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DISPERSION AND FRACTIONAL LIE GROUP ANALYSIS OF TIME FRACTIONAL EQUATION FROM BURGERS HIERARCHY

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Abstract The paper presents the analysis of time fractional 5^{th} order equation from Burgers hierarchy. We discuss the dispersion relation and provide the complete analysis of the phase velocity and group velocity along with the nature of wave dispersion. Similarity reductions are carried out using infinitesimal symmetries to obtain nonlinear fractional ordinary differential equations having Erdélyi-Kober fractional differential operator. The explicit power series solution is obtained for reduced fractional ordinary differential equation and its convergence is discussed. The solution is appeared in the form of singular kink wave and further analysed graphically for various values of fractional order α. The new conservation theorem is applied to derive the conservation laws.

Keywords Fractional differential equations, symmetry analysis, explicit solutions, dispersion, conservation laws.

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1. Introduction

Fractional differential equations (FDEs) arise generally in various fields of science and engineering [37, 44, 47, 49, 55] and they portray the nonlinear physical phenomenon judiciously than the integer order differential equations. The linear dispersion analysis of FDEs gives the dispersion relation that relates the wave number and frequency. It provides the phase and group velocities whose relation predict the nature of dispersion/damping of the waves [13, 21]. The nonlinear analysis of FDEs to obtain solutions can be done by various methodologies such as symmetry method, the fractional sub-equation method, (G'/G)-expansion method, exp function method, fractional complex transformation method, the first integral method, etc. [6–8, 23, 50, 54, 60, 66]. Among all these successfully applied methodologies to FDEs, symmetry method [18, 19, 31–34, 56–58] is one such way that gives not only the symmetries of physical systems spanned by FDEs but also gives the solution as well as associated conservation laws [22, 25–29, 51, 53, 59, 64]. The existence of higher order conservation laws reveals the integrability of differential equations [10, 11, 48] and there are number of methods to obtain the conservation laws such as Noether's theorem [45], direct method [4], partial Lagrangian method [30], new conservation

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theorem method [24] and so on. The formulation of Noether's theorem and fractional generalization of the Noether operators using new conservation theorem for FDEs are given in [17,40].

In this paper, we try to develop the systematic algorithm for linear analysis, symmetries, explicit solutions and conservation laws of equation obtained from time fractional Burgers hierarchy [1,9,20,62,65]. The time fractional Burgers hierarchy can be determined from following

$$D_t^\alpha u + \mu D_x (D_x + u)^s u = 0, \ s = 0, 1, 2, ..., \tag{1.1}$$

where μ is arbitrary constant and α $(0 < \alpha < 1)$ is the fractional order of derivative w.r.t. time. Fractional Burgers hierarchy (1.1) of order α is the generalization of Burgers hierarchy of integer order [14,38,39,62] given by

$$D_t u + \mu D_x (D_x + u)^s u = 0, \ s = 0, 1, 2, ..., \tag{1.2}$$

Burger's equation balances the dissipation and nonlinear convection processes [61]. It is used to describe the model of fluid mechanics, traffic flow, nonlinear acoustic transmission and gas dynamics [61]. The 2^{nd} and 3^{rd} order time fractional equations from Burgers hierarchy have been investigated for invariant solutions by Lie group method [54,60]. Also, the multiwave solutions are investigated of 4^{th} and 5^{th} order space-time FDEs from Burgers hierarchy [1]. The time fractional 5^{th} order equation (TFFB) from Burger's hierarchy has not been investigated yet for linear dispersion properties, symmetries, explicit solutions and conservation laws. So, our main thrust in this paper is to solve the TFFB equation for all above mentioned properties.

The TFFB equation can be found by substituting $s = 4$ in the Eq. (1.1)

$$\begin{aligned} \Xi \equiv & D_t^\alpha u + \mu \Big(u_{5x} + 10\, {u_{2x}}^2 + 15\, u_x u_{3x} + 5\, u u_{4x} + 15\, {u_x}^3 + 50\, u u_x u_{2x} \\ & + 10\, u^2 u_{3x} + 30\, u^2 {u_x}^2 + 10\, u^3 u_{2x} + 5\, u^4 u_x \Big) = 0, \ 0 < \alpha < 1, \end{aligned} \tag{1.3}$$

where $u_{ix} = \frac{\partial^i u}{\partial x^i}$, $i = 2, ..., 5$. The analysis in the paper is divided into various sections in which section 2 describes the linear analysis of Eq. (1.3) including various properties of dispersion. In section 3, symmetries and reductions are performed. The section 4 includes the way to explicit power series solution of reduced fractional ordinary differential equations and its convergence. The conservation laws are obtained in section 5 and finally in the last section, concluding remarks are given.

2. Linear analysis of TFFB equation

The linear analysis of the TFFB equation gives the dispersion relation and it helps to find the phase velocity v_p and group velocity v_g. For the linear analysis the dispersive waves are usually taken in the form of a sinusoidal wave having periodic spatial and time dependence [2,3,13,21,35,41,42,63].

Let us consider $\psi(x,t)$ as the wave function for a (1+1)-dimensional system and sinusoidal wave form can be represented as

$$\psi(x,t) = \mathtt{Re}\{A e^{i(\omega t - kx)}\}, \tag{2.1}$$

where A is known as the complex amplitude and the parameters ω and k satisfy following equation called the dispersion relation

$$\mathcal{D}(\omega, k) = 0, \tag{2.2}$$

where $\mathcal{D}$ is the suitable real function of ω and k. Such a relation is, in general, satisfied by certain ω, $k \in \mathbb{C}$. This equation can be solved explicitly in terms of a real parameter (ω or k) by means of following two conditions

$$\begin{aligned} &\bar{\omega}_l(k) \in \mathbb{C},\ k \in \mathbb{R}, \\ &\bar{k}_m(\omega) \in \mathbb{C},\ \omega \in \mathbb{R}, \end{aligned} \tag{2.3}$$

where l and m are positive integers called mode indices. These branches are then related to the normal mode solutions of the dynamical equations for the physical system, i.e.,

$$\begin{aligned} \psi_l(x,t;k) &= \mathtt{Re}\{A_l(k)\exp\left[i(\bar{\omega}_l t - kx)\right]\}, \\ \psi_m(x,t;\omega) &= \mathtt{Re}\{A_m(k)\exp\left[i(\omega t - \bar{k}_m x)\right]\}. \end{aligned} \tag{2.4}$$

Discard the mode labels for sake of simplicity and equation (2.4) gives the phase velocity as

$$v_p(k) := \frac{\mathtt{Re}\bar{\omega}(k)}{k}. \tag{2.5}$$

Furthermore, the group velocity can be defined as follows

$$v_g(k) = \frac{\partial}{\partial k}\mathtt{Re}\bar{\omega}(k). \tag{2.6}$$

Now, the linear from of TFFB is given by

$$D_t^\alpha u_t + \mu u_{5x} = 0. \tag{2.7}$$

Here using the fractional Caputo derivative of order α, defined by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{-\infty}^{t}\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, \tag{2.8}$$

where $n \in \mathbb{N}$ such that $n-1 < \alpha < n$. In this case, we consider $0 < \alpha < 1$, so $n = 1$. The Fourier transform for $D_t^\alpha f(t)$ [36] provides the dispersion relation corresponding to the Eq. (2.7) by following relation

$$(i\bar{\omega})^\alpha + \mu(-ik)^5 = 0. \tag{2.9}$$

Solution of Eq. (2.9) becomes

$$\bar{\omega}(k) = \mu^{\frac{1}{\alpha}} k^{\frac{5}{\alpha}} i^{-1+\frac{1}{\alpha}},\ \alpha \neq 1. \tag{2.10}$$

