for $(\hat{\alpha}_\delta,\hat{\beta}_\delta)$ and $(\hat{\alpha}_{\delta^*_\beta},\hat{\beta}_{\delta^*_\beta})$ under SEL and LINEX loss function. The rest of the involved expressions for the computation of Bayes estimates can be obtained easily likewise in the case of α. Furthermore, the complete expressions for terms l_{20}, l_{02} and l_{11} are reported in the Appendix. It is to be noticed that this method is not useful in interval estimation. Thus we consider Metropolis-Hastings (MH) algorithm for this purpose, and the procedure is illustrated in the next section.

5.2. MH algorithm

This section discusses MH algorithm. This algorithm is very much useful particularly when posterior distribution does not admit analytically tractable form, and it has been widely used in Bayesian inference. In our case also the posterior distribution does not admit a closed form. Therefore we first consider a symmetric proposal distribution of type $J((\acute{\alpha},\acute{\beta})\mid(\alpha,\beta))=J((\alpha,\beta)\mid(\acute{\alpha},\acute{\beta}))$ to approximate the posterior distribution. Notice that a bivariate normal density $N_2((\alpha,\beta),I_X^{-1}(\alpha,\beta))$ where $I_X^{-1}(\alpha,\beta)$ is the inverse of the observed Fisher information matrix can be an ideal choice. However, considering a bivariate normal distribution there are possibilities that negative observations may occur for β, but here $\beta>0$ so it can not be accepted. To avoid this situation we consider the algorithm as discussed by Dey *et al.* [12]. Now suppose that we have total N generated samples of (α,β) according to the algorithm. Then from these samples, we discard some of the initial samples (burn-in), say N_0, and consider the remaining M number of samples such that $M=N-N_0$ for further utilization to compute Bayes estimates. Subsequently the Bayes estimates of α and β under the SEL can be obtained as

$$\hat{\alpha}_{SEL}=\frac{1}{M}\sum_{k=1}^{M}\alpha_k \quad\text{and}\quad \hat{\beta}_{SEL}=\frac{1}{M}\sum_{k=1}^{M}\beta_k.$$

Likewise, the Bayes estimates of α and β under the LINEX loss can be respectively obtained as

$$\hat{\alpha}_{LINEX}=-\frac{1}{\nu}\ln\left(\frac{1}{M}\sum_{k=1}^{M}e^{-\nu\alpha_k}\right)\quad\text{and}\quad \hat{\beta}_{LINEX}=-\frac{1}{\nu}\ln\left(\frac{1}{M}\sum_{k=1}^{M}e^{-\nu\beta_k}\right).$$

Furthermore, the method of Chen and Shao [9] can be used to construct the highest posterior density (HPD) intervals.

6. Bayesian prediction

This section deals with the problem of Bayesian prediction in which m number of lower record values, say, $\boldsymbol{r} = (r_1, r_2, \ldots, r_m)$ are available, and our interest is to predict future s-th lower record value and predictive interval estimate for the s-th record, $1 < m < s$. Notice that the conditional PDF of future record $R_s = r_s$ given the record values $\boldsymbol{r} = (r_1, r_2, \ldots, r_m)$ can be written as [32]

$$f_1(r_s \mid \boldsymbol{r}, \alpha, \beta) = \frac{1}{F(r_m)\Gamma(s-m)} \sum_{i=0}^{s-m-1} \binom{s-m-1}{i} (\ln F(r_m))^i (-\ln F(r_s))^{s-m-1-i} f(r_s),$$
$$0 < r_s < r_m < \infty. \tag{19}$$

Here $f(\cdot) = f(\cdot; \alpha, \beta)$ and $F(\cdot) = F(\cdot; \alpha, \beta)$. Assume that a prior $\pi(\alpha, \beta)$ for (α, β) is considered. Then by making use of the likelihood function $L(\alpha, \beta \mid \boldsymbol{r})$, the associated posterior predictive density can be written as

$$f_1^*(r_s \mid \boldsymbol{r}) = \int_0^\infty \int_0^\infty f_1(r_s \mid \boldsymbol{r}, \alpha, \beta) \pi(\alpha, \beta \mid \boldsymbol{r}) \, \mathrm{d}\alpha \, \mathrm{d}\beta.$$

