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ORIGINAL ARTICLE

The Singh–Maddala distribution: properties and estimation

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Abstract The Singh–Maddala distribution is very flexible and most widely used for modeling the income, wage, expenditure and wealth distribution of the country. Several mathematical and statistical properties of this distribution (such as quantiles, moments, moment generating function, hazard rate, mean residual lifetime, mean deviation about mean and median, Bonferroni and Lorenz curves and various entropies) are derived. We establish relations for the single and product moments of generalized order statistics from the Singh–Maddala distribution and then we use these results to compute the first four moments and variance of order statistics and record values for sample different sizes for various values of the shape and scale parameters. For this distribution, two characterizing results based on conditional moments of generalized order statistics and recurrence relations for single moments are established. The method of maximum likelihood is adopted for estimating the unknown parameters. For different parameters settings and sample sizes, the various simulation studies are performed and compared to the performance of the Singh–Maddala distribution. An application of the model to a real data set is presented and compared with the fit attained by some other well-known two and three parameters distributions.

Keywords Generalized order statistics · Order statistics · Record values · Single moments · Product moments · Recurrence relations · Singh–Maddala distribution · Characterization · Estimation

Mathematics Subject Classification 62G30 · 62E10 · 62E15

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1 Introduction

The beta distribution is widely used in statistical modeling of bounded random variables. It is easily calculated, can take on a variety of shapes, and, perhaps as importantly, none of the other commonly used distribution functions have compact support. However, its application is limited. First, as a two parameter distribution, it can provide only limited precision in fitting data. It is desirable to have more parametrically flexible versions of the beta to allow a richer empirical description of data while still offering more structure than a nonparametric estimator. Second, the beta distribution does not offer a natural and convenient means of introducing explanatory variables. In a $B(\alpha, \beta)$ distribution, the parameters (α, β) jointly determine both the shape and moments of the distribution. There is no satisfactory way of conditioning the mean by specifying α and β as functions of explanatory variables and regression coefficients. Third, the beta is inconvenient for use in Bayesian analysis. It is conjugate for binomial signals, but not for signals of any continuous distribution. Kotz and Dorp (2004) and Nadarajah and Gupta (2004) provided a comprehensive account of statistical properties of beta distribution and it remains fair to say that the beta distribution provides the premier family of continuous distributions on bounded support (0, 1). The beta distribution, $B(\alpha, \beta)$, has probability density function (*pdf*)

$$f(x)=\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1},\quad 0<x<1,\quad \alpha,\beta>0,$$

where

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt,$$

denotes the beta function. More details and generalizations of standard beta distribution involving algebraic and exponential functions have been proposed in the literature; see in Johnson et al. (1995) and Gupta and Nadarajah (2004).

McDonld (1984) introduced the four parameters generalized Beta distribution of second kind with *pdf*

$$f(x)=\frac{\alpha x^{\delta\alpha-1}}{\beta^{\delta\alpha}B(\delta,\lambda)[1+(x/\beta)^{\alpha}]^{\lambda+\delta}},\quad x>0,\quad \alpha,\beta>0,\quad \lambda,\delta>0$$

and cumulative distribution function (*cdf*)

$$F(x)=\frac{1}{\delta B(\delta,\lambda)}\left[\frac{(x/\beta)^{\alpha}}{1+(x/\beta)^{\alpha}}\right]^{\delta}{}_2F_1\left[\begin{matrix}\delta,1-\lambda;\dfrac{(x/\beta)^{\alpha}}{1+(x/\beta)^{\alpha}}\\ \alpha+1;\end{matrix}\right]\quad x>0,$$

where ${}_pF_q$ denotes the generalized hypergeometric series defined as

$${}_pF_q=\left[\begin{matrix}a_1,a_2,\ldots,a_p;x\\ b_1,b_2,\ldots,b_q;\end{matrix}\right]=\sum_{i=0}^{\infty}\frac{(a_1)_i,(a_2)_i,\ldots,(a_p)_i}{(b_1)_i,(b_2)_i,\ldots,(b_q)_i}\frac{x^i}{i!},$$

where $(a)_i=a(a+1)\cdots(a+i-1)$ denotes the ascending factorial.

Singh–Maddala (SM) distribution is the member of generalized Beta distribution of second kind if $\delta=1$. McDonld (1984) showed that the SM distribution provided better fits than gamma and lognormal. Shahzad and Asghar (2013) used the L-moments and TL-moments methods to derive the point estimators of the parameters for SM distribution.

The family of distributions proposed by Singh and Maddala (1976), whose core distribution was generalized beta distribution, became popular distribution for fitting the distribution on income and expenditure. The three parameters SM distribution has the following *pdf*

$$f(x)=\alpha\lambda\beta^{-\alpha}x^{\alpha-1}[1+(x/\beta)^{\alpha}]^{-(\lambda+1)},\quad x>0,\quad \alpha,\beta>0,\quad \lambda>0 \tag{1.1}$$

and the corresponding *cdf* is

$$F(x)=1-[1+(x/\beta)^{\alpha}]^{-\lambda},\quad x>0,\quad \alpha,\beta>0,\quad \lambda>0. \tag{1.2}$$

The survival function is

$$S(x)=[1+(x/\beta)^{\alpha}]^{-\lambda},\quad x>0 \tag{1.3}$$

and the hazard function

$$H(x)=\frac{f(x)}{S(x)}=\frac{\alpha\lambda\beta^{-\alpha}x^{\alpha-1}}{[1+(x/\beta)^{\alpha}]},\quad x>0. \tag{1.4}$$

Here α, λ are the shapes parameters and β is the scale parameter respectively. Hereafter, a random variable X that follows the distribution (1.2) is denoted by $X\sim SM(\alpha,\beta,\lambda)$.

Special cases Let $X\sim SM(\alpha,\beta,\lambda)$.

i. If $\lambda\to 1$, then X reduces to the Fisk distribution.
ii. If $\lambda\to\infty$, then X reduces to the Weibull distribution.

Order statistics and record values play an important role in a wide range of theoretical and practical problems such as characterization of probability distributions and goodness of fit tests, entropy estimation, analysis of censored samples, reliability analysis, quality control and strength of materials. Their distributional and stochastic properties have been studied extensively in the literature. However, they can be considered as special cases of generalized order statistics (GOS) that have been introduced and extensively studied by Kamps (1995) to unify several concepts that have been used in statistics such as order statistics, record values, progressively Type II censoring order statistics, Pfeifer records and sequential order statistics. The statistical properties and the estimation problems based on generalized order statistics for some life time distributions are studied by several researchers. For instance, Wu et al. (2007) obtained maximum likelihood estimator (MLE) of lifetime performance index for the Burr XII distribution with progressively type II right censored sample and Kim and Han (2014) obtained Bayesian estimators and highest posterior density credible intervals for the scale parameter of Rayleigh distribution based GOS. Also, they derived the Bayesian predictive estimator and the highest posterior density predictive interval for independent future observations. Kumar (2015a, b) obtained the relations for moments and moment generating function of type-II exponentiated log-logistic and extended generalized half logistic distribution based on GOS respectively.

The moments of order statistics play an important role in many inferential problems. For example, they are useful in the derivation of best linear unbiased estimators for scale and location-scale families of distributions based on complete and type-II censored samples. They are also useful in evaluating the performance of the maximum likelihood estimators and L-moment estimators, and in developing point prediction and goodness-of-fit tests. Order statistics and their moments have received considerable interest in recent years and the moments of order statistics have been tabulated quite extensively for several distributions, for example, see Arnold et al. (1992) and David (1981). The recurrence relations and identities have great significance because they are useful in reducing the number of operations necessary to obtain a general form for the function under consideration and they reduce the amount of direct computation, time and labour. This concept is well-established in the statistical literature, see Arnold et al. (1992).

The motivation of the paper is two fold: first is to study the properties and relations for GOS of the SM distribution, and second is to estimate the parameters of the model by using maximum likelihood method assuming different sample sizes and different parameters values. The uniqueness of this study comes from the fact that we provide explicit expressions for single and product moments using GOS for SM distribution. Also, to the best of our knowledge thus far, no attempt has been made to estimate the unknown parameters by using maximum likelihood method.

The remaining of the article is organized as follows. Various mathematical and statistical properties of SM distribution are presented in Sect. 2. Section 3 describes the relations for single and product moments of generalized order statistics from SM distribution. The obtained relations were used to compute first four moments and variances of order statistics and record values. We obtained the characterization of this distribution by using conditional moments and recurrence relation for single moment of generalized order statistics in Sect. 4. In Sect. 5, we derive maximum likelihood estimation of SM distribution. In Sect. 6, various simulations are performed for different sample sizes. The importance of the model is illustrated by means of an application to a real data set in Sect. 6. Finally, Sect. 7 ends up with some general concluding remarks.

2 Mathematical and statistical properties of SM distribution

Let $x_p = Q(p) = F^{-1}(p)$, for $0 < p < 1$ denote the quantile function of the SM distribution. Then

$$x_p = \beta\left[(1-p)^{1/\lambda} - 1\right]^{1/\alpha}. \tag{2.1}$$

In particular, the first three quantiles, Q_1, Q_2 and Q_3, can be obtain by setting $p = 0.25$, $p = 0.5$ and $p = 0.75$ in Eq. (2.1) respectively.

