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Alpha power transformed inverse Lindley distribution: A distribution with an upside-down bathtub-shaped hazard function

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ABSTRACT

The inverse Lindley distribution has been generalized by many authors in recent years. Here, we introduce a new generalization called alpha power transformed inverse Lindley (APTIL) distribution that provides better fits than the inverse Lindley distribution and some of its known generalizations. The new model includes the inverse Lindley distribution as a special case. Various properties of the proposed distribution, including explicit expressions for the mode, moments, conditional moments, mean residual lifetime, Bonferroni and Lorenz curves, entropies, stochastic ordering, stress–strength reliability and order statistics are derived. The new distribution can have an upside-down bathtub failure rate function depending on its parameters. The model parameters are obtained by the method of maximum likelihood estimation. The approximate confidence intervals of the model parameters are also obtained. A simulation study is carried out to examine the performance of the maximum likelihood estimators of the parameters. Finally, two data sets have been analyzed to show how the proposed model works in practice.

1. Introduction

Most of the standard distributions are incapable of modeling a variety of complex real data sets; particularly, lifetime ones. This is a matter of grave concern among distribution users and researchers and has resulted in enormous research attention over the last two decades. Fortunately, several research breakthroughs have been made by many researchers in their pursuit for solution to model complex real data sets. In the recent past, researchers proposed various ways of generating new continuous distributions in lifetime data analysis to enhance its capability to fit diverse lifetime data which have a high degree of skewness and kurtosis. A detailed survey of methods for generating distributions was discussed by Lee et al. [1] and Jones [2]. Most of these distributions are special cases of the T-X class defined by Alzaatreh et al. [3]. This class of distributions extends some recent families such as the beta-G pioneered by Eugene et al. [4], the gamma-G defined by Zografos and Balakrishnan [5], the Kw-G family proposed by Cordeiro and Castro [6] and the Weibull-G introduced by Bourguignon et al. [7] and so on.

The one parameter Lindley distribution was originally introduced by Lindley [8] in the context of Bayesian statistics, as a counter example of fiducial statistics. Lindley distribution has only one scale parameter and is capable of modeling the data with monotonic increasing failure rate and as such the applicability of this distribution may be limited to some real life

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phenomenon which has non-monotone failure rates. Since the shape parameter plays a vital role in describing the various behavior of the distribution, many generalizations of the Lindley distribution have been attempted by researchers under different scenarios. Notable among these generalizations which we are aware of are: Sankaran [9] discussed the discrete Poisson–Lindley distribution by compounding the Poisson distribution and the Lindley distribution. Ghitany et al. [10] investigated the properties of the zero-truncated Poisson–Lindley distribution. Zakerzadeh and Dolati [11] introduced and analyzed a three-parameter generalization of the Lindley distribution. Weighted Lindley distribution is due to Ghitany et al. [12]. Nadarajah et al. [13] proposed a generalized Lindley distribution and provided comprehensive account of the mathematical properties of the distribution. Bakouch et al. [14] extended the Lindley distribution by exponentiation. Exponential Poisson–Lindley distribution is due to Barreto-Souza and Bakouch [15]. Power Lindley distribution is due to Ghitany et al. [16]. Shanker et al. [17] introduced a two-parameter Lindley distribution of which the one-parameter Lindley distribution is a particular case, for modeling waiting and survival times data. A new weighted Lindley distribution is due to Asgharzadeh et al. [18] and the generalized inverse Lindley distribution is due to Sharma et al. [19].

Many authors have discussed the situations where the data shows the upside-down bathtub (UBT) shapes hazard rates. For example: Efron [20] analyzed the data set in the context of head and neck cancer, in which the hazard rate initially increased, attained a maximum and then decreased before it stabilized owing to a therapy. Bennette [21] analyzed lung cancer trial data which showed that failure rates were unimodal in nature. Langlands et al. [22] have studied the breast carcinoma data and found that the mortality reached a peak after some finite period, and then declined gradually. It is interesting to know that the hazard rates of inverse versions of the probability distributions show the UBT shapes. A few inverse Statistical distributions namely inverse Weibull, inverse Gaussian, inverse Gamma and inverse Lindley etc., are used to model such UBT data in various real life applications.

Comprehending such a unique applicability of inverse distributions to UBT data, we propose a new two-parameter distribution, referred to as APTIL distribution using a similar idea to Mahdavi and Kundu [23] which will definitely add a new dimension to this direction, see also [24]. We are motivated to introduce the APTIL distribution because (i) it is capable of modeling upside down bathtub hazard rates; (ii) it can be viewed as a suitable model for fitting the skewed data which may not be properly fitted by other common distributions and can also be used in a variety of problems in different areas such as public health, biomedical studies and industrial reliability and survival analysis; and (iii) two real data applications show that it compares well with other competing lifetime distributions in modeling survival data and failure data.

The rest of the paper is organized as follows. In Sections 2 and 3, we introduce the APTIL distribution, and discuss some properties of this distribution. In Section 4, maximum likelihood estimators of the unknown parameters are obtained. In Section 5, we investigate the maximum likelihood estimation procedure to estimate the model parameters. The analysis of two real data sets has been presented in Section 6. Finally, in Section 7, we conclude the paper.

2. Definition and statistical properties

Sharma et al. [25] suggested the inverse Lindley distribution with the probability density function (pdf)

$$g(x) = \frac{\lambda^2}{\lambda+1}\left(\frac{1+x}{x^3}\right) e^{-\frac{\lambda}{x}};\ x>0,\ \lambda>0,$$

with a cumulative distribution (cdf) of the form

$$G(x) = \left[1+\frac{\lambda}{(\lambda+1)x}\right] e^{-\frac{\lambda}{x}};\ x>0,\ \lambda>0.$$

We now introduce the notion of Alpha-Power Transformed inverse Lindley Distribution.

Definition. A random variable X is said to have APTIL distribution if its pdf is of the form

$$f_{APTIL}(x;\alpha,\lambda) = \begin{cases} \dfrac{\log(\alpha)}{\alpha-1}\left(\dfrac{\lambda^2}{\lambda+1}\right)\left(\dfrac{1+x}{x^3}\right) e^{-\frac{\lambda}{x}}\, \alpha^{\left[1+\frac{\lambda}{(\lambda+1)x}\right] e^{-\frac{\lambda}{x}}}, & \text{if } \alpha>0,\ \alpha\neq 1, \\ \dfrac{\lambda^2}{\lambda+1}\left(\dfrac{1+x}{x^3}\right) e^{-\frac{\lambda}{x}}, & \text{if } \alpha = 1, \end{cases} \tag{1}$$

where $x > 0$ and $\alpha, \lambda > 0$. The corresponding cdf, survival function and hazard rate functions are, respectively, given by

$$F_{APTIL}(x;\alpha,\lambda) = \begin{cases} \dfrac{\alpha^{\left[1+\frac{\lambda}{(\lambda+1)x}\right] e^{-\frac{\lambda}{x}}}-1}{\alpha-1}, & \text{if } \alpha>0,\ \alpha\neq 1 \\ \left[1+\dfrac{\lambda}{(\lambda+1)x}\right] e^{-\frac{\lambda}{x}} & \text{if } \alpha = 1, \end{cases} \tag{2}$$

$$S_{APTIL}(x;\alpha,\lambda)=\begin{cases}\dfrac{\alpha-\alpha^{\left[1+\frac{\lambda}{(\lambda+1)x}\right]e^{-\frac{\lambda}{x}}}}{\alpha-1} & \text{if } \alpha>0,\ \alpha\neq 1\\ 1-\left[1+\dfrac{\lambda}{(\lambda+1)x}\right]e^{-\frac{\lambda}{x}} & \text{if } \alpha=1\end{cases} \tag{3}$$

and

$$h_{APTIL}(x;\alpha,\lambda)=\begin{cases}\dfrac{\lambda^2\log(\alpha)\,(1+x)\,e^{-\frac{\lambda}{x}}\,\alpha^{\left[1+\frac{\lambda}{(\lambda+1)x}\right]e^{-\frac{\lambda}{x}}}}{(\lambda+1)\,x^3\left\{\alpha-\alpha^{\left[1+\frac{\lambda}{(\lambda+1)x}\right]e^{-\frac{\lambda}{x}}}\right\}}, & \text{if } \alpha>0,\ \alpha\neq 1\\ \dfrac{\lambda^2\,(1+x)\,e^{-\frac{\lambda}{x}}}{(\lambda+1)\,x^3\left\{1-[1+\frac{\lambda}{(\lambda+1)\,x}]e^{-\frac{\lambda}{x}}\right\}}, & \text{if } \alpha=1.\end{cases} \tag{4}$$

Hereafter, a random variable X that follows the distribution in (1) is denoted by $X\sim APTIL(\alpha,\lambda)$.

