Optical Soliton Solutions of the Conformable Time Fractional Radhakrishnan–Kundu–Lakshmanan Model

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Research Article

Keywords: Radhakrishnan–Kundu–Lakshmanan model,Conformable fractional derivative,Generalized Jacobi elliptic function method,Optical soliton solutions

Posted Date: July 14th, 2022

DOI: https://doi.org/10.21203/rs.3.rs-1567824/v1

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Received: date / Accepted: date

Abstract In the present study, we have obtained different kinds of wave solutions which possess distinctive physical characteristics of the nonlinear conformable time fractional Radhakrishnan–Kundu–Lakshmanan model by utilizing the generalized Jacobi elliptic function (GJEF) method. 3-D surfaces to some of the reported solutions are plotted and the dependence of the behaviour of the solutions on the fractional derivative has also been analyzed in the present study. In addition to providing physical explanations of Radhakrishnan-Kundu-Lakshmanan equation, the solutions presented here may also provide an insight into the the study of wave propagation in various conformable fractional nonlinear models arising in nonlinear sciences.

Keywords Radhakrishnan–Kundu–Lakshmanan model · Conformable fractional derivative · Generalized Jacobi elliptic function method · Optical soliton solutions

PACS 42.65.Tg · 02.90.+p · 02.30.Jr

MSC 2020 35B30 · 35C08 · 26A33

1 Introduction

Over the last few years, significant progress has been made in the field of nonlinear partial differential equations (NLPDEs). A variety of NLPDEs are used in the mathematical formulation of complex phenomenon in several nonlinear physics fields like plasma physics, nonlinear fiber optics, electromagnetism, fluid dynamics and optics [1, 2]. One interesting topic to investigate is the theory of optical solitons and their propagation through nonlinear optical fibers. The optical soliton represents a pulse which travels without any distortion due to dispersion or other factors. A soliton in optics is used to illustrate any optical field that does not change during propagation as a result of the delicate balance between nonlinear and linear phenomena in the medium. Throughout a nonlinear medium, electromagnetic pulses propagate in multiple dimensions. Different physical factors such as dispersion, material dispersion, diffraction and nonlinear response affect the pulse dynamics [3]. Soliton propagation has been represented mathematically by a variety of models such as Fokas–Lenells equation [4], Sasa–Satsuma equation [5], Chen–Lee–Liu equation [6], nonlinear Schrödinger's equation [7], Kundu–Mukherjee–Naskar equation [8], Kundu–Eckhaus equation [9], Schrödinger–Hirota equation [10] and Ginzburg–Landau model [11]

Obtaining the exact solutions of the NLPDEs helps in understanding the relationship between a differential equation and its mathematical and physical applications. The exact traveling-wave solutions of NLPDEs have been constructed via various effective approaches which include exponential rational function method [12], extended trial function method [13], improved Bernoulli sub equation function method [14, 15], Riccati-Bernoulli sub-ODE method [16].

This study aims to examine numerous exact solutions including bright-dark soliton solutions, Jacobi elliptic solutions of the conformable time-fractional perturbed Radhakrishnan–Kundu–Lakshmanan (RKL) equation [17–19] via GJEF method for the first time in the conformable time fractional RKL model. The dynamics of the light pulses were studied and represented by various NLPDEs including RKL equation. According to the Kerr law nonlinearity, conformable time-fractional perturbed RKL equation has the dimensionless form as follows [20]:

$$iD_t^{\alpha}\Psi + a\Psi_{xx} + b|\Psi|^2\Psi - i\Omega\Psi_x - i\Lambda\left(|\Psi|^2\Psi\right)_x - i\sigma\left(|\Psi|^2\right)_x\Psi - i\Upsilon\Psi_{xxx} = 0, \quad 0 < \alpha \leq 1 \tag{1}$$

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here Ψ is the complex-valued wave function of x; space and t; time. The fractional temporal evolution of the nonlinear wave is represented by the term $D_t^\alpha \Psi$. The coefficient a represents the group-velocity dispersion (GVD), b represents the coefficient of nonlinearity, Ω symbolises the inter-modal dispersion (IMD) and Λ, σ, Υ signify the coefficient of self-steepening for short pulses, the higher-order dispersion coefficient and the third order dispersion term respectively.
As soon as we substitue $\alpha = 1$ in the time fractional perturbed RKL equation, we obtain the original Radhakrishnan–Kundu–Lakshmanan equation [18].

This paper is structured as follows: In sec. 2, we have given a brief overview of the conformable fractional derivative. In sec. 3, the GJEF method has been described. In sec. 4, exact solution of conformable time-fractional RKL model have been obtained. In sec. 5, we have briefly analysed of the obtained exact solutions using their graphs finally concluding remarks have been given in sec. 6

2 Conformable fractional derivative

In this section, we present some overview on the conformable fractional derivative. In 2014, Khalil et al. [21] gave a new definition of the fractional derivative called the conformable fractional derivative (CFD). In recent times, many authors have used this CFD in various ways. Abdeljawad worked the conception of fractional versions of chain rule, Gronwall's inequality, exponential functions, Taylor power series expansions, integration by parts, Laplace transforms and linear differential systems [22]. Oqielat et al. [23] compared conformable and Caputo derivatives in the solutions of nonlinear time-fractional version of Schrödinger equations. Some new properties and theorems of this definition of CFD were invetigated and some new definitions were by Atangana et al. [24]. The CFD of order α is defined for a function $g : (0, \infty) \to \mathbb{R}$ by [21] :

$$D^{m\alpha} g(t) = \lim_{\varepsilon \to 0} \frac{g^{\lfloor \alpha \rfloor - 1}\left(t + \varepsilon t^{\lfloor \alpha \rfloor - \alpha}\right) - g^{\lfloor \alpha \rfloor - 1}(t)}{\varepsilon}, \qquad m - 1 < \alpha \leqslant m, t > 0$$

where $m \in \mathbb{N}$ and $\lfloor \alpha \rfloor$ is the smallest integer number greater than or equal α. Provided that $D^{m\alpha} g(0) = \lim_{\varepsilon \to 0^+} D^{m\alpha} g(t), g(t)$ is m-differentiable and $D^{m\alpha} g(0) = \lim_{\varepsilon \to 0^+} D^{\alpha} g(t)$ exists. As a special case, if $0 < \alpha \leqslant 1$, then we have:

$$D^\alpha g(t) = \lim_{\varepsilon \to 0} \frac{g\left(t + \varepsilon t^{1-\alpha}\right) - g(t)}{\varepsilon}, \qquad t > 0$$

provided $D^\alpha g(0) = \lim_{\varepsilon \to 0^+} D^\alpha g(t)$, $g(t)$ is differentiable and $D^\alpha f(0) = \lim_{\varepsilon \to 0^+} D^\alpha f(x)$ exists.

For functions $g(t)$ and $h(t)$ the conformable fractional derivative bears the following properties for $0 < \alpha \leqslant 1$, [21]:

(1) $D^\alpha(pg + qh) = pD^\alpha(g) + qD^\alpha(h), p, q \in \mathbb{R}$
(2) $D^\alpha\left(t^\Lambda\right) = \Lambda t^{\Lambda - \alpha}, \Lambda \in \mathbb{R}$
(3) $D^\alpha(gh) = gD^\alpha(h) + hD^\alpha(g)$,
(4) $D^\alpha\left(\frac{g}{h}\right) = \frac{hD^\alpha(g) - gD^\alpha(h)}{h^2}$
A significant and useful rule is that: If $g(t)$ is an m-differentiable function at $t > 0$ and $m - 1 < \alpha \leqslant m$, then: $D^{m\alpha} g(t) = t^{\lfloor \alpha \rfloor - \alpha} g^{\lfloor \alpha \rfloor}(t)$

The main advantage of using the CFD is that it complies with all the rules and concepts of an ordinary derivative, like product, quotient and chain rules, whereas the other fractional definitions do not. It also helps in the generalisation of some of the well known transforms like the Sumudu and Laplace transforms and can be utilized as a tool for solving the fractional differential equations It opens the door to extend and create new definitions such as the non-conformable fractional derivative [25], M-conformable fractional derivative [26], class conformable fractional derivatives [27], Fuzzy generalized conformable fractional derivative [28], modified of conformable fractional derivative [29] and truncated Ω fractional derivative [32].

