Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Exact Solutions of Benjamin-Bona-Mahoney-Burgers Equation with Dual Power-Law Nonlinearity by Modified Exp-Function Method

Manjeet Sharma, Rajesh Kumar Gupta*

Department of Mathematics, School of Basic Sciences, Central University of Haryana, Mahendergarh, Haryana 123031, India
Email: rajeshateli@gmail.com, rajeshgupta@cuh.ac.in

Received: 7 February 2023; **Revised:** 27 March 2023; **Accepted:** 5 April 2023

Abstract: In this work, the Benjamin-Bona-Mahoney-Burgers equation has been examined, which includes the dual power-law nonlinearity and diffraction term. By using the modified exp-function method, the exact solutions of the governing equation have been obtained. The resulting traveling wave solutions have been found to exhibit various characteristics, such as being dark solitons, periodic, and singular, depending on the values of certain constants. To further illustrate these solutions, 3D, 2D, and contour graphs have been displayed. To the best of our knowledge, this is the first time in literature that the dark solitons, periodic soliton, and singular soliton solutions of considered equations have been obtained by utilizing the modified exp-function method.

Keywords: Benjamin-Bona-Mahoney-Burgers equation, modified exp-function method, traveling wave solution

MSC: 35C07, 35C09, 35E20

1. Introduction

It is completely obvious that nonlinear partial differential equations (NLPDEs) have been used to explain a great deal of the mathematical and physical models that arise in the many disciplines of science, including quantum mechanics, classical mechanics, optical fibers, plasma physics, and biological science. Finding solutions to many distinct types of NLPDEs has caught the interest of numerous researchers. Since NLPDEs have such a broad variety of applications in applied mathematics, it is essential to look for different sorts of solutions to NLPDEs. There are different methods that have been utilized to derive the analytic solutions of NLPDEs, like tanh method [1, 2], Jacobi elliptic function method [3, 4], G'/G method [5-7], F-expansion method [8, 9], modified simple equation method [10, 11], etc. The Lie classical method [12-14], nonclassical method [15-20], and direct method [21] can also be used to find the exact solutions of NLPDEs. Also, the fractional natural decomposition method [22] is a strong approach to finding iterative solutions to partial differential equations (PDEs). The q-homotopy analysis transform technique [23] is used to analyze the nature and capture the corresponding consequences of the solutions obtained for PDEs. The modified exp-function method [24] is also an efficient method that can be used to find singular, periodic, dark, and bright solitary wave solutions.

In this manuscript, we consider the Benjamin-Bona-Mahoney-Burgers (BBMB) equation with dual power-law nonlinearity,

DOI: https://doi.org/10.37256/cm.5120242434

$$v_t + \epsilon_1 v_x + \left(\epsilon_2 v^{2n} + \epsilon_3 v\right) v_x + \epsilon_4 v_{xx} + \epsilon_5 v_{xxt} = 0. \tag{1}$$

This equation is a variant of the more general Benjamin-Bona-Mahoney (BBM) equation [25], which is itself a modification of the Korteweg-de Vries (KdV) equation [26, 27]. The BBM equation and its variants are used to model a wide range of physical phenomena, including fluid dynamics, plasma physics, and nonlinear optics. This equation explains the unidirectional propagation of long waves with small amplitudes in water. The Benjamin equation [28] explains the one-directional movement of low-amplitude, long internal waves that occur at the interface between two layers of fluid, subject to the combined influences of surface tension and gravity. Khater et al. [29] obtained the solitary wave solutions of the BBMB equation with dual power-law nonlinearity by using the extended tanh function method. Wang et al. [30] derived the shock wave solutions of the BBMB equation with the Lie symmetry analysis.

The modified exp-function method is a powerful mathematical tool used to obtain exact solutions for NLPDEs that arise in various fields of physics, engineering, and applied mathematics. The main idea behind this method is to transform the original NLPDE into an ordinary differential equation (ODE) by introducing a wave transformation. This transformation leads to a reduction in the number of independent variables, which simplifies the problem considerably. The traveling wave solutions of the reduced equation are assumed to be in the form of an exponential function exp($-\phi$) solution, where ϕ is a solution of the first-order differential equation. The obtained solutions are then represented graphically, enabling a better understanding of the physical properties exhibited by the considered equation. By studying the graphical representations, various physical parameters can be extracted, such as the speed of propagation of the wave, the amplitude, and the shape of the wave. The computational software MAPLE is used to solve the system of nonlinear equations and to depict the graphical representation of the exact solutions obtained.