2.1. Shape

In this section we discuss shape characteristics of pdf $f_{APTIL}(x)$ and hazard rate function $h_{APTIL}(x)$. The pdf and hazard rate function of the APTIL obey the following end behavior $f(0)=f(+\infty)=0$, $h(0)=h(+\infty)=0$.

Theorem 2.1. *The pdf of APTIL is unimodal.*

Proof. It may be noted that

$$f'_{APTIL}(x)=\begin{cases}\dfrac{\lambda^2\log(\alpha)e^{-\frac{2\lambda}{x}}\alpha^{\frac{e^{-\frac{\lambda}{x}}(\lambda+\lambda x+x)}{(\lambda+1)x}}\left\{\lambda^2(x+1)^2\log(\alpha)-(\lambda+1)xe^{\lambda/x}\left[-\lambda+2x^2-(\lambda-3)x\right]\right\}}{(\alpha-1)(\lambda+1)^2x^6} & \text{if } \alpha\neq 1\\ \dfrac{\lambda^2e^{-\frac{\lambda}{x}}\left[\lambda-2x^2+(\lambda-3)x\right]}{(\lambda+1)x^5} & \text{if } \alpha=1\end{cases}$$

and thus the derivative becomes zero if $A(x,\alpha,\lambda)=\lambda^2(x+1)^2\log(\alpha)-(\lambda+1)xe^{\lambda/x}\left(-\lambda+2x^2-(\lambda-3)x\right)$, $B(x,\lambda)=\lambda-2x^2+(\lambda-3)x$ is zero for the cases $\alpha\neq 1$, $\alpha=1$, respectively. This may be readily visible that the expression A is a strictly decreasing, continuous function in x and $A(0+,\alpha,\lambda)$ is positive, while A takes negative values as x approaches $+\infty$. Hence by intermediate value theorem A has only one zero. It is quite obvious that B has only one zero at $x=\frac{1}{4}\left(\sqrt{\lambda^2+2\lambda+9}+\lambda-3\right)$. Hence the conclusion follows. ■

Fig. 1 shows various curves for the pdf of APTIL distribution with various values of the parameters α,λ.

Similarly, one can show that the hazard rate function of APTIL is unimodal. Fig. 2 shows various curves for the hazard rate function of APTIL distribution with various values of the parameters α,λ.

Special Cases: Let $X\sim APTIL(\alpha,\lambda)$.

i. If $\alpha\to 1$, then X reduces to the inverse Lindley distribution proposed by Sharma et al. [25].
ii. If $\alpha=e$, then X reduces to Inverse Poisson–Lindley distribution in the form:

$$f_{ILP}(x;\lambda)=\frac{1}{(e-1)}\left(\frac{\lambda^2}{\lambda+1}\right)\left(\frac{1+x}{x^3}\right)e^{-\frac{\lambda}{x}}\,e^{[1+\frac{\lambda}{(\lambda+1)x}]e^{-\frac{\lambda}{x}}}$$

Result 1. *APTIL*(α,λ) distribution has the following mixture representation for $\alpha>1$. $\frac{\log(\alpha)}{(\alpha-1)}$ is a decreasing function from 1 to 0, as α varies from 1 to ∞. If $X\sim APTIL(\alpha,\lambda)$, then it can be represented as follows:

$$X=\begin{cases}X_1 & \text{with probability } \left(\dfrac{\log\alpha}{\alpha-1}\right)\\ X_2 & \text{with probability } 1-\left(\dfrac{\log\alpha}{\alpha-1}\right)\end{cases} \tag{5}$$

where X_1 and X_2 have the following pdfs

$$f(X_1)=\frac{\lambda^2}{\lambda+1}\left(\frac{1+x}{x^3}\right)e^{-\frac{\lambda}{x}} \tag{6}$$

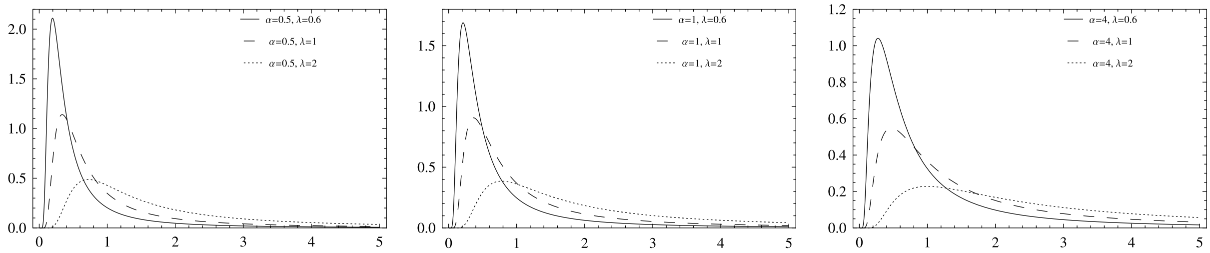

Fig. 1. Density plots of APTIL distribution.

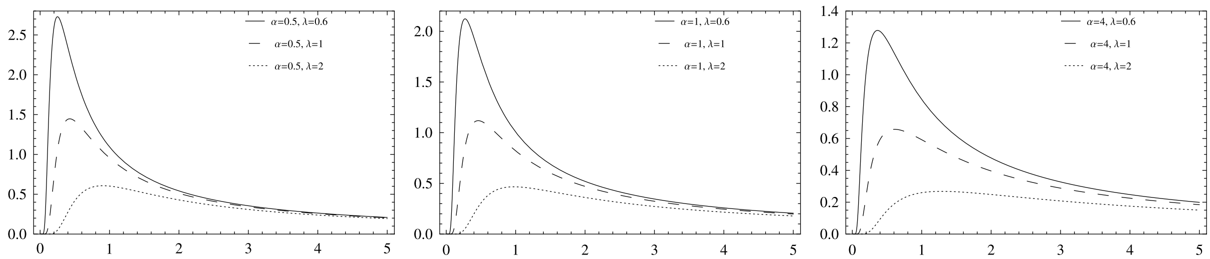

Fig. 2. Hazard rate function of APTIL distribution.

$$f(X_2) = \left[\frac{\log\alpha}{(\alpha - 1 - \log\alpha)}\right]\left[\frac{\lambda^2}{\lambda+1}\,\left(\frac{1+x}{x^3}\right)\,e^{-\frac{\lambda}{x}}\right]\left[\alpha^{\left[1+\frac{\lambda}{(\lambda+1)x}\right]\,e^{-\frac{\lambda}{x}}} - 1\right] \tag{7}$$

respectively. It is clear from the representation (7) that as α approaches 1, X_1 behaves like an inverse Lindley distribution, and α increases, it behaves like X_2.

3. Mathematical properties

The formulae derived throughout the paper can be easily handled in analytical soft wares such as Maple and Mathematica which have the ability to deal with analytic expressions of formidable size and complexity. Established algebraic expansions to determine some mathematical properties of the APTIL family can be more efficient than computing those directly by numerical integration of its density function, which can be prone to rounding off errors among others. Here, we present nth moment and moment generating function of APTIL distribution. Also, we provide expressions for the incomplete moments, conditional moments, Bonferroni and Lorenz curves, Rényi and cumulative residual entropy, mean residual life and order statistics of this distribution.