3 Description of generalized Jacobi elliptic function method

In the following section, we have briefly described the steps of generalized Jacobi elliptic function method (GJEFM) [30] for conformable fractional equation.
Step 1: Consider a general time conformable fractional evolution equation as follows:

$$F\left(\Psi, \frac{\partial^\alpha \Psi}{\partial t^\alpha}, \frac{\partial \Psi}{\partial x}, \frac{\partial^{2\alpha} \Psi}{\partial t^{2\alpha}}, \frac{\partial^2 \Psi}{\partial x^2}, \cdots\right) = 0 \tag{2}$$

By using the wave transformation

$$\Psi(x,t) = \chi(\zeta) e^{i\Phi}, \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right), \Phi = -\nu x + \varpi \frac{t^\alpha}{\alpha} + \theta.$$

and using the chain rule [22], (2) reduces into an ordinary differential equation (ODE).

$$f\left(\chi, \chi', \chi'', \cdots\right) = 0 \tag{3}$$

Table 1: Solutions of (5) for some values of ξ, τ and η [30]

ξ	τ	η	F
m^2	$-(1+m^2)$	1	sn, cd
$-m^2$	$2m^2-1$	$1-m^2$	cn
-1	$2-m^2$	m^2-1	dn
1	$-(1+m^2)$	m^2	ns, dc
$1-m^2$	$2m^2-1$	$-m^2$	nc
m^2-1	$2-m^2$	-1	nd
$1-m^2$	$2-m^2$	1	sc
$-m^2(1-m^2)$	$2m^2-1$	1	sd
1	$2-m^2$	$1-m^2$	cs
1	$2m^2-1$	$-m^2(1-m^2)$	ds
$\frac{-1}{4}$	$\frac{m^2+1}{2}$	$\frac{-(1-m^2)^2}{4}$	$mcn \mp dn$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$ns \mp cs$
$\frac{1-m^2}{4}$	$\frac{m^2+1}{2}$	$\frac{1-m^2}{4}$	$nc \mp sc$
$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$ns \mp ds$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$sn \mp \iota cn, \frac{dn}{\sqrt{1-m^2}sn \mp cn}$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$mcn \mp \iota dn, \frac{sn}{1 \mp cn}$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$	$\frac{sn}{1 \mp dn}$
$\frac{m^2-1}{4}$	$\frac{m^2+1}{2}$	$\frac{m^2-1}{4}$	$\frac{dn}{1 \mp msn}$
$\frac{1-m^2}{4}$	$\frac{m^2+1}{2}$	$\frac{1-m^2}{4}$	$\frac{cn}{1 \mp sn}$
$\frac{(1-m^2)^2}{4}$	$\frac{m^2+1}{2}$	$\frac{1}{4}$	$\frac{sn}{dn \mp cn}$
$\frac{(m^4}{4}$	$\frac{m^2-1}{2}$	$\frac{1}{4}$	$\frac{cn}{\sqrt{1-m^2} \mp dn}$

where $'$ denotes first order derivative w.r.t. ζ.
Step 2: Assume that the solution of (3) takes the following form:

$$\chi(\zeta) = a_0 + \sum_{i=1}^{s} a_i F^i(\zeta) + \sum_{i=1}^{s} b_i F^{-i}(\zeta) \tag{4}$$

Here, the functions $F(\zeta)$ is the solutions of the following ODE :

$$\left(F'\right)^2(\zeta) = \xi F^4(\zeta) + \tau F^2(\zeta) + \eta \tag{5}$$

which are given in Table 1 for different values of constants ξ, τ and η.
Step 3: By using the balancing principle, s can be determined. Then inserting (4) into (3), and equating the coefficients of polynomial in F to zero, a system of algebraic equations is yielded. On solving that system, we can determine a_i, b_i.
Step 4: By utilizing all values of ξ, τ, η and gathering all the results given in Table 1 , the exact solutions of (2) can be obtained.

In this table 1, we have

$sn = sn(\zeta,m), cd = cd(\zeta,m), cn = cn(\zeta,m), dn = dn(\zeta,m), ns = ns(\zeta,m), cs = cs(\zeta,m), ds = ds(\zeta,m), sc = sc(\zeta,m), sd = sd(\zeta,m)$ are the Jacobi elliptic functions with the modulus $0 < m < 1$. These functions degenerate into hyperbolic functions, when $m \to 1$ as follows:

$$\begin{array}{l} sn(\zeta,1) = tanh(\zeta),\ cn(\zeta,1) = sech(\zeta),\ dn(\zeta,1) = sech(\zeta),\ ns(\zeta,1) = coth(\zeta),\ cs(\zeta,1) = csch(\zeta) \\ ds(\zeta,1) = csch(\zeta),\ \ sc(\zeta,1) = sinh(\zeta),\ sd(\zeta,1) = sinh(\zeta),\ nc(\zeta,1) = cosh(\zeta), \quad cd(\zeta,1) = 1 \end{array}$$

and into trigonometric functions, when $m \to 0$, as follows:

$$\begin{array}{l} sn(\zeta,0) = sin(\zeta),\ cd(\zeta,0) = cos(\zeta),\ cn(\zeta,0) = cos(\zeta),\ ns(\zeta,0) = csc(\zeta),\ cs(\zeta,0) = cot(\zeta) \\ ds(\zeta,0) = csc(\zeta),\ \ sc(\zeta,0) = tan(\zeta),\ sd(\zeta,0) = sin(\zeta),\ nc(\zeta,0) = sec(\zeta),\ dn(\zeta,0) = 1 \end{array}$$

4 Exact solution of conformable time-fractional RKL equation by GJEFM

In this section, we use the GJEFM to the conformable time-fractional perturbed RKL equation. Consider the complex conformable time-fractional travelling wave transformation

$$\Psi(x,t) = \chi(\zeta)e^{i\Phi}, \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right), \Phi = -\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta. \tag{6}$$

Substituting (6) into (1) and comparing the real and imaginary parts, we get

$$\kappa^2(a + 3\nu\Upsilon)\chi'' + (b - \nu\Lambda)\chi^3 - \left(\varpi + a\nu^2 + \Omega\nu + \Upsilon\nu^3\right)\chi = 0 \tag{7}$$

from the real part, and

$$\kappa^2\Upsilon\chi'' - \left(c + 2a\nu + \Omega + 3\nu^2\Upsilon\right)\chi' - (3\Lambda + 2\vartheta)\chi^2\chi' = 0 \tag{8}$$

from the imaginary part. Integrating (8) once, we have

$$3\kappa^2\Upsilon\chi'' - 3\left(c + 2a\nu + \Omega + 3\nu^2\Upsilon\right)\chi - (3\Lambda + 2\vartheta)\chi^3 = 0 \tag{9}$$

As the function χ satisfies both (7) and (9), the following constraint condition is given:

$$\frac{a + 3\nu\Upsilon}{3\Upsilon} = \frac{\varpi + a\nu^2 + \Omega\nu + \Upsilon\nu^3}{3\left(c + 2a\nu + \Omega + 3\nu^2\Upsilon\right)} = -\frac{b - \nu\Lambda}{3\Lambda + 2\vartheta} \tag{10}$$

Solving (10) for ν and c, we obtain

$$\begin{aligned} \nu &= -\frac{(3b\Upsilon + 2a\vartheta + 3a\Lambda)}{6\Upsilon(\Lambda + \vartheta)}, \\ c &= \frac{\Upsilon\left(\varpi + a\nu^2 + \Omega\nu + \Upsilon\nu^3\right)}{a + 3\nu\Upsilon} - \left(2a\nu + \Omega + 3\Upsilon\nu^2\right). \end{aligned} \tag{11}$$