The structure of this manuscript is as follows: in Section 1, the introduction provides a brief overview of the paper's objectives and motivation. It includes a description of the problem under consideration and its significance in the field of mathematics. In Section 2, a description of the modified exp-function method to find the exact solutions of NLPDE is given. In Section 3, the modified exp-function method is successfully applied to the equation (1). The exact solutions obtained using the method are presented and discussed. In Section 4, graphical representations of the derived analytic solutions for some specific values of constants are given. The figures help to illustrate the behavior of the solutions and provide insight into the physical interpretation of the results. In Section 5, the paper's key findings and conclusions are summarized. The authors provide their final remarks on the modified exp-function method and its usefulness in solving NLPDEs. They may also suggest avenues for further research in the field.

2. Modified exp-function method [31]

Consider the following NLPDE

$$F\left(V, V_x, V_t, V_{xt}, V_{tt}, V_{xx}, \ldots\right) = 0, \tag{2}$$

in which $V = V(x, t)$ is function of x and t, F is polynomial of $V(x, t)$, and its derivatives. The following are the key concepts behind this method:

Step 1: Applying the traveling wave transformation to equation (2), which is clearly described as

$$V(x,t) = V(\xi), \xi = (x - c_1 t), \tag{3}$$

where c_1 describes the wave velocity. Equation (2) is changed into the nonlinear ODE using the above wave transformation

$$V(x,t) = V(\xi), \xi = (x - c_1 t), \tag{4}$$

where prime denotes the derivatives with respect to ξ.

Step 2: Suppose that equation (4) has the following form of solution:

$$V(\xi)=\sum_{i=0}^{N}A_i[\exp(-\phi(\xi))]^i=A_0+\epsilon_1\exp(-\phi)+\ldots+A_N\exp(N(-\phi)), \tag{5}$$

where $A_i^{'}s(0\le i\le N)$ are the constants to be determined later such that $A_N\neq 0$ and $\phi=\phi(\xi)$ is the solution of first-order ODE

$$\phi^{'}(\xi)=\exp(-\phi(\xi))+\delta\exp(\phi(\xi))+\theta, \tag{6}$$

where δ and θ are real constants.

Equation (6) has five exact solutions, which are given below:

- When $\delta\neq 0,\ \theta^2-4\delta>0,$

$$\phi(\xi)=\ln\left(\frac{-\sqrt{\theta^2-4\delta}}{2\delta}\tanh\left(\frac{\sqrt{\theta^2-4\delta}}{2}(\xi+E)\right)-\frac{\theta}{2\delta}\right). \tag{7}$$

- When $\delta\neq 0,\ \theta^2-4\delta>0,$

$$\phi(\xi)=\ln\left(\frac{\sqrt{-\theta^2+4\delta}}{2\delta}\tan\left(\frac{\sqrt{-\theta^2+4\delta}}{2}(\xi+E)\right)-\frac{\theta}{2\delta}\right). \tag{8}$$

- $\delta=0,\theta\neq 0,\theta^2-4\delta>0,$

$$\phi(\xi)=-\ln\left(-\frac{\theta}{\exp(\theta(\xi+E))-1}\right). \tag{9}$$

- $\delta=0,\theta\neq 0,\theta^2-4\delta>0,$

$$\phi(\xi)=-\ln\left(-\frac{2\theta(\xi+E)+4}{\theta^2(\xi+E)}\right). \tag{10}$$

- When $\delta=0,\theta=0,\theta^2-4\delta=0,$

$$\phi(\xi)=\ln(\xi+E). \tag{11}$$

The constants to be obtained subsequently are the coefficients $A_0,\epsilon_1,A_2,\ldots A_N,\theta,$ and δ. Comparing the nonlinear term with the highest order derivative term, the value of N is obtained.

Step 3: Substituting the equation (5) with the obtained value of N into equation (4) and using equation (6), a system of nonlinear equations is obtained by comparing the coefficients of various powers of $\exp(-\phi(\xi))$ to zero.

Step 4: The obtained system is solved for the constants $A_0,\epsilon_1,A_2,\ldots A_N,\theta,$ and δ to obtain the traveling wave solutions of equation (2).

3. Exact solutions of the BBMB equation with dual power-law nonlinearity

Consider the BBMB equation with dual power-law nonlinearity,

$$v_t+\epsilon_1 v_x+\left(\epsilon_2 v^{2n}+\epsilon_3 v\right)v_x+\epsilon_4 v_{xx}+\epsilon_5 v_{xxt}=0, \tag{12}$$

where ϵ_1 indicates the intensity of defection or drifting, the parameters ϵ_4, ϵ_5 are the dissipative diffraction coefficients, the exponent n is the power-law nonlinearity parameter and ϵ_2, ϵ_3 quantify the strength of the two nonlinear terms.