The effects of the shape parameters α and λ on the skewness and kurtosis can be considered based on quantile measures. The Bowley skewness (Kenney & Keeping 1962) is one of the earliest skewness measures defined by

$$B = \frac{Q\left(\frac{3}{4}\right) + Q\left(\frac{1}{4}\right) - 2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}.$$

Since only the middle two quartiles are considered and the outer two quartiles are ignored, this adds robustness to the measure. The Moors kurtosis (Moors 1988) is defined as

$$M = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}.$$

Clearly, M > 0 and there is good concordance with the classical kurtosis measures for some distributions. These measures are less sensitive to outliers and they exist even for distributions without moments. For the standard normal distribution, these measures are 0 (Bowley) and 1.2331 (Moors).

In Figs. 1 and 2, various graphs of the *pdf*, *cdf* and hazard rate and survival function for the SM distribution

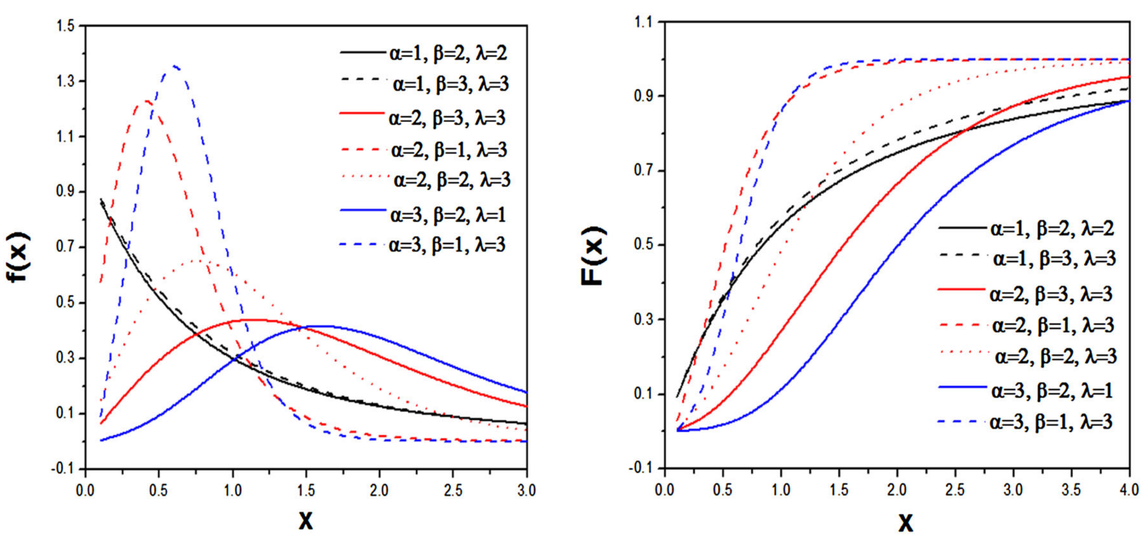

Fig. 1 The *pdf* and *cdf* of the SM distribution for various parameter values

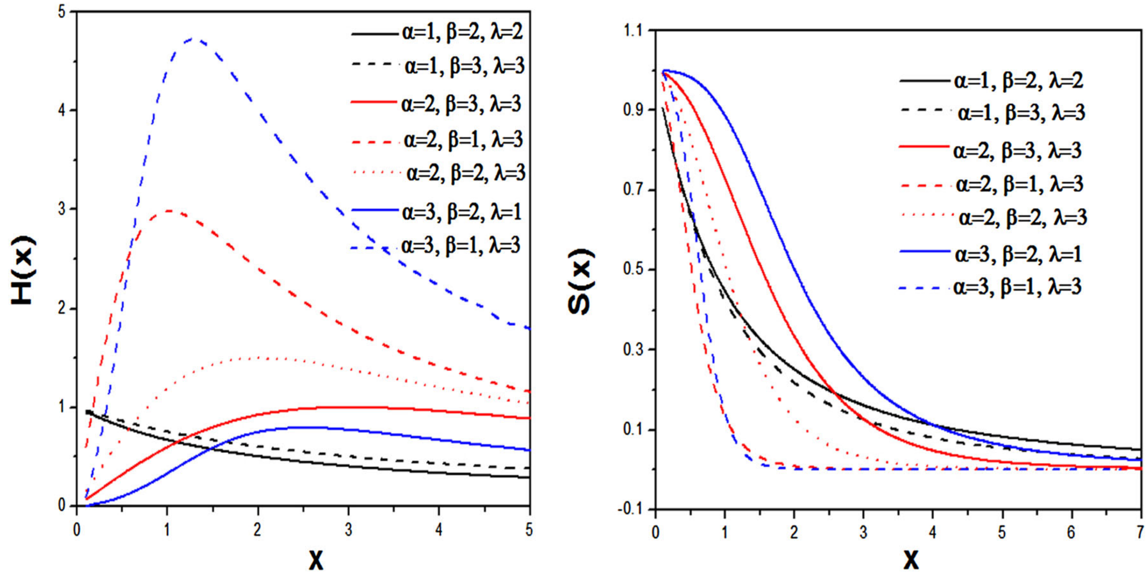

Fig. 2 The hazard rate function and survival function of the SM distribution for various parameter values

for the different parameters values. These plots shows that the *pdf* can be right skewed, approximately symmetric or reversed J-shape. The plots in Fig. 2 indicate that the hazard rate for the SM distribution is very flexible. It can have increasing failure rate (IFR), decreasing failure rate (DFR) functions.

2.1 Moments

We hardly need to emphasize the necessity and importance of the moments in any statistical analysis especially in applied work. Some of the most important features and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness, and kurtosis).

Theorem 2.1 *Let the random variable X follow the SM distribution, then its nth moment has the following form*

$$E[X^n] = \lambda\beta^n B\left(\frac{\alpha+n}{\alpha}, \lambda - \frac{n}{\alpha}\right), \tag{2.2}$$

where B(a, b) denotes the beta function defined by $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt.$

Proof The nth moments is given by

$$\begin{aligned}\mu'_n = E(X^n) &= \int_0^\infty x^n f(x)dx \\ &= \alpha\lambda\beta^{-\alpha}\int_0^\infty x^{\alpha+n-1}[1+(x/\beta)^\alpha]^{-(\lambda+1)}dx.\end{aligned} \tag{2.3}$$

The result follows by using Eq. (3.252.11) in Gradshteyn and Ryzhik (2007) to calculate the integral in (2.3). The proof is complete.

The central moments (μ_n) of X can be determined from (2.2) as $\mu_1 = \mu'_1$. Thus $\mu_2 = \mu'_2 - \mu'^2_1$, $\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1$, $\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu'^2_2 + 12\mu'_2\mu'^2_1 - 6\mu'^4_1$, etc.

The variance, skewness and kurtosis measures can now be calculated using the relations

$$Var(X) = \mu'_2 - \mu'^2_1,$$

$$\text{Skewness } (X) = \frac{\mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1}{Var^{3/2}(X)},$$

$$\text{Kurtosis } (X) = \frac{\mu'_4 - 4\mu'_3\mu'_1 - 3\mu'^2_2 + 12\mu'_2\mu'^2_1 - 6\mu'^4_1}{Var^2(X)}.$$

Theorem 2.2 *The moment generating function of X, $M_X(t)$ when random variable follows the SM distribution is*

$$M_X(t) = \sum_{r=0}^{\infty}\frac{\lambda\beta^r t^r}{r!}B\left(\frac{\alpha+r}{\alpha}, \lambda - \frac{r}{\alpha}\right). \tag{2.4}$$

Proof Let the moment generating function for X is given by

$$\begin{aligned}M_X(t) = E[e^{tx}] &= \int_0^\infty e^{tx}f(x)dx \\ &= \int_0^\infty \left(1 + tx + \frac{t^2x^2}{2!} + \cdots + \frac{t^nx^n}{n!} + \cdots\right)f(x)dx \\ &= \sum_{r=0}^{\infty}\frac{t^rE[X^r]}{r!} \\ &= \sum_{r=0}^{\infty}\frac{\lambda\beta^r t^r}{r!}B\left(\frac{\alpha+r}{\alpha}, \lambda - \frac{r}{\alpha}\right).\end{aligned}$$

2.2 Conditional moments

For the SM distribution, it can be easily seen that the conditional moments, $E(X^n|X > x)$ can be written as

$$E(X^n|X > x) = \frac{1}{S(x)} J_n(x),$$

where

$$\begin{aligned} J_n(x) &= \int_x^\infty y^n f(y)dy \\ &= \alpha\lambda\beta^{-\alpha} \int_x^\infty y^{n+\alpha-1}[1 + (y/\beta)^\alpha]^{-(\lambda+1)}dy \\ &= \frac{\lambda\beta^n[(\beta/x)^\alpha]^{\lambda-(n/\alpha)}}{(\lambda - \frac{n}{\alpha})} {}_2F_1\left(\lambda - \frac{n}{\alpha}, \quad 1+\lambda; \quad \lambda - \frac{n}{\alpha} + 1; \quad -\frac{\beta^\alpha}{x^\alpha}\right), \end{aligned} \tag{2.5}$$

where $S(x) = 1 - F(x)$ define in (1.3) and ${}_2F_1(a, b; c; x)$ denotes the Gauss hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

where $(e)_k = e\,(e+1)\cdots(e+k-1)$ denotes the ascending factorial.