By using the homogeneous balance principle [31] and balancing χ'' and χ^3 in (7), yields $s = 1$. Therefore, we assume that the solution of (7) is of the form

$$\chi(\zeta) = a_0 + a_1F(\zeta) + \frac{b_1}{F(\zeta)} \tag{12}$$

Substituting (12) into (7) and comparing the coefficients of various powers of $F(\zeta)$, the following system of equations is obtained

$$\begin{aligned}
&-{b_2}^3(\nu\Lambda - b) = 0,\\
&-3\,b_1{b_2}^2(\nu\Lambda - b) = 0,\\
&18\,b_2\left(\left(\Upsilon\,\eta\kappa^2 - 1/6\,\Lambda\,a_0b_2 - 1/6\,\Lambda\,{b_1}^2\right)\nu + 1/3\,\eta a\kappa^2 + 1/6\,a_0b_2b + 1/6\,{b_1}^2b\right) = 0,\\
&\left(6\,\eta\Upsilon\,\kappa^2b_1 - 6\,\Lambda\,a_0b_1b_2 - 3\,\Lambda\,a_1{b_2}^2 - \Lambda\,{b_1}^3\right)\nu + 3\,a_1{b_2}^2b + 6\,a_0b_1b_2b + b_1\left(2\,\eta\kappa^2a + {b_1}^2b\right) = 0,\\
&-b_2\Upsilon\,\nu^3 - b_2a\nu^2 + \left(-3\,\Lambda\,a_2{b_2}^2 + \left(12\,\Upsilon\,\tau\kappa^2 - 3\,{a_0}^2\Lambda - 6\,\Lambda\,a_1b_1 - \Omega\right)b_2 - 3\,\Lambda\,a_0{b_1}^2\right)\nu + 3\,a_2{b_2}^2b\\
&\qquad + \left(4\,\tau a\kappa^2 + 3\,b{a_0}^2 + 6\,ba_1b_1 - \varpi\right)b_2 + 3\,a_0{b_1}^2b = 0,\\
&-b_1\Upsilon\,\nu^3 - b_1a\nu^2 + \left(-6\,\Lambda\,a_2b_1b_2 - 6\,\Lambda\,a_0a_1b_2 + 3\,b_1\left(\Upsilon\,\tau\kappa^2 - {a_0}^2\Lambda - \Lambda\,a_1b_1 - \Omega/3\right)\right)\nu + 6\,b_1a_2b_2b\\
&\qquad + 6\,a_0a_1b_2b - b_1\left(-\tau a\kappa^2 - 3\,b{a_0}^2 - 3\,ba_1b_1 + \varpi\right) = 0,\\
&-a_0\Upsilon\,\nu^3 - a_0a\nu^2 + \left(\left(6\,\Upsilon\,\eta\kappa^2 - 6\,\Lambda\,a_0b_2 - 3\,{b_1}^2\Lambda\right)a_2 + \left(6\,\xi\Upsilon\,\kappa^2 - 3\,\Lambda\,{a_1}^2\right)b_2 - a_0\left({a_0}^2\Lambda + 6\,\Lambda\,a_1b_1 + \Omega\right)\right)\nu\\
&\qquad + \left(2\,\eta\kappa^2a + 6\,b_2ba_0 + 3\,{b_1}^2b\right)a_2 + \left(2\,\xi a\kappa^2 + 3\,b{a_1}^2\right)b_2 - a_0\left(-b{a_0}^2 - 6\,ba_1b_1 + \varpi\right) = 0,\\
&-a_1\Upsilon\,\nu^3 - a_1a\nu^2 + \left(-6\,\Lambda\,(a_0b_1 + a_1b_2)\,a_2 + 3\,a_1\left(\Upsilon\,\tau\kappa^2 - {a_0}^2\Lambda - \Lambda\,a_1b_1 - \Omega/3\right)\right)\nu + 6\,b\,(a_0b_1 + a_1b_2)\,a_2\\
&\qquad - a_1\left(-\tau a\kappa^2 - 3\,b{a_0}^2 - 3\,ba_1b_1 + \varpi\right) = 0,\\
&-\Upsilon\,\nu^3a_2 - a\nu^2a_2 + \left(-3\,\Lambda\,{a_2}^2b_2 + \left(12\,\Upsilon\,\tau\kappa^2 - 3\,{a_0}^2\Lambda - 6\,\Lambda\,a_1b_1 - \Omega\right)a_2 - 3\,\Lambda\,a_0{a_1}^2\right)\nu + 3\,b{a_2}^2b_2 + \left(4\,\tau a\kappa^2\right.\\
&\qquad \left. + 3\,b{a_0}^2 + 6\,ba_1b_1 - \varpi\right)a_2 + 3\,ba_0{a_1}^2 = 0,\\
&\left(6\,\xi\Upsilon\,\kappa^2a_1 - 6\,\Lambda\,a_0a_1a_2 - \Lambda\,{a_1}^3 - 3\,\Lambda\,{a_2}^2b_1\right)\nu + 2\,\xi a\kappa^2a_1 + 6\,ba_0a_1a_2 + b{a_1}^3 + 3\,b{a_2}^2b_1 = 0,\\
&18\,a_2\left(\left(\xi\Upsilon\,\kappa^2 - 1/6\,\Lambda\,a_0a_2 - 1/6\,\Lambda\,{a_1}^2\right)\nu + 1/3\,\xi a\kappa^2 + 1/6\,a_0ba_2 + 1/6\,b{a_1}^2\right) = 0,
\end{aligned} \tag{13}$$

$$\begin{aligned}
&-3\,a_1{a_2}^2(\nu\Lambda - b) = 0,\\
&-{a_2}^3(\nu\Lambda - b) = 0,
\end{aligned} \tag{14}$$

On solving the above obtained system with the aid of Maple, the following values are obtained

SET I

$$a_0 = 0, \quad a_1 = \iota\sqrt{\frac{6\,\xi\Upsilon\,\nu + 2\,\xi a}{b-\nu\Lambda}}\kappa, \quad b_1 = 0, \quad \varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu, \tag{15}$$

SET II

$$a_0 = 0, \quad a_1 = 0, \quad b_1 = \iota\sqrt{\frac{6\,\eta\Upsilon\,\nu + 2\,\eta a}{b-\nu\Lambda}}\kappa, \quad \varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu, \tag{16}$$

SET III

$$a_0 = 0, \quad a_1 = \iota\sqrt{\frac{6\,\xi\Upsilon\,\nu + 2\,\xi a}{b-\nu\Lambda}}\kappa, \quad b_1 = \iota\sqrt{\frac{6\,\eta\Upsilon\,\nu + 2\,\eta a}{b-\nu\Lambda}}\kappa \text{ and}$$

$$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu, \text{where } \vartheta_1 = \iota\sqrt{\frac{6\,\xi\Upsilon\,\nu + 2\,\xi a}{b-\nu\Lambda}}, \text{ and} \quad \vartheta_2 = \iota\sqrt{\frac{6\,\eta\Upsilon\,\nu + 2\,\eta a}{b-\nu\Lambda}}. \tag{17}$$

Subsequently, using the corresponding values of ξ , τ and η from Table 1 and values of ν and c from (11) and utilising them together with (11), we obtain the following solutions of (1)

Case (i) $\xi = m^2, \quad \tau = -(1+m^2), \quad \eta = 1, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$$\varpi = -3\,(1+m^2)\Upsilon\,\nu\kappa^2 - (1+m^2)a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{1,1}(x,t) = \left(\iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2\,m^2 a}{b-\nu\Lambda}}\kappa sn(\zeta)\right)e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{18}$$

and

$$\Psi_{1,2}(x,t) = \left(\iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2\,m^2 a}{b-\nu\Lambda}}\kappa cd(\zeta)\right)e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{19}$$

Solution corresponding to the values obtained in **set (II)**

$$\varpi = -3\,(1+m^2)\Upsilon\,\nu\kappa^2 - (1+m^2)a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{1,3}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa\frac{1}{sn(\zeta)}\right)e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{20}$$

and

$$\Psi_{1,4}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa\frac{1}{cd(\zeta)}\right)e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{21}$$