Substituting $n = 1$ in above equation, we get the following form of equation (12)

$$v_t + \epsilon_1 v_x + \left(\epsilon_2 v^2 + \epsilon_3 v\right) v_x + \epsilon_4 v_{xx} + \epsilon_5 v_{xxt} = 0. \tag{13}$$

Using $v(x,t) = u(\xi)$, where $(\xi = x - c_1 t)$, above equation becomes

$$\left(\epsilon_1 - c_1\right) u' + \left(\epsilon_2 u^2 + \epsilon_3 u\right) u' + \epsilon_4 u'' - \epsilon_5 c_1 u''' = 0. \tag{14}$$

Suppose that equation (14) has the following form of traveling wave solution

$$u(\xi) = \sum_{i=0}^{N} A_i [\exp(-\phi(\xi))]^i = A_0 + A_1 \exp(-\phi) + \ldots + A_N \exp(N(-\phi)). \tag{15}$$

Balancing u^2u' and u''' in equation (14), we obtained $(2N + N + 1 = N + 3) \Rightarrow (N = 1)$, so the equation (14) can be rewritten as

$$v(\xi) = A_0 + A_1 \exp(-\phi(\xi)), \tag{16}$$

where $A_1 \neq 0$. Substituting equation (16) along with equation (6), we obtained a system of nonlinear equations after comparing the various powers of $\exp(-\phi(\xi))$, solving the obtained system of nonlinear equations, the following values of the constants are obtained.

Set 1:

$$\begin{aligned}&\theta = \theta, \delta = \frac{\theta^2 A_1^4 \epsilon_2^2 \epsilon_5 - 2A_1^4 \epsilon_2^2 + 12A_1^2 \epsilon_1 \epsilon_2 \epsilon_5 - 3A_1^2 \epsilon_3^2 \epsilon_5 + 12\epsilon_4^2 \epsilon_5}{4A_1^4 \epsilon_2^2 \epsilon_5}, A_0 = \frac{\theta A_1^2 \epsilon_2 - A_1 \epsilon_3 + 2\epsilon_4}{2A_1 \epsilon_2},\\ &A_1 = A_1, c_1 = \frac{A_1^2 \epsilon_2}{6\epsilon_5}.\end{aligned} \tag{17}$$

Taking the values of constant from Set 1 and using equations (7-11), we obtained the following solutions of the considered equation:

- When $\delta \neq 0$, $\theta^2 - 4\delta > 0$,

$$u_1(x,t) = \frac{\theta A_1^2 \epsilon_2 - A_1 \epsilon_3 + 2\epsilon_4}{2A_1 \epsilon_2} - \frac{2A_1 \delta}{\sqrt{\theta^2 - 4\delta} \tanh\left(\dfrac{\sqrt{\theta^2 - 4\delta}\left(x - \dfrac{A_1^2 \epsilon_2}{6\epsilon_5 t} + E\right)}{2}\right) + \theta}, \tag{18}$$

where $\delta = \dfrac{\theta^2 A_1^4 \epsilon_2^2 \epsilon_5 - 2A_1^4 \epsilon_2^2 + 12A_1^2 \epsilon_1 \epsilon_2 \epsilon_5 - 3A_1^2 \epsilon_3^2 \epsilon_5 + 12\epsilon_4^2 \epsilon_5}{4A_1^4 \epsilon_2^2 \epsilon_5}$.

- When $\delta \neq 0$, $\theta^2 - 4\delta > 0$,

$$u_2(x,t)=\frac{\theta A_1^2\epsilon_2-A_1\epsilon_3+2\epsilon_4}{2A_1\epsilon_2}-\frac{2A_1\delta}{-\sqrt{-\theta^2+4\delta}\tan\left(\frac{\sqrt{-\theta^2+4\delta}\left(-\frac{A_1^2\epsilon_2 t}{6\epsilon_5}+E+x\right)}{2}\right)+\theta}, \tag{19}$$

where $\delta=\frac{\theta^2A_1^4\epsilon_2^2\epsilon_5-2A_1^4\epsilon_2^2+12A_1^2\epsilon_1\epsilon_2\epsilon_5-3A_1^2\epsilon_3^2\epsilon_5+12\epsilon_4^2\epsilon_5}{4A_1^4\epsilon_2^2\epsilon_5}$.

- $\delta=0, \theta\neq 0, \theta^2-4\delta>0$,

If system of nonlinear equations is solved with $\delta = 0$, we obtained the following solution of the considered equation.

$$u_3(x,t)=\frac{\sqrt{-\frac{-2A_1^4\epsilon_2^2+12A_1^2\epsilon_1\epsilon_2\epsilon_5-3A_1^2\epsilon_3^2\epsilon_5+12\epsilon_4^2\epsilon_5}{\epsilon_5}}-A_1\epsilon_3+2\epsilon_4}{2A_1\epsilon_2}+\frac{A_1\theta}{e^{\theta\left(-\frac{A_1^2\epsilon_2 t}{6\epsilon_5}+E+x\right)}}, \tag{20}$$

where $\theta=\frac{\sqrt{-\frac{-2A_1^4\epsilon_2^2+12A_1^2\epsilon_1\epsilon_2\epsilon_5-3A_1^2\epsilon_3^2\epsilon_5+12\epsilon_4^2\epsilon_5}{\epsilon_5}}}{\epsilon_2A_1^2}$.