3.1. Moments

Now we present an infinite sum representation for the nth moment $\mu_n' = E[X^n]$, and consequently find the mean and variance for the APTIL distribution. Let X denote a random variable with the probability density function (1). Calculating moments of X requires the following

Lemma 1. *Let $f(x)$ and $F(x)$ be given by (1) and (2), respectively. For $a > 0$, $b > 0$, $c > 0$ and $\delta > 0$, let*

$$K(a,b,c,\delta) = \int_0^\infty x^{c-3}(1+x)\,a^{\left[1+\frac{b}{(1+b)x}\right]e^{-\frac{b}{x}}}\,e^{-\frac{\delta}{x}}\,dx.$$

We have

$$K(a,b,c,\delta) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\left(\frac{b}{1+b}\right)^j\frac{(\log(a))^i}{i!}\;\frac{(j-c)!\,(j-c+1+bi+\delta)}{(bi+\delta)^{j-c+2}}.$$

Proof. Using the power series expansion, (1), one can write

$$K(a,b,c,\delta)=\sum_{i=0}^{\infty}\frac{(\log(a))^i}{i!}\int_0^{\infty}x^{c-3}(1+x)\left[1+\frac{b}{(1+b)\,x}\right]^i e^{-\frac{(bi+\delta)}{x}}dx$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\left(\frac{b}{1+b}\right)^j\frac{(\log(a))^i}{i!}\int_0^{\infty}x^{c-j-3}(1+x)e^{-\frac{(bi+\delta)}{x}}dx$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\left(\frac{b}{1+b}\right)^j\frac{(\log(a))^i}{i!}(bi+\delta)^{c-j-2}\int_0^{\infty}y^{j-c+1}\,e^{-y}dy$$

$$+\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\left(\frac{b}{1+b}\right)^j\frac{(\log(a))^i}{i!}(bi+\delta)^{c-j-1}\int_0^{\infty}y^{j-c}\,e^{-y}dy,$$

where $y=\frac{bi+\delta}{x}$. The result of the lemma follows by the definition of the gamma function. ∎

It follows from Lemma 1 that

$$E(X^n)=\frac{\log(\alpha)}{\alpha-1}\left(\frac{\lambda^2}{\lambda+1}\right)K(\alpha,\lambda,n,\lambda). \tag{8}$$

In particular, the mean and variance of X are

$$E(X)=\frac{\log(\alpha)}{\alpha-1}\left(\frac{\lambda^2}{\lambda+1}\right)K(\alpha,\lambda,1,\lambda),$$

and

$$V(X)=\frac{\log(\alpha)}{\alpha-1}\left(\frac{\lambda^2}{\lambda+1}\right)\left[K(\alpha,\lambda,2,\lambda)-(K(\alpha,\lambda,1,\lambda))^2\right].$$

The expression (8) can be readily computed numerically using standard statistical software. In numerical applications, a large natural number N can be used in the sums instead of infinity. Several quantities of X (central moments, variance, skewness and kurtosis) can be derived using (8).

The central moments μ_r and cumulants k_r of X can be determined from (8) as

$$\mu_r=\sum_{k=0}^{r}(-1)^k\binom{r}{k}\mu_1'^{r}\mu_{r-k}',$$

and

$$k_r=\mu_r'-\sum_{k=1}^{r-1}\binom{r-1}{k-1}k_r\mu_{r-k}',$$

where $k_1=\mu_1'$. Thus $k_2=\mu_2'-\mu_1'^2$, $k_3=\mu_3'-3\mu_2'\mu_1'+2\mu_1'^3$, $k_4=\mu_4'-4\mu_3'\mu_1'-3\mu_2'^2+12\mu_2'\mu_1'^2-6\mu_1'^4$, etc. The skewness $\gamma_1=k_3/k_2^{3/2}$ and kurtosis $\gamma_2=k_4/k_2^2$ can be calculated from the second, third and fourth standardized cumulants.

Incomplete moments of a distribution are used in measuring inequality: for example, the Lorenz and Bonferroni curves. The nth incomplete moment of APTIL distribution is defined by

$$m_n(y)=E\left[X^n|x<y\right]=\int_0^{y}x^n f(x)dx$$

$$=\frac{1}{(\alpha-1)}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(\log(\alpha))^{i+1}\,\lambda^{j+2}}{i!(1+\lambda)^{j+1}}\binom{i}{j}\int_0^{y}x^{n-j-3}(1+x)e^{-\frac{(\lambda i+\delta)}{x}}dx$$

$$=\frac{1}{(\alpha-1)}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(\log(\alpha))^{i+1}\,\lambda^{j+2}}{i!(1+\lambda)^{j+1}(\lambda i+\delta)^{j-n+2}}\binom{i}{j}\int_{\frac{\lambda i+\delta}{y}}^{\infty}t^{j-n+1}e^{-t}dt$$

$$+\frac{1}{(\alpha-1)}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(\log(\alpha))^{i+1}\,\lambda^{j+2}}{i!(1+\lambda)^{j+1}(\lambda i+\delta)^{j-n+2}}\binom{i}{j}\int_{\frac{\lambda i+\delta}{y}}^{\infty}t^{j-n}e^{-t}dt$$

$$= \frac{1}{(\alpha-1)}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(\log(\alpha))^{i+1}\,\lambda^{j+2}}{i!(1+\lambda)^{j+1}(\lambda i+\delta)^{j-n+2}}\binom{i}{j}$$
$$\times\ (\lambda i+\delta)^{n-j-2}\left[\Gamma\left(j-n+2,\ \frac{\lambda i+\delta}{y}\right)+\frac{\Gamma\left(j-n+1,\ \frac{\lambda i+\delta}{y}\right)}{\lambda i+\delta}\right],$$

where $t=\frac{\lambda i+\delta}{y}$ and $\Gamma(a,\ x)=\int_x^{\infty}t^{a-1}exp(-t)dt$ denote the complementary incomplete gamma function.

3.2. Conditional moments

For lifetime models, it is also of interest to know what $E(X^n|X>x)$ is. Calculating these moments requires the following

Lemma 2. *Let $f(x)$ and $F(x)$ be given by (1) and (2), respectively. For $a>0, b>0, c>0$ and $\delta>0$, let*

$$L(a,b,c,\delta,t)=\int_t^{\infty}x^{c-3}(1+x)\,a^{\left[1+\frac{b}{(1+b)x}\right]e^{-\frac{b}{x}}}e^{-\frac{\delta}{x}}dx.$$

We have

$$L(a,b,c,\delta,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\left(\frac{b}{1+b}\right)^{j}\frac{(\log(a))^{i}}{i!}(bi+\delta)^{c-j-2}$$
$$\times\left[\Gamma\left(j-c+2,\ \frac{bi+\delta}{t}\right)+\frac{1}{bi+\delta}\Gamma\left(j-c+1,\ \frac{bi+\delta}{t}\right)\right]. \tag{9}$$

If c is an integer then (9) can be simplified to

$$L(a,b,c,\delta,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\left(\frac{b}{1+b}\right)^{j}\frac{(\log(a))^{i}}{i!}(j-c)!(bi+\delta)^{c-j-2}$$
$$\times\ e^{-\frac{bi+\delta}{t}}\left[(j-c+1)\sum_{k=0}^{j-c+1}\frac{\left(\frac{bi+\delta}{k!}\right)^{k}}{k!}+\frac{1}{bi+\delta}\sum_{k=0}^{j-c}\frac{\left(\frac{bi+\delta}{k!}\right)^{k}}{k!}\right].$$

Proof. The proof of (9) is similar to the proof of Lemma 1, but using the definition of the complementary incomplete gamma function. The final relation follows by using the fact

$$\Gamma(a,x)=(a-1)!e^{-x}\sum_{i=0}^{a-1}\frac{x^{i}}{i!}.$$

Using Lemma 2, it is easily seen that

$$E(X^n|X>x)=\frac{\log(\alpha)}{\alpha-1}\left(\frac{\lambda^2}{1+\lambda}\right)\frac{1}{[1-C(x)]}L(\alpha,\lambda,n,\lambda,x), \tag{10}$$

where

$$C(x)=\frac{\alpha^{1-\left[1+\frac{\lambda}{(1+\lambda)x}\right]e^{-\frac{\lambda}{x}}}-1}{\alpha-1}. \tag{11}$$

An application of the conditional moments is the mean residual life (MRL). In life testing experiments, the expected additional lifetime given that an item has survived until time x is called the MRL. The MRL function of the APTIL distribution can be written as follows

$$m_X(x)=E(X-x|X>x)=\frac{\log(\alpha)}{\alpha-1}\left(\frac{\lambda^2}{1+\lambda}\right)\frac{1}{[1-C(x)]}L(\alpha,\lambda,1,\lambda,x)-x$$

where $L(\alpha,\lambda,1,\lambda,x)$ can be obtained from (9) with $n=1$ and $C(x)$ is defined in (11).