Solution corresponding to the values obtained in **set (III)**

$$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$$

$$\Psi_{1,5}(x,t) = \left(\iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2\,m^2 a}{b-\nu\Lambda}}\kappa sn(\zeta) + \iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{sn(\zeta)}\right)\right)e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{22}$$

and

$$\Psi_{1,6}(x,t) = \left(\iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2\,m^2 a}{b-\nu\Lambda}}\kappa cd(\zeta) + \iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{cd(\zeta)}\right)\right)e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{23}$$

Case (ii) $\xi = -m^2, \quad \tau = 2m^2 - 1, \quad \eta = 1 - m^2, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$$\varpi = 3\,(2m^2-1)\Upsilon\,\nu\kappa^2 + (2m^2-1)a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{2,1}(x,t) = \left(-\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2\,m^2 a}{b-\nu\Lambda}}\kappa cn(\zeta)\right)e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{24}$$

Solution corresponding to the values obtained in **set (II)**

$$\varpi = 3\,(2m^2-1)\Upsilon\,\nu\kappa^2 + (2m^2-1)a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{2,2}(x,t) = \left(\iota\sqrt{\frac{6(1-m^2)\,\Upsilon\,\nu + 2\,(1-m^2)a}{b-\nu\Lambda}}\kappa\left(\frac{1}{cn(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{25}$$

Solution corresponding to the values obtained in **set (III)**

$$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$$

$$\Psi_{2,3}(x,t) = \left(-\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2\,m^2a}{b-\nu\Lambda}}\kappa cn(\zeta) + \iota\sqrt{\frac{6(1-m^2)\,\Upsilon\,\nu + 2\,(1-m^2)a}{b-\nu\Lambda}}\kappa\left(\frac{1}{cn(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{26}$$

Case (iii) $\xi = -1, \quad \tau = (2-m^2), \quad \eta = m^2-1, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$$\varpi = 3\,(2-m^2)\Upsilon\,\nu\kappa^2 + (2-m^2)a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{3,1}(x,t) = \left(-\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa dn(\zeta)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{27}$$

Solution corresponding to the values obtained in **set (II)**

$$\varpi = 3\,(2-m^2)\Upsilon\,\nu\kappa^2 + (2-m^2)a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{3,1}(x,t) = \left(\iota\sqrt{\frac{6(m^2-1)\,\Upsilon\,\nu + 2(m^2-1)\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{dn(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{28}$$

Solution corresponding to the values obtained in **set (III)**

$$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$$

$$\Psi_{3,2}(x,t) = \left(-\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa dn(\zeta) + \iota\sqrt{\frac{6(m^2-1)\,\Upsilon\,\nu + 2(m^2-1)\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{dn(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{29}$$

Case (iv) $\xi = 1, \quad \tau = -(1+m^2), \quad \eta = m^2, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$$\varpi = -3\,(1+m^2)\Upsilon\,\nu\kappa^2 - (1+m^2)a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{4,1}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa ns(\zeta)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{30}$$

and

$$\Psi_{4,2}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa dc(\zeta)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{31}$$

Solution corresponding to the values obtained in **set (II)**

$$\varpi = -3\,(1+m^2)\Upsilon\,\nu\kappa^2 - (1+m^2)a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{4,3}(x,t) = \left(\iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2m^2\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{ns(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{32}$$

and

$$\Psi_{4,4}(x,t) = \left(\iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2m^2\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{dc(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{33}$$

Solution corresponding to the values obtained in **set (III)**

$$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$$

$$\Psi_{4,5}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa ns(\zeta) + \iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2m^2\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{ns(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{34}$$

and

$$\Psi_{4,6}(x,t)=\left(\iota\sqrt{\frac{6\,\Upsilon\,\nu+2\,a}{b-\nu\Lambda}}\kappa dc(\zeta)+\iota\sqrt{\frac{6m^2\,\Upsilon\,\nu+2m^2\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{dc(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{35}$$

Case (v) $\xi=1-m^2,\quad \tau=2m^2-1,\quad \eta=-m^2,\quad \zeta=\kappa\left(x-c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$\varpi=3\,(2m^2-1)\Upsilon\,\nu\kappa^2+(2m^2-1)a\kappa^2-\Upsilon\,\nu^3-a\nu^2-\Omega\,\nu$

$$\Psi_{5,1}(x,t)=\left(\iota\sqrt{\frac{6(1-m^2)\,\Upsilon\,\nu+2(1-m^2)\,a}{b-\nu\Lambda}}\kappa nc(\zeta)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{36}$$

Solution corresponding to the values obtained in **set (II)**

$\varpi=3\,(2m^2-1)\Upsilon\,\nu\kappa^2+(2m^2-1)a\kappa^2-\Upsilon\,\nu^3-a\nu^2-\Omega\,\nu$

$$\Psi_{5,2}(x,t)=\left(-\sqrt{\frac{6m^2\,\Upsilon\,\nu+2m^2\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{nc(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{37}$$

Solution corresponding to the values obtained in **set (III)**

$\varpi=3\,\tau\Upsilon\,\nu\kappa^2+\tau a\kappa^2-\Upsilon\,\nu^3-3\nu\Lambda\vartheta_1\kappa^2\vartheta_2-a\nu^2+3b\vartheta_1\kappa^2\vartheta_2-\Omega\,\nu,$

$$\Psi_{5,3}(x,t)=\left(\iota\sqrt{\frac{6(1-m^2)\,\Upsilon\,\nu+2(1-m^2)\,a}{b-\nu\Lambda}}\kappa nc(\zeta)-\sqrt{\frac{6m^2\,\Upsilon\,\nu+2m^2\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{nc(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{38}$$

Case (vi) $\xi=m^2-1,\quad \tau=2-m^2,\quad \eta=-1,\quad \zeta=\kappa\left(x-c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$\varpi=3\,(2-m^2)\Upsilon\,\nu\kappa^2+(2-m^2)a\kappa^2-\Upsilon\,\nu^3-a\nu^2-\Omega\,\nu$

$$\Psi_{6,1}(x,t)=\left(\iota\sqrt{\frac{6(m^2-1)\,\Upsilon\,\nu+2(m^2-1)\,a}{b-\nu\Lambda}}\kappa nd(\zeta)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{39}$$

Solution corresponding to the values obtained in **set (II)**

$\varpi=3\,(2-m^2)\Upsilon\,\nu\kappa^2+(2-m^2)a\kappa^2-\Upsilon\,\nu^3-a\nu^2-\Omega\,\nu$

$$\Psi_{6,2}(x,t)=\left(-\sqrt{\frac{6\,\Upsilon\,\nu+2\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{nd(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{40}$$

Solution corresponding to the values obtained in **set (III)**

$\varpi=3\,\tau\Upsilon\,\nu\kappa^2+\tau a\kappa^2-\Upsilon\,\nu^3-3\nu\Lambda\vartheta_1\kappa^2\vartheta_2-a\nu^2+3b\vartheta_1\kappa^2\vartheta_2-\Omega\,\nu,$

$$\Psi_{6,3}(x,t)=\left(\iota\sqrt{\frac{6(m^2-1)\,\Upsilon\,\nu+2(m^2-1)\,a}{b-\nu\Lambda}}\kappa nd(\zeta)-\sqrt{\frac{6\,\Upsilon\,\nu+2\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{nd(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{41}$$

Case (vii) $\xi=1-m^2,\quad \tau=2-m^2,\quad \eta=1,\quad \zeta=\kappa\left(x-c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$\varpi=3\,(2-m^2)\Upsilon\,\nu\kappa^2+(2-m^2)a\kappa^2-\Upsilon\,\nu^3-a\nu^2-\Omega\,\nu$

$$\Psi_{7,1}(x,t)=\left(\iota\sqrt{\frac{6(1-m^2)\,\Upsilon\,\nu+2(1-m^2)\,a}{b-\nu\Lambda}}\kappa sc(\zeta)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{42}$$

Solution corresponding to the values obtained in **set (II)**

$\varpi=3\,(2-m^2)\Upsilon\,\nu\kappa^2+(2-m^2)a\kappa^2-\Upsilon\,\nu^3-a\nu^2-\Omega\,\nu$

$$\Psi_{7,2}(x,t)=\left(\iota\sqrt{\frac{6\,\Upsilon\,\nu+2\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{sc(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{43}$$