- $\delta\neq 0, \theta\neq 0, \theta^2-4\delta=0$, solving the system of nonlinear equation using the condition $\theta^2-4\delta=0$, the following solution of the original equation is obtained

$$u_4(x,t)=A_0-\frac{A_1\theta^2\left(-\frac{\left(12\epsilon_1\epsilon_5\epsilon_2-3\epsilon_5\epsilon_3^2+\sqrt{144\epsilon_1^2\epsilon_2^2\epsilon_5^2-72\epsilon_1\epsilon_2\epsilon_3^2\epsilon_5^2+9\epsilon_3^4\epsilon_5^2+96\epsilon_2^2\epsilon_4^2\epsilon_5}\right)t}{24\epsilon_2\epsilon_5}+E+x\right)}{4+2\theta\left(-\frac{\left(12\epsilon_1\epsilon_5\epsilon_2-3\epsilon_5\epsilon_3^2+\sqrt{144\epsilon_1^2\epsilon_2^2\epsilon_5^2-72\epsilon_1\epsilon_2\epsilon_3^2\epsilon_5^2+9\epsilon_3^4\epsilon_5^2+96\epsilon_2^2\epsilon_4^2\epsilon_5}\right)t}{24\epsilon_2\epsilon_5}+E+x\right)}, \tag{21}$$

where A_1 is given by

$$A_1=\frac{\sqrt{12\epsilon_1\epsilon_5\epsilon_2-3\epsilon_5\epsilon_3^2+\sqrt{144\epsilon_1^2\epsilon_2^2\epsilon_5^2-72\epsilon_1\epsilon_2\epsilon_3^2\epsilon_5^2+9\epsilon_3^4\epsilon_5^2+96\epsilon_2^2\epsilon_4^2\epsilon_5}}}{2\epsilon_2}.$$

- When $\delta=0, \theta=0, \theta^2-4\delta=0$,

$$u_5(x,t)=\frac{A_0\left(-\frac{\left(12\epsilon_1\epsilon_5\epsilon_2-3\epsilon_5\epsilon_3^2+\sqrt{144\epsilon_1^2\epsilon_2^2\epsilon_5^2-72\epsilon_1\epsilon_2\epsilon_3^2\epsilon_5^2+9\epsilon_3^4\epsilon_5^2+96\epsilon_2^2\epsilon_4^2\epsilon_5}\right)t}{24\epsilon_2\epsilon_5}+E+x\right)+A_1}{-\frac{\left(12\epsilon_1\epsilon_5\epsilon_2-3\epsilon_5\epsilon_3^2+\sqrt{144\epsilon_1^2\epsilon_2^2\epsilon_5^2-72\epsilon_1\epsilon_2\epsilon_3^2\epsilon_5^2+9\epsilon_3^4\epsilon_5^2+96\epsilon_2^2\epsilon_4^2\epsilon_5}\right)t}{24\epsilon_2\epsilon_5}+E+x}, \tag{22}$$

where A_0 and A_1 are given by

$$A_0=\frac{-\dfrac{\sqrt{12\epsilon_1\epsilon_5\epsilon_2-3\epsilon_5\epsilon_3^2+\sqrt{144\epsilon_1^2\epsilon_2^2\epsilon_5^2-72\epsilon_1\epsilon_2\epsilon_3^2\epsilon_5^2+9\epsilon_3^4\epsilon_5^2+96\epsilon_2^2\epsilon_4^2\epsilon_5}\,\epsilon_3}}{2\epsilon_2}+2\epsilon_4}{\sqrt{12\epsilon_1\epsilon_5\epsilon_2-3\epsilon_5\epsilon_3^2+\sqrt{144\epsilon_1^2\epsilon_2^2\epsilon_5^2-72\epsilon_1\epsilon_2\epsilon_3^2\epsilon_5^2+9\epsilon_3^4\epsilon_5^2+96\epsilon_2^2\epsilon_4^2\epsilon_5}}}.$$

$$A_1=\frac{\sqrt{12\epsilon_1\epsilon_5\epsilon_2-3\epsilon_5\epsilon_3^2+\sqrt{144\epsilon_1^2\epsilon_2^2\epsilon_5^2-72\epsilon_1\epsilon_2\epsilon_3^2\epsilon_5^2+9\epsilon_3^4\epsilon_5^2+96\epsilon_2^2\epsilon_4^2\epsilon_5}}}{2\epsilon_2}$$