The mean deviations about the mean and the median of the APTIL distribution can be obtained from (10). Let μ and M denote the mean and the median of the APTIL distribution, respectively, then the mean deviations about the mean and the median can be calculated as

$$\delta_\mu=\int_0^{\infty}|x-\mu|f(x)dx=2\mu F(\mu)-2\mu+2\frac{\log(\alpha)}{\alpha-1}\left(\frac{\lambda^2}{1+\lambda}\right)L(\alpha,\lambda,1,\lambda,\mu)$$

and

$$\delta_M = \int_0^\infty |x - M| f(x)dx = 2\frac{\log(\alpha)}{\alpha - 1}\left(\frac{\lambda^2}{1+\lambda}\right) L(\alpha, \lambda, 1, \lambda, M) - \mu$$

respectively, where $L(\alpha, \lambda, 1, \lambda, \mu)$ and $L(\alpha, \lambda, 1, \lambda, M)$ can be obtained from (9). Also, $F(\mu)$ and $F(M)$ are easily calculated from (2).

3.3. L-moments

Some other important measures useful for lifetime models are the L-moments due to Hosking [26]. It can be shown using Lemma 1 that the kth L-moment is

$$L_r = \sum_{j=0}^{r-1} (-1)^{r-1-j} \binom{r-1}{j} \binom{r-1+j}{j} \lambda_j,$$

where

$$\lambda_r = \frac{\log(\alpha)}{\alpha - 1}\left(\frac{\lambda^2}{1+\lambda}\right) K(\alpha(r+1),\ \lambda,\ 1,\ \lambda).$$

L-moments possess several advantages compared to conventional moments. Hosking [26] proved that if the mean of a distribution exists, then all its L-moments exist and the distribution is uniquely characterized by its L-moments.

3.4. MGF, CHF and CGF

Let X denote a random variable with the probability density function (1). It follows from Lemma 1 that the moment generating function of X, $M(t) = E[e^{(tx)}]$, is given by

$$M(t) = \frac{\log(\alpha)}{\alpha - 1}\left(\frac{\lambda^2}{\lambda+1}\right) K(\alpha,\ \lambda,\ 0,\ \lambda - t),$$

for $t < \lambda$. The characteristic function of X, $\phi(t) = E[e^{(itX)}]$, and the cumulant generating function of X, $K(t) = \log \phi(t)$, are given by

$$\phi(t) = \frac{\log(\alpha)}{\alpha - 1}\left(\frac{\lambda^2}{\lambda+1}\right) K(\alpha,\ \lambda,\ 0,\ \lambda - it),$$

and

$$K(t) = \log\left(\frac{\log(\alpha)}{\alpha - 1}\right) + \log\left(\frac{\lambda^2}{\lambda+1}\right) + \log[K(\alpha,\ \lambda,\ 0,\ \lambda - it)],$$

respectively, where $i = \sqrt{-1}$.

3.5. Bonferroni and Lorenz curves

Bonferroni curve proposed by Bonferroni [27] and Lorenz curve by Lorenz [28] are used to measure the inequality of the distribution of a random variable X. They are applied in many fields such as economics, reliability, demography, insurance, etc. These indices are defined as:

$$B(p) = \frac{1}{p\mu}\int_0^q x f(x)dx$$

and

$$L(p) = \frac{1}{\mu}\int_0^q x f(x)dx,$$

respectively, where $q = F^{-1}(p)$. If X has the pdf in (1), then, by Lemma 2, one can calculate Bonferroni curve of the APTIL distribution as

$$B(p) = \frac{1}{p\mu}\left[\mu - \frac{\log(\alpha)}{\alpha - 1}\left(\frac{\lambda^2}{\lambda+1}\right) L(\alpha,\ \lambda,\ 1,\ \lambda, q)\right].$$

The Lorenz curve of the APTIL distribution is

$$L(p) = \frac{1}{\mu}\left[\mu - \frac{\log(\alpha)}{\alpha - 1}\left(\frac{\lambda^2}{\lambda+1}\right) L(\alpha,\ \lambda,\ 1,\ \lambda, q)\right].$$

The area between the line $L(F(x)) = F(x)$ and the Lorenz curve, known as the area of concentration, may be regarded as a measure of inequality of income, so it is important in insurance, economics and other fields like reliability and medicine.

3.6. Entropies

Entropy is used to measure the variation of the uncertainty of the random variable X. If X has the probability distribution function $f(\cdot)$, then Rényi entropy (Rényi [29]) is defined by

$$H_\delta(x) = \frac{1}{1-\delta}\log\left(\int_{-\infty}^{\infty} f^\delta(x)dx\right), \qquad \delta > 0, \qquad \delta \neq 1. \tag{12}$$

Suppose X has the pdf in (1). Then, one can calculate

$$\begin{aligned}
\int_0^\infty f^\delta(x)dx &= \left[\frac{\lambda^2\log(\alpha)}{(\alpha-1)(1+\lambda)}\right]^\delta \int_0^\infty (1+x)^\delta\, x^{-3\delta}\left\{\alpha^{\left[1+\frac{\lambda}{(\lambda+1)\,x}\right]e^{-\frac{\lambda}{x}}}\right\}^\delta e^{-\frac{\delta\lambda}{x}}dx \\
&= \left[\frac{\lambda^2\log(\alpha)}{(\alpha-1)(1+\lambda)}\right]^\delta \sum_{i=0}^{\infty}\frac{(\delta\log(\alpha))^i}{i!}\int_0^\infty \frac{(1+x)^\delta}{x^{3\delta}}\left[1+\frac{\lambda}{(\lambda+1)\,x}\right]^i e^{-\frac{(i+\delta)\lambda}{x}}dx \\
&= \left[\frac{\lambda^2\log(\alpha)}{(\alpha-1)(1+\lambda)}\right]^\delta \sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(\delta\log(\alpha))^i}{i!}\binom{i}{j} \\
&\times \int_0^\infty \frac{(1+x)^\delta}{x^{3\delta}}\left[\frac{\lambda}{(\lambda+1)\,x}\right]^j e^{-\frac{(i+\delta)\lambda}{x}}dx \\
&= \left[\frac{\lambda^2\log(\alpha)}{(\alpha-1)(1+\lambda)}\right]^\delta \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\delta}\frac{(\delta\log(\alpha))^i}{i!}\left(\frac{\lambda}{\lambda+1}\right)^j\binom{i}{j} \\
&\times \binom{\delta}{k}\int_0^\infty x^{k-j-3\delta}e^{-\frac{(i+\delta)\lambda}{x}}dx \\
&= \left[\frac{\lambda^2\log(\alpha)}{(\alpha-1)(1+\lambda)}\right]^\delta \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\delta}\frac{(\delta\log(\alpha))^i}{i!}\left(\frac{\lambda}{\lambda+1}\right)^j\binom{i}{j} \\
&\times \binom{\delta}{k}[\lambda(i+\delta)]^{k-j-3\delta+1}\int_0^\infty y^{k-j-3\delta}\, e^{-y}dy \\
&= \left[\frac{\lambda^2\log(\alpha)}{(\alpha-1)(1+\lambda)}\right]^\delta \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\delta}\frac{(\delta\log(\alpha))^i}{i!}\left(\frac{\lambda}{\lambda+1}\right)^j\binom{i}{j} \\
&\times \frac{\Gamma(k+3\delta-j-1)}{[\lambda(i+\delta)]^{k-3\delta-j-1}},
\end{aligned}$$

where $y = \frac{\lambda(i+\delta)}{x}$. Now, the Rényi entropy of the APTIL distribution can be obtained as

$$\begin{aligned}
H_\delta(x) &= \frac{\delta}{1-\delta}\log\left[\frac{\lambda^2 log\alpha}{(\alpha-1)(1+\lambda)}\right] \\
&+ \frac{1}{1-\delta}\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\delta}\frac{(-1)^j(\delta\log(\alpha))^i}{i!}\left(\frac{\lambda}{\lambda+1}\right)^j\binom{i}{j}\right. \\
&\times \left.\binom{\delta}{k}\frac{\Gamma(k+3\delta-j-1)}{[\lambda(i+\delta)]^{k-3\delta-j-1}}\right\}.
\end{aligned} \tag{13}$$