Furthermore, making use of the above posterior predictive density, a predictive estimate for s-th lower record under SEL can be obtained as $E(r_s \mid \boldsymbol{r})$, and is given by

$$\hat{r}_s = \int_0^{r_m} r_s f_1^*(r_s \mid \boldsymbol{r}) \, \mathrm{d}r_s = \int_0^\infty \int_0^\infty \left[\int_0^{r_m} r_s f_1(r_s \mid \boldsymbol{r}, \alpha, \beta) \, \mathrm{d}r_s \right] \pi(\alpha, \beta \mid \boldsymbol{r}) \, \mathrm{d}\alpha \, \mathrm{d}\beta,$$
$$= \int_0^\infty \int_0^\infty I_1(\alpha, \beta) \pi(\alpha, \beta \mid \boldsymbol{r}) \, \mathrm{d}\alpha \, \mathrm{d}\beta, \tag{20}$$

where

$$I_1(\alpha, \beta) = \frac{1}{F(r_m)\Gamma(s-m)} \sum_{i=0}^{s-m-1} \binom{s-m-1}{i} [\ln F(r_m)]^i$$
$$\times \int_0^{r_m} r_s (-\ln F(r_s))^{s-m-1-i} f(r_s) \, \mathrm{d}r_s.$$

Observe that the expression given by (20) can be seen as $E(I_1(\alpha, \beta) \mid \boldsymbol{r})$ which can be computed easily based on the samples drawn using the MH algorithm, and subsequently we get $\hat{r}_s = [\sum_{i=1}^M I_1(\alpha_i, \beta_i)]/M$. Likewise a predictive estimate under LINEX loss function can be obtained as $-(1/\nu) \ln[E(\mathrm{e}^{-\nu r_s} \mid \boldsymbol{r})]$.

Now based on given record values $\boldsymbol{r} = (r_1, r_2, \ldots, r_m)$, CDF for the future record $R_s = r_s$ can be written as

$$F_1(t \mid \boldsymbol{r}, \alpha, \beta) = \frac{P(R_s \leq t \mid \boldsymbol{r}, \alpha, \beta)}{P(R_s \leq r_m \mid \boldsymbol{r}, \alpha, \beta)},$$

where

$$P(R_s \leq w \mid \boldsymbol{r}, \alpha, \beta) = \int_0^w f_1(r_s \mid \boldsymbol{r}, \alpha, \beta)\, \mathrm{d}r_s$$
$$= \frac{1}{F(r_m)\,\Gamma(s-m)} \sum_{i=0}^{s-m-1} \binom{s-m-1}{i} (\ln F(r_m))^i \Gamma(s-m-i)$$
$$[1 - G(-\ln F(w), s-m-i)].$$

Here $G(x, \alpha)$ denotes the CDF of a gamma distribution. Thus under the prior $\pi(\alpha, \beta)$, associated posterior predictive distribution function $F_1^*(t \mid \boldsymbol{r})$ can be obtained which will be further used to compute predictive interval estimate for r_s-th record. An equal-tail symmetric predictive interval, say (L, U) with $1-\gamma$ degree of belief can be obtained on solving the following non-linear equations for L and U:

$$F_1^*(L \mid \boldsymbol{r}) = \frac{\gamma}{2} \quad \text{and} \quad F_1^*(U \mid \boldsymbol{r}) = 1 - \frac{\gamma}{2}.$$