Thus dispersion relation is found to be complex in nature and its real and imaginary parts for $k > 0$ are obtained as follows

$$\begin{aligned} \mathtt{Re}(\bar{\omega}(k)) &= \cos\left(\left(\frac{1}{\alpha}-1\right)\frac{\pi}{2}\right)\mu^{\frac{1}{\alpha}}k^{\frac{5}{\alpha}}, \\ \mathtt{Im}(\bar{\omega}(k)) &= \sin\left(\left(\frac{1}{\alpha}-1\right)\frac{\pi}{2}\right)\mu^{\frac{1}{\alpha}}k^{\frac{5}{\alpha}}. \end{aligned} \tag{2.11}$$

The complex form of phase and group velocities will appeared in the following form

$$\begin{aligned} \bar{v_p}(k) &= \frac{\bar{\omega}(k)}{k}, \\ &= \mu^{\frac{1}{\alpha}} k^{\frac{5}{\alpha}-1} i^{\frac{1}{\alpha}-1}, \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \bar{v_g}(k) &= \frac{\partial}{\partial k}\bar{\omega}(k), \\ &= \frac{5}{\alpha}\mu^{\frac{1}{\alpha}} k^{\frac{5}{\alpha}-1} i^{\frac{1}{\alpha}-1}. \end{aligned} \tag{2.13}$$

The real and imaginary parts of phase and group velocities are obtained as follows

$$\begin{aligned} v_p(k) = \mathtt{Re}(\bar{v_p}(k)) &= \cos((\frac{1}{\alpha}-1)\frac{\pi}{2})\mu^{\frac{1}{\alpha}} k^{\frac{5}{\alpha}-1}, \\ = \mathtt{Im}(\bar{v_p}(k)) &= \sin((\frac{1}{\alpha}-1)\frac{\pi}{2})\mu^{\frac{1}{\alpha}} k^{\frac{5}{\alpha}-1}, \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} v_g(k) = \mathtt{Re}(\bar{v_g}(k)) &= \frac{5}{\alpha}\cos((\frac{1}{\alpha}-1)\frac{\pi}{2})\mu^{\frac{1}{\alpha}} k^{\frac{5}{\alpha}-1}, \\ = \mathtt{Im}(\bar{v_g}(k)) &= \frac{5}{\alpha}\sin((\frac{1}{\alpha}-1)\frac{\pi}{2})\mu^{\frac{1}{\alpha}} k^{\frac{5}{\alpha}-1}. \end{aligned} \tag{2.15}$$

The variation of phase and group velocities with k for TFFB equation is shown in Figure 1 and 2 for $\alpha = 0.75$ and $\alpha = 0.5$, respectively. It has been found that group velocity is greater than phase velocity for all k values. The phase velocity and group velocity is related to each other by relation $v_g = v_p + k\,\frac{dv_p}{dk}$. Thus waves follow anomalous dispersion and longer wavelengths propagate slower than the waves with shorter wavelength. The variation of v_p and v_g with α for $k = 1$, $\mu = 1$ is shown in Figure 3. It reveals that there are number of α values lies between 0 and 1 for which real part of v_p and v_g approaches zero and it leads to the propagation of waves in opposite direction. The real part of phase and group velocity vanishes at $\alpha = \frac{1}{2(m+1)}$, where m is an integer having values 0, 1, 2,... and the corresponding α values are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, up to so on. Similarly the α values for which the imaginary part of the v_p and v_g becomes zero is given by $\frac{1}{2n+1}$, where n is an integer. It is to be noted that imaginary part of the $\bar{\omega}(k)$ corresponds to the damping of the waves as they propagates in space with time. Damping will take place except the points $\alpha = \frac{1}{2n+1}$, $n = 1, 2, 3, ...$

3. Symmetry analysis of TFFB equation

This section provides symmetry and reductions [18,19,23,54,60] of TFFB equation with Riemann-Liouville fractional derivative [37,49] of order α and defined by the following expression

$$D_t^{\alpha} u = \begin{cases} \frac{\partial^n u}{\partial t^n}, & \alpha = n, \text{ where } n \in N; \\ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(x,\underline{\theta})}{(t-\underline{\theta})^{\alpha+1-n}} d\underline{\theta}, & n-1 < \alpha < n,\ n \in N. \end{cases} \tag{3.1}$$

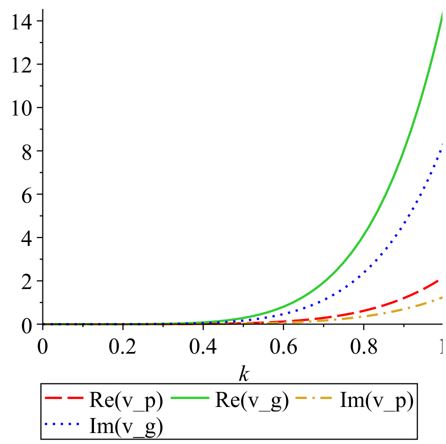

Figure 1. Variation of phase and group velocities with k at $\alpha = 0.75$.

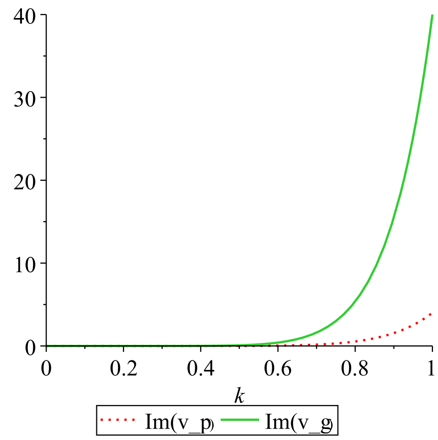

Figure 2. Variation of phase and group velocities with k at $\alpha = 0.5$. Here real part of phase and group velocities are vanishes and velocities are purely imaginary functions of wave number k.

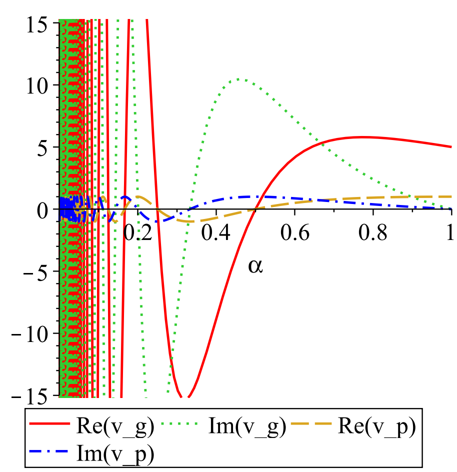

Figure 3. Variation of phase and group velocities with α having $\mu = 1$, $k = 1$.

The admitted Lie algebra for the Eq. (1.3) under a Lie group of transformations is spanned by following infinitesimal generator

$$\Omega = X\partial_x + T\partial_t + U\partial_u, \tag{3.2}$$

where X, T and U are infinitesimals corresponding to x, t and u, respectively.
If the infinitesimal generator (3.2) is a Lie point symmetry of Eq. (1.3) then it must satisfy following condition

$$Pr^{(\alpha,5)}\Omega(\Xi)|_{\Xi=0} = 0, \tag{3.3}$$

and gives prolonged vector as follows

$$Pr^{(\alpha,5)}\Omega = \Omega + U^{\alpha,t}\partial_{\partial_t^\alpha u} + U^x\partial_{u_x} + U^{2x}\partial_{u_{2x}} + U^{3x}\partial_{u_{3x}} + U^{4x}\partial_{u_{4x}} + U^{5x}\partial_{u_{5x}}, \tag{3.4}$$

where U^x, U^{2x}, U^{3x}, U^{4x}, U^{5x}, $U^{\alpha,t}$ are extended infinitesimals [18,19,48,60]. The invariance condition (3.3) for the TFFB equation produces some determining equations by equating the coefficients of various derivatives of u to zero. The solution of various determining equations is established as follows

$$X = c_1x + c_2,\ T = \frac{5t}{\alpha}c_1,\ U = -uc_1, \tag{3.5}$$

where c_1 and c_2 are arbitrary constants. The associated Lie algebra from above infinitesimals for the TFFB equation is spanned by following two vector fields

$$\Omega_1 = \partial_x,\ \Omega_2 = x\partial_x + \frac{5t}{\alpha}\partial_t - u\partial_u. \tag{3.6}$$