Set 2:

$$\theta=\frac{\sqrt{-\dfrac{-2A_1^4\epsilon_2^2+12A_1^2\epsilon_1\epsilon_2\epsilon_5-3A_1^2\epsilon_3^2\epsilon_5+12\epsilon_4^2\epsilon_5}{\epsilon_5}}}{\epsilon_2A_1^2},\ \delta=0,$$

$$A_0=\frac{\sqrt{-\dfrac{-2A_1^4\epsilon_2^2+12A_1^2\epsilon_1\epsilon_2\epsilon_5-3A_1^2\epsilon_3^2\epsilon_5+12\epsilon_4^2\epsilon_5}{\epsilon_5}}-A_1\epsilon_3+2\epsilon_4}{2A_1\epsilon_2},\ A_1=A_1,\ c_1=\frac{A_1^2\epsilon_2}{6\epsilon_5} \tag{23}$$

- When $\delta=0,\ \theta=0,\ \theta^2-4\delta=0,$ the following solution of original equation is obtained.

$$u_6(x,t)=\frac{A_0\mathrm{e}^{\theta\left(-\frac{A_1^2\epsilon_2t}{6\epsilon_5}+E+x\right)}+A_1\theta-A_0}{\theta\left(-\dfrac{A_1^2\epsilon_2t}{6\epsilon_5}+E+x\right)}, \tag{24}$$

where values of θ and A_0 are from Set 2.

4. Physical interpretation

One of the most effective means of comprehending the physical dynamics of real-world systems is through visual representation. In this regard, we have illustrated the graphical characteristics of the analytical solutions derived in the previous section. By selecting appropriate parameter values, we have rendered the dark, singular, and periodic solutions in various graphical forms, including 3D, 2D, and contour plots. In order to illustrate various dynamical wave patterns of derived exact solutions to the considered problem, this study extensively uses the modified exp-function technique. Figure 1 represents the singular dark soliton solution $v_1(x, t)$ with values of parameters $A_1=3$, $\epsilon_1=3.5$, $\epsilon_2=9$, $\epsilon_3=-7$, $\epsilon_4=5$, $\epsilon_5=4$, $E=1$, $\theta=1$. Figure 2 displays the periodic singular solution $v2(x, t)$ with values of parameters $A_1=1$, $\epsilon_1=3$, $\epsilon_2=-4$, $\epsilon_3=4$, $\epsilon_4=5$, $\epsilon_5=4$, $E=1$, $\theta=1$. The 2D plot in Figure 2 shows the periodic behavior of the obtained traveling wave solution. Figure 3 shows the 3D, 2D, and contour plots of the exponential singular solution $v_3(x, t)$ with the values of the constants $A_1=1$, $\epsilon_1=4$, $\epsilon_2=3$, $\epsilon_3=11$, $\epsilon_4=3$, $\epsilon_5=5$, $E=1$. From the 2D plot of Figure 3, it can be observed that the line $y=-1$ is an asymptote of the obtained traveling wave solution. Figure 4 shows the singular soliton solution $u_6(x, t)$ with unknown values $A_1=1$, $\epsilon_1=-0.4$, $\epsilon_2=9$, $\epsilon_3=-1.5$, $\epsilon_4=3$, $\epsilon_5=2.5$, $E=1$. By exploiting the traveling wave solutions obtained in this manuscript, it is possible to predict the shape and speed of waves in water as they propagate through these environments. This information can be useful in designing structures such as breakwaters and seawalls to mitigate the impact of waves on coastal communities.