The algorithm given by Singh and Tripathi [31] can also be implemented to solve the above expressions using the samples drawn by the MH algorithm. Furthermore, to obtain HPD predictive interval, one may refer to the algorithm discussed in Turkkan and Pham-Gia [34]. However, in case of the uni-modal posterior predictive density, alternatively the HPD predictive interval (L, U) can also be obtained by simultaneously solving the following equations:

$$\int_L^U f_1^*(r_s \mid \boldsymbol{r})\, \mathrm{d}r_s = 1 - \gamma \quad \text{and} \quad f_1^*(L \mid \boldsymbol{r}) = f_1^*(U \mid \boldsymbol{r}).$$

7. Simulation study and data analysis

7.1. Data analysis

To demonstrate how the proposed methods can be used in practice, we consider the following data set on the total annual rainfall (in inches) during March recorded at Los Angeles Civic Center from 1973 to 2006 (see the website of Los Angeles Almanac: www.laalman-ac.com/weather/we08aa.htm).

```
2.70  3.78  4.83  1.81  1.89  8.02  5.85  4.79  4.10  3.54
8.37  0.28  1.29  5.27  0.95  0.26  0.81  0.17  5.92  7.12
2.74  1.86  6.98  2.16  0.00  4.06  1.24  2.82  1.17  0.32
4.31  1.17  2.14  2.87.
```

We notice that in the given data set, rain fall data in the year 1997 is 0.00 which means no rain in that year. Now if one wants to proceed with the same data with GL distribution then either a small value for the year need to be considered or to omit this observation as this distribution's support is greater than zero. We further mention that one may also proceed with the rainfall data till the 2.16 observation of year 1996 as after that no lower record value is observed. For a comparison purpose, we take into account Lindley (L) distribution, Lognormal (LN) distribution, and power Lindley (PL) distribution, and consider

Table 3. Goodness-of-fit test criterion values.

Model	PDF	$\hat{\alpha}$	$\hat{\beta}$	NL	AIC	BIC
GL	$f(x;\alpha,\beta)$	0.7134	0.4463	70.414	144.8292	147.8819
L	$\frac{\alpha^2}{\alpha+1}(1+x)e^{-\alpha x}$	0.5321		71.654	145.3087	146.8350
PL	$\frac{\alpha\beta^2}{\beta+1}(1+x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}}$	0.9098	0.5952	71.384	146.7682	149.8209
LN	$\frac{1}{x\sqrt{\beta}}\phi\left(\frac{\ln x-\alpha}{\sqrt{\beta}}\right)$	0.4876	3.8599	87.7864	179.5728	182.6256

Table 4. Maximum likelihood, Bayes and associated interval estimates.

Maximum likelihood estimation		Bayesian estimation						
		TK method			MH algorithm			HPD interval
MLE	Asymptotic C.I.	SEL	LINEX loss		SEL	LINEX loss		
1.9556	(0, 4.1306)	1.0862	1.3029[a]	1.3032[b]	1.4775	1.5904[a]	1.6239[b]	(0.6758, 3.5384)
0.9407	(0, 2.0876)	1.3357	0.9651	0.9633	1.0894	1.0592	1.1094	(0.3271, 2.0334)

Note: *a* and *b* respectively correspond to the values for $\nu = -0.15$ and $\nu = 0.15$.