An application of the conditional moments is the mean residual life (MRL). MRL function is the expected remaining life, $X - x$, given that the item has survived to time x. Thus, in life testing situations, the expected additional lifetime given that a component has survived until time x is called the (MRL). The MRL function in terms of the first conditional moment as

$$m_X(x) = E[X - x|X > x] = \frac{1}{S(x)} J_1(x) - x,$$

where $J_1(x)$ can be obtained from (2.5) where $n = 1$.

Another application of the conditional moments is the mean deviations about the mean and the median. They are used to measure the dispersion and the spread in a population from the center. If we denote the median by M, then the mean deviations about the mean and the median can be calculated as

$$\delta_\mu = \int_0^\infty |x - \mu| f(x)\,dx = 2\mu F(\mu) - 2\mu + 2J_1(\mu)$$

and

$$\delta_M = \int_0^\infty |x - M| f(x)\,dx = 2J_1(M) - \mu$$

respectively. Where $J_1(\mu)$ and $J_1(M)$ can be obtained from (2.5). Also, $F(\mu)$ and $F(M)$ are easily calculated from (1.2).

2.3 Bonferroni and Lorenz curve

The Lorenz curve concept was introduced by Lorenz (1905), who investigated the problem of measuring concentration of wealth. The Lorenz curve has played a basic role, for example, in the analysis of income and earnings inequality (Sen 1973; Slottje 1989; Doiron and Barrett 1996), industrial concentration (Hart 1971, 1975), reliability (Chandra and Singpurwalla 1978, 1981). It has also been used as a goodness of t test for exponentiality (Gail and Gastwirth 1978; Nikitin and Tchirina 1996). Another income inequality analysis tools, is the Bonferroni Curve that was introduced by Bonferroni (1930). Bonferroni curve have assumed relief not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. Csörgő et al. (1998) discussed the asymptotic confidence bands for the Lorenz and Bonferroni curves based on the empirical Lorenz curve.

Let X be a continuous random variable with probability density function $f(x)$ cumulative distribution function $F(x)$. Let $F^{-1}(.)$ denote the quantile function then the Boneferroni and Lorenz curves of a random variable X are defined by

$$B(p) = \frac{1}{p\mu} \int_0^q x f(x) dx \tag{2.6}$$

and

$$L(p) = \frac{1}{\mu} \int_0^q x f(x) dx, \tag{2.7}$$

respectively, where $\mu = E(x)$ and $q = F^{-1}(p)$. By using (2.5), one can reduce (2.6) and (2.7) to

$$B(p) = \frac{\alpha\lambda\, q^{1+\alpha}}{p\mu\,\beta^\alpha(1+\alpha)} {}_2F_1\left(1 + \frac{1}{\alpha}, \ 1+\lambda; \ \frac{1}{\alpha} + 2; \ -\frac{q^\alpha}{\beta^\alpha}\right)$$

and

$$L(p) = \frac{\alpha\lambda q^{1+\alpha}}{\mu\beta^\alpha(1+\alpha)} {}_2F_1\left(1 + \frac{1}{\alpha}, 1+\lambda; \frac{1}{\alpha} + 2; -\frac{q^\alpha}{\beta^\alpha}\right),$$

respectively.

2.4 Renyi entropy

The entropy of a random variable X with the density function $f(x)$ is a measure of variation of the uncertainty. Renyi entropy is defined as $I_R(\rho) = (1-\rho)^{-1}\log[I(\rho)]$, where $I(\rho) = \int_{\Re} f^\rho(x)dx$, $\rho > 0$ and $\rho \neq 1$. If a random variable X has a SM distribution, then, we have

$$\begin{aligned} I(\rho) &= (\alpha\lambda\beta^{-\alpha})^\rho \int_0^\infty x^{\rho(\alpha-1)}[1 + (x/\beta)^\alpha]^{-\rho(\lambda+1)}dx \\ &= \lambda^\rho\alpha^{\rho-1}\beta^{1-\rho} B\left(\frac{\rho(\alpha-1)+1}{\alpha}, \rho(\lambda+1) - \frac{\rho(\alpha-1)+1}{\alpha}\right), \end{aligned}$$

[see Gradshteyn and Ryzhik (2007), p. 325], where

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt,$$

denotes the beta function. Hence, the Renyi entropy reduces to $I_R(\rho) = \frac{\rho \log \lambda}{1-\rho} - \log \alpha + \log \beta + \frac{1}{1-\rho} \log B \left(\frac{\rho(\alpha-1)+1}{\alpha}, \rho(\lambda+1) - \frac{\rho(\alpha-1)+1}{\alpha}\right)$.

3 Generalized order statistics

The concept of GOS was introduced by Kamps (1995) as a general framework for models of ordered random variables. Moreover, many other models of ordered random variables, such as, order statistics, k-th upper record values, upper record values, progressively Type II censoring order statistics, Pfeifer records and sequential order statistics are seen to be particular cases of GOS. These models can be effectively applied, e.g., in reliability theory. Suppose $X(1, n, m, k), \ldots, X(n, n, m, k)$, $(k \geq 1,\ m$ is a real number), are nGOS from an absolutely continuous cumulative distribution function (*cdf*) $F(x)$ with probability density function (*pdf*) $f(x)$, if their joint *pdf* is of the form

$$k\left(\prod_{j=1}^{n-1} \gamma_j\right)\left(\prod_{i=1}^{n-1} [1-F(x_i)]^m f(x_i)\right)(1-F(x_n))^{k-1} f(x_n) \qquad (3.1)$$

on the cone $F^{-1}(0) < x_1 \leq \cdots \leq x_n < F^{-1}(1)$, where $\gamma_j = k + (n-j)(m+1) > 0$ for all j, $1 \leq j \leq n$, k is a positive integer and $m \geq -1$.

If $m = 0$ and $k = 1$, then this model reduces to the ordinary r-th order statistic and (3.1) will be the joint *pdf* of n order statistics $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ from *cdf* $F(x)$. If $k = 1$ and $m = -1$, then (3.1) will be the joint *pdf*of the first n record values of the identically and independently distributed (*iid*) random variables with *cdf* $F(x)$ and corresponding *pdf* $f(x)$.

In view of (3.1), the marginal *pdf* of the r-th GOS, $X(r, n, m, k)$, $1 \leq r \leq n$, is

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)), \qquad (3.2)$$

and the joint *pdf* of $X(r, n, m, k)$ and $X(s, n, m, k)$, $1 \leq r < s \leq n$, is

$$\begin{aligned} & f_{X(r,n,m,k)X(s,n,m,k)}(x,y) \\ & = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) \\ & \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1} f(y), \quad x < y. \end{aligned} \qquad (3.3)$$

Let $X(r, n, m, k)$, $r = 1, 2, \ldots, n$ be GOS, then the conditional *pdf* of $X(s, n, m, k)$ given $X(r, n, m, k) = x$, $1 \leq r < s \leq n$, in view of (3.2) and (3.3), is

$$\begin{aligned} f_{X(s,n,m,k)|X(r,n,m,k)}(y|x) &= \frac{C_{s-1}}{(s-r-1)!C_{r-1}} \\ & \times \frac{[(h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}}{[\bar{F}(x)]^{\gamma_{r+1}}} f(y), \quad x < y. \end{aligned} \qquad (3.4)$$

where

$$\bar{F}(x) = 1 - F(x), \quad C_{r-1} = \prod_{i=1}^{r} \gamma_i,$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -\ln(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0), \quad x \in [0, 1).$$

3.1 Relations for single moments of GOS

We shall first establish explicit expression for single moments of jth GOS, $E[X^j(r, n, m, k)] = \mu^{(j)}_{r,n,m,k}$. Theorem 1 gives an explicit expression for $1 \leq r \leq n$ and $j = 0, 1, 2, \ldots$.

Theorem 1 *For the SM distribution as given in (1.2) and $1 \leq r \leq n$, $k \geq 1$, $m \geq -1$ and $j = 0, 1, 2, \ldots$,*

$$\mu^{(j)}_{r,n,m,k} = \beta^j \sum_{p=0}^{\infty} \frac{(-1)^p \Gamma\left(\frac{j}{\alpha}+1\right)}{p!\Gamma\left(\frac{j}{\alpha}+1-p\right)\prod_{a=1}^{r}\left(1+\frac{p-(j/\alpha)}{\lambda\gamma_a}\right)}, \quad p < \frac{j}{\alpha}+1, \qquad (3.5)$$

where $\Gamma(.\ )$ denotes the complete gamma function defined by $\Gamma(a) = \int_0^\infty t^{a-1} e^t\, dt$.