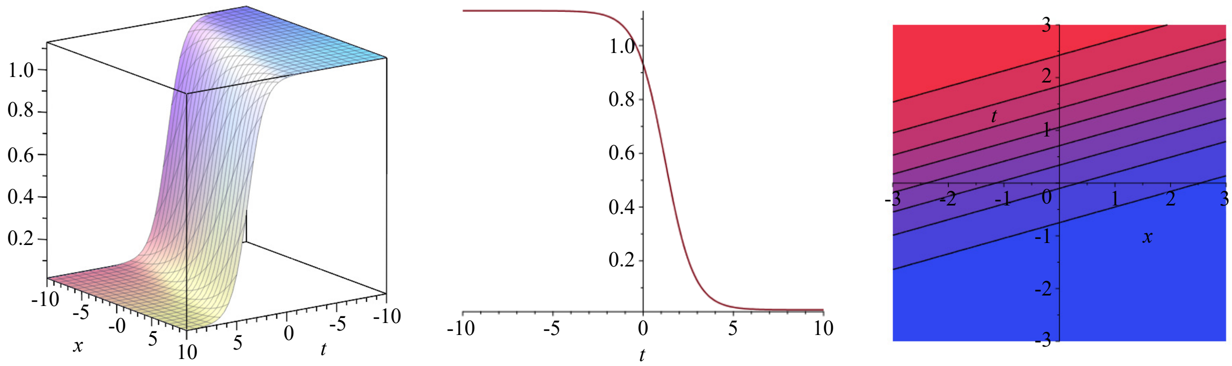

Figure 1. 3D, 2D, and contour plot of $u_1(x, t)$

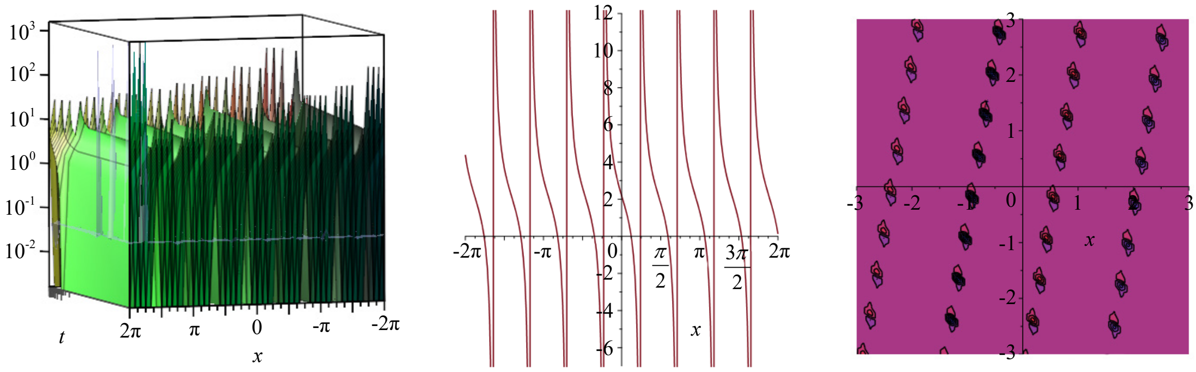

Figure 2. 3D, 2D, and contour plot of $u_2(x, t)$

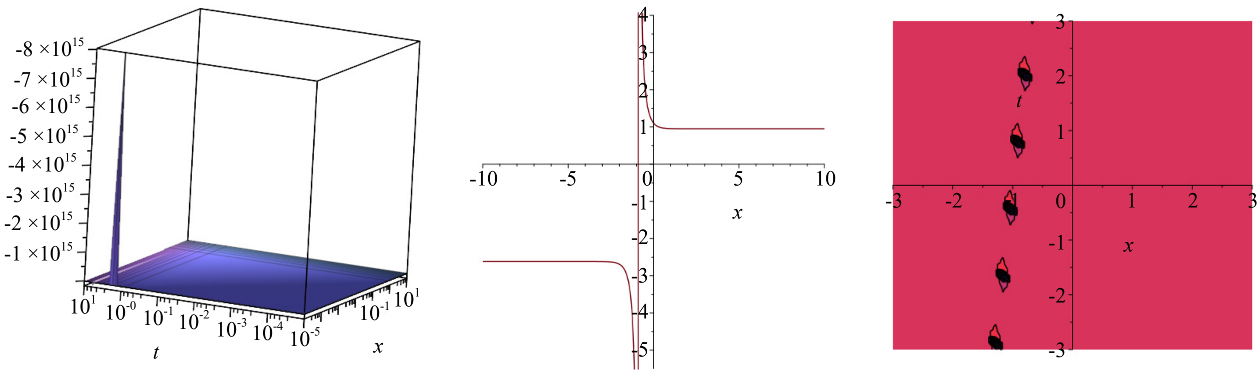

Figure 3. 3D, 2D, and contour plot of $u_3(x, t)$

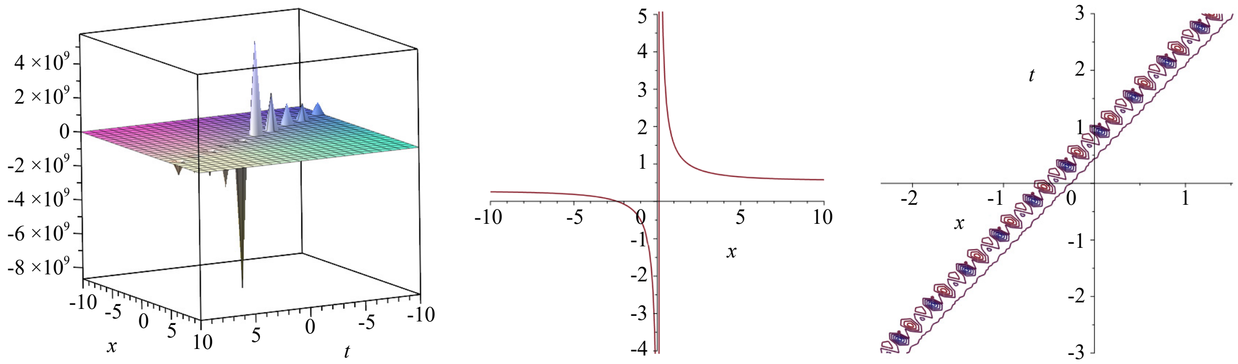

Figure 4. 3D, 2D, and contour plot of $u_6(x, t)$

5. Conclusion

The modified exp-function method has been utilized to obtain the traveling wave solution of equation (12). The considered equation has been reduced to an ODE by assuming the wave variable. The periodic, singular, and dark soliton solutions of the considered equation have been obtained by considering the solutions of the reduced ODE in the form of an exponential function. The obtained solutions have been validated by plugging them back into the original equation, and their behavior has been depicted through a graphical representation of the resulting traveling wave solutions. The computational software Maple has been utilized to illustrate the visual depictions of the acquired solutions. By making certain modifications to this method, it can be utilized to tackle other intricate models that arise in the fields of mathematics, physics, and applied mathematics.

Acknowledgments

Manjeet Sharma is grateful to the Council of Scientific and Industrial Research, New Delhi, India (CSIR file No. 09/1152(0017)/2019-EMR-1) for the financial support.

Conflict of interest

There is no conflict of interest in this study.

References

[1] Malfliet W, Hereman W. The tanh method: I. Exact solutions of nonlinear evolution and wave equations. *Physica Scripta*. 1996; 54(6): 563-568. Available from: https://doi.org/10.1088/0031-8949/54/6/003.

[2] Wazwaz A-M. The tanh method for traveling wave solutions of nonlinear equations. *Applied Mathematics and Computation.* 2004; 154(3): 713-723. Available from: https://doi.org/10.1016/S0096-3003(03)00745-8.

[3] Fu Z, Liu S, Liu S, Zhao Q. New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations. *Physics Letters A*. 2001; 290(1-2): 72-76. Available from: https://doi.org/10.1016/S0375-9601(01)00644-2.