Thus the invariant solutions and similarity reductions are discussed in the following cases.
Case 3.1.1: Vector field $\Omega_1 = \partial_x$ gives the invariant solution for Eq. (1.3) as follows

$$u(x,t) = \Psi(t). \tag{3.7}$$

The reduced fractional ordinary differential equation (ODE) from invariant solution is retrieved as

$$\frac{\partial^\alpha\Psi(t)}{\partial t^\alpha} = 0. \tag{3.8}$$

The solution of reduced fractional ODE is obtained as follows

$$u(x,t) = d_1\, t^{\alpha-1}, \tag{3.9}$$

where d_1 is arbitrary constant.
Case 3.1.2: For the vector field $\Omega_2 = x\partial_x + \frac{5t}{\alpha}\partial_t - u\partial_u$, the associated characteristic equations are given as follows

$$\frac{dx}{x} = \frac{dt}{\frac{5t}{\alpha}} = \frac{du}{-u}. \tag{3.10}$$

From the solution of above characteristic equations, we get following invariants

$$\zeta = x\,t^{-\frac{\alpha}{5}},\ u(x,t) = t^{-\frac{\alpha}{5}}\,\Psi(\zeta). \tag{3.11}$$

Substitution of above invariants into TFFB equation (1.3) provides the nonlinear ODE of fractional order by following procedure.

The fractional derivative of $u(x,t)=t^{-\frac{\alpha}{5}}\Psi(\zeta)$ w.r.t. t of order α using (3.1) for $n-1<\alpha<n$, $n=1,2,3,...$ is given by

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\Big[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\underline{\theta})^{n-\alpha-1}\underline{\theta}^{\frac{-\alpha}{5}}\Psi(\underline{\theta}^{\frac{-\alpha}{5}}x)d\underline{\theta}\Big]. \tag{3.12}$$

Let us consider $w=t/\underline{\theta}$. Then above equation can be drafted as follows

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha}&=\frac{\partial^n}{\partial t^n}\Big[\frac{1}{\Gamma(n-\alpha)}t^{n-\frac{6\alpha}{5}}\int_1^\infty (w-1)^{n-\alpha-1}w^{-(n+1-\frac{6\alpha}{5})}\Psi(\zeta w^{\frac{\alpha}{5}})\Big]dw\\ &=\frac{\partial^n}{\partial t^n}\Big[t^{n-\frac{6\alpha}{5}}\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}\Psi\Big)(\zeta)\Big],\end{aligned} \tag{3.13}$$

where $K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}$ is Erdélyi-Kober fractional integral operator [37], On account of the relation $\zeta=x\,t^{-\frac{\alpha}{5}}$, $\varphi\in C^1(0,\infty)$, we get

$$t\frac{\partial}{\partial t}\varphi(\zeta)=tx(-\frac{\alpha}{5})t^{-\frac{\alpha}{5}-1}\varphi'(\zeta)=-\frac{\alpha}{5}\zeta\varphi'(\zeta). \tag{3.14}$$

Using relation (3.14), we have

$$\begin{aligned}\frac{\partial^n}{\partial t^n}\Big[t^{n-\frac{6\alpha}{5}}\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}\Psi\Big)(\zeta)\Big]&=\frac{\partial^{n-1}}{\partial t^{n-1}}\Big[\frac{\partial}{\partial t}\Big(t^{n-\frac{6\alpha}{5}}\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}\Psi\Big)(\zeta)\Big)\Big]\\ &=\frac{\partial^{n-1}}{\partial t^{n-1}}\Big[t^{n-\frac{6\alpha}{5}-1}\Big(n-\frac{6\alpha}{5}-\frac{\alpha}{5}\zeta\frac{\partial}{\partial\zeta}\Big)\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}\Psi\Big)(\zeta)\Big].\end{aligned} \tag{3.15}$$

By repeating above procedure for $n-1$ times, we obtain

$$\begin{aligned}\frac{\partial^n}{\partial t^n}\Big[t^{n-\frac{6\alpha}{5}}\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}\Psi\Big)(\zeta)\Big]&=\frac{\partial^{n-1}}{\partial t^{n-1}}\Big[\frac{\partial}{\partial t}\Big(t^{n-\frac{6\alpha}{5}}\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}\Psi\Big)(\zeta)\Big)\Big]\\ &=\frac{\partial^{n-1}}{\partial t^{n-1}}\Big[t^{n-\frac{6\alpha}{5}-1}\Big(n-\frac{6\alpha}{5}-\frac{\alpha}{5}\zeta\frac{\partial}{\partial\zeta}\Big)\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}\Psi\Big)(\zeta)\Big]\\ &\ \ \vdots\\ &=t^{-\frac{6\alpha}{5}}\prod_{j=0}^{n-1}\Big(1-\frac{6\alpha}{5}+j-\frac{\alpha}{5}\zeta\frac{\partial}{\partial\zeta}\Big)\Big(K_{\frac{5}{\alpha}}^{1-\frac{\alpha}{5},n-\alpha}\Psi\Big)(\zeta)\\ &=t^{-\frac{6\alpha}{5}}\Big(P_{\frac{5}{\alpha}}^{1-\frac{6\alpha}{5},\alpha}\Psi\Big)(\zeta),\end{aligned} \tag{3.16}$$

where $\Big(P_{\frac{5}{\alpha}}^{1-\frac{6\alpha}{5},\alpha}\Psi\Big)$ is Erdélyi-Kober fractional differential operator [37], Thus, we have

$$\frac{\partial^\alpha u}{\partial t^\alpha}=t^{-\frac{6\alpha}{5}}\Big(P_{\frac{5}{\alpha}}^{1-\frac{6\alpha}{5},\alpha}\Psi\Big)(\zeta). \tag{3.17}$$

Hence, the TFFB equation is reduced to nonlinear fractional ODE of the following form

$$\begin{aligned}&(P_{\frac{5}{\alpha}}^{1-6\frac{\alpha}{5},\alpha}\Psi)(\zeta)+\Big(10\,\Psi^3\Psi_{2\zeta}+5\,\Psi^4\Psi_\zeta+10\,\Psi^2\Psi_{3\zeta}+30\,\Psi^2{\Psi_\zeta}^2+\Psi_{5\zeta}+10\,{\Psi_{2\zeta}}^2\\ &+15\,\Psi_\zeta\Psi_{3\zeta}+5\,\Psi\Psi_{4\zeta}+15\,{\Psi_\zeta}^3+50\,\Psi\Psi_\zeta\Psi_{2\zeta}\Big)\mu=0.\end{aligned} \tag{3.18}$$