Shannon entropy (Shannon [30]) defined by $E[-\log f(x)]$ is the particular case of (13) for $\delta \uparrow 1$. Finally, consider the cumulative residual entropy (Rao *et al.* [31]) defined by

$$\Im_c = -\int Pr(X > x)\,\log[Pr(X > x)]dx. \tag{14}$$

Using the series expansions,

$$(1-x)^{n-1} = \sum_{p=0}^{\infty}(-1)^p\binom{n-1}{p}x^p \tag{15}$$

and

$$\log(1-x) = -\sum_{p=1}^{\infty}\frac{x^p}{p},$$

one calculates (14) as

$$\begin{aligned}
\Im_c &= \sum_{i=1}^{\infty}\frac{1}{i}\int_0^{\infty}\left\{\frac{\alpha^{\left[1+\frac{\lambda}{(1+\lambda)x}\right]e^{-\frac{\lambda}{x}}}-1}{\alpha-1}\right\}^i\left\{1-\frac{\alpha^{\left[1+\frac{\lambda}{(1+\lambda)x}\right]e^{-\frac{\lambda}{x}}}-1}{\alpha-1}\right\}dx\\
&= \sum_{i=1}^{\infty}\frac{1}{i}\left\{\int_0^{\infty}\left[\frac{\alpha^{\left[1+\frac{\lambda}{(1+\lambda)x}\right]e^{-\frac{\lambda}{x}}}-1}{\alpha-1}\right]^i-\left[\frac{\alpha^{\left[1+\frac{\lambda}{(1+\lambda)x}\right]e^{-\frac{\lambda}{x}}}-1}{\alpha-1}\right]^{i+1}\right\}dx\\
&= \left[\sum_{i=1}^{\infty}\frac{1}{i(\alpha-1)^i}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}-\sum_{i=1}^{\infty}\frac{1}{i(\alpha-1)^{i+1}}\sum_{j=0}^{i}(-1)^{i+1-j}\binom{i+1}{j}\right]\\
&\times\int_0^{\infty}\alpha^{j\left[1+\frac{\lambda}{(1+\lambda)\,x}\right]e^{-\frac{\lambda}{x}}}dx\\
&= \left[\sum_{i=1}^{\infty}\frac{1}{i(\alpha-1)^i}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}-\sum_{i=1}^{\infty}\frac{1}{i(\alpha-1)^{i+1}}\sum_{j=0}^{i}(-1)^{i+1-j}\binom{i+1}{j}\right]\\
&\times\sum_{k=0}^{\infty}\sum_{l=0}^{k}\binom{k}{l}\left(\frac{\lambda}{1+\lambda}\right)^k\frac{[j\log(\alpha)]^k\,\Gamma(l-1)}{k!\,(\lambda k)^{l-1}}.
\end{aligned}$$

3.7. Stochastic ordering

If X and Y are independent random variables with cdfs F_X and F_Y respectively, then X is said to be smaller than Y in the

- stochastic order ($X\le_{st}(Y)$) if $F_X(x)\ge F_Y(x)$ for all x
- hazard rate order ($X\le_{hr}(Y)$) if $h_X(x)\ge h_Y(x)$ for all x
- mean residual life order ($X\le_{mrl}(Y)$) if $m_X(x)\ge m_Y(x)$ for all x
- likelihood ratio order ($X\le_{lr}(Y)$) if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The APTIL distribution is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem. It shows the flexibility of two parameter APTIL distribution.

Theorem 1. *Let $X\sim APTIL(\alpha_1,\lambda_1)$ and $Y\sim APTIL(\alpha_2,\lambda_2)$. If $\alpha_1=\alpha_1=\alpha$ and $\lambda_1\ge\lambda_2$, then $X\le_{lr}Y$, $X\le_{hr}Y$, $X\le_{mrl}Y$ and $X\le_{st}Y$.*

Proof. The likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)}=\frac{\lambda_1^2\,\log(\alpha_1)\,(\alpha_2-1)\,(\lambda_2+1)\,e^{-\frac{\lambda_1}{x}}\,\alpha_1^{\left[1+\frac{\lambda_1}{(\lambda_1+1)\,x}\right]e^{-\frac{\lambda_1}{x}}}}{\lambda_2^2\,\log(\alpha_2)\,(\alpha_1-1)\,(\lambda_1+1)\,e^{-\frac{\lambda_2}{x}}\,\alpha_2^{\left[1+\frac{\lambda_2}{(\lambda_2+1)\,x}\right]e^{-\frac{\lambda_2}{x}}}}$$

thus,

$$\begin{aligned}
\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} &= \frac{\lambda_1-\lambda_2}{x^2}-\frac{\lambda_1}{x^2}e^{-\frac{\lambda_1}{x}}\left(1+\frac{\lambda_1}{(\lambda_1+1)x}\right)-e^{-\frac{\lambda_1}{x}}\frac{\lambda_1}{(\lambda_1+1)x^2}\\
&+\frac{\lambda_2}{x^2}e^{-\frac{\lambda_2}{x}}\left(1+\frac{\lambda_2}{(\lambda_2+1)x}\right)+e^{-\frac{\lambda_2}{x}}\frac{\lambda_2}{(\lambda_2+1)x^2}.
\end{aligned}$$

Now if $\alpha_1=\alpha_2=\alpha$ and $\lambda_1\ge\lambda_2$ then $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)}\le 0$, which implies that $X\le_{lr}Y$ and hence $X\le_{lr}Y$, $X\le_{hr}Y$, $X\le_{mrl}Y$ and $X\le_{st}Y$.

3.8. Order statistics

Suppose $X_1,X_2,\ldots,X_n$ is a random sample from (1). Let $X_{1:n},X_{2:n},\ldots,X_{n:n}$ denote the corresponding order statistics. It is well known that the pdf and the cdf of the rth order statistic, say $Y=X_{r:n}$, are given by

$$\begin{aligned}
f_Y(y) &= \frac{n!}{(r-1)!(n-r)!}F^r(y)[1-F^r(y)]^{n-r}f(y)\\
&= \frac{n!}{(r-1)!(n-r)!}\sum_{u=0}^{n-r}(-1)^u\binom{n-r}{u}F^{r-1+u}(y)f(y)
\end{aligned}$$

and

$$F_Y(y) = \sum_{j=r}^{n} \binom{n}{j} F^j(y)[1-F(y)]^{n-j} = \sum_{j=r}^{n}\sum_{u=0}^{n-j} \binom{n}{j}\binom{n-j}{u} (-1)^u F^{j+u}(y),$$

respectively, for $r = 1, 2, \ldots, n$. It follows from (1) and (2) that

$$f_Y(y) = \frac{\frac{\log(\alpha)}{(\alpha-1)}\left(\frac{\lambda^2}{1+\lambda}\right) n!}{(r-1)!(n-r)!} \sum_{u=0}^{n-r}(-1)^u \binom{n-r}{u} \frac{(1+y)}{y^3}\, e^{-\frac{\lambda}{y}}\, \alpha^{\left[1+\frac{\lambda}{(1+\lambda)x}\right]e^{-\frac{\lambda}{x}}}\, [C(y)]^{r-1+u}$$

and

$$F_Y(y) = \sum_{j=r}^{n}\sum_{u=0}^{n-j} \binom{n}{j}\binom{n-j}{u} (-1)^u [C(y)]^{j+u},$$

where $C(\,.\,)$ is given by (11). The mth moment of Y can be expressed as

$$\begin{aligned} E\left(Y^m\right) &= \frac{\log(\alpha)\, n!}{(r-1)!(n-1)!} \sum_{u=0}^{n-r}\sum_{v=0}^{r-1+u} \binom{n-r}{u}\binom{r-1+u}{v} \frac{(-1)^{u+r-1+v}}{(\alpha-1)^{r+u}} \\ &\times \sum_{k=0}^{\infty}\sum_{l=0}^{k}\binom{k}{l} \frac{[(1+v)\log(\alpha)]^k}{k!} \frac{\lambda^{l+2}}{(1+\lambda)^{l+1}} \\ &\times \left[\frac{\Gamma(l-m+2)}{[(l+1)\lambda]^{l-m+2}} + \frac{\Gamma(l-q+1)}{[(l+1)\lambda]^{l-m+1}}\right], \end{aligned}$$

for $m \geq 1$.