Proof Using (4), we have

$$\begin{aligned} \mu^{(j)}_{r,n,m,k} &= \int_0^\infty x^j f_{X(r,n,m,k)}(x)dx \\ &= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_{r-u}-1} f(x)dx \\ &= \frac{\lambda\beta^j C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} \int_0^1 (1-t)^{j/\alpha} t^{\lambda\gamma_{r-u}-(j/\alpha)-1} dt \\ &= \frac{\lambda\beta^j C_{r-1}}{(r-1)!(m+1)^r} \sum_{u=0}^{r-1}\sum_{p=0}^{\infty}(-1)^{u+p}\binom{r-1}{u}\frac{\Gamma\left(\frac{j}{\alpha}+1\right)}{p!\Gamma\left(\frac{j}{\alpha}+1-p\right)} \\ & \int_0^1 t^{\lambda\gamma_{r-u}+p-(j/\alpha)-1} dt \\ &= \frac{\lambda\beta^j C_{r-1}}{(r-1)!(m+1)^r} \sum_{u=0}^{r-1}\sum_{p=0}^{\infty}(-1)^{u+p}\binom{r-1}{u}\frac{\Gamma\left(\frac{j}{\alpha}+1\right)}{p!\Gamma\left(\frac{j}{\alpha}+1-p\right)} \\ & \times B\left(\frac{k}{m+1}+n-r+u+\frac{(p-(j/\alpha))/\lambda}{m+1}, 1\right). \end{aligned} \qquad (3.6)$$

where $t = [\bar{F}(x)]^{1/\lambda}$ and using the relation $\sum_{a=0}^{b}(-1)^a \binom{b}{a} B(a+k,\ c) = B(k,\ c+b)$ in (3.6) we get

$$\mu_{r,n,m,k}^{(j)} = \frac{\beta^j C_{r-1}}{(m+1)^r}\sum_{p=0}^{\infty}(-1)^p \frac{\Gamma\left(\frac{j}{\alpha}+1\right)\Gamma\left[\frac{k+(n-r)(m+1)+\{p-(j/\alpha)\}/\lambda}{(m+1)}\right]}{p!\Gamma\left(\frac{j}{\alpha}+1-p\right)\Gamma\left[\frac{k+n(m+1)+\{p-(j/\alpha)\}/\lambda}{(m+1)}\right]}.$$

The result follows from the definition of the complete gamma function.

In particular, the mean and variance of GOS are

$$\mu_{r,n,m,k}^{(1)} = \beta\sum_{p=0}^{\infty}\frac{(-1)^p\Gamma\left(\frac{1}{\alpha}+1\right)}{p!\Gamma\left(\frac{1}{\alpha}+1-p\right)\prod_{a=1}^{r}\left(1+\frac{p-(1/\alpha)}{\lambda\gamma_a}\right)}$$

and

$$\sigma_{r,n,m,k}^2 = \mu_{r,n,m,k}^{(2)} - \left(\mu_{r,n,m,k}^{(1)}\right)^2$$
$$= \beta^2\sum_{p=0}^{\infty}(-1)^p\frac{\Gamma\left(\frac{2}{\alpha}+1\right)}{p!\Gamma\left(\frac{2}{\alpha}+1-p\right)\prod_{a=1}^{r}\left(1+\frac{p-(2/\alpha)}{\lambda\gamma_a}\right)} - \left(\mu_{r,n,m,k}^{(1)}\right)^2,$$

respectively.

3.2 Special Cases

1. Putting $m = 0$, $k = 1$ in (3.5), the explicit formula for single moments of order statistics from the SM distribution can be obtained as

$$\mu_{r:n}^{(j)} = \frac{\beta^j n!}{(n-r)!}\sum_{p=0}^{\infty}\frac{(-1)^p\Gamma\left(\frac{j}{\alpha}+1\right)}{p!\Gamma\left(\frac{j}{\alpha}+1-p\right)}\frac{\Gamma[n-r+1+(p-(j/\alpha))/\lambda]}{\Gamma[n+1+(p-(j/\alpha))/\lambda]}, \tag{3.7}$$

   for $r = 1$

$$\mu_{1:n}^{(j)} = n\beta^j\sum_{p=0}^{\infty}\frac{(-1)^p\Gamma\left(\frac{j}{\alpha}+1\right)}{p!\Gamma\left(\frac{j}{\alpha}+1-p\right)[n+(p-(j/\alpha))/\lambda]}.$$

2. Setting $m = -1$ in (3.5), we get the explicit expression for single moments of upper k record values from the SM distribution can be obtained as

$$\mu_{U(r):k}^{(j)} = \beta^j\sum_{p=0}^{\infty}\frac{(-1)^p\Gamma\left(\frac{j}{\alpha}+1\right)}{p!\Gamma\left(\frac{j}{\alpha}+1-p\right)\left(1+\frac{p-(j/\alpha)}{\lambda k}\right)^r},$$

   and hence for upper records

$$\mu_{U(r)}^{(j)} = \beta^j\sum_{p=0}^{\infty}\frac{(-1)^p\Gamma\left(\frac{j}{\alpha}+1\right)}{p!\Gamma\left(\frac{j}{\alpha}+1-p\right)\left(1+\frac{p-(j/\alpha)}{\lambda}\right)^r}. \tag{3.8}$$

Theorem 2 establishes a recurrence relations for $\mu_{r,n,m,k}^{(j)}$ which can help us to obtain the higher moments.

Theorem 2 *For the distribution given in (1.2) and for $1 \le r \le n$, $k \ge 1$, $m \ge -1$,*

$$\left(1-\frac{j}{\alpha\lambda\gamma_r}\right)\mu_{r,n,m,k}^{(j)} = \mu_{r-1,n,m,k}^{(j)} + \frac{j\beta^\alpha}{\alpha\lambda\gamma_r}\mu_{r,n,m,k}^{(j-\alpha)}. \tag{3.9}$$

Through, we follow the conventions that $\mu_{0,n,m,k}^{(j)} = 0$ for $n \ge 1$ and $\mu_{r,n,m,k}^{(0)} = 1$ for $1 \le r \le n$.

Proof Clearly, from (1.1) and (1.2), we see that

$$\alpha\lambda x^{\alpha-1}\bar{F}(x) = \beta^\alpha[1+(x/\beta)^\alpha]f(x). \tag{3.10}$$

Therefore, from (3.2), we have

$$\mu_{r,n,m,k}^{(j)} = \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx.$$

By integrating by parts, we obtain

$$\mu_{r,n,m,k}^{(j)} = \mu_{r-1,n,m,k}^{(j)} + \frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}[\bar{F}(x)]^{\gamma_r}g_m^{r-1}(F(x))dx$$
$$= \mu_{r-1,n,m,k}^{(j)} + \frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}[\bar{F}(x)]^{\gamma_{r-1}}\left(\frac{\beta^\alpha+x^\alpha}{\alpha\lambda x^{\alpha-1}}\right) f(x)g_m^{r-1}(F(x))dx$$
$$= \mu_{r-1,n,m,k}^{(j)} + \frac{j\beta^\alpha C_{r-1}}{\alpha\lambda\gamma_r(r-1)!}\int_0^\infty x^{j-\alpha}[\bar{F}(x)]^{\gamma_{r-1}}f(x)g_m^{r-1}(F(x))dx + \frac{jC_{r-1}}{\alpha\lambda\gamma_r(r-1)!}\int_0^\infty x^j[\bar{F}(x)]^{\gamma_{r-1}}f(x)g_m^{r-1}(F(x))dx.$$

The result follows.

Remark 3.1 Under the assumption of Theorem 2 with $m = 0$, $k = 1$ we shall deduced the recurrence relations for single moments of ordinary order statistics of the SM distribution.

Remark 3.2 Putting $k = 0$, $m = -1$ in Theorem 2 we obtain the recurrence relations for single moments of record values of the *SM* distribution, which is in agreement with the corresponding result obtained by Kumar and Khan (2012).

3.3 Relations for product moments of GOS

We shall first establish explicit expressions for the product moments of ith and jth GOS, $E[X_{r,s,n,m,k}^{(i,j)}] = \mu_{r,s,n,m,k}^{(i,j)}$. Theorem 3 gives an explicit expression for $1 \le r < s \le n$ and $i, j = 0, 1, 2, \ldots$

Theorem 3 For the distribution given in (1.2) and $1 \le r < s \le n$, $n \ge 1$, $k = 1, 2, \ldots$ $i, j = 0, 1, 2, \ldots$ and $p < \frac{i}{\alpha}+1$ and $q < \frac{i}{\alpha}+1$

$$\mu_{r,s,n,m,k}^{(i,j)} = \beta^{i+j}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{p+q}\Gamma\left(\frac{i}{\alpha}+1\right)\Gamma\left(\frac{i}{\alpha}+1\right)}{p!q!\Gamma\left(\frac{i}{\alpha}+1-p\right)\Gamma\left(\frac{i}{\alpha}+1-q\right)} \times \frac{1}{\prod_{a=1}^{r}\left(1+\frac{p+q-(i+j)/\alpha}{\lambda\gamma_a}\right)\prod_{b=r+1}^{s}\left(1+\frac{p-(j/\alpha)}{\lambda\gamma_b}\right)}. \quad (3.11)$$