negative log-likelihood (NL) criterion, AIC and BIC. All the values are reported in Table 3 which suggest the GL distribution is more appropriate for this data set. Now the lower record values from the given data set are $2.70, 1.81, 0.28, 0.26, 0.17$, and based on these record values the uniqueness, existence and further convergence of iteration have been shown in Section 4, also see Figure 1. We get the maximum likelihood estimates of (α, β) as $(1.9556, 0.9407)$ with NL value 0.1344. However, for Lindley distribution maximum likelihood estimate is 0.5872 with NL value 0.7526, for PL distribution maximum likelihood estimates are (0.8217, 0.8852) with NL value 1.4102, and finally for LN distribution maximum likelihood estimates are (0.6278, 0.9667) with NL value 3.0213. We observe that the maximum likelihood estimates under GL distribution maximizes the associated likelihood function as compared to other considered distributions, so still based on the generated lower record values GL distribution can be the best choice. We next consider these five lower records, and based on these observations, we compute maximum likelihood estimates and associated interval estimates, Bayes estimates using the TK method and MH algorithm based on squared error and LINEX loss functions, and HPD interval estimates using the method of Chen and Shao [9]. We mention that for LINEX loss function, we have considered two values of ν as $\nu = -0.15$ and $\nu = 0.15$. Furthermore, to estimate the Bayes estimates, hyper-parameter values are considered as $a = 2, b = 1, c = 2$ and $d = 1$, the values are chosen in such a way that the prior means $\alpha = c/d$ and $\beta = a/(b\alpha)$ remain sufficiently close to the maximum likelihood estimated values of the unknown parameters, see Kundu [20] and Singh and Tripathi [31]. All the estimated values are reported in Table 4. It can be observed that the Bayes estimates obtained using the MH algorithm have larger values compared to the TK method. Next, to illustrate prediction problem, we consider the lower record values and provide inference about the next three lower record values. The computed results are given in Table 5. From tabulated values, it is seen that a higher value of s leads to wider predictive intervals. It is also observed that the lengths of HPD predictive intervals are smaller than those of equal-tail predictive intervals.

Table 5. Bayesian predictive and associated predictive interval estimates.

	SEL	LINEX loss		Credible interval	HPD interval
s	$\hat{r}_s$	$\hat{r}_s$	$\hat{r}_s$	(L, U)/ AIL	(L, U)/ AIL
$m+1$	0.1607	0.1616[a]	0.1604[b]	(0.1508, 0.1698)/ 0.0190	(0.1553, 0.1700)/ 0.0147
$m+2$	0.1498	0.1508	0.1519	(0.1395, 0.1645)/ 0.0250	(0.1401, 0.1657)/ 0.0256
$m+3$	0.1119	0.1203	0.1258	(0.0976, 0.1605)/ 0.0629	(0.1029, 0.1585)/ 0.0556

Note: a and b respectively correspond to the values for $\nu = -0.15$ and $\nu = 0.15$.

Table 6. Average estimates and associated confidence interval estimates, AILS and CPs.

		MLE			Asymptotic confidence interval	
m		α	β		α	β
5	Average	1.3702	1.9254	Average interval	(0.3597, 4.3189)	(0.6971, 6.0826)
	MSE	0.3567	0.3880	AIL/ CP	3.9592/ 90.71	5.3855/ 93.09
8	Average	1.3278	1.8219	Average interval	(0.4089, 4.1683)	(0.6819, 5.9389)
	MSE	0.2437	0.2908	AIL/ CP	3.7594/ 92.87	5.2570/ 93.96
10	Average	1.1745	1.6597	Average interval	(0.4218, 3.9815)	(0.7292, 5.6814)
	MSE	0.1510	0.1829	AIL/ CP	3.5597/ 94.98	4.9522/ 95.47

7.2. Simulation study

In this section, we conduct a simulation study to observe the behavior of different proposed methods of estimation and prediction. We first simulate n number of observations from the GL(1, 1.5) distribution, and from generated observations we then compute the m number of record values which are used to compute MLEs, Bayes estimates and associated interval estimates. We mention that the simulation study may be influenced by the generated records, however the results based on a large number of repetitions may represent the same phenomenon. In this work, the results are based on 5000 repetitions using R-statistical software, and are reported in Table 6. To compare the performance of proposed estimators, average estimates and means square error (MSE) values are taken into account. Tabulated values suggest that with more number of record values, behavior of estimates improve in terms of smaller MSE values and tend to close the true parameter values. Furthermore, the average interval lengths (AILs) for asymptotic confidence intervals decrease but associated 95% coverage percentages (CPs) do improve. It can be seen that reported asymptotic confidence intervals contain the true and estimate values, and further the associated CPs are near the nominal level. Table 7 report Bayes estimates obtained using the TK method and MH algorithm under SEL and LINEX loss functions correspond to the two values of ν such that $\nu = -0.15$ and $\nu = 0.15$. The hyper-parameter values are considered as $a = 3, b = 2, c = 2$ and $d = 2$ so the prior means remain close to the true parameter values. Tabulated values observe similar type of behavior, as reported for Table 6 in case of large number of record values. Furthermore, average estimated values of α are higher, in general, obtained using the MH algorithm as compared to the TK method but an opposite behavior can be seen for the estimated values of β. However, MSE values are smaller associated to estimates obtained using the MH algorithm. Furthermore, AILs and CPs associated to HPD intervals obtained using the idea of Chen and Shao, are smaller for α as compared to β. AILs tend to decrease and associated CPs tend to increase with a large number of record values. Finally for predictive interval estimates, Tabulated values reported in Table 8