4. Explicit power series solution of reduced nonlinear fractional ODE (3.18)

This section presents a way to obtain explicit convergent power series solutions to Eq. (3.18) as this method is an excellent way to obtain solution for fractional ODEs [16,25–27,51]. Let us consider the solution of Eq. (3.18) in power series form

$$\Psi(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n. \tag{4.1}$$

Substituting Eq. (4.1) in to Eq. (3.18), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\Gamma\left(1-\frac{1}{5}\alpha-\frac{1}{5}n\alpha\right) a_n\zeta^n}{\Gamma\left(1-\frac{6}{5}\alpha-\frac{1}{5}n\alpha\right)} + \mu \left(10\left(\sum_{n=0}^{\infty} a_n\zeta^n\right)^3 \sum_{n=0}^{\infty} (n+2)(n+1)\,a_{n+2}\zeta^n\right.\\
&+5\left(\sum_{n=0}^{\infty} a_n\zeta^n\right)^4 \sum_{n=0}^{\infty} (n+1)\,a_{n+1}\zeta^n\\
&+10\left(\sum_{n=0}^{\infty} a_n\zeta^n\right)^2 \sum_{n=0}^{\infty} (n+3)(n+2)(n+1)\,a_{n+3}\zeta^n\\
&+30\left(\sum_{n=0}^{\infty} a_n\zeta^n\right)^2 \left(\sum_{n=0}^{\infty} (n+1)\,a_{n+1}\zeta^n\right)^2\\
&+\sum_{n=0}^{\infty} (n+5)(n+4)(n+3)(n+2)(n+1)\,a_{n+5}\zeta^n\\
&+10\left(\sum_{n=0}^{\infty} (n+2)(n+1)\,a_{n+2}\zeta^n\right)^2\\
&+15\sum_{n=0}^{\infty} (n+1)\,a_{n+1}\zeta^n \sum_{n=0}^{\infty} (n+3)(n+2)(n+1)\,a_{n+3}\zeta^n\\
&+5\sum_{n=0}^{\infty} a_n\zeta^n \sum_{n=0}^{\infty} (n+4)(n+3)(n+2)(n+1)\,a_{n+4}\zeta^n\\
&+15\left(\sum_{n=0}^{\infty} (n+1)\,a_{n+1}\zeta^n\right)^3\\
&\left.+50\sum_{n=0}^{\infty} a_n\zeta^n \sum_{n=0}^{\infty} (n+1)\,a_{n+1}\zeta^n \sum_{n=0}^{\infty} (n+2)(n+1)\,a_{n+2}\zeta^n\right) = 0.
\end{aligned} \tag{4.2}$$

From Eq. (4.2), equating the coefficients of various powers of ζ^n, to zero. For $n = 0$ we get

$$\begin{aligned}
a_5 = &-\frac{1}{120}\frac{\Gamma\left(1-\frac{1}{5}\alpha\right) a_0}{\mu\,\Gamma\left(1-\frac{6}{5}\alpha\right)} - \frac{1}{120}\left(20\,{a_0}^3 a_2 + 5\,{a_0}^4 a_1 + 60\,{a_0}^2 a_3\right.\\
&\left.+30\,{a_0}^2{a_1}^2 + 40\,{a_2}^2 + 90\,a_1a_3 + 120\,a_0a_4 + 15\,{a_1}^3 + 100\,a_0a_1a_2\right).
\end{aligned} \tag{4.3}$$

For $n \geq 1$, we have

$$
\begin{aligned}
a_{n+5} = & -\frac{1}{(n+5)(n+4)(n+3)(n+2)(n+1)}\left(\frac{\Gamma\left(1-\frac{1}{5}\,\alpha-\frac{1}{5}\,n\alpha\right)a_n}{\mu\,\Gamma\left(1-\frac{6}{5}\,\alpha-\frac{1}{5}\,n\alpha\right)}\right.\\
& +10\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}a_l a_{j-l}a_{k-j}\left(n-k+2\right)\left(n-k+1\right)a_{n-k+2}\\
& +5\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}\sum_{m=0}^{l}a_m a_{l-m}a_{j-l}a_{k-j}\left(n-k+1\right)a_{n-k+1}\\
& +10\sum_{k=0}^{n}\sum_{j=0}^{k}a_j a_{k-j}\left(n-k+3\right)\left(n-k+2\right)\left(n-k+1\right)a_{n-k+3}\\
& +30\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}a_l a_{j-l}\left(k-j+1\right)a_{k-j+1}\left(n-k+1\right)a_{n-k+1}\\
& +10\sum_{k=0}^{n}\left(k+2\right)\left(k+1\right)a_{k+2}\left(n-k+2\right)\left(n-k+1\right)a_{n-k+2}\\
& +15\sum_{k=0}^{n}\left(k+1\right)a_{k+1}\left(n-k+3\right)\left(n-k+2\right)\left(n-k+1\right)a_{n-k+3}\\
& +5\sum_{k=0}^{n}a_k\left(n-k+4\right)\left(n-k+3\right)\left(n-k+2\right)\left(n-k+1\right)a_{n-k+4}\\
& +15\sum_{k=0}^{n}\sum_{j=0}^{k}\left(j+1\right)a_{j+1}\left(k-j+1\right)a_{k-j+1}\left(n-k+1\right)a_{n-k+1}\\
& \left.+50\sum_{k=0}^{n}\sum_{j=0}^{k}a_j a_{k-j+1}\left(k-j+1\right)\left(n-k+2\right)\left(n-k+1\right)a_{n-k+2}\right).
\end{aligned}
\tag{4.4}
$$

Thus, for arbitrary chosen constant numbers a_0, a_1, a_2, a_3, a_4 and the other terms of the sequence $\{a_n\}_{n=0}^{\infty}$ can be determined successively from Eqs. (4.3) and (4.4) in a unique manner. Thus, power series solution for the Eq. (3.18) with the coefficients given by Eqs (4.3) and (4.4) can be represented as follows

$$
\begin{aligned}
\Psi(\zeta) = & a_0+a_1\zeta+a_2\zeta^2+a_3\zeta^3+a_4\zeta^4+a_5\zeta^5+\sum_{n=1}^{\infty}a_{n+5}\zeta^{n+5}\\
= & a_0+a_1\zeta+a_2\zeta^2+a_3\zeta^3+a_4\zeta^4-\frac{1}{120}\left(\frac{\Gamma\left(1-\frac{1}{5}\,\alpha\right)a_0}{\mu\,\Gamma\left(1-\frac{6}{5}\,\alpha\right)}\right.\\
& +20\,{a_0}^3a_2+5\,{a_0}^4a_1+60\,{a_0}^2a_3+30\,{a_0}^2{a_1}^2+40\,{a_2}^2+90\,a_1a_3\\
& \left.+120\,a_0a_4+15\,{a_1}^3+100\,a_0a_1a_2\right)\zeta^5\\
& -\left(\sum_{n=1}^{\infty}\frac{1}{(n+5)(n+4)(n+3)(n+2)(n+1)}\left(\frac{\Gamma\left(1-\frac{1}{5}\,\alpha-\frac{1}{5}\,n\alpha\right)a_n}{\mu\,\Gamma\left(1-\frac{6}{5}\,\alpha-\frac{1}{5}\,n\alpha\right)}\right.\right.\\
& +10\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}a_l a_{j-l}a_{k-j}\left(n-k+2\right)\left(n-k+1\right)a_{n-k+2}
\end{aligned}
$$

$$
\begin{aligned}
&+5\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}\sum_{m=0}^{l}a_m a_{l-m}a_{j-l}a_{k-j}\left(n-k+1\right)a_{n-k+1}\\
&+10\sum_{k=0}^{n}\sum_{j=0}^{k}a_j a_{k-j}\left(n-k+3\right)\left(n-k+2\right)\left(n-k+1\right)a_{n-k+3}\\
&+30\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}a_l a_{j-l}\left(k-j+1\right)a_{k-j+1}\left(n-k+1\right)a_{n-k+1}\\
&+10\sum_{k=0}^{n}\left(k+2\right)\left(k+1\right)a_{k+2}\left(n-k+2\right)\left(n-k+1\right)a_{n-k+2}\\
&+15\sum_{k=0}^{n}\left(k+1\right)a_{k+1}\left(n-k+3\right)\left(n-k+2\right)\left(n-k+1\right)a_{n-k+3}\\
&+5\sum_{k=0}^{n}a_k\left(n-k+4\right)\left(n-k+3\right)\left(n-k+2\right)\left(n-k+1\right)a_{n-k+4}\\
&+15\sum_{k=0}^{n}\sum_{j=0}^{k}\left(j+1\right)a_{j+1}\left(k-j+1\right)a_{k-j+1}\left(n-k+1\right)a_{n-k+1}\\
&+50\sum_{k=0}^{n}\sum_{j=0}^{k}a_j a_{k-j+1}\left(k-j+1\right)\left(n-k+2\right)\left(n-k+1\right)a_{n-k+2}\zeta^{n+5}\Bigg)\Bigg).
\end{aligned}
\tag{4.5}
$$