Proof Using (3.3), we have

$$\begin{aligned}\mu_{r,s,n,m,k}^{(i,j)} &= \int_0^\infty\int_x^\infty x^i y^j f_{X(r,n,m,k)X(s,n,m,k)}(x,y)dxdy\\ &= \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^i y^j[\bar{F}(x)]^m f(x)g_m^{r-1}(F(x))\\ &\quad\times [h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y)dydx\\ &= \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\\ &\quad\times\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}\\ &\quad\times\int_0^\infty\int_x^\infty x^i y^j[\bar{F}(x)]^{(s-r+u-v)(m+1)-1}[\bar{F}(y)]^{\gamma_{s-v}-1}f(x)f(y)dydx\\ &= \frac{\beta^j C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\\ &\quad\times\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{\infty}(-1)^{u+v+q}\binom{r-1}{u}\binom{s-r-1}{v}\\ &\quad\times\frac{\Gamma\left(\frac{j}{\alpha}+1\right)}{p!\Gamma\left(\frac{j}{\alpha}+1-q\right)[\gamma_{s-v}+(p-(j/\alpha))/\lambda]}\\ &\quad\times\int_0^\infty x^i[\bar{F}(x)]^{\gamma_{r-u}-1+(p-(j/\alpha))/\lambda}f(x)\\ &= \frac{\beta^{i+j}C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s}}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{p+q}\Gamma\left(\frac{i}{\alpha}+1\right)\Gamma\left(\frac{i}{\alpha}+1\right)}{p!q!\Gamma\left(\frac{i}{\alpha}+1-p\right)\Gamma\left(\frac{i}{\alpha}+1-q\right)}\\ &\quad\times\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}B\left(\frac{k}{m+1}+n-r+u+\frac{[p+q-(i+j)/\alpha]/\lambda}{m+1},1\right)\\ &\quad\times\sum_{v=0}^{s-r-1}(-1)^v\binom{s-r-1}{v}B\left(\frac{k}{m+1}+n-s+v+\frac{(p-(j/\alpha))/\lambda}{m+1},1\right).\end{aligned}$$

The proof is complete.

As a check, put $j=0$ in (3.10) and use (3.5), we have $\mu_{r,s,n,m,k}^{(i,0)} = \mu_{r,n,m,k}^{(i)}$.

For simplicity, we denote the (1, 1)th moments of $X(r, n, m, k)$and $X(s, n, m, k)$, which are also called the simple product moments of these GOS, by $\mu_{r,s,n,m,k}$. The simple product moments are used for evaluating the covariances, in other words

$$\begin{aligned}\sigma_{r,s,n,m,k} &= Cov[X(r,n,m,k),X(s,n,m,k)]\\ &= \mu_{r,s,n,m,k}-\mu_{r,n,m,k}\mu_{s,n,m,k}.\end{aligned}$$

3.4 Special cases

1. Putting $m=0$, $k=1$ in (3.11), we shall deduced the explicit formula for product moments of ordinary order statistics of SM distribution.
2. Setting $m=-1$ in (3.11), we obtain the explicit expression for product moments of k record values of SM distribution.

Theorem 4 establishes a recurrence relations for product moments of $X(r, n, m, k)$ and $X(s, n, m, k)$.

Theorem 4 *For* $1 \le r < s \le n-1$, $n \ge 2$ *and* $k = 1, 2, \ldots$

$$\left(1-\frac{j}{\alpha\lambda\gamma_s}\right)\mu_{r,s,n,m,k}^{(i,j)} = \mu_{r,s-1,n,m,k}^{(i,j)} + \frac{j\beta^\alpha}{\alpha\lambda\gamma_s}\mu_{r,s,n,m,k}^{(i,j-\alpha)}. \quad (3.12)$$

Proof From (3.3), we have

$$\mu_{r,s,n,m,k}^{(i,j)} = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty x^i[\bar{F}(x)]^m f(x)g_m^{r-1}(F(x))I(x)dx, \quad (3.13)$$

where $I(x)=\int_x^\infty y^j[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y)dy$.

Solving the integral in $I(x)$ by parts and substituting the resulting expression in (3.13), we get

$$\begin{aligned}\mu_{r,s,n,m,k}^{(i,j)} &= \mu_{r,s-1,n,m,k}^{(i,j)} + \frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\\ &\quad\times\int_0^\infty\int_x^\infty x^i y^{j-1}[\bar{F}(x)]^m f(x)g_m^{r-1}(F(x))\\ &\quad\times [h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s}dydx\\ &= \mu_{r,s-1,n,m,k}^{(i,j)} + \frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\\ &\quad\times\int_0^\infty\int_x^\infty x^i y^{j-1}[\bar{F}(x)]^m f(x)g_m^{r-1}(F(x))\\ &\quad\times [h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}\left(\frac{\beta^\alpha+y^\alpha}{\alpha\lambda y^{\alpha-1}}\right)f(y)dydx\\ &= \mu_{r,s-1,n,m,k}^{(i,j)} + \frac{j\beta^\alpha C_{s-1}}{\alpha\lambda\gamma_s(r-1)!(s-r-1)!}\\ &\quad\times\int_0^\infty\int_x^\infty x^i y^{j-\alpha}[\bar{F}(x)]^m f(x)g_m^{r-1}(F(x))\\ &\quad\times [h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y)dydx\\ &\quad+\frac{jC_{s-1}}{\alpha\lambda\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^i y^j[\bar{F}(x)]^m f(x)g_m^{r-1}(F(x))\\ &\quad\times [h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y)dydx.\end{aligned}$$

The result follows.

Remark 3.4 Under the assumption of Theorem 4 with $m=0$, $k=1$ we shall deduced the recurrence relations for product moments of order statistics of the SM distribution.

Remark 3.5 Putting $k=0$, $m=-1$ in Theorem 4 we obtain the recurrence relations for product moments of k-th record values from SM distribution, which is in agreement with the corresponding result obtained by Kumar and Khan (2012).

4 Characterization

In this section, we shall characterize SM distribution based on conditional expectation of GOS and recurrence relation for single moment of GOS. Using Theorem 5 we provide the stronger version of Theorem 2.

Let $L(a, b)$ stand for the space of all integrable functions on (a, b). A sequence $(h_n) \subset L(a, b)$ is called complete on $L(a, b)$ if for all functions $g \in L(a, b)$ the condition

$$\int_a^b g(x) f_n(x) dx = 0, \quad n \in \mathrm{N},$$

implies $g(x) = 0$ a.e. on (a, b). We start with the following result of Lin (1986).

Proposition 1 *Let n_0 be any fixed non-negative integer, $-\infty \le a < b \le \infty$ and $g(x) \ge 0$ an absolutely continuous function with $g'(x) \ne 0$ a.e. on (a, b). Then the sequence of functions $\{(g(x))^n e^{-g(x)},\ n \ge n_0\}$ is complete in L(a, b)iff g(x) is strictly monotone on (a, b).*

Using the above Proposition we get a stronger version of Theorem 2.

Theorem 5 *Let X be a non-negative random variable having an absolutely continuous distribution function F(x) with F(0) = 0 and 0 < F(x) < 1 for all x > 0, then*

$$\left(1 - \frac{j}{\alpha\lambda\gamma_r}\right) \mu_{r,n,m,k}^{(j)} = \mu_{r-1,n,m,k}^{(j)} + \frac{j\beta^{\alpha}}{\alpha\lambda\gamma_r} \mu_{r,n,m,k}^{(j-\alpha)}. \tag{4.1}$$

if and only if

$$\bar{F}(x; \alpha, \beta, \lambda) = [1 + (x/\beta)^{\alpha}]^{-\lambda}, \quad x > 0, \quad \alpha, \beta > 0, \text{ and } \lambda > 0.$$

Proof The necessary part follows immediately from Eq. (3.8). By the other hand if the recurrence relation in Eq. (4.1) is satisfied, then by using Eq. (3.2), we have

$$\begin{aligned} &\frac{C_{r-1}}{(r-1)!} \int_0^{\infty} x^j [\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx \\ &= \frac{(r-1) C_{r-1}}{\gamma_r (r-1)!} \int_0^{\infty} x^j [\bar{F}(x)]^{\gamma_r + m} f(x) g_m^{r-2}(F(x)) dx \\ &\quad + \frac{j\beta^{\alpha} C_{r-1}}{\alpha\lambda\gamma_r (r-1)!} \int_0^{\infty} x^{j-\alpha} [\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx \\ &\quad + \frac{j C_{r-1}}{\alpha\lambda\gamma_r (r-1)!} \int_0^{\infty} x^{j-1} [\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx \end{aligned} \tag{4.2}$$

Integrating the first integral on the right hand side of Eq. (4.2) by parts and simplifying the resulting expression, we get

$$\frac{j C_{r-1}}{\gamma_r (r-1)!} \int_0^{\infty} x^{j-1} [F(x)]^{\gamma_r - 1} g_m^{r-1}(F(x)) dx \times \left\{ \bar{F}(x) - \frac{\beta^{\alpha}}{\alpha\lambda x^{\alpha-1}} f(x) - \frac{x}{\alpha\lambda} f(x) \right\} dx = 0.$$

It now follows from Proposition 1

$$\bar{F}(x) = \frac{\beta^{\alpha}[1 + (x/\beta)^{\alpha}]}{\alpha\lambda x^{\alpha-1}} f(x),$$

which prove that $\bar{F}(x; \alpha, \beta, \lambda) = [1 + (x/\beta)^{\alpha}]^{-\lambda}, \quad x > 0, \alpha, \beta > 0,$ and $\lambda > 0$.