[4] Fan E, Zhang J. Applications of the Jacobi elliptic function method to special-type nonlinear equations. *Physics Letters A*. 2002; 305(6): 383-392. Available from: https://doi.org/10.1016/S0375-9601(02)01516-5.

[5] Bekir A. Application of the $\left(\frac{G'}{G}\right)$-expansion method for nonlinear evolution equations. *Physics Letters A*. 2008; 372(19): 3400-3406. Available from: https://doi.org/10.1016/j.physleta.2008.01.057.

[6] Kumar S, Singh K, Gupta RK. Coupled Higgs field equation and Hamiltonian amplitude equation: Lie classical approach and (G'/G)-expansion method. *Pramana - Journal of Physics*. 2012; 79(1): 41-60. Available from: https://doi.org/10.1007/s12043-012-0284-7.

[7] Wang M, Li X, Zhang J. The $\left(\frac{G'}{G}\right)$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Physics Letters A*. 2008; 372(4): 417-423. Available from: https://doi.org/10.1016/j.physleta.2007.07.051.

[8] Abdou MA. The extended F-expansion method and its application for a class of nonlinear evolution equations. *Chaos, Solitons and Fractals*. 2007; 31(1): 95-104. Available from: https://doi.org/10.1016/j.chaos.2005.09.030.

[9] Kaur B, Gupta RK. Dispersion analysis and improved F-expansion method for space–time fractional differential equations. *Nonlinear Dynamics*. 2019; 96(2): 837-852. Available from: https://doi.org/10.1007/s11071-019-04825-w.

[10] Jawad AJM, Petković MD, Biswas A. Modified simple equation method for nonlinear evolution equations. *Applied Mathematics and Computation.* 2010; 217(2): 869-877. Available from: https://doi.org/10.1016/j.amc.2010.06.030.

[11] Zayed EME. A note on the modified simple equation method applied to Sharma-Tasso-Olver equation. *Applied Mathematics and Computation.* 2011; 218(7): 3962-3964. Available from: https://doi.org/10.1016/j.amc.2011.09.025.

[12] Olver PJ. *Applications of Lie groups to differential equations*. New York: Springer-Verlag; 1986. Available from: https://doi.org/10.1007/978-1-4684-0274-2.

[13] Bluman GW, Anco S. *Symmetry and integration methods for differential equations.* Springer Verlag, New York; 2002. Available from: https://doi.org/10.1007/b97380.