Hence, explicit power series solution for TFFB equation can be expressed as follows

$$
\begin{aligned}
u(x,t) =&a_0 t^{\frac{-\alpha}{5}}+a_1 x t^{\frac{-2\alpha}{5}}+a_2x^2t^{\frac{-3\alpha}{5}}+a_3x^3t^{\frac{-4\alpha}{5}}+a_4x^4t^{-\alpha}\\
&+a_5x^5t^{\frac{-6\alpha}{5}}+\sum_{n=1}^{\infty}a_{n+5}x^{n+5}t^{\frac{-(n+5)\alpha}{5}}\\
=&a_0 t^{\frac{-\alpha}{5}}+a_1 x t^{\frac{-2\alpha}{5}}+a_2x^2t^{\frac{-3\alpha}{5}}+a_3x^3t^{\frac{-4\alpha}{5}}+a_4x^4t^{-\alpha}\\
&-\frac{1}{120}\left(\frac{\Gamma\left(1-\frac{1}{5}\alpha\right)a_0}{\mu\Gamma\left(1-\frac{6}{5}\alpha\right)}+20{a_0}^3a_2+5{a_0}^4a_1+60{a_0}^2a_3+30{a_0}^2{a_1}^2+40{a_2}^2\right.\\
&\left.+90\,a_1a_3+120\,a_0a_4+15\,{a_1}^3+100\,a_0a_1a_2\right)x^5t^{\frac{-6\alpha}{5}}\\
&-\left(\sum_{n=1}^{\infty}\frac{1}{\left(n+5\right)\left(n+4\right)\left(n+3\right)\left(n+2\right)\left(n+1\right)}\left(\frac{\Gamma\left(1-\frac{1}{5}\,\alpha-\frac{1}{5}\,n\alpha\right)a_n}{\mu\,\Gamma\left(1-\frac{6}{5}\,\alpha-\frac{1}{5}\,n\alpha\right)}\right.\right.\\
&+10\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}a_l a_{j-l}a_{k-j}\left(n-k+2\right)\left(n-k+1\right)a_{n-k+2}\\
&+5\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}\sum_{m=0}^{l}a_m a_{l-m}a_{j-l}a_{k-j}\left(n-k+1\right)a_{n-k+1}\\
&+10\sum_{k=0}^{n}\sum_{j=0}^{k}a_j a_{k-j}\left(n-k+3\right)\left(n-k+2\right)\left(n-k+1\right)a_{n-k+3}
\end{aligned}
$$

$$
\begin{aligned}
&+30\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}a_l a_{j-l}\left(k-j+1\right)a_{k-j+1}\left(n-k+1\right)a_{n-k+1}\\
&+10\sum_{k=0}^{n}\left(k+2\right)\left(k+1\right)a_{k+2}\left(n-k+2\right)\left(n-k+1\right)a_{n-k+2}\\
&+15\sum_{k=0}^{n}\left(k+1\right)a_{k+1}\left(n-k+3\right)\left(n-k+2\right)\left(n-k+1\right)a_{n-k+3}\\
&+5\sum_{k=0}^{n}a_k\left(n-k+4\right)\left(n-k+3\right)\left(n-k+2\right)\left(n-k+1\right)a_{n-k+4}\\
&+15\sum_{k=0}^{n}\sum_{j=0}^{k}\left(j+1\right)a_{j+1}\left(k-j+1\right)a_{k-j+1}\left(n-k+1\right)a_{n-k+1}\\
&+50\sum_{k=0}^{n}\sum_{j=0}^{k}a_j a_{k-j+1}\left(k-j+1\right)(n-k+2)(n-k+1)\,a_{n-k+2}\Bigg)x^{n+5}t^{\frac{-(n+5)\alpha}{5}}\Bigg)\,.
\end{aligned}
\tag{4.6}
$$

The obtained power series solution (4.6) is analyzed graphically by plotting its two-dimensional (2D) and three-dimensional (3D) curves. The figure caption provides the various parameters selected for plotting. Figure 4 and 5 show 3D and 2D plots respectively of the solution for $\alpha = 0.5$ and it reveals singular kink wave profile. The combination of Figures 6-7 and Figure 8-9 also represent singular kink wave pattern at $\alpha = 0.75$ and $\alpha = 0.9$, respectively by 3D and 2D plots.

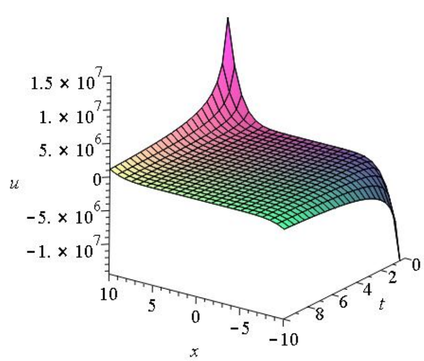

Figure 4. 3D plot of the solution (4.6) with $a_0 = 1, a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 1, a_5 = -4, a_6 = 0.17, a_7 = 0.76$, $\mu = 2$, $\alpha = 0.5$, $n = 0$ to 7.

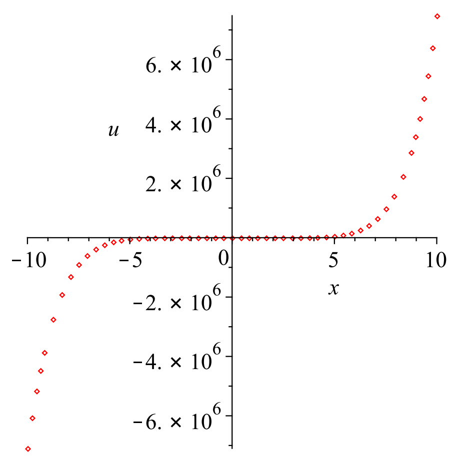

Figure 5. 2D plot of the solution (4.6) with $a_0 = 1, a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 1, a_5 = -4, a_6 = 0.17, a_7 = 0.76$, $\mu = 2$, $\alpha = 0.5$, $t = 1$, $n = 0$ to 7

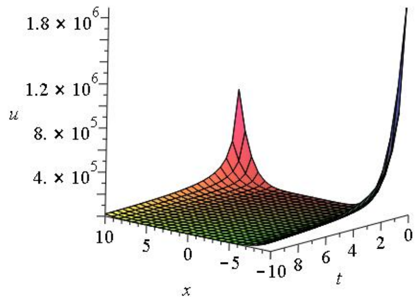

Figure 6. 3D plot of the solution (4.6) with $a_0 = 1, a_1 = 0.5, a_2 = 0.25, a_3 = 0.125, a_4 = 2, a_5 = -2.41, a_6 = 0.53$, $\mu = 2$, $\alpha = 0.75$, $n = 0$ to 6.

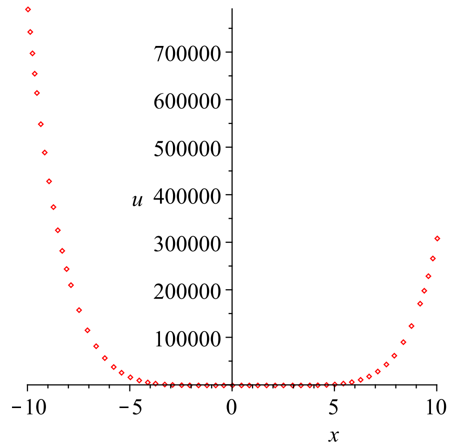

Figure 7. 2D plot of the solution (4.6) with $a_0 = 1, a_1 = 0.5, a_2 = 0.25, a_3 = 0.125, a_4 = 2, a_5 = -2.41, a_6 = 0.53$, $\mu = 2$, $\alpha = 0.75$, $t = 1$, $n = 0$ to 6

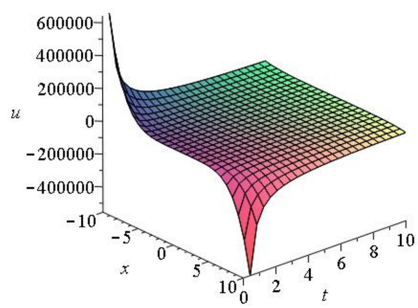

Figure 8. 3D plot of the solution (4.6) with $a_0 = 1, a_1 = 0.5, a_2 = 0.25, a_3 = 0.125, a_4 = 2, a_5 = -2.41, a_6 = 0.53$, $\mu = 2$, $\alpha = 0.9$, $n = 0$ to 5.