Table 7. Bayes estimates and associated intervals, AILs and CPs.

			TK			MH			HPD interval	
m			SEL	LINEX loss		SEL	LINEX loss			HPD interval
5	α	Average	1.3318	1.3251[a]	1.3408[b]	1.3812	1.3527[a]	1.3906[b]	Average interval	(0.3791, 4.1892)
		MSE	0.2736	0.2682	0.2509	0.2387	0.2438	0.2216	AIL/ CP	3.8101/ 94.06
	β	Average	1.8698	1.8198	1.9120	1.8257	1.7898	1.8592	Average interval	(0.6595, 5.7591)
		MSE	0.3598	0.3189	0.3291	0.3172	0.3098	0.2981	AIL/ CP	5.0996/ 95.18
8	α	Average	1.2992	1.2896	1.3153	1.3481	1.3269	1.3685	Average interval	(0.3973, 3.9129)
		MSE	0.2340	0.2298	0.2164	0.2141	0.2204	0.2047	AIL/ CP	3.5156/ 95.83
	β	Average	1.8387	1.7482	1.8571	1.7489	1.7159	1.7579	Average interval	(0.6972, 5.6939)
		MSE	0.2974	0.2792	0.2869	0.2284	0.2185	0.2091	AIL/ CP	4.9967/ 96.09
10	α	Average	1.1678	1.1654	1.1701	1.1571	1.1474	1.1984	Average interval	(0.3686, 3.7892)
		MSE	0.1491	0.1401	0.1318	0.1468	0.1398	0.1309	AIL/ CP	3.4206/ 97.77
	β	Average	1.7156	1.6594	1.7549	1.6792	1.6598	1.7018	Average interval	(0.6794, 5.2896)
		MSE	0.1921	0.1910	0.1982	0.1782	0.1871	0.1820	AIL/ CP	4.6102/ 98.18

Note: a and b respectively correspond to the values for $\nu = -0.15$ and $\nu = 0.15$.

Table 8. Predictive interval estimates and associated AILs and CPs.

		$m+1$		$m+2$	
m	$s=$	ET	HPD	ET	HPD
5	Average interval	(0.0591, 5.0332)	(0.0395, 3.0612)	(0.0450, 5.5427)	(0.0348, 3.0724)
	AIL/ CP	4.9741/ 95.6	3.0217/ 96.3	5.4977/ 96.8	3.0376/ 97.3
8	Average interval	(0.0061, 2.9414)	(0.0052, 2.0914)	(0.0056, 2.9681)	(0.0041, 2.0916)
	AIL/ CP	2.9353/ 96.4	2.0862/ 97.1	2.9625/ 97.4	2.0875/ 98.1
10	Average interval	(0.0008, 1.3956)	(0.0007, 1.3571)	(0.0006, 1.3956)	(0.0004, 1.3519)
	AIL/ CP	1.3948/ 97.5	1.3564/ 98.3	1.3950/ 97.5	1.3515/ 98.5

suggest the performance of HPD interval estimates appreciable as compared to equal-tail interval estimates both in the terms of AILs and associated CPs. It can be observed that AILs for predictive interval estimates are higher for the bigger value of s, and based on more number of records AILs and CPs do improve.