3.9. Stress–strength reliability

Here, we derive the reliability parameter, denoted by R, when X_1 and X_2 are independent variables distributed according to (1) with parameters (α_1, λ_1) and (α_2, λ_2) respectively. The stress–strength parameter R, of the APTIL distribution can be obtained as from (1) and (2) that

$$\begin{aligned} R &= \frac{\log(\alpha_1)}{(\alpha_1-1)}\left(\frac{\lambda_1^2}{1+\lambda_1}\right)\int_0^\infty \frac{(1+x)}{x^3}\, e^{-\frac{\lambda_1}{x}}\, \alpha_1^{1-\left[1+\frac{\lambda_1}{(1+\lambda_1)\,x}\right]e^{-\frac{\lambda_1}{x}}} \left\{\frac{\alpha_2^{1-\left[1+\frac{\lambda_2}{(1+\lambda_2)\,x}\right]e^{-\frac{\lambda_2}{x}}}-1}{\alpha_2-1}\right\} dx \\ &= \frac{\log(\alpha_1)}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\lambda_1^2}{1+\lambda_1}\right)\int_0^\infty \frac{(1+x)}{x^3}\, e^{-\frac{\lambda_1}{x}}\, \alpha_1^{\left[1+\frac{\lambda_1}{(1+\lambda_1)\,x}\right]e^{-\frac{\lambda_1}{x}}}\, \alpha_2^{\left[1+\frac{\lambda_2}{(1+\lambda_2)\,x}\right]e^{-\frac{\lambda_2}{x}}} dx \\ &- \frac{\log(\alpha_1)}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\lambda_1^2}{1+\lambda_1}\right)\int_0^\infty \frac{(1+x)}{x^3}\, e^{-\frac{\lambda_1}{x}}\, \alpha_1^{\left[1+\frac{\lambda_1}{(1+\lambda_1)\,x}\right]e^{-\frac{\lambda_1}{x}}} dx \\ &= \frac{\log(\alpha_1)}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\lambda_1^2}{1+\lambda_1}\right)[I_1 - I_2], \end{aligned}$$

where

$$I_1 = \int_0^\infty \frac{(1+x)}{x^3}\, e^{-\frac{\lambda_1}{x}}\, \alpha_1^{\left[1+\frac{\lambda_1}{(1+\lambda_1)\,x}\right]e^{-\frac{\lambda_1}{x}}}\, \alpha_2^{\left[1+\frac{\lambda_2}{(1+\lambda_2)\,x}\right]e^{-\frac{\lambda_2}{x}}} dx.$$

Applying the power series expansion, one obtains the representation

$$\begin{aligned} I_1 &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\log(\alpha_1))^i\,(\log(\alpha_2))^j}{i!\,j!}\sum_{k=0}^{i}\sum_{p=0}^{k}\binom{i}{k}\binom{k}{p}\left(\frac{\lambda_1}{1+\lambda_1}\right)^k\left(\frac{\lambda_2}{1+\lambda_2}\right)^p \\ &\times \int_0^\infty \frac{(1+x)}{x^{k+p+3}} e^{-\frac{\lambda_1(1+k)-\lambda_2 l}{x}} dx \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\log(\alpha_1))^i\,(\log(\alpha_2))^j}{i!\,j!}\sum_{k=0}^{i}\sum_{p=0}^{k}\binom{i}{k}\binom{k}{p}\left(\frac{\lambda_1}{1+\lambda_1}\right)^k\left(\frac{\lambda_2}{1+\lambda_2}\right)^p \\ &\times \left[\int_0^\infty x^{-k-p-3} e^{-\frac{\lambda_1(1+k)+\lambda_2 l}{x}} dx + \int_0^\infty x^{-k-p-2} e^{-\frac{\lambda_1(1+k)+\lambda_2 l}{x}} dx\right] \end{aligned}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\log(\alpha_1))^i(\log(\alpha_2))^j}{i!\,j!}\sum_{k=0}^{i}\sum_{p=0}^{k}\binom{i}{k}\binom{k}{p}\left(\frac{\lambda_1}{1+\lambda_1}\right)^k\left(\frac{\lambda_2}{1+\lambda_2}\right)^p$$
$$\times\frac{(k+p)!}{[\lambda_1(1+k)+\lambda_2 l]^{k+p+1}}\left[1+\frac{k+p+1}{\lambda_1(1+k)+\lambda_2 l}\right],$$

where the final step follows by the definition of the gamma function. Similarly, we obtain

$$I_2=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\frac{(\log(\alpha_1))^i}{i!}\left(\frac{\lambda_1}{1+\lambda_1}\right)^j\frac{j!}{[\lambda_1(i+1)]^{j+1}}\left[1+\frac{j+1}{\lambda_1(i+1)}\right].$$

4. Maximum likelihood estimation

In this section, the maximum likelihood estimates (MLEs) of the unknown parameters of the APTIL distribution are obtained. Also the observed information matrix is obtained and used to construct the approximate confidence intervals of the unknown parameters. Let $x_1,\ldots,x_n$ be a random sample of size n from (1), then the log-likelihood function of (1) is given by

$$\ell(\alpha,\lambda;\underline{x}) = \log L(\alpha,\lambda;\underline{x}) = n\log(\log(\alpha)) - n\log(\alpha-1) + n\log\left(\frac{\lambda^2}{\lambda+1}\right) + \sum_{i=1}^{n}\log\left(\frac{1+x_i}{x_i^3}\right)$$
$$-\lambda\sum_{i=1}^{n}x_i^{-1} + \log(\alpha)\sum_{i=1}^{n}\left[1+\frac{\lambda}{(1+\lambda)x_i}\right]e^{-\frac{\lambda}{x_i}}. \tag{16}$$

To obtain the likelihood equations for the unknown parameters, we differentiate (16) partially with respect to α and λ and equate to zero. The resulting equations are

$$0=\frac{\partial\ell(\alpha,\lambda;\underline{x})}{\partial\alpha} = \frac{n}{\alpha\log(\alpha)} - \frac{n}{\alpha-1} + \frac{1}{\alpha}\sum_{i=1}^{n}\left[1+\frac{\lambda}{(1+\lambda)x_i}\right]e^{-\frac{\lambda}{x_i}}$$

and

$$0=\frac{\partial\ell(\alpha,\lambda;\underline{x})}{\partial\lambda} = \frac{2n}{\lambda} - \frac{n}{\lambda+1} - \sum_{i=1}^{n}x_i^{-1}$$
$$-\log(\alpha)\sum_{i=1}^{n}\left\{\left[1+\frac{\lambda}{(1+\lambda)x_i}\right]\frac{e^{-\frac{\lambda}{x_i}}}{x_i} - \frac{x_ie^{-\frac{\lambda}{x_i}}}{[(1+\lambda)x_i]^2}\right\}.$$

The MLEs of α and λ denoted by $\hat{\alpha}$ and $\hat{\lambda}$ are obtained by solving the above nonlinear system of equations. Mathcad program or R package can be used to solve these equations numerically.