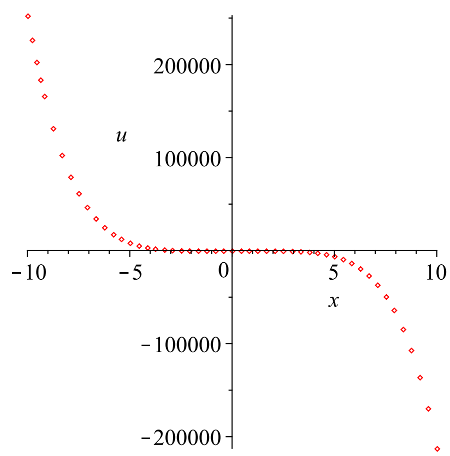

Figure 9. 2D plot of the solution (4.6) with $a_0 = 1, a_1 = 0.5, a_2 = 0.25, a_3 = 0.125, a_4 = 2, a_5 = -2.41, a_6 = 0.53$, $\mu = 2$, $\alpha = 0.9$, $t = 1$, $n = 0$ to 5

4.1. Convergence analysis

In this subsection the convergence [25–27, 51] of the power series solution (4.6) is tested. The modulus of the general recurrence relation given by Eq. (4.4) is represented as follows

$$\begin{aligned}
&|a_{n+5}| \\
\leq &\left(\frac{|\Gamma\left(1-\frac{1}{5}\alpha-\frac{1}{5}n\alpha\right)||a_n|}{|\mu||\Gamma\left(1-\frac{6}{5}\alpha-\frac{1}{5}\,n\alpha\right)|} + 10\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}|a_l||a_{j-l}||a_{k-j}||a_{n-k+2}| \right. \\
&+5\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}\sum_{m=0}^{l}|a_m||a_{l-m}||a_{j-l}||a_{k-j}||a_{n-k+1}| + 10\sum_{k=0}^{n}\sum_{j=0}^{k}|a_j||a_{k-j}||a_{n-k+3}| \\
&+30\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}|a_l||a_{j-l}||a_{k-j+1}||a_{n-k+1}| + 10\sum_{k=0}^{n}|a_{k+2}||a_{n-k+2}| \\
&+15\sum_{k=0}^{n}|a_{k+1}||a_{n-k+3}| + 5\sum_{k=0}^{n}|a_k||a_{n-k+4}| + 15\sum_{k=0}^{n}\sum_{j=0}^{k}|a_{j+1}||a_{k-j+1}||a_{n-k+1}| \\
&\left. +50\sum_{k=0}^{n}\sum_{j=0}^{k}|a_j||a_{k-j+1}||a_{n-k+2}| \right).
\end{aligned} \tag{4.7}$$

By utilizing the property of Γ function, $\frac{|\Gamma\left(1-\frac{1}{5}\,\alpha-\frac{1}{5}\,n\alpha\right)|}{|\Gamma\left(1-\frac{6}{5}\,\alpha-\frac{1}{5}\,n\alpha\right)|} < 1$ for arbitrary n, the Eq. (4.7) can be written as

$$\begin{aligned}
&|a_{n+5}| \\
\leq &M\left(|a_n| + \sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}|a_l||a_{j-l}||a_{k-j}||a_{n-k+2}| \right. \\
&+\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}\sum_{m=0}^{l}|a_m||a_{l-m}||a_{j-l}||a_{k-j}||a_{n-k+1}| + \sum_{k=0}^{n}\sum_{j=0}^{k}|a_j||a_{k-j}||a_{n-k+3}| \\
&+\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}|a_l||a_{j-l}||a_{k-j+1}||a_{n-k+1}| + \sum_{k=0}^{n}|a_{k+2}||a_{n-k+2}| + \sum_{k=0}^{n}|a_{k+1}||a_{n-k+3}| \\
&\left. +\sum_{k=0}^{n}|a_k||a_{n-k+4}| + \sum_{k=0}^{n}\sum_{j=0}^{k}|a_{j+1}||a_{k-j+1}||a_{n-k+1}| + \sum_{k=0}^{n}\sum_{j=0}^{k}|a_j||a_{k-j+1}||a_{n-k+2}| \right),
\end{aligned} \tag{4.8}$$

where $M = \max(\frac{1}{|mu|}, 5, 10, 15, 30, 50)$.

Consider another power series of the form

$$B(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^n, \tag{4.9}$$

where the expansion coefficients b_n are related to the coefficients of the (4.6) as

$b_i = |a_i|$, $i = 0, ..., 5$. Thus the Eq. (4.8) reads

$$
\begin{aligned}
|a_{n+5}| \leq M \Bigg(& b_n + \sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j} b_l b_{j-l} b_{k-j} b_{n-k+2} + \sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}\sum_{m=0}^{l} b_m b_{l-m} b_{j-l} b_{k-j} b_{n-k+1} \\
& + \sum_{k=0}^{n}\sum_{j=0}^{k} b_j b_{k-j} b_{n-k+3} + \sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j} b_l b_{j-l} b_{k-j+1} b_{n-k+1} + \sum_{k=0}^{n} b_{k+2} b_{n-k+2} \\
& + \sum_{k=0}^{n} b_{k+1} b_{n-k+3} + \sum_{k=0}^{n} b_k b_{n-k+4} + \sum_{k=0}^{n}\sum_{j=0}^{k} b_{j+1} b_{k-j+1} b_{n-k+1} \\
& + \sum_{k=0}^{n}\sum_{j=0}^{k} b_j b_{k-j+1} b_{n-k+2} \Bigg), \qquad (4.10)
\end{aligned}
$$

where $n = 0, 1, 2, ...$ Therefore, it is easily seen that $|b_n| \leq a_n$, n=0,1,2,... In other words, the series $B = B(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^n$ is majorant series of Eq. (4.1). Now we will prove that the series $B = B(\zeta)$ has positive radius of convergence and hence our obtained power series solution of TFFB is convergent. For this rewrite the eq. (4.9) for $B(\zeta)$ as

$$
\begin{aligned}
B(\zeta) = & b_0 + b_1\zeta + b_2\zeta^2 + b_3\zeta^3 + b_4\zeta^4 + M \Bigg(\sum_{n=0}^{\infty} b_n \zeta^{n+5} \\
& + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j} b_l b_{j-l} b_{k-j} b_{n-k+2} \zeta^{n+5} \\
& + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j}\sum_{m=0}^{l} b_m b_{l-m} b_{j-l} b_{k-j} b_{n-k+1} \zeta^{n+5} \\
& + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k} b_j b_{k-j} b_{n-k+3} \zeta^{n+5} + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{l=0}^{j} b_l b_{j-l} b_{k-j+1} b_{n-k+1} \zeta^{n+5} \\
& + \sum_{n=0}^{\infty}\sum_{k=0}^{n} b_{k+2} b_{n-k+2} \zeta^{n+5} + \sum_{k=0}^{n} b_{k+1} b_{n-k+3} \zeta^{n+5} + \sum_{n=0}^{\infty}\sum_{k=0}^{n} b_k b_{n-k+4} \zeta^{n+5} \\
& + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k} b_{j+1} b_{k-j+1} b_{n-k+1} \zeta^{n+5} + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k} b_j b_{k-j+1} b_{n-k+2} \zeta^{n+5} \Bigg) \\
= & b_0 + b_1\zeta + b_2\zeta^2 + b_3\zeta^3 + b_4\zeta^4 + M \left(\zeta^5 B(\zeta) + B^3(B - b_0 - b_1\zeta) \right. \\
& + B^4(B - b_0) + B^2(B - b_0 - b_1\zeta - b_2\zeta^2) + B^2(B - b_0)(B - b_0) \\
& + (B - b_0 - b_1\zeta)(B - b_0 - b_1\zeta) + (B - b_0)(B - b_0 - b_1\zeta - b_2\zeta^2) \\
& + B(B - b_0 - b_1\zeta - b_2\zeta^2 - b_3\zeta^3) + (B - a_0)(B - a_0)(B - a_0) \\
& \left. + B(B - a_0)(B - a_0 - b_1\zeta) \right). \qquad (4.11)
\end{aligned}
$$