Solution corresponding to the values obtained in **set (III)**

$$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$$

$$\Psi_{7,3}(x,t) = \left(\iota\sqrt{\frac{6(1-m^2)\,\Upsilon\,\nu + 2(1-m^2)\,a}{b-\nu\Lambda}}\kappa sc(\zeta) + \iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{sc(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{44}$$

Case (viii) $\xi = -m^2(1-m^2), \quad \tau = 2m^2-1, \quad \eta = 1, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$$\varpi = 3\,(2m^2-1)\Upsilon\,\nu\kappa^2 + (2m^2-1)a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{8,1}(x,t) = \left(-\sqrt{\frac{6m^2(1-m^2)\,\Upsilon\,\nu + 2m^2(1-m^2)\,a}{b-\nu\Lambda}}\kappa sd(\zeta)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{45}$$

Solution corresponding to the values obtained in **set (II)**

$$\varpi = 3\,(2m^2-1)\Upsilon\,\nu\kappa^2 + (2m^2-1)a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{8,2}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{sd(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{46}$$

Solution corresponding to the values obtained in **set (III)**

$$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$$

$$\Psi_{8,3}(x,t) = \left(-\sqrt{\frac{6m^2(1-m^2)\,\Upsilon\,\nu + 2m^2(1-m^2)\,a}{b-\nu\Lambda}}\kappa sd(\zeta) + \iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{sd(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{47}$$

Case (ix) $\xi = 1, \quad \tau = 2-m^2, \quad \eta = 1-m^2, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$$\varpi = 3\,(2-m^2)\Upsilon\,\nu\kappa^2 + (2-m^2)a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{9,1}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa cs(\zeta)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{48}$$

Solution corresponding to the values obtained in **set (II)**

$$\varpi = 3\,(2-m^2)\Upsilon\,\nu\kappa^2 + (2-m^2)a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{9,2}(x,t) = \left(\iota\sqrt{\frac{6(1-m^2)\,\Upsilon\,\nu + 2(1-m^2)\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{cs(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{49}$$

Solution corresponding to the values obtained in **set (III)**

$$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$$

$$\Psi_{9,3}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa cs(\zeta) + \iota\sqrt{\frac{6(1-m^2)\,\Upsilon\,\nu + 2(1-m^2)\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{cs(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{50}$$

Case (x) $\xi = 1, \quad \tau = 2m^2-1, \quad \eta = -m^2(1-m^2), \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$$\varpi = 3\,(2m^2-1)\Upsilon\,\nu\kappa^2 + (2m^2-1)a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{10,1}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa ds(\zeta)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{51}$$

Solution corresponding to the values obtained in **set (II)**

$$\varpi = 3\,(2m^2-1)\Upsilon\,\nu\kappa^2 + (2m^2-1)a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{10,2}(x,t) = \left(-\sqrt{\frac{6m^2(1-m^2)\,\Upsilon\,\nu + 2m^2(1-m^2)\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{ds(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{52}$$

Solution corresponding to the values obtained in **set (III)**

$$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$$

$$\Psi_{10,3}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa ds(\zeta) - \sqrt{\frac{6m^2(1-m^2)\,\Upsilon\,\nu + 2m^2(1-m^2)\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{ds(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{53}$$

Case (xi) $\xi = \frac{-1}{4}, \quad \tau = \frac{m^2+1}{2}, \quad \eta = \frac{-(1-m^2)^2}{4}, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$$\varpi = 3\,(\tfrac{m^2+1}{2})\Upsilon\,\nu\kappa^2 + (\tfrac{m^2+1}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{11,1}(x,t) = \left(-\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa(mcn(\zeta) \mp dn(\zeta))\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{54}$$

Solution corresponding to the values obtained in **set (II)**

$$\varpi = 3\,(\tfrac{m^2+1}{2})\Upsilon\,\nu\kappa^2 + (\tfrac{m^2+1}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{11,2}(x,t) = \left(-\sqrt{\frac{6(1-m^2)^2\,\Upsilon\,\nu + 2(1-m^2)^2\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{1}{(mcn(\zeta) \mp dn(\zeta))}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{55}$$

Solution corresponding to the values obtained in **set (III)**

$$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$$

$$\Psi_{11,3}(x,t) = \left(-\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa(mcn(\zeta) \mp dn(\zeta)) - \sqrt{\frac{6(1-m^2)^2\,\Upsilon\,\nu + 2(1-m^2)^2\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{1}{(mcn(\zeta) \mp dn(\zeta))}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{56}$$

Case (xii) $\xi = \frac{1}{4}, \quad \tau = \frac{1-2m^2}{2}, \quad \eta = \frac{1}{4}, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$$\varpi = 3\,(\tfrac{1-2m^2}{2})\Upsilon\,\nu\kappa^2 + (\tfrac{1-2m^2}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{12,1}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa(ns(\zeta) \mp cs(\zeta))\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{57}$$

Solution corresponding to the values obtained in **set (II)**

$$\varpi = 3\,(\tfrac{1-2m^2}{2})\Upsilon\,\nu\kappa^2 + (\tfrac{1-2m^2}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$$

$$\Psi_{12,2}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{1}{(ns(\zeta) \mp cs(\zeta))}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{58}$$

Solution corresponding to the values obtained in **set (III)**

$$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$$

$$\Psi_{12,3}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa(ns(\zeta) \mp cs(\zeta)) + \iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{1}{(ns(\zeta) \mp cs(\zeta))}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{59}$$

Case (xiii) $\xi = \frac{1-m^2}{4}, \quad \tau = \frac{m^2+1}{2}, \quad \eta = \frac{1-m^2}{4}, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$\varpi = 3\,(\frac{m^2+1}{2})\Upsilon\,\nu\kappa^2 + (\frac{m^2+1}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{13,1}(x,t) = \left(\iota\sqrt{\frac{6(1-m^2)\,\Upsilon\,\nu + 2(1-m^2)\,a}{4(b-\nu\Lambda)}}\kappa(nc(\zeta) \mp sc(\zeta))\right) e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{60}$$

Solution corresponding to the values obtained in **set (II)**

$\varpi = 3\,(\frac{m^2+1}{2})\Upsilon\,\nu\kappa^2 + (\frac{m^2+1}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{13,2}(x,t) = \left(\iota\sqrt{\frac{6(1-m^2)\,\Upsilon\,\nu + 2(1-m^2)\,a}{4(b-\nu\Lambda)}}\kappa\frac{1}{(nc(\zeta) \mp sc(\zeta))}\right) e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{61}$$

Solution corresponding to the values obtained in **set (III)**

$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$

$$\Psi_{13,3}(x,t) = \left(\iota\sqrt{\frac{6(1-m^2)\,\Upsilon\,\nu + 2(1-m^2)\,a}{4(b-\nu\Lambda)}}\kappa(nc(\zeta) \mp sc(\zeta)) + \iota\sqrt{\frac{6(1-m^2)\,\Upsilon\,\nu + 2(1-m^2)\,a}{4(b-\nu\Lambda)}}\kappa\frac{1}{(nc(\zeta) \mp sc(\zeta))}\right) e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{62}$$

Case (xiv) $\xi = \frac{1}{4}, \quad \tau = \frac{m^2-2}{2}, \quad \eta = \frac{m^2}{4}, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$\varpi = 3\,(\frac{m^2-2}{2})\Upsilon\,\nu\kappa^2 + (\frac{m^2-2}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{14,1}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa(ns(\zeta) \mp ds(\zeta))\right) e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{63}$$

Solution corresponding to the values obtained in **set (II)**

$\varpi = 3\,(\frac{m^2-2}{2})\Upsilon\,\nu\kappa^2 + (\frac{m^2-2}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{14,2}(x,t) = \left(\iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2m^2\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{1}{(ns(\zeta) \mp ds(\zeta))}\right)\right) e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{64}$$

Solution corresponding to the values obtained in **set (III)**

$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$

$$\Psi_{14,3}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa(ns(\zeta) \mp ds(\zeta) + \iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2m^2\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{1}{(ns(\zeta) \mp ds(\zeta))}\right)\right) e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{65}$$