4.1. Approximate confidence intervals

Using large sample approximation, the asymptotic distribution of the MLEs is $[\sqrt{n}(\hat{\alpha}-\alpha), \sqrt{n}(\hat{\lambda}-\lambda)] \to N_2(0, I^{-1}(\alpha,\lambda))$, see [32]. This result can be used for constructing the approximate confidence intervals for the parameters α and λ. For this purpose, the inverse of observed information matrix, $I^{-1}(\alpha,\lambda)$, of the unknown parameters is required as follows

$$I^{-1}(\alpha,\lambda) = \begin{pmatrix} -\frac{\partial^2\ell(\alpha,\lambda;\underline{x})}{\partial\alpha^2} & -\frac{\partial^2\ell(\alpha,\lambda;\underline{x})}{\partial\alpha\partial\lambda} \\ -\frac{\partial^2\ell(\alpha,\lambda;\underline{x})}{\partial\lambda\partial\alpha} & -\frac{\partial^2\ell(\alpha,\lambda;\underline{x})}{\partial\lambda^2}\end{pmatrix}^{-1}\Bigg|_{(\alpha,\lambda)=(\hat{\alpha},\hat{\lambda})} = \begin{pmatrix} var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\lambda}) \\ cov(\hat{\lambda},\hat{\alpha}) & var(\hat{\lambda})\end{pmatrix},$$

where

$$\frac{\partial^2\ell(\alpha,\lambda;\underline{x})}{\partial\alpha^2} = -\frac{n(1+\log(\alpha))}{(\alpha\log(\alpha))^2} + \frac{n}{(\alpha-1)^2} - \frac{1}{\alpha^2}\sum_{i=1}^{n}\left(1+\frac{\lambda}{(1+\lambda)x_i}\right)e^{-\frac{\lambda}{x_i}}$$

$$\frac{\partial^2\ell(\alpha,\lambda;\underline{x})}{\partial\alpha\,\partial\lambda} = -\frac{1}{\alpha}\sum_{i=1}^{n}\left\{\left(1+\frac{\lambda}{(1+\lambda)x_i}\right)\frac{e^{-\frac{\lambda}{x_i}}}{x_i} - \frac{x_ie^{-\frac{\lambda}{x_i}}}{[(1+\lambda)x_i]^2}\right\}$$

$$\frac{\partial^2 \ell(\alpha,\lambda;\underline{x})}{\partial\lambda^2} = -\frac{2n}{\lambda^2} + \frac{n}{(\lambda+1)^2} - \log(\alpha)\sum_{i=1}^{n}\left\{\frac{e^{-\frac{\lambda}{x_i}}}{[(\lambda+1)x_i]^2} - \frac{e^{-\frac{\lambda}{x_i}}}{x_i^2}\left(1+\frac{\lambda}{(\lambda+1)x_i}\right) + \frac{(\lambda+1)^2\frac{1}{x_i}e^{-\frac{\lambda}{x_i}} + 2(\lambda+1)e^{-\frac{\lambda}{x_i}}}{x_i(\lambda+1)^4}\right\}.$$

Now, approximate $100(1-\tau)\%$ confidence intervals of the parameters α and λ of the forms

$$\hat{\alpha} \pm z_{\tau/2}\sqrt{var(\hat{\alpha})}$$

and

$$\hat{\lambda} \pm z_{\tau/2}\sqrt{var(\hat{\lambda})},$$

where $z_{\tau/2}$ is the upper $(\tau/2)$th percentile of the standard normal distribution.

5. Monte Carlo simulation study

In this section, we investigate the performance of the MLEs of the APTIL parameters by using Monte Carlo simulations. The simulation study is repeated 1000 times each with sample sizes $n = 25, 50, 100$ under the scenarios: I: $\lambda = 2.0, \alpha = 0.5$, 0.8, 1.5 and 2.5 and II: $\lambda = 4.0, \alpha = 0.5$, 0.8, 1.5 and 2.5. Table 1 presents the average value of the estimates, mean squared error (MSE), the approximate confidence intervals (CIs) of MLEs of the parameters α and λ for each parameters combinations and different sample sizes. From the results in this table, we can conclude that the maximum likelihood method performs well for estimating the APTIL parameters. In addition, the approximate CIs lengths decrease as the sample size increases in all cases. Also, for fixed n and λ, as α increases the MSEs of λ decrease and the CIs length of λ increases. For fixed n and α, as λ increases, the MSEs of α increase. We use the inverse cdf method to generate the random variate from (2). For the APTIL distribution, the inverse cdf cannot be obtained in explicit form. For this reason, we propose to use the Newton's method to solve the inverse cdf of APTIL distribution. We use the following algorithm to generate the random sample from APTIL distribution:

Step 1: Set the values of n, α, λ and the initial value x^0.
Step 2: Generate U, where $U \sim uniform(0, 1)$.
Step 3: Use Newton's method to update x^0 by

$$x^* = x^0 - \frac{F(x^0,\alpha,\lambda) - U}{f(x^0,\alpha,\lambda)},$$

where $f(.)$ and $F(.)$ are given by (1) and (2), respectively.
Step 4: If $| x^0 - x^* | \leq \epsilon$ ($\epsilon > 0$ is the tolerance limit), then store $x = x^*$ as a sample from APTIL distribution. Otherwise, set $x^0 = x^*$ and go to step 3.
Step 5: Repeat steps 2–4, n times to get the random sample $x_1, \ldots, x_n$.

6. Applications

In this section, the APTIL distribution is applied to model two complete data sets. The first data set considered here was initially reported by Efron [20] and Makkar et al. [33] and recently by Sharma et al. [25]. The data set represents the survival times of a group of patients suffering from head and neck cancer disease. The patients were treated using radiotherapy. The data are:

Data set 1. 6.53, 7, 10.42, 14.48, 16.1, 22.7, 34, 41.55, 42, 45.28, 49.4, 53.62, 63, 64, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 146, 149, 154, 157, 160, 160, 165, 173, 176, 218, 225, 241, 248, 273, 277, 297, 405, 417, 420, 440, 523, 583, 594, 1101, 1146, 1417.

The second data represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli. The data were observed and reported by Bjerkedal [34]. The data are:

Data set 2. 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

We compare the fits of the APTIL model with the inverse Lindley (IL), generalized inverse Lindley (GIL), exponentiated generalized inverse Lindley (EGIL), exponentiated inverse Lindley (EtIL) and inverse Weibull (IW) distributions. We estimate the model parameters by using the maximum likelihood method. We compare the goodness-of-fit of the models using $-2L$

Table 1
Average values of estimates, MSEs (in parentheses) and CIs.

Parameters		MLE		CIs	
λ	α	$\hat{\lambda}$	$\hat{\alpha}$	λ	α
$n = 25$					
2	0.5	2.0153 (0.0633)	0.7755 (0.5563)	(0.8936, 3.1370)	(0.2992, 1.2517)
	0.8	2.0147 (0.0550)	1.1753 (1.1367)	(0.6968, 3.3325)	(0.3525, 1.9982)
	1.5	2.0122 (0.0418)	2.1091 (3.3746)	(0.5342, 3.4901)	(0.5461, 3.6720)
	2.5	2.0094 (0.0308)	3.4730 (9.4320)	(0.0498, 3.9690)	(0.4667, 6.4794)
4	0.5	4.0389 (0.3375)	0.7959 (0.6121)	(1.7405, 6.3374)	(0.3195, 1.2724)
	0.8	4.0374 (0.2976)	1.2005 (1.2315)	(1.2371, 6.8377)	(0.3732, 2.0277)
	1.5	4.0314 (0.2309)	2.1417 (3.5649)	(0.9278, 7.1349)	(0.5491, 3.7343)
	2.5	4.0246 (0.1731)	3.5121 (9.7266)	(0.2868, 7.7624)	(0.6799, 6.3443)
$n = 50$					
2	0.5	2.0085 (0.0251)	0.6588 (0.2610)	(1.1125, 2.9045)	(0.4189, 0.8988)
	0.8	2.0076 (0.0213)	1.0191 (0.5355)	(0.9595, 3.0557)	(0.6249, 1.4132)
	1.5	2.0055 (0.0154)	1.8607 (1.5399)	(0.8123, 3.1986)	(1.0985, 2.6229)
	2.5	2.0034 (0.0109)	3.0789 (4.0035)	(0.6169, 3.3899)	(1.7028, 4.4549)
4	0.5	4.0212 (0.1309)	0.6699 (0.2860)	(2.2134, 5.8289)	(0.4371, 0.9026)
	0.8	4.0191 (0.1125)	1.0324 (0.5802)	(1.9275, 6.1107)	(0.6456, 1.4192)
	1.5	4.0142 (0.0832)	1.8774 (1.6380)	(1.6098, 6.4185)	(1.1587, 2.5960)
	2.5	4.0093 (0.0596)	3.0980 (4.1851)	(1.1982, 6.8205)	(1.7847, 4.4113)
$n = 100$					
2	0.5	2.0093 (0.0146)	0.5740 (0.1155)	(1.3156, 2.7031)	(0.4598, 0.6882)
	0.8	2.0084 (0.0123)	0.8986 (0.2358)	(1.1347, 2.8822)	(0.6939, 0.8990)
	1.5	2.0068 (0.0088)	1.6539 (0.6487)	(1.0869, 2.9267)	(1.2915, 2.0163)
	2.5	2.0054 (0.0062)	2.7373 (1.5825)	(0.9582, 3.0525)	(2.0946, 3.3800)
4	0.5	4.0222 (0.0757)	0.5801 (0.1267)	(2.5218, 5.5226)	(0.4607, 0.6994)
	0.8	4.0202 (0.0645)	0.9059 (0.2564)	(2.2517, 5.7887)	(0.7082, 1.1036)
	1.5	4.0165 (0.0473)	1.6630 (0.6954)	(2.1023, 5.9306)	(1.3065, 2.0196)
	2.5	4.0131 (0.0338)	2.7479 (1.6716)	(1.8927, 6.1336)	(2.1324, 3.3634)