Theorem 6 *Let X(r, n, m, k), r = 1, 2, …, n be GOS based on continuous cumulative distribution function F(x) with F(0) = 0 and 0 < F(x) < 1 for all x > 0, then for two consecutive values r and r + 1, 2 ≤ r + 1 < s ≤ n, the conditional expectation of GOS X(s, n, m, k) given X(l, n, m, k) = x, is given as*

$$E[X(s,n,m,k) | X(l,n,m,k) = x] = \beta \sum_{p=0}^{\infty} \frac{(-1)^{p+(1/\alpha)} \Gamma\left(\frac{1}{\alpha} + 1\right)}{p! \Gamma\left(\frac{1}{\alpha} + 1 - p\right)} \times [1 + (x/\beta)^{\alpha}]^p \prod_{j=1}^{s-l} \left(\frac{\gamma_{l+j}}{\gamma_{l+j} - p/\lambda}\right), \quad l = r, r+1 \tag{4.3}$$

if and only if X has the cdf

$$\bar{F}(x; \alpha, \beta, \lambda) = [1 + (x/\beta)^{\alpha}]^{-\lambda}, \quad x > 0, \quad \alpha, \beta > 0 \text{ and } \lambda > 0.$$

Proof We have from (3.4)

$$E[X(s,n,m,k) | X(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)! C_{r-1} (m+1)^{s-r-1}} \times \int_x^{\infty} y \left[1 - \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{m+1}\right]^{s-r-1} \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_s - 1} \frac{f(y)}{\bar{F}(x)} dy. \tag{4.4}$$

By setting $u = \frac{\bar{F}(y)}{\bar{F}(x)}$ from (1.2) in (4.4), we obtain

$$E[X(s,n,m,k) | X(r,n,m,k) = x] = \frac{\beta C_{s-1}}{(s-r-1)! C_{r-1} (m+1)^{s-r-1}} \times \sum_{p=0}^{\infty} \frac{(-1)^{p+(1/\alpha)} \Gamma\left(\frac{1}{\alpha} + 1\right) [1 + (x/\beta)^{\alpha}]^p}{p! \Gamma\left(\frac{1}{\alpha} + 1 - p\right)} \times \int_0^1 u^{\gamma_s - (p/\lambda) - 1} (1 - u^{m+1})^{s-r-1} du \tag{4.5}$$

Again by setting $t = u^{m+1}$ in (4.5), we get

$$E[X(s, n, m, k) | X(r,n,m,k) = x] = \frac{\beta C_{s-1}}{C_{r-1}} \sum_{p=0}^{\infty} \frac{(-1)^{p+(1/\alpha)} \Gamma\left(\frac{1}{\alpha} + 1\right) [1 + (x/\beta)^{\alpha}]^p}{p! \Gamma\left(\frac{1}{\alpha} + 1 - p\right) \prod_{j=1}^{s-r} (\gamma_{r+j} - p/\lambda)}$$

and hence the result given in (4.3).

To prove sufficient part, we have from (3.4) and (4.3)

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_x^\infty y\left[(\bar{F}(x))^{m+1}-(\bar{F}(y))^{m+1}\right]^{s-r-1} \times [\bar{F}(y)]^{\gamma_s-1}f(y)dy=[\bar{F}(x)]^{\gamma_{r+1}}H_r(x), \tag{4.6}$$

where

$$H_r(x)=\beta\sum_{p=0}^{\infty}\frac{(-1)^{p+(1/\alpha)}\Gamma\left(\frac{1}{\alpha}+1\right)[1+(x/\beta)^\alpha]^p}{p!\Gamma\left(\frac{1}{\alpha}+1-p\right)}\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}-p/\lambda}\right).$$

Differentiating (4.6) both sides with respect to x and re-arranging the terms, we get

$$-\frac{C_{s-1}[\bar{F}(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}}\int_x^\infty y\left[(\bar{F}(x))^{m+1}-(\bar{F}(y))^{m+1}\right]^{s-r-2} \times [\bar{F}(y)]^{\gamma_s-1}f(y)dy=H'_r(x)[\bar{F}(x)]^{\gamma_{r+1}}-\gamma_{r+1}H_r(x)[\bar{F}(x)]^{\gamma_{r+1}-1}f(x)$$

or

$$-\gamma_{r+1}H_{r+1}(x)[\bar{F}(x)]^{\gamma_{r+2}+m}f(x)=H'_r(x)[\bar{F}(x)]^{\gamma_{r+1}} -\gamma_{r+1}H_r(x)[\bar{F}(x)]^{\gamma_{r+1}-1}f(x).$$

Therefore,

$$\frac{f(x)}{\bar{F}(x)}=-\frac{H'_r(x)}{\gamma_{r+1}[H_{r+1}(x)-H_r(x)]}=\frac{\alpha\lambda x^{\alpha-1}}{\beta^\alpha[1+(x/\beta)^\alpha]}$$

which gives

$$\bar{F}(x;\alpha,\beta,\lambda)=[1+(x/\beta)^\alpha]^{-\lambda},\quad x>0, \quad \alpha,\beta>0 \text{ and } \lambda>0.$$

Remark 4.1 For $k=1$, $m=0$ and $k=1$, $m=-1$, in Theorems 5–6, we obtain the characterization results of the SM distribution based on order statistics and record values, respectively.

5 Estimation

In this section we discuss the process of obtaining the maximum likelihood estimators of the parameters α, β and λ. Let $X_1, X_2, \ldots X_n$ be the random sample with observed values $x_1, x_2, \ldots, x_n$ from SM distribution. Let $\Theta=(\alpha,\beta,\lambda)$ be the parameter vector. The likelihood function based on the random sample of size n is obtained from

$$L(\alpha,\beta,\lambda;x)=\frac{\alpha^n\lambda^n\prod_{i=1}^n x_i^{\alpha-1}}{\beta^{n\alpha}\prod_{i=1}^n[1+(x_i/\beta)^\alpha]^{\lambda+1}}, \tag{5.1}$$

The maximum likelihood estimates are the values of α, β and λ that maximize this likelihood function.

5.1 Maximum likelihood estimation

The log likelihood function $l(\alpha,\beta,\lambda|\,x)=\log L(\alpha,\beta,\lambda\,|\,x)$, dropping terms that do not involve α, β and λ is

$$l(\alpha,\beta,\lambda|x)=n\ln\alpha-n\alpha\ln\beta+n\ln\lambda+(\alpha-1)\sum_{i=1}^n\ln x_i -(\lambda+1)\sum_{i=1}^n\ln[1+(x_i/\beta)^\alpha]. \tag{5.2}$$

To obtain the normal equations for the unknown parameters, we differentiate (5.2) partially with respect to α, β and λ and equate to zero. The resulting equations are

$$0=\frac{\partial l(\alpha,\beta,\lambda|x)}{\partial\alpha}=\frac{n}{\alpha}-n\ln\beta+\sum_{i=1}^n\ln x_i-(\lambda+1)\sum_{i=1}^n\frac{(x_i/\beta)^\alpha\ln(x_i/\beta)}{[1+(x_i/\beta)^\alpha]}, \tag{5.3}$$

$$0=\frac{\partial l(\alpha,\beta,\lambda|x)}{\partial\beta}=-\frac{n\alpha}{\beta}+\alpha(\lambda+1)\sum_{i=1}^n\frac{x_i^\alpha}{\beta^{\alpha+1}[1+(x_i/\beta)^\alpha]} \tag{5.4}$$

and

$$0=\frac{\partial l(\alpha,\beta,\lambda|x)}{\partial\lambda}=\frac{n}{\lambda}-(\alpha+1)\sum_{i=1}^n\ln[1+(x_i/\beta)^\alpha]. \tag{5.5}$$

The solutions of the above equations are the maximum likelihood estimators of the parameters α, β and λ denoted by $\hat{\alpha}_{MLE}$, $\hat{\beta}_{MLE}$ and $\hat{\lambda}_{MLE}$ respectively. As the equations expressed in (5.3), (5.4) and (5.5) cannot be solved analytically. It is usually more convenient to use nonlinear optimization algorithms such as the quasi-Newton algorithm to numerically maximize the sample likelihood function.

5.2 Approximate confidence intervals

In this section, we present the asymptotic confidence intervals for the parameters of the SM distribution.