Case (xv) $\xi = \frac{m^2}{4}, \quad \tau = \frac{m^2-2}{2}, \quad \eta = \frac{m^2}{4}, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$\varpi = 3\,(\frac{m^2-2}{2})\Upsilon\,\nu\kappa^2 + (\frac{m^2-2}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{15,1}(x,t) = \left(\iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2m^2\,a}{4(b-\nu\Lambda)}}\kappa(sn(\zeta) \mp \iota cn(\zeta))\right) e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{66}$$

and

$$\Psi_{15,2}(x,t) = \left(\iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2m^2\,a}{4(b-\nu\Lambda)}}\kappa(\frac{dn(\zeta)}{\sqrt{1-m^2}sn(\zeta) \mp cn(\zeta)}\right) e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{67}$$

Solution corresponding to the values obtained in **set (II)**

$\varpi = 3\,(\frac{m^2-2}{2})\Upsilon\,\nu\kappa^2 + (\frac{m^2-2}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{15,3}(x,t) = \left(\iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2m^2\,a}{4(b-\nu\Lambda)}}\kappa\frac{1}{(sn(\zeta) \mp \iota cn(\zeta))}\right) e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{68}$$

and

$$\Psi_{15,4}(x,t) = \left(\iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2m^2\,a}{4(b-\nu\Lambda)}}\kappa(\frac{\sqrt{1-m^2}sn(\zeta) \mp cn(\zeta)}{dn(\zeta)}\right) e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{69}$$

Solution corresponding to the values obtained in **set (III)**

$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$

$$\Psi_{15,5}(x,t) = \left(\iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2m^2\,a}{4(b-\nu\Lambda)}}\kappa(sn(\zeta)\mp\iota cn(\zeta)) + \iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2m^2\,a}{4(b-\nu\Lambda)}}\kappa\frac{1}{(sn(\zeta)\mp\iota cn(\zeta))}\right) e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{70}$$

and

$$\Psi_{15,6}(x,t) = \left(\iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2m^2\,a}{4(b-\nu\Lambda)}}\kappa(\frac{dn(\zeta)}{\sqrt{1-m^2}sn(\zeta)\mp cn(\zeta)} + \iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2m^2\,a}{4(b-\nu\Lambda)}}\kappa(\frac{\sqrt{1-m^2}sn(\zeta)\mp cn(\zeta)}{dn(\zeta)}\right) e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{71}$$

Case (xvi) $\xi = \frac{1}{4}, \quad \tau = \frac{1-2m^2}{2}, \quad \eta = \frac{1}{4}, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$\varpi = 3\,(\frac{1-2m^2}{2})\Upsilon\,\nu\kappa^2 + (\frac{1-2m^2}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{16,1}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa(mcn(\zeta)\mp\iota dn(\zeta))\right) e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{72}$$

and

$$\Psi_{16,2}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa(\frac{sn(\zeta)}{1\mp cn(\zeta)}\right) e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{73}$$

Solution corresponding to the values obtained in **set (II)**

$\varpi = 3\,(\frac{1-2m^2}{2})\Upsilon\,\nu\kappa^2 + (\frac{1-2m^2}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{16,3}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{1}{(mcn(\zeta)\mp\iota dn(\zeta))}\right)\right) e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{74}$$

and

$$\Psi_{16,4}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa(\frac{1\mp cn(\zeta)}{sn(\zeta)}\right) e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{75}$$

Solution corresponding to the values obtained in **set (III)**

$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$

$$\Psi_{16,5}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa(mcn(\zeta)\mp\iota dn(\zeta)) + \iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{1}{(mcn(\zeta)\mp\iota dn(\zeta))}\right)\right) e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{76}$$

and

$$\Psi_{16,6}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa(\frac{sn(\zeta)}{1\mp cn(\zeta)} + \iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa(\frac{1\mp cn(\zeta)}{sn(\zeta)}\right) e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{77}$$

Case (xvii) $\xi = \frac{m^2}{4}, \quad \tau = \frac{m^2-2}{2}, \quad \eta = \frac{1}{4}, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$\varpi = 3\,(\frac{m^2-2}{2})\Upsilon\,\nu\kappa^2 + (\frac{m^2-2}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{17,1}(x,t) = \left(\iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2m^2\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{sn(\zeta)}{1\mp dn(\zeta)}\right)\right) e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{78}$$

Solution corresponding to the values obtained in **set (II)**

$\varpi = 3\,(\frac{m^2-2}{2})\Upsilon\,\nu\kappa^2 + (\frac{m^2-2}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{17,2}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{1\mp dn(\zeta)}{sn(\zeta)}\right)\right) e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{79}$$

Solution corresponding to the values obtained in **set (III)**

$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$

$$\Psi_{17,3}(x,t) = \left(\iota\sqrt{\frac{6m^2\,\Upsilon\,\nu + 2m^2\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{sn(\zeta)}{1\mp dn(\zeta)}\right) + \iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{1\mp dn(\zeta)}{sn(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{80}$$

Case (xviii) $\xi = \frac{m^2-1}{4}, \quad \tau = \frac{m^2+1}{2}, \quad \eta = \frac{m^2-1}{4}, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$\varpi = 3\,(\frac{m^2+1}{2})\Upsilon\,\nu\kappa^2 + (\frac{m^2+1}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{18,1}(x,t) = \left(\iota\sqrt{\frac{6(m^2-1)\,\Upsilon\,\nu + 2(m^2-1)\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{dn(\zeta)}{1\mp msn(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{81}$$

Solution corresponding to the values obtained in **set (II)**

$\varpi = 3\,(\frac{m^2+1}{2})\Upsilon\,\nu\kappa^2 + (\frac{m^2+1}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{18,2}(x,t) = \left(\iota\sqrt{\frac{6(m^2-1)\,\Upsilon\,\nu + 2(m^2-1)\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{1\mp msn(\zeta)}{dn(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{82}$$

Solution corresponding to the values obtained in **set (III)**

$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$

$$\Psi_{18,3}(x,t) = \left(\iota\sqrt{\frac{6(m^2-1)\,\Upsilon\,\nu + 2(m^2-1)\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{dn(\zeta)}{1\mp msn(\zeta)}\right) + \iota\sqrt{\frac{6(m^2-1)\,\Upsilon\,\nu + 2(m^2-1)\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{1\mp msn(\zeta)}{dn(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{83}$$

Case (xix) $\xi = \frac{1-m^2}{4}, \quad \tau = \frac{m^2+1}{2}, \quad \eta = \frac{1-m^2}{4}, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$\varpi = 3\,(\frac{m^2+1}{2})\Upsilon\,\nu\kappa^2 + (\frac{m^2+1}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{19,1}(x,t) = \left(\iota\sqrt{\frac{6(1-m^2)\,\Upsilon\,\nu + 2(1-m^2)\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{cn(\zeta)}{1\mp sn(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{84}$$

Solution corresponding to the values obtained in **set (II)**

$\varpi = 3\,(\frac{m^2+1}{2})\Upsilon\,\nu\kappa^2 + (\frac{m^2+1}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{19,2}(x,t) = \left(\iota\sqrt{\frac{6(1-m^2)\,\Upsilon\,\nu + 2(1-m^2)\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{1\mp sn(\zeta)}{cn(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{85}$$

Solution corresponding to the values obtained in **set (III)**

$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$

$$\Psi_{19,3}(x,t) = \left(\iota\sqrt{\frac{6(1-m^2)\,\Upsilon\,\nu + 2(1-m^2)\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{cn(\zeta)}{1\mp sn(\zeta)}\right) + \iota\sqrt{\frac{6(1-m^2)\,\Upsilon\,\nu + 2(1-m^2)\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{1\mp sn(\zeta)}{cn(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{86}$$

Case (xx) $\xi = \frac{(1-m^2)^2}{4}, \quad \tau = \frac{m^2+1}{2}, \quad \eta = \frac{1}{4}, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$\varpi = 3\,(\frac{m^2+1}{2})\Upsilon\,\nu\kappa^2 + (\frac{m^2+1}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{20,1}(x,t) = \left(\iota\sqrt{\frac{6((1-m^2)^2)\,\Upsilon\,\nu + 2((1-m^2)^2)\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{sn(\zeta)}{dn(\zeta)\mp cn(\zeta)}\right)\right)e^{\iota(-\nu x+\varpi\frac{t^\alpha}{\alpha}+\theta)}, \tag{87}$$