Remark 7.1: In many practical applications, the total number of attempts made before making/breaking a record become available. For statistical analysis, such data are studied in the form of $(\boldsymbol{R}, \boldsymbol{K}) = (R_1, K_1, R_2, K_2, \ldots, R_m, K_m)$. Here R_i represent the i-th record value, and K_i denote the number of trials following the R_i observations that are required to obtain a new record value R_{i+1}, called inter-record time. The study of record values with inter-record times have also considered attention of many researchers, see Kizilaslan and Nadar [17] and [18] on generalized exponential and Kumaraswamy distributions, Pak and Dey [24] on power Lindley distribution, and references cited therein. Suppose that from a distribution having PDF and CDF respectively given by $f(x;\theta)$ and $F(x;\theta)$, record values, say, $\boldsymbol{r} = (r_1, r_2, \ldots, r_m)$ are observed with inter-record times, say, $\boldsymbol{k} = (k_1, k_2, \ldots, k_m)$ in the form $(\boldsymbol{r}, \boldsymbol{k})$. Then the associated likelihood function of θ given the data $(\boldsymbol{r}, \boldsymbol{k})$ can be written as [26]

$$L_1(\theta \mid \boldsymbol{r}, \boldsymbol{k}) = \prod_{i=1}^{m} f(r_i;\theta)[1 - F(r_i;\theta)]^{k_i-1} I_{(-\infty, r_{i-1})}(r_i),$$

where $I_A(r)$ is the indicator function of the set A. Notice that under inverse sampling scheme $k_m \equiv 1$, and under random sampling scheme $k_m \equiv n - \sum_{i=1}^{m-1} k_i$ where m denote the random number of records observed from the first n (pre-fixed) number of observations. So if we assume that record with inter-record times are observed from $GL(\alpha, \beta)$ distribution, then MLEs of α and β and associated asymptotic confidence intervals can be obtained using the likelihood function $L_1(\theta \mid \boldsymbol{r}, \boldsymbol{k})$ with PDF and CDF as defined in (1) and (2), and proceeding likewise in Section 4. In a similar way, the procedures for Bayesian estimation and prediction, and associated interval estimates will be same as discussed in Sections 5 and 6 but with the likelihood function $L_1(\theta \mid \boldsymbol{r}, \boldsymbol{k})$.

8. Conclusion

In this paper, we have considered GL distribution when data are available in the form of lower record values. We first presented expressions for the single and product moments, and by making use of them we have obtained means, variance and covariances for the lower record values. In the simulation study, we observed that means and variances do decrease when a high number of lower records are taken into account, however, the behavior is found opposite when a large value of shape parameter is considered correspond to a fixed value of scale parameter. In practice, the reported values can also be used for best linear unbiased estimation and prediction. We next considered the problem of estimating the unknown parameters of the GL distribution and obtained maximum likelihood estimators and associated interval estimates for the unknown parameters of the distribution. In the simulation study, we observed that with more number of record values behavior of estimates improve in terms of smaller mean square error values and tend to close the true parameter values. Furthermore, the average interval lengths for asymptotic confidence intervals decrease but associated 95% coverage percentages do improve. It is also observed that asymptotic confidence intervals contain the true and estimates values, and further the associated CPs are near the nominal level. Next, we considered the problem of Bayesian estimation and proposed gamma priors for the shape and the scale parameters of the distribution. We made use of the TK method and MH algorithm to compute Bayes estimates under squared error and LINEX loss functions. In the simulation study, we found the performance of the TK method appreciable. However, the MH algorithm helped to compute HPD interval estimates, and further in predicting the future lower records and the computation of associated predictive interval estimates under Bayesian framework. We observed that predictive intervals contain the predictive estimates, and the performance of the intervals do better with the consideration of a higher sample size. We also illustrated all the proposed methods of estimation and prediction using a real data set. Finally, a remark on the consideration of lower record values with inter-record times is presented. The proposed methodologies can also be considered for upper record values.