Since the *MLE*s of the unknown parameters α, β and λ cannot be derived in closed form, it is not easy to derive the exact distributions of the *MLE*s. Hence, we cannot obtain exact confidence intervals for the parameters. We must use the large sample approximation. It is known that the asymptotic distribution of the *MLEs* is $[\sqrt{n}(\hat{\alpha}_{MLE}-\alpha), \sqrt{n}(\hat{\beta}_{MLE}-\beta), \sqrt{n}(\hat{\lambda}_{MLE}-\lambda)]\rightarrow N_3(\underline{0}, I^{-1}(\Theta))$, where $I(\Theta)$ is the observed Fisher information matrix of the unknown parameters $\Theta=(\alpha,\beta,\lambda)$ and is given by

$$I(\Theta) = \begin{pmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\lambda} \\ & I_{\beta\beta} & I_{\beta\lambda} \\ & & I_{\lambda\lambda} \end{pmatrix}_{\Theta=\hat{\Theta}}$$

The elements of $I(\Theta)$ are defined in "Appendix". The multivariate normal distribution $N_3(\underline{0}, I(\Theta)^{-1})$, where the mean vector $\underline{0} = (0,0,0)^T$, can be used to construct confidence intervals.

The approximate $100(1-\tau)\%$ two-sided confidence intervals for α, β and λ of the form

$$\hat{\alpha} \pm z_{\tau/2}\sqrt{Var(\hat{\alpha})},$$

$$\hat{\beta} \pm z_{\tau/2}\sqrt{Var(\hat{\beta})}$$

and

$$\hat{\lambda} \pm z_{\tau/2}\sqrt{Var(\hat{\lambda})}$$

respectively, where $z_{\tau/2}$ is the upper $(\tau/2)$-th percentile of a standard normal distribution.

6 Numerical results

6.1 Tabulations of first four moments and variances

The recurrence relations obtained in the preceding sections allow us to evaluate the means, variances and covariances of all order statistics for all sample sizes in a simple recursive manner. Means, variances and covariances of all order statistics can be used for various inferential purposes; for example, they are useful in determining BLUEs of location/scale parameters and best linear unbiased predictors (BLUPs) of censored failure times. More details on BLUEs and BLUPs based on order statistics can be seen in Balakrishnan and Cohen (1991) and Arnold et al. (1992). In Tables 1 and 2, we have presented the first four moments and variances of order statistics and upper record values up to five decimal places, for sample sizes $n = 1(1)10$, $\alpha = 3$, $\beta = 2$ and $\lambda = 1$.

6.2 Simulation study

This section deals with the simulation study to evaluate the performance of maximum likelihood estimator (MLEs) of the three parameters SM distribution in terms of the sample size n. We use the inverse *cdf* method to generate the random variate from (1.2). For the SM distribution, the inverse *cdf* can not be obtain in explicit form. For this reason, we propose to use the Newton's method to solve the inverse *cdf* SM distribution. We consider different values for sample size where $n = 3, 5, 7, 10, 15, 30, 50, 100, 150$ and different parameter values I: $\alpha = 1.0$, $\beta = 0.5$, $\lambda = 1.0$ and II: $\alpha = 1.2$, $\beta = 1.0$, $\lambda = 1.5$. We repeat the process 1000 times. For a total of 2 parameter combinations, we obtain the average value of estimate and root mean square error (RMEs). The simulation results are displayed in Table 3. From Table 3, it is noted that the MLEs are quite close to the true parameter values and RMEs decrease as the sample size increase in both the cases. It indicate that the deduce asymptotically unbiased and consistent estimator of the parameters α, β and λ.

Table 1 First four moments, variances, some order statistics from Eq. (3.7)

$X_{r:n}$ ↓		j -th moment→	$j = 1$	$j = 2$	$j = 3$	$j = 4$	Variance
$n = 1$	$r = 1$	Expression (3.7)	3.63264	15.58368	10.00498	8.28967	2.387607
$n = 2$	$r = 1$	Expression (3.7)	3.16346	10.00498	9.03490	6.93246	0.002510
$n = 2$	$r = 2$	Expression (3.7)	7.26528	31.16736	20.00996	16.57933	2.616960
$n = 3$	$r = 1$	Expression (3.7)	3.06700	9.30036	8.81236	6.67184	0.106130
$n = 3$	$r = 2$	Expression (3.7)	4.84420	19.37115	17.69737	11.23843	1.095129
$n = 3$	$r = 3$	Expression (3.7)	10.89793	46.75105	30.01493	24.86900	1.013810
$n = 4$	$r = 1$	Expression (3.7)	3.02745	9.03490	8.71451	6.56659	0.130550
$n = 4$	$r = 2$	Expression (3.7)	5.48588	22.22948	21.52619	12.82423	2.865400
$n = 4$	$r = 3$	Expression (3.7)	0.39606	17.45473	16.58011	3.35900	0.297870
$n = 4$	$r = 4$	Expression (3.7)	14.53057	62.33473	40.01991	33.15866	1.803780
$n = 5$	$r = 1$	Expression (3.7)	3.00616	8.89679	8.65953	6.51034	0.140210
$n = 5$	$r = 2$	Expression (3.7)	6.23441	25.91985	25.56853	14.70145	3.948701
$n = 5$	$r = 3$	Expression (3.7)	14.79631	71.52458	71.46460	36.46482	3.406680
$n = 5$	$r = 4$	Expression (3.7)	14.82714	6.38806	3.72421	26.26477	4.232745
$n = 5$	$r = 5$	Expression (3.7)	18.16321	77.91841	50.02489	41.44833	4.984290

Table 2 First four moments and variances of some upper record values from Eq. (3.8)

$r\downarrow$	j -th moment→	$j=1$	$j=2$	$j=3$	$j=4$	Variance
1	Expression (3.8)	0.806522	0.806190	0.833070	0.855186	0.155712
2	Expression (3.8)	1.194722	1.700648	1.916390	2.072625	0.273287
3	Expression (3.8)	1.560754	2.923529	3.541159	4.003828	0.487576
4	Expression (3.8)	1.955091	4.685094	6.069290	7.155188	0.862713
5	Expression (3.8)	2.402608	7.275322	10.06014	12.35472	1.502797
6	Expression (3.8)	2.923235	11.12048	16.40166	20.97667	2.575177
7	Expression (3.8)	3.536888	16.85596	26.51133	33.16200	4.346383
8	Expression (3.8)	4.265492	25.43269	42.65533	64.06124	7.238268
9	Expression (3.8)	5.134243	38.27570	80.48108	114.4830	11.91525
10	Expression (3.8)	6.172692	57.52164	131.2132	124.5281	19.41951

Table 3 Average values of estimates and RMSEs of $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$

n	Parameter	I		II	
		Estimate	RMSE	Estimate	RMSE
3	α	1.019976	0.000399	1.179724	0.034892
	β	0.355183	0.415789	1.151882	0.084756
	λ	0.833650	1.360371	1.682543	0.294209
5	α	0.963012	0.000284	1.158834	0.025229
	β	0.525402	0.225243	1.083712	0.080412
	λ	1.054736	0.893524	1.521267	0.229186
7	α	0.975461	0.000264	1.150615	0.024162
	β	0.408437	0.215294	1.040182	0.070820
	λ	0.854063	0.091748	1.539384	0.212168
10	α	1.016478	0.000188	1.160315	0.023154
	β	0.323468	0.210142	1.2112301	0.070618
	λ	0.687501	0.081378	1.742335	0.066392
15	α	1.028461	0.000164	1.221310	0.021253
	β	0.445282	0.204011	0.906684	0.070329
	λ	0.607570	0.079646	1.281032	0.023140
30	α	0.869796	0.000148	1.143027	0.011548
	β	0.617787	0.146086	1.132319	0.070126
	λ	0.092576	0.062123	1.018425	0.022184
50	α	1.004798	0.000082	1.121414	0.010152
	β	0.287672	0.131981	1.214386	0.060132
	λ	0.745971	0.059676	1.683743	0.021342
100	α	0.938863	0.000045	1.201253	0.010105
	β	0.630241	0.130216	1.128741	0.058281
	λ	0.138276	0.058627	1.585772	0.020573
150	α	1.021824	0.000039	1.106251	0.010101
	β	0.338515	0.099864	0.841054	0.049202
	λ	0.158362	0.053465	1.267843	0.020112

6.3 Real data analysis

In this section, a real data set is considered for the illustration of the usefulness and applicability of the SM distribution. The data set represents the survival times (in days) of guinea pigs injected with different doses of tubercle bacilli. The regimen number is the common logarithm of the number of bacillary units in 0.5 ml. of challenge solution; i.e., regimen 6.6 corresponds to 4.0×10^6 bacillary units per 0.5 ml. ($\log_{10}(4.0\times10^6)=6.6$). This data set was originally reported by Bjerkedal (1960). This data set has also been analyzed by Kundu and Howlader (2010). They showed that inverse Weibull distribution is good fitted model for this data set. Corresponding to regimen 6.6, there were 72 observations which are listed below:

12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

The SM distribution is fitted to the data set and its fitting is compared with some well known lifetime distributions namely, gamma (Gm), Weibull (We) and inverse Weibull (IW) distributions with their respective pdfs given by

$$\text{Gm}: f(x)=\frac{\beta^{-\alpha}}{\Gamma(\alpha)}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right), x>0, \alpha,\beta>0,$$

$$\text{We}: f(x)=\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\}, x>0, \alpha,\beta>0,$$

and

$$\text{IW}: f(x)=\alpha\beta x^{-(\alpha+1)}\exp(-\beta x^{-\alpha}), x>0, \alpha,\beta>0.$$