Solution corresponding to the values obtained in **set (II)**

$\varpi = 3\,(\frac{m^2+1}{2})\Upsilon\,\nu\kappa^2 + (\frac{m^2+1}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{20,2}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{dn(\zeta) \mp cn(\zeta)}{sn(\zeta)}\right)\right)e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{88}$$

Solution corresponding to the values obtained in **set (III)**

$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$

$$\Psi_{20,3}(x,t) = \left(\iota\sqrt{\frac{6((1-m^2)^2)\,\Upsilon\,\nu + 2((1-m^2)^2)\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{sn(\zeta)}{dn(\zeta) \mp cn(\zeta)}\right) + \iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{dn(\zeta) \mp cn(\zeta)}{sn(\zeta)}\right)\right)e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{89}$$

Case (xxi) $\xi = \frac{m^4}{4}, \quad \tau = \frac{m^2-1}{2}, \quad \eta = \frac{1}{4}, \quad \zeta = \kappa\left(x - c\frac{t^\alpha}{\alpha}\right)$

Solution corresponding to the values obtained in **set (I)**

$\varpi = 3\,(\frac{m^2-1}{2})\Upsilon\,\nu\kappa^2 + (\frac{m^2-1}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{21,1}(x,t) = \left(\iota\sqrt{\frac{6m^4\,\Upsilon\,\nu + 2m^4\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{cn(\zeta)}{\sqrt{1-m^2} \mp dn(\zeta)}\right)\right)e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{90}$$

Solution corresponding to the values obtained in **set (II)**

$\varpi = 3\,(\frac{m^2-1}{2})\Upsilon\,\nu\kappa^2 + (\frac{m^2-1}{2})a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi_{21,2}(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{\sqrt{1-m^2} \mp dn(\zeta)}{cn(\zeta)}\right)\right)e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{91}$$

Solution corresponding to the values obtained in **set (III)**

$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu,$

$$\Psi_{21,3}(x,t) = \left(\iota\sqrt{\frac{6m^4\,\Upsilon\,\nu + 2m^4\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{cn(\zeta)}{\sqrt{1-m^2} \mp dn(\zeta)}\right) + \iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa\left(\frac{\sqrt{1-m^2} \mp dn(\zeta)}{cn(\zeta)}\right)\right)e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{92}$$

Note: By using the limiting value of m, we can obtain both solitonic and trigonometric solutions. Listed below are a few such solutions.

Bright soliton solution
$\varpi = 3\,\Upsilon\,\nu\kappa^2 + a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi(x,t) = \left(-\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa sech(\zeta)\right)e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{93}$$

Dark soliton solution

$$\varpi = 3\,\tau\Upsilon\,\nu\kappa^2 + \tau a\kappa^2 - \Upsilon\,\nu^3 - 3\nu\Lambda\vartheta_1\kappa^2\vartheta_2 - a\nu^2 + 3b\vartheta_1\kappa^2\vartheta_2 - \Omega\,\nu, \text{where } \vartheta_1 = \iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}, \text{ and } \vartheta_2 = \iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}. \tag{94}$$

$$\Psi(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa tanh(\zeta) + \iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa\left(\frac{1}{tanh(\zeta)}\right)\right)e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{95}$$

Singular soliton solution
$\varpi = 3\,\Upsilon\,\nu\kappa^2 + a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{b-\nu\Lambda}}\kappa csch(\zeta)\right)e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{96}$$

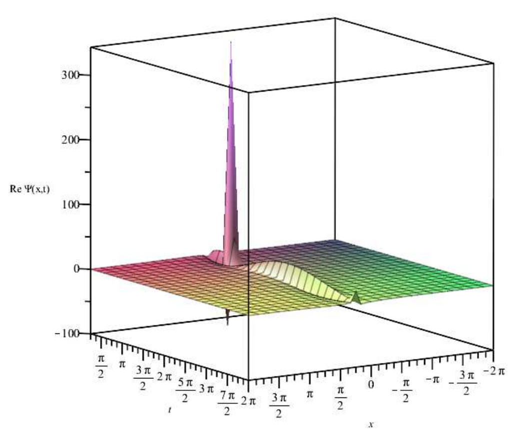
Combined singular soliton solution
$\varpi = \frac{-3}{2}\Upsilon\,\nu\kappa^2 + \frac{-1}{2}a\kappa^2 - \Upsilon\,\nu^3 - a\nu^2 - \Omega\,\nu$

$$\Psi(x,t) = \left(\iota\sqrt{\frac{6\,\Upsilon\,\nu + 2\,a}{4(b-\nu\Lambda)}}\kappa(coth(\zeta) \mp csch(\zeta))\right)e^{\iota(-\nu x + \varpi\frac{t^\alpha}{\alpha} + \theta)}, \tag{97}$$

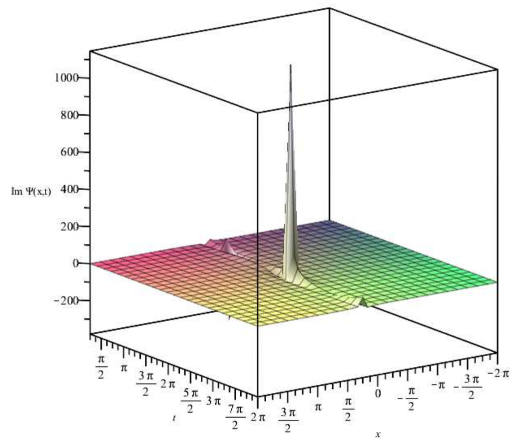
5 Analysis of the exact solution

In this section we summaries the analysis of the exact solution obtained in the sec. 4. The graphical representation of these solutions has been done by choosing the suitable values of various parameters to depict the spatio temporal extension of the derived wave solutions.

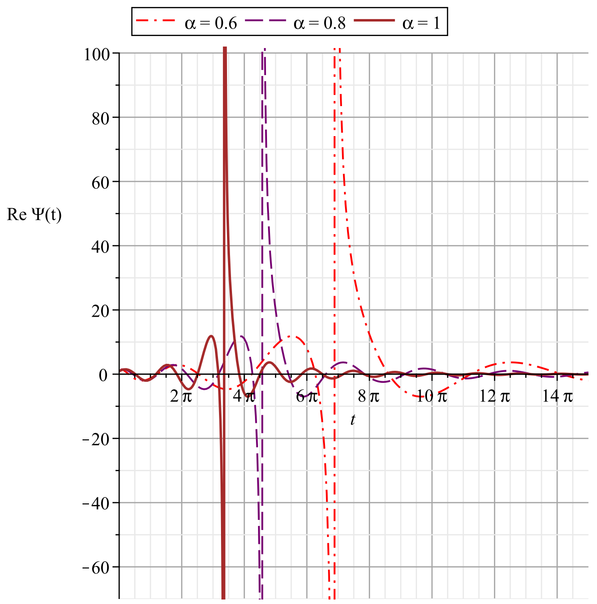
3D plots of the solution represented by (96) are shown in the Fig. 1a, 1b show that the solution exhibits lump solitons for these parametric values. It is evident from Fig. 1c, 1d that the width of the wave decreases as the value of α increases and the wave decays towards 0 rapidly for higher values of α as compared to lower values. This shows the dependence of the behaviour of the wave on the value of the fractional derivative i.e α.