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References

[1] M. Ahsanullah, *Record Statistics*, Nova Science Publishers, Commack, NY, 1995.
[2] J. Ahmadi and M. Doostparast, *Bayesian estimation and prediction for some life distributions based on record values*, Stat. Pap. 47 (2006), pp. 373–392.
[3] B.C. Arnold, N. Balakrishnan, and H.N. Nagaraja, *Record*, Wiley, New York, 1998.
[4] B.C. Arnold and S.J. Press, *Bayesian inference for Pareto populations*, J. Econom. 21 (1983), pp. 287–306.
[5] N. Balakrishnan and M. Ahsanullah, *Recurrence relations for single and product moments of record values from generalized Pareto distribution*, Commun. Stat. – Theory Methods 23 (1994), pp. 2841–2852.
[6] N. Balakrishnan and P.S. Chan, *On the normal record values and associated inference*, Stat. Probab. Lett. 39 (1998), pp. 73–80.
[7] M. Bebbington, C.D. Lai, and R. Zitikis, *Bathtub-type curves in reliability and beyond*, Austral. N. Z. J. Stat. 49 (2007), pp. 251–265.
[8] K.N. Chandler, *The distribution and frequency of record values*, J. R. Stat. Soc. Ser. B 14 (1952), pp. 220–228.
[9] M.H. Chen and Q.M. Shao, *Monte Carlo estimation of Bayesian credible and HPD intervals*, J. Comput. Graph. Stat. 8 (1999), pp. 69–92.
[10] S. Dey, T. Dey, and D.J. Luckett, *Statistical inference for the generalized inverted exponential distribution based on upper record values*, Math. Comput. Simul. 120 (2016), pp. 64–78.
[11] S. Dey, T. Dey, M. Salehi, and J. Ahmadi, *Bayesian inference of generalized exponential distribution based on lower record values*, Am. J. Math. Manag. Sci. 32 (2013), pp. 1–18.
[12] S. Dey, S. Singh, Y.M. Tripathi, and A. Asgharzadeh, *Estimation and prediction for a progressively censored generalized inverted exponential distribution*, Stat. Methodol. 32 (2016), pp. 185–202.
[13] M. Doostparast, S. Deepak, and A. Zangoie, *Estimation with the lognormal distribution on the basis of records*, J. Stat. Comput. Simul. 83 (2013), pp. 2339–2351.
[14] S. Gulati and W.J. Padgett, *Smooth nonparametric estimation of the distribution and density functions from record-breaking data*, Commun. Stat. Theory Methods 23 (1994), pp. 1259–1274.
[15] N. Glick, *Breaking records and breaking boards*, Am. Math. Mon. 85 (1978), pp. 2–26.
[16] S.K. MirMostafaee, A. Asgharzadeh, and A. Fallah, *Record values from NH distribution and associated inference*, Metron 74 (2016), pp. 37–59.
[17] F. Kizilaslan and M. Nadar, *Estimation with the generalized exponential distribution based on record values and inter-record times*, J. Stat. Comput. Simul. 85 (2015), pp. 978–999.
[18] F. Kizilaslan and M. Nadar, *Estimation and prediction of the Kumaraswamy distribution based on record values and inter-record times*, J. Stat. Comput. Simul. 86 (2016), pp. 2471–2493.
[19] D. Kumar, T. Dey, and S. Dey, *Statistical inference of exponentiated moment exponential distribution based on lower record values*, Commun. Math. Stat. 5 (2017), pp. 231–260.
[20] D. Kundu, *Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring*, Technometrics 50 (2008), pp. 144–154.
[21] D. Kundu and R.D. Gupta, *Generalized exponential distribution: Bayesian estimations*, Comput. Stat. Data. Anal. 52 (2008), pp. 1873–1883.
[22] D.V. Lindley, *Approximate Bayesian methods*, Trabajos De Estadistica Y De Investigacn Operativa 31 (1980), pp. 223–245.
[23] S. Nadarajah, H.S. Bakouch, and R. Tahmasbi, *A generalized Lindley distribution*, Sankhya B 73 (2011), pp. 331–359.
[24] A. Pak and S. Dey, *Statistical inference for the power Lindley model based on record values and inter-record times*, J. Comput. Appl. Math. 347 (2019), pp. 156–172.
[25] M.Z. Raqab, *Inferences for generalized exponential distribution based on record statistics*, J. Stat. Plann. Inference 104 (2002), pp. 339–350.