Consider the implicit functional equation with respect to the independent variable ζ

$$
F(\zeta, B) = B - b_0 - b_1\zeta + b_2\zeta^2 - b_3\zeta^3 - b_4\zeta^4 - M \left(\zeta^5 B + B^3(B - b_0 - b_1\zeta) \right.
$$

$$
\begin{aligned}
&+B^4(B-b_0)+B^2(B-b_0-b_1\zeta-b_2\zeta^2)+B^2(B-b_0)(B-b_0)\\
&+(B-b_0-b_1\zeta)(B-b_0-b_1\zeta)+(B-b_0)(B-b_0-b_1\zeta-b_2\zeta^2)\\
&+B(B-b_0-b_1\zeta-b_2\zeta^2-b_3\zeta^3)+(B-b_0)(B-b_0)(B-b_0)\\
&+B(B-b_0)(B-b_0-b_1\zeta))\,. \qquad (4.12)
\end{aligned}
$$

From the above formula it can be easily proved that F is analytical in the neighborhood of $(0, b_0)$ and

$$F(0,b_0)=0,\ F'_B(0,b_0)=1-M\,b_0\neq 0. \qquad (4.13)$$

Thus based on the theorem given in [52], we see that $B = B(\zeta)$ is analytical in a neighborhood of the point $(0, b_0)$ and possess the positive radius. Hence power series (4.1) is convergent in a neighborhood of the point $(0, b_0)$. The TFFB equation has been not analyzed yet for linear dispersion analysis, symmetry analysis and explicit power series solutions. For $\alpha = 1$, the multiple kink solutions and multiple singular kink solutions are obtained and reported in [62]. The convergent power series solution of TFFB obtained in the present discussion has not been obtained and reported by anyone else.

5. Conservation Laws for TFFB

In this section, we obtain the conservation laws [5, 12, 15, 40, 43, 46] of the TFFB equation by using new conservation theorem [24]. For this the Riemann-Liouville left-sided time-fractional derivative [40] and left-sided time-fractional integral of order $n-\alpha$ are defined as follows

$${}_0D_t^{\alpha}u = D_t^n\left({}_0I_t^{n-\alpha}u\right) \qquad (5.1)$$

and

$$\left({}_0I_t^{n-\alpha}u\right)(x,t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u(x,\underline{\theta})}{(t-\underline{\theta})^{1-n+\alpha}}d\underline{\theta}, \qquad (5.2)$$

where $\Gamma(z)$ is the Gamma function, D_t is the operator of differentiation with respect to t and $n = [\alpha] + 1$.

The vector $C = (C^t, C^x)$ provides a conservation law if it satisfies following equation

$$D_t(C^t)+D_x(C^x)|_{(1.3)}=0. \qquad (5.3)$$

For the construction of conservation laws, the formal Lagrangian of TFFB can be written as follows

$$
\begin{aligned}
L =&\Upsilon(x,t)\Big(D_t^{\alpha}u+\mu\left(u_{5x}+10\,{u_{2x}}^2+15\,u_xu_{3x}+5\,uu_{4x}+15\,{u_x}^3+50\,uu_xu_{2x}\right.\\
&\left.+10\,u^2u_{3x}+30\,u^2{u_x}^2+10\,u^3u_{2x}+5\,u^4u_x\right)\Big),
\end{aligned} \qquad (5.4)
$$

where $\Upsilon(x,t)$ is a new dependent variable. With the formal Lagrangian, the action integral is given by

$$\int_0^{\Phi}\int_{\Theta} L(x,t,u,\Upsilon,D_t^{\alpha}u,u_x,...)dxdt. \qquad (5.5)$$

The adjoint equation [24] to the TFFB is defined as follows

$$\frac{\delta L}{\delta u} = 0, \tag{5.6}$$

where $\dfrac{\delta}{\delta u}$ is the Euler-Lagrange operator [17, 40] of the following form

$$\frac{\delta}{\delta u} = \partial_u + (D_t^\alpha)^* \partial_{\partial_t^\alpha u} - D_x \partial_{u_x} + D_x^2 \partial_{u_{2x}} - D_x^3 \partial_{u_{3x}} + D_x^4 \partial_{u_{4x}} - D_x^5 \partial_{u_{5x}}, \tag{5.7}$$

where $(D_t^\alpha)^*$ is the adjoint operator of (D_t^α). Using Eqs (5.4) and (5.6), the adjoint equation to the Eq. (1.3) can be written as follows

$$\begin{aligned}&\big(-10\, v_{2x} u u_x - 10\, v_x u u_{2x} + 5\, v_{3x} u_x + 5\, v_{4x} u + 5\, v_{2x} u_{2x} - v_{5x} + 10\, v_{2x} u^3 \\ &-5\, v_x u^4 - 10\, v_{3x} u^2 - 5\, v_x {u_x}^2\big)\, \mu - (D_t^\alpha)^* \Upsilon = 0.\end{aligned} \tag{5.8}$$

As the Eq. TFFB has two independent variables x, t and one dependent variable u so, we have following relation in accordance with [24]

$$\bar{\Omega} + D_t(T)e + D_x(X)e = W \frac{\delta}{\delta u} + D_t N^t + D_x N^x, \tag{5.9}$$

where e is identity operator, N^t and N^x are the Noether operators, and $\bar{\Omega}$ is given by

$$\bar{\Omega} = T\partial_t + X\partial_x + U\partial_u + U^{\alpha t}\partial_{\partial_t^\alpha u} + U^x \partial_{u_x} + U^{2x}\partial_{u_{2x}} + \ldots + U^{5x}\partial_{u_{5x}}. \tag{5.10}$$

The Lie characteristic function W is defined as follows

$$W = U - T u_t - X u_x. \tag{5.11}$$

The operator N^t with the use of Riemann-Liouville time-fractional derivative to the TFFB equation is given by [17, 40, 59]

$$N^t = T\, e + \sum_{k=0}^{n-1} (-1)^k_0 D_t^{\alpha-1-k}(W) D_t^k \frac{\partial}{\partial_0 D_t^\alpha u} - (-1)^n J\left(W, D_t^n \frac{\partial}{\partial_0 D_t^\alpha u}\right), \tag{5.12}$$

where J is the integral given by

$$J(F, G) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^\vartheta \frac{F(x,\phi) G(x,q)}{(q-\phi)^{\alpha+1-n}} dq d\phi. \tag{5.13}$$