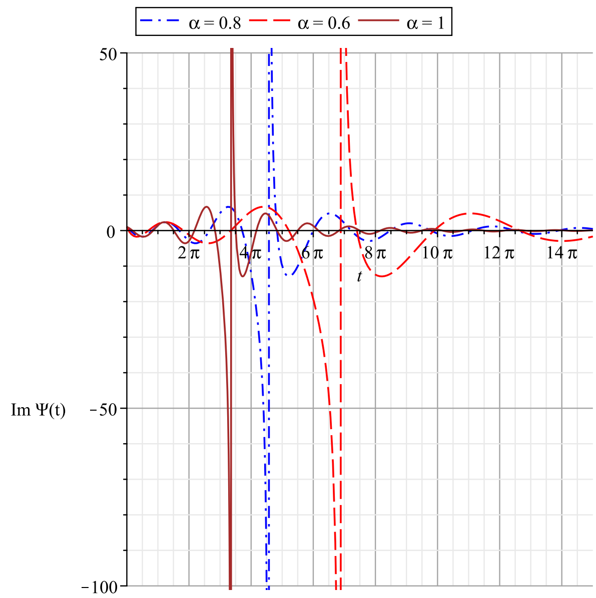
(a)



(b)



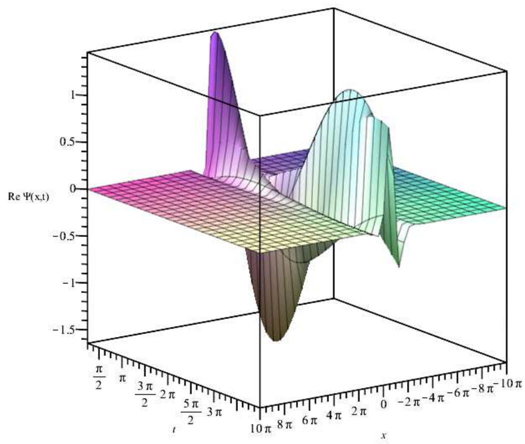
(c) 2D plot of 1a with x=1 to show that the behaviour of the solution changes with the values of fractional derivative α.



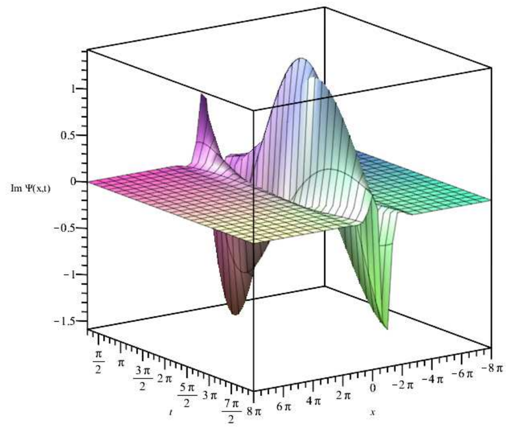
(d) 2D plot of 1b with x=1 to show that the behaviour of the solution changes with the values of fractional derivative α.

Fig. 1: Plots of (96) with the parametric values $\Upsilon = 0.75$, $a = 0.65$, $b = 0.55$, $m = 1$, $\alpha = 0.65$, $\Lambda = 0.45$, $\sigma = 0.3$, $\Omega = 0.5$, $\theta = 0$, $\kappa = 1$.

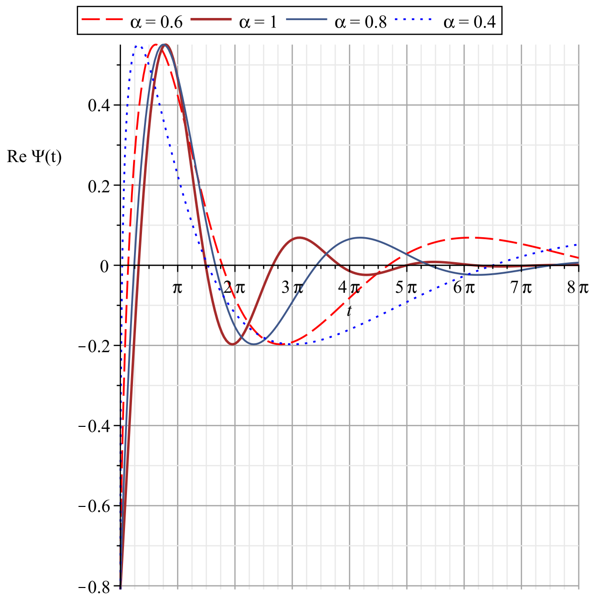
3D plots of $\Psi(x, t)$ given in (93) are presented in the Fig. 2a, 2b. Fig. 2c, 2d shows the change in curvature and the rate of decay of the wave solution which increases as the value of α increases. 3D plots of the solution $\Psi(x, t)$ in (57) are presented in the Fig. 3a, 3b show that two parallel breather type waves travel parallel in case of real part of the solution and in case of the imaginary part exhibits a single breather type wave.



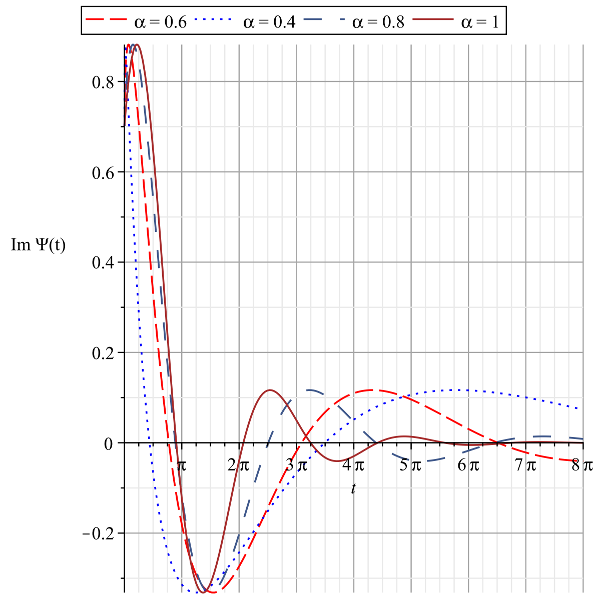
(a)



(b)

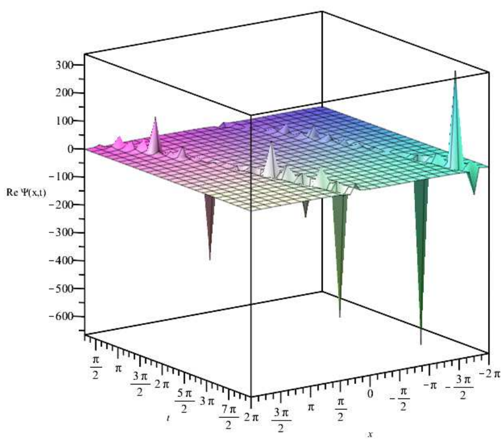


(c) 2D plot of 2a with x fixed at 1

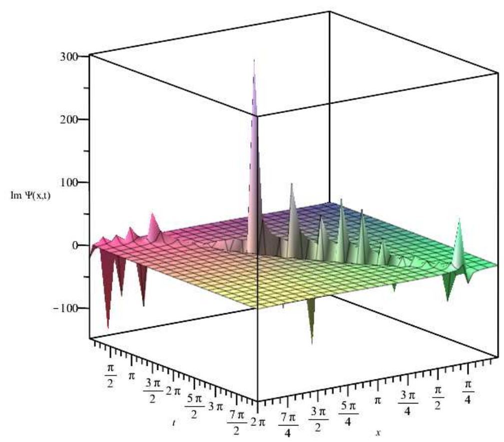


(d) 2D plot of 2b with x fixed at 1

Fig. 2: Plots of (93) with the parametric values $\varUpsilon = 0.75$, $a = 0.65$, $b = 0.55$, $\alpha = 0.8$, $\varLambda = 0.45$, $\sigma = 0.3$, $\varOmega = 0.5$, $\theta = 0$, $\kappa = 1$.



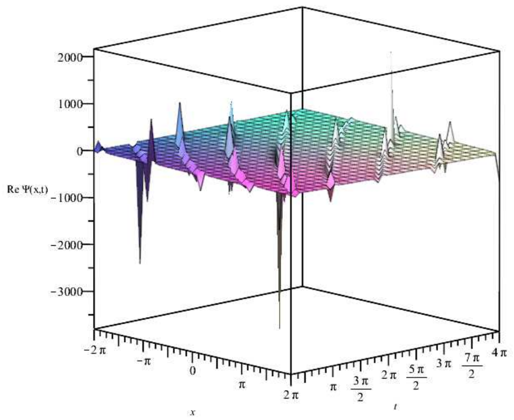
(a)