[26] F.J. Samaniego and L.R. Whitaker, *On estimating population characteristics from record breaking observations. I. Parametric results*, Nav. Res. Logist. Q. 33 (1986), pp. 531–543.
[27] A.I. Shawky and R.A. Bakoban, *Characterization from exponentiated gamma distribution based on record values*, J. Stat. Theory Appl. 7 (2008), pp. 263–277.
[28] S.K. Singh, U. Singh, and V.K. Sharma, *Expected total test time and Bayesian estimation for generalized Lindley distribution under progressively type-II censored sample where removals follow the beta-binomial probability law*, Appl. Math. Comput. 222 (2013), pp. 402–419.
[29] S.K. Singh, U. Singh, and V.K. Sharma, *Bayesian estimation and prediction for the generalized Lindley distribution under asymmetric loss function*, Hacettepe J. Math. Stat. 43 (2014), pp. 661–678.
[30] S.K. Singh, U. Singh, and V.K. Sharma, *Estimation and prediction for type-I hybrid censored data from generalized Lindley distribution*, J. Stat. Manag. Syst. 19 (2016), pp. 367–396.
[31] S. Singh and Y.M. Tripathi, *Bayesian estimation and prediction for a hybrid censored lognormal distribution*, IEEE Trans. Reliability 65 (2016), pp. 782–795.
[32] S. Singh, Y.M. Tripathi, and S.J. Wu, *Bayesian estimation and prediction based on lognormal record values*, J. Appl. Stat. 44 (2017), pp. 916–940.
[33] L. Tierney and J.B. Kadane, *Accurate approximations for posterior moments and marginal densities*, J. Am. Stat. Assoc. 81 (1986), pp. 82–86.
[34] N. Turkkan and T. Pham-Gia, *Computation of the highest posterior density interval in Bayesian analysis*, J. Stat. Comput. Simul. 44 (1993), pp. 243–250.
[35] S. Yoon, Y. Cho, and K. Lee, *Estimation based on lower record values from exponentiated Pareto distribution*, J. Korean Data Inform. Sci. Soc. 28 (2017), pp. 1205–1215.

Appendix

Notice that we denote

$$A(\beta,x) = 1 - \left(1 + \frac{\beta}{1+\beta}x\right)\mathrm{e}^{-\beta x},$$
$$A_\beta(\beta,x) = x\mathrm{e}^{-\beta x}\left[\left(1 + \frac{\beta}{1+\beta}x\right) - \frac{1}{(1+\beta)^2}\right],$$

and

$$A_{\beta\beta}(\beta,x) = -xA_\beta(\beta,x) + x\mathrm{e}^{-\beta x}\left[\frac{\beta x}{(1+\beta)^2} + \frac{2}{(1+\beta)^3}\right].$$

Furthermore, the expressions for l_{20}, l_{02} and l_{11} are given below

$$l_{20} = -m/\alpha^2,$$
$$l_{02} = -\frac{2m}{\beta^2} + \frac{m}{(1+\beta)^2} - \sum_{i=1}^{m}\left[\frac{A_{\beta\beta}(\beta,r_i)}{A(\beta,r_i)} - \frac{(A_\beta(\beta,r_i))^2}{(A(\beta,r_i))^2}\right] + \alpha\left[\frac{A_{\beta\beta}(\beta,r_m)}{A(\beta,r_m)} - \frac{(A_\beta(\beta,r_m))^2}{(A(\beta,r_m))^2}\right],$$
$$l_{11} = \frac{A_\beta(\beta,r_m)}{A(\beta,r_m)}.$$