The operator N^x is defined as follows

$$\begin{aligned} N^x =& Xe + W\Big(\partial_{u_x} - D_x \partial_{u_{2x}} + D_x^2 \partial_{u_{3x}} - D_x^3 \partial_{u_{4x}} + D_x^4 \partial_{u_{5x}}\Big) \\ &+ D_x(W)\Big(\partial_{u_{2x}} - D_x \partial_{u_{3x}} + D_x^2 \partial_{u_{4x}} - D_x^3 \partial_{u_{5x}}\Big) \\ &+ D_x^2(W)\Big(\partial_{u_{3x}} - D_x \partial_{u_{4x}} + D_x^2 \partial_{u_{5x}}\Big) \\ &+ D_x^3(W)\Big(\partial_{u_{4x}} - D_x \partial_{u_{5x}}\Big) + D_x^4(W)\partial_{u_{5x}}. \end{aligned} \tag{5.14}$$

The generator Ω should hold following

$$(\bar{\Omega}L + D_t(T)L + D_x(X)L)|_{(1.3)} = 0. \tag{5.15}$$

Thus the conservation law for TFFB equation can be written as follows

$$D_t(N^t L) + D_x(N^x L) = 0. \tag{5.16}$$

Thus t and x components of conserved vectors corresponding to Ω_1 and Ω_2 using (5.12) and (5.14) to the Eq. (1.3) are calculated as follows

$$\begin{aligned} C_1^t &= TL + (-1)^0\, {}_0D_t^{\alpha-1}(W_1)D_t^0 \frac{\partial L}{\partial_0 D_t^\alpha u} - (-1)^1 J\Big(W_1, D_t^1 \frac{\partial L}{\partial_0 D_t^\alpha u}\Big) \\ &= {}_0D_t^{\alpha-1}(W_1)\Upsilon + J(W_1, \Upsilon_t), \\ &= {}_0D_t^{\alpha-1}(-u_x)\Upsilon + J(-u_x, \Upsilon_t), \end{aligned} \tag{5.17}$$

$$\begin{aligned} C_1^x = \big(&-u_{3x}\Upsilon_{2x} - 50\, u\Upsilon u_x u_{2x} - 15\, u_x u_{3x}\Upsilon - 30\, u^2\Upsilon {u_x}^2 - 5\, u^4\Upsilon u_x \\ &+10\, u_x u_{2x}\Upsilon_x + 10\, u\Upsilon_x {u_x}^2 + 10\, u^3\Upsilon_x u_x - 10\, u_x\Upsilon_{2x}u^2 + 5\, uu_x\Upsilon_{3x} \\ &-10\, u^3\Upsilon u_{2x} + 10\, u^2\Upsilon_x u_{2x} - 5\, uu_{2x}\Upsilon_{2x} - 10\, u^2\Upsilon u_{3x} + 5\, uu_{3x}\Upsilon_x \\ &-5\, uu_{4x}\Upsilon - u_{5x}\Upsilon - u_x\Upsilon_{4x} - 10\, {u_{2x}}^2\Upsilon - 15\, {u_x}^3\Upsilon + u_{2x}\Upsilon_{3x} \\ &+u_{4x}\Upsilon_x\big)\,\mu, \end{aligned} \tag{5.18}$$

$$\begin{aligned} C_2^t &= TL + (-1)^0\, {}_0D_t^{\alpha-1}(W_2)D_t^0 \frac{\partial L}{\partial_0 D_t^\alpha u} - (-1)^1 J\Big(W_2, D_t^1 \frac{\partial L}{\partial_0 D_t^\alpha u}\Big) \\ &= {}_0D_t^{\alpha-1}(W_2)\Upsilon + J(W_2, \Upsilon_t), \\ &= {}_0D_t^{\alpha-1}(-u - xu_x - \frac{5t}{\alpha}u_t)\Upsilon + J(-u - xu_x - \frac{5t}{\alpha}u_t, \Upsilon_t), \end{aligned} \tag{5.19}$$

$$\begin{aligned} C_2^x = -\,\mu\, \big(&5\,\Upsilon u_{4x} - 4\, u_{3x}\Upsilon_x + 3\, u_{2x}\Upsilon_{2x} - 2\, u_x\Upsilon_{3x} - 10\, {u_x}^2\Upsilon_x \\ &+u\Upsilon_{4x} - 5\, u^2\Upsilon_{3x} + 10\, u^3\Upsilon_{2x} - 10\, u^4\Upsilon_x + 5\, u^5\Upsilon + 15\, xu_x\Upsilon u_{3x} \\ &+30\, x{u_x}^2\Upsilon u^2 + 5\, xu_x\Upsilon u^4 - 10\, xu_x\Upsilon_x u_{2x} - 10\, x{u_x}^2\Upsilon_x u - 10\, xu_x\Upsilon_x u^3 \\ &+10\, xu_x\Upsilon_{2x}u^2 - 5\, xu_x\Upsilon_{3x}u + 10\, xu_{2x}\Upsilon u^3 - 10\, xu_{2x}\Upsilon_x u^2 + 5\, xu_{2x}\Upsilon_{2x}u \\ &+10\, xu_{3x}\Upsilon u^2 - 5\, xu_{3x}\Upsilon_x u + 5\, xu_{4x}\Upsilon u + 50\, xu_x\Upsilon uu_{2x} - 20\, u\Upsilon_x u_{2x} \\ &+25\, u\Upsilon u_{3x} + xu_x\Upsilon_{4x} + 50\, u_x\Upsilon u_{2x} + 10\, u_x\Upsilon_{2x}u + 50\, u^3\Upsilon u_x \\ &+50\, u^2\Upsilon u_{2x} - xu_{4x}\Upsilon_x + 15\, x{u_x}^3\Upsilon + 75\, u\Upsilon {u_x}^2 - 30\, u^2\Upsilon_x u_x \\ &-xu_{2x}\Upsilon_{3x} + xu_{3x}\Upsilon_{2x} + 10\, x{u_{2x}}^2\Upsilon + \Upsilon xu_{5x}\big) \\ -\,\frac{\mu}{\alpha}\big(&25\, tu_{3xt}\Upsilon u + 25\, tu_t\Upsilon u_{3x} + 75\, tu_t\Upsilon {u_x}^2 + 25\, tu_t\Upsilon u^4 - 25\, tu_t\Upsilon_x u_{2x} \\ &-50\, tu_t\Upsilon_x u^3 + 50\, tu_t\Upsilon_{2x}u^2 - 25\, tu_t\Upsilon_{3x}u + 50\, tu_{tx}\Upsilon u_{2x} + 50\, tu_{tx}\Upsilon u^3 \\ &-25\, tu_{tx}\Upsilon_x u_x - 50\, tu_{tx}\Upsilon_x u^2 + 25\, tu_{tx}\Upsilon_{2x}u + 50\, tu_{2x}\Upsilon u_x + 50\, tu_{2xt}\Upsilon u^2 \\ &-25\, tu_{2xt}\Upsilon_x u + 5\,\Upsilon tu_{4xt} - 5\, tu_{3xt}\Upsilon_x + 5\, tu_{2xt}\Upsilon_{2x} \\ &-5\, tu_{tx}\Upsilon_{3x} + 5\, tu_t\Upsilon_{4x} + 100\, tu_t\Upsilon uu_{2x} + 150\, tu_t\Upsilon u^2 u_x - 50\, tu_t\Upsilon_x uu_x \\ &+150\, tu_{tx}\Upsilon uu_x\big)\,. \end{aligned} \tag{5.20}$$

Thus we have obtained conservation laws associated with TFFB equation and this equation has not been explored for conservation laws in literature.

6. Conclusion

In this paper, we present an algorithm to systematically analyze the 5^{th} order fractional equation from Burgers hierarchy. The linear analysis of the equation gives the dispersion relation whose real and imaginary parts correspond to the dispersion and damping of waves. The relation between phase and group velocity signifies anomalous dispersion of waves and velocities are found to be a function of time fractional derivative order. The convergent power series solution is obtained of the reduced fractional ODE in Lie symmetry analysis. The graphical analysis of the solution for different α values reveals singular kink wave profile. The new conservation theorem has been applied to derive conservation laws corresponding to infinitesimal symmetries.

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