We used different methods to test the goodness of fit of the above distributions based on maximum likelihood estimation method. We obtain estimated negative log likelihood function −lnL, the Akaike information criterion (AIC), proposed by Akaike (1974), defined by $AIC=2\times(k-\ln(L))$, Bayesian information criterion (BIC) proposed by Schwarz (1978), defined by $BIC=k\times\ln(n)-2\times\log(L)$, where k is the number of parameters in the distribution, n is the number of observations in the given data set, and L is the maximized value

Table 4 The MLEs, the values of log-likelihood function, AIC, BIC and K-S statistic and associated p-values for the real data set

S. no.	Distribution	MLEs	$-lnL$	AIC	BIC	$K–S$	p value
1.	Gamma	$\hat{\alpha} = 2.0815$ $\hat{\beta} = 47.9564$	394.2476	792.4952	797.0485	0.1384	0.12664
2.	Weibull	$\hat{\alpha} = 1.3932$ $\hat{\beta} = 110.5551$	397.1477	798.2953	802.8487	0.1465	0.09107
3.	Inverse Weibull	$\hat{\alpha} = 1.4148$ $\hat{\beta} = 283.8312$	395.6491	795.2982	799.8515	0.1520	0.07184
4.	SM	$\hat{\alpha} = 3.0165$	389.4698	784.9395	791.7695	0.0837	0.69368

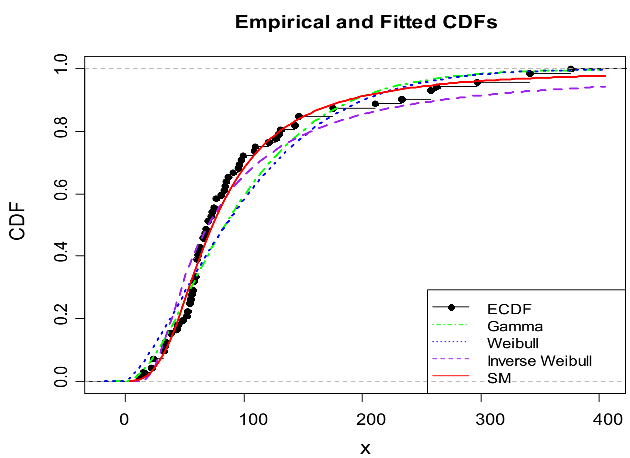

Fig. 3 Estimated cdfs of different distributions and the empirical cdf for the real data

of the likelihood function for the estimated model and Kolmogorov–Smirnov (K–S) statistics with its p value. The best distribution corresponds to lowest $-lnL$, AIC, BIC and $K–S$ statistic values.

The values of MLEs of the parameters of the considered lifetime distributions, $-lnL$, AIC, BIC, $K–S$ statistic with its *p-values* are presented in Table 4. This Table shows that the SM distribution out-performs the other three well known distributions, since it has the smallest $-lnL$, AIC, BIC and $K–S$ statistic values and highest *p-values*. More information is provided by visual comparisons using fitted cumulative distribution functions of the competing distributions with the empirical cdf (ECDF) and quantile–quantile (Q–Q) plots. Figure 3 shows the estimated cdfs of the competing models with ECDF. From Fig. 3 one can observe that the estimated cdf of the SM distribution is

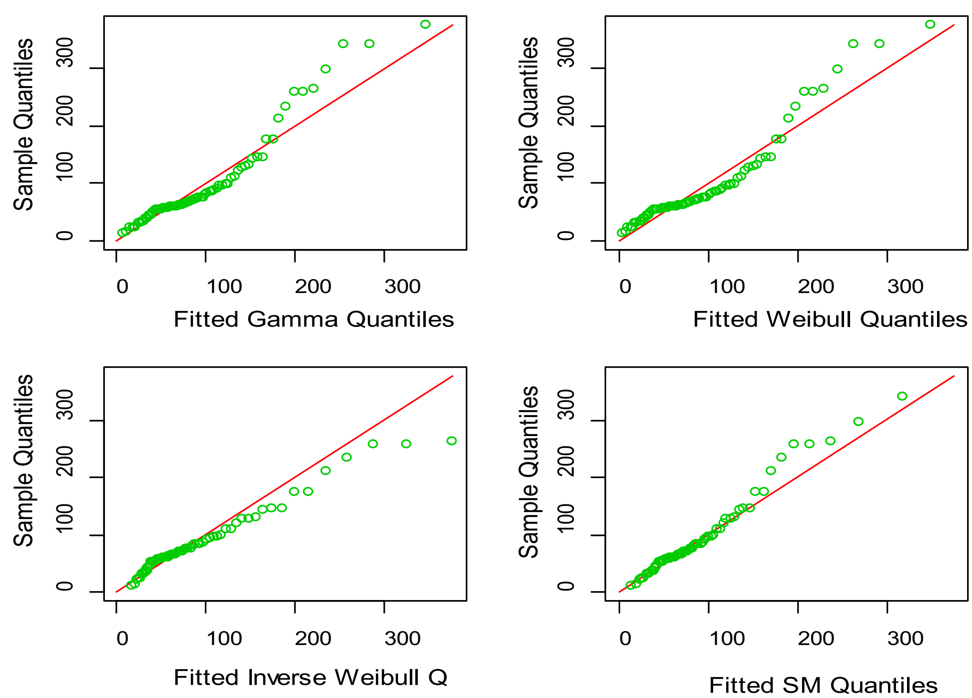

Fig. 4 Quantile–quantile plots for the real data set

closest to the ECDF. Also, a Q–Q plot consists of plots of the sample quantiles and the fitted quantiles. The Q–Q plots for the competing distributions are shown in Fig. 4 and from this Figure one can observe that the model based on SM distribution has the points closest to the diagonal line. Thus, these plots indicate that the SM distribution provide a good fit for the given data set and therefore, the SM distribution can be used effectively in the analysis of the given data set.

Substituting the MLEs of the unknown parameters in Fisher information matrix given in Sect. 5.2, the observed variance–covariance matrix comes out

$$I^{-1}(\Theta) = \begin{bmatrix} 0.3480 & -5.2457 & -0.1240 \\ -5.2457 & 137.5071 & 2.6533 \\ -0.1240 & 2.6533 & 0.0610 \end{bmatrix}.$$

Thus, using the diagonal elements of the matrix $I^{-1}(\Theta)$, the two sided 95% asymptotic confidence intervals of the parameters α, β, and λ are (1.8603, 4.1728), (37.8448, 83.8121), and (0.1890, 1.1571), respectively.

7 Concluding Remarks

In this paper, the various structural properties of the distribution are derived including explicit expressions for moments, mean deviation, Bonferroni and Lorenz curves, Renyi entropy and quantile function. The explicit expressions and recurrence relations for single and product moments of GOS from the SM distribution are derived. The two characterizing results of SM distribution have been obtained on using conditional moments of GOS and a recurrence relation for single moments. The method of maximum likelihood is adopted for estimating the model parameters. For different parameter settings and sample sizes, the various simulation studies are performed and compared to assess the performance of the SM distribution. We have analyzed one real data set and the proposed SM distribution provides a very good fit to the data set.

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Appendix

By differentiating (5.2), the elements of the Fisher information matrix $I(\Theta)$ for the parameters (α, β, λ) are:

$$I_{\alpha\alpha} = -\frac{n}{\alpha^2} - \alpha(\alpha+1)(\lambda+1)\sum_{i=1}^{n}\frac{(x_i/\beta)^{\alpha-2}}{[1+(x_i/\beta)^{\alpha}]} - (\lambda+1)\sum_{i=1}^{n}\frac{(x_i/\beta)^{2\alpha}\ln(x_i/\beta)}{[1+(x_i/\beta)^{\alpha}]^2}.$$

$$I_{\alpha\beta} = -\frac{n}{\beta} - \alpha(\lambda+1)\sum_{i=1}^{n}\frac{(x_i/\beta)^{\alpha-1}\ln(x_i/\beta)}{[1+(x_i/\beta)^{\alpha}]} + \alpha(\lambda+1)\sum_{i=1}^{n}\frac{(x_i/\beta)^{2\alpha-1}\ln(x_i/\beta)}{[1+(x_i/\beta)^{\alpha}]^2}.$$

$$I_{\alpha\lambda} = -\sum_{i=1}^{n}\frac{(x_i/\beta)^{\alpha}\ln(x_i/\beta)}{[1+(x_i/\beta)^{\alpha}]}.$$

$$I_{\beta\beta} = \frac{n\alpha}{\beta^2} - \alpha(\lambda+1)(\alpha+1)\sum_{i=1}^{n}\frac{x_i^{\alpha}}{\beta^{\alpha+2}[1+(x_i/\beta)^{\alpha}]} + \alpha^2(\lambda+1)\sum_{i=1}^{n}\frac{x_i^{2\alpha}}{\beta^{\alpha+1}[1+(x_i/\beta)^{\alpha}]^2}.$$

$$I_{\beta\lambda} = -\alpha\sum_{i=1}^{n}\frac{x_i^{\alpha}}{\beta^{\alpha+1}[1+(x_i/\beta)^{\alpha}]}.$$

$$I_{\lambda\lambda} = -\frac{n}{\lambda^2}.$$

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