[14] Singh K, Gupta RK, Kumar S. Benjamin-Bona-Mahony (BBM) equation with variable coefficients: similarity reductions and Painlevé analysis. *Applied Mathematics and Computation*. 2011; 217(16): 7021-7027. Available from: https://doi.org/10.1016/j.amc.2011.02.003.

[15] Bluman GW, Cole JD. The general similarity solution of the heat equation. *Journal of Mathematics and Mechanics*. 1969; 18(11): 1025-1042. Available from: http://www.jstor.org/stable/24893142.

[16] Gupta RK, Singh M. Nonclassical symmetries and similarity solutions of variable coefficient coupled KdV system using compatibility method. *Nonlinear Dynamics.* 2017; 87(3): 1543-1552. Available from: https://doi.org/10.1007/s11071-016-3132-3.

[17] Arrigo DJ, Broadbridge P, Hill JM. Nonclassical symmetry solutions and the methods of Bluman-Cole and Clarkson-Kruskal. *Journal of Mathematical Physics*. 1993; 34(10): 4692-4703. Available from: https://doi.org/10.1063/1.530365.

[18] Levi D, Winternitz P. Non-classical symmetry reduction: example of the Boussinesq equation. *Journal of Physics A: Mathematical and General.* 1989; 22(15): 2915. Available from: https://doi.org/10.1088/0305-4470/22/15/010.

[19] Clarkson PA. Nonclassical symmetry reductions of the Boussinesq equation. *Chaos, Solitons and Fractals*. 1995; 5(12): 2261-2301. Available from: https://doi.org/10.1016/0960-0779(94)E0099-B.

[20] Manjeet, Gupta RK. On nonclassical symmetries, Painlevé analysis and singular, periodic and solitary wave solutions of generalized Hirota-Satsuma coupled KdV system. *Communications in Nonlinear Science and Numerical Simulation.* 2022; 115: 106710. Available from: https://doi.org/10.1016/j.cnsns.2022.106710.

[21] Hood S. New exact solutions of Burgerss equation—an extension to the direct method of Clarkson and Kruskal. *Journal of Mathematical Physics*. 1995; 36(4): 1971-1990. Available from: https://doi.org/10.1063/1.531097.

[22] Yao SW, Ilhan E, Veeresha P, Baskonus HM. A powerful iterative approach for quintic complex Ginzburg-Landau equation within the frame of fractional operator. *Fractals*. 2021; 29(5): 2140023. Available from: https://doi.org/10.1142/S0218348X21400235.

[23] Veeresha P, Baskonus HM, Gao W. Strong interacting internal waves in rotating ocean: Novel fractional approach. *Axioms.* 2021; 10(2): 123. Available from: https://doi.org/10.3390/axioms10020123.

[24] He Y, Li S, Long Y. Exact solutions of the Klein-Gordon equation by modified exp-function method. *International Mathematical Forum.* 2012; 7(1-4): 175-182.

[25] Bona JL, Tzvetkov N. Sharp well-posedness results for the BBM equation. *Discrete and Continuous Dynamical Systems.* 2009; 23(4): 1241-1252. Available from: https://doi.org/10.3934/dcds.2009.23.1241.

[26] Wang M. Exact solutions for a compound KdV-Burgers equation. *Physics Letters A*. 1996; 213(5-6): 279-287.

Available from: https://doi.org/10.1016/0375-9601(96)00103-X.
[27] Veeresha P, Yavuz M, Baishya C. A computational approach for shallow water forced Korteweg–De Vries equation on critical flow over a hole with three fractional operators. *An International Journal of Optimization and Control: Theories & Applications*. 2021; 11(3): 52-67. Available from: https://doi.org/10.11121/ijocta.2021.1177.
[28] Veeresha P, Prakasha D, Magesh N, Christopher AJ, Sarwe DU. Solution for fractional potential KdV and Benjamin equations using the novel technique. *Journal of Ocean Engineering and Science*. 2021; 6(3): 265-275. Available from: https://doi.org/10.1016/j.joes.2021.01.003.
[29] Khater MMA, Lu D, Zahran EHM. Solitary wave solutions of the Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity. *Applied Mathematics and Information Sciences*. 2017; 11(5): 1-5. Available from: https://doi.org/10.18576/amis/110511.
[30] Wang G, Xu T, Abazari R, Jovanoski Z, Biswas A. Shock waves and other solutions to the Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity. *Acta Physica Polonica A*. 2014; 126(6): 1221-1225. Available from: https://doi.org/10.12693/APhysPolA.126.1221.
[31] Biazar J, Ayati Z. Exp and modified exp function methods for nonlinear Drinfeld-Sokolov system. *Journal of King Saud University Science*. 2012; 24(4): 315-318. Available from: https://doi.org/10.1016/j.jksus.2010.08.013.