where (L denotes the log-likelihood function evaluated at the maximum likelihood estimates), Kolmogorov–Smirnov (K–S) statistic, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). The pdfs of the IL, GIL, EGIL, EtIL and IW models are, respectively, given by

$$\text{IL}: f(x;\lambda) = \frac{\lambda^2}{\lambda+1}\left(\frac{1+x}{x^3}\right) e^{-\frac{\lambda}{x}},$$

$$\text{GIL}: f(x;\alpha,\lambda) = \frac{\alpha\lambda^2}{1+\lambda}\left(\frac{1+x^\alpha}{x^{2\alpha+1}}\right)e^{-\frac{\lambda}{x^\alpha}},$$

$$\text{EIL}: f(x;\beta,\lambda) = \frac{\beta\lambda^2}{1+\lambda}\left(\frac{1+x}{x^3}\right)\left\{1+\frac{\lambda}{(1+\lambda)x}\right\}^{\beta-1}e^{-\frac{\beta\lambda}{x}},$$

$$\text{EGIL}: f(x;\alpha,\beta,\lambda) = \frac{\beta\alpha\lambda^2}{1+\lambda}\left(\frac{1+x^\alpha}{x^{2\alpha+1}}\right)\left\{1+\frac{\lambda}{(1+\lambda)x^\alpha}\right\}^{\beta-1}e^{-\frac{\beta\lambda}{x^\alpha}},$$

$$\text{IW}: f(x;\alpha,\lambda) = \alpha\lambda^\alpha x^{-(\alpha+1)}e^{-(\lambda/x)^\alpha}.$$

Tables 2 and 4 list the MLEs (and the corresponding standard errors in parentheses) of the parameters. The values of $-2L$, K–S, AIC and BIC are displayed in Tables 3 and 5. Among all other competitive models, it is to be noted that the APTIL distribution has the lowest values of $-2L$, AIC, BIC and K–S, and so it could be chosen as the best model to fit the given data sets. The relative histogram and the fitted densities and the plot of the fitted survival and the empirical survival functions are displayed in Figs. 3 and 4 for data 1 and data 2, respectively.

7. Conclusion

In this paper, we proposed a new two-parameter family of distributions, so-called the APTIL distribution. The proposed APTIL distribution has one shape parameter and one scale parameter. The APTIL density function is unimodal and its hazard rate function upside down bathtub shaped. Therefore, it can be used quite effectively in analyzing lifetime data. Additionally, the new APTIL model can be used as an alternative to the generalized form of the inverse Lindley distributions and inverse

Table 2
MLEs, standard errors (in parentheses) for data 1.

Model	Estimates		
IL (λ)	60.007 (7.7542)		
APTIL (α, λ)	50.756 (64.127)	24.800 (7.7207)	
EIL (λ)	3.9940 (8.2021)	15.696 (30.438)	
GIL (α, λ)	0.7856 (0.0716)	29.409 (8.2370)	
IW (α, λ)	0.7857 (0.0714)	70.962 (12.612)	
EGIL (α, β, λ)	0.7842 (0.0720)	5.3499 (19.465)	6.1230 (19.6605)

Table 3
Goodness-of fit statistics for data 1.

Model	$-2L$	AIC	BIC	K-S
APTIL	753.2427	757.243	761.364	0.16090
IL	771.4063	773.406	775.467	0.28799
EIL	771.4918	775.492	780.045	0.28837
GIL	763.2041	767.204	771.325	0.19032
EGIL	763.3393	769.339	775.521	0.19099
IW	763.1635	767.163	771.284	0.19030

Table 4
MLEs, standard errors (in parentheses) for data 2.

Model	Estimates		
IL (λ)	1.5767 (0.1457)		
APTIL (α, λ)	0.0322 (0.0010)	2.8084 (0.1173)	
EIL (λ)	0.6966 (0.9926)	2.1041 (2.4529)	
GIL (α, λ)	1.0713 (0.0764)	1.5487 (0.1468)	
IW (α, λ)	1.1731 (0.0842)	1.0583 (0.1132)	
EGIL (α, β, λ)	1.1632 (0.0848)	0.0534 (0.07913)	20.6388 (29.62438)

Table 5
Goodness-of fit statistics for data 2.

Model	$-2L$	AIC	BIC	K-S
APTIL	230.817	234.817	239.37	0.14570
IL	239.5689	241.569	243.846	0.19414
EIL	239.4938	243.494	247.615	0.19900
GIL	238.6981	242.698	247.251	0.18107
EGIL	236.5008	242.501	249.331	0.18357
IW	236.3320	240.332	244.885	0.18272

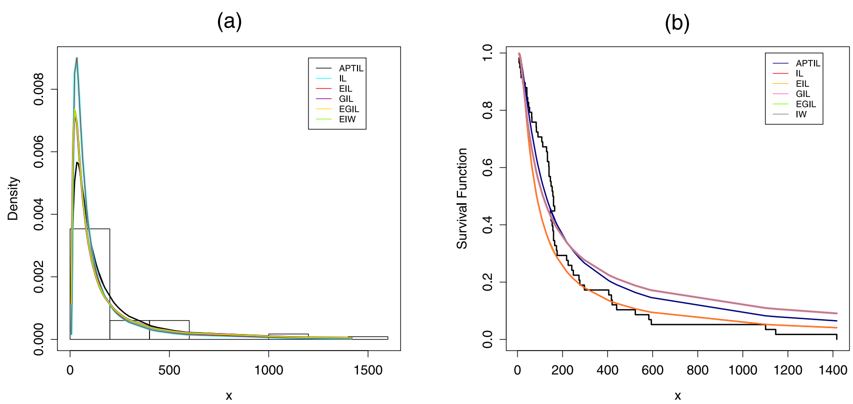

Fig. 3. (a) The relative histogram and the fitted densities, (b) the fitted and empirical survival functions for data 1.

Weibull distribution and is expected that in some situations it might work better (in terms of model fitting) than the models stated, although it cannot be always guaranteed. In this paper, the APTIL distribution shows its ability to model survival times of a group of patients suffering from head and neck cancer disease and survival times of guinea pigs infected with virulent tubercle bacilli. Finally, we hope that the APTIL distribution attracts wider sets of applications such as medical, engineering and social sciences etc.

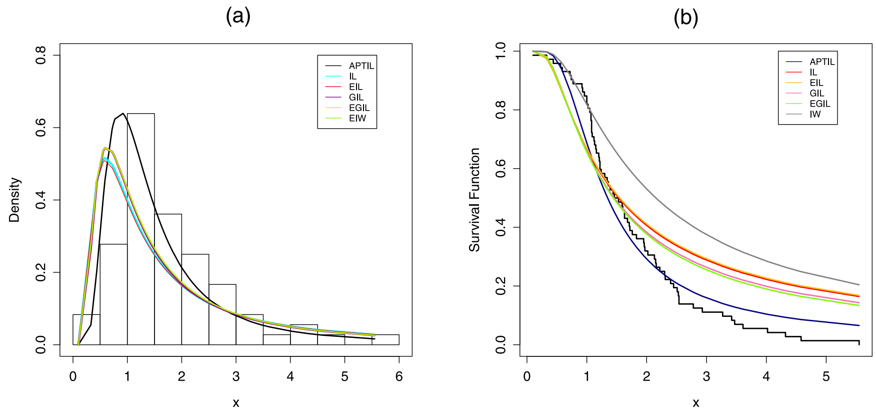

Fig. 4. (a) The relative histogram and the fitted densities, (b) the fitted and empirical survival functions for data 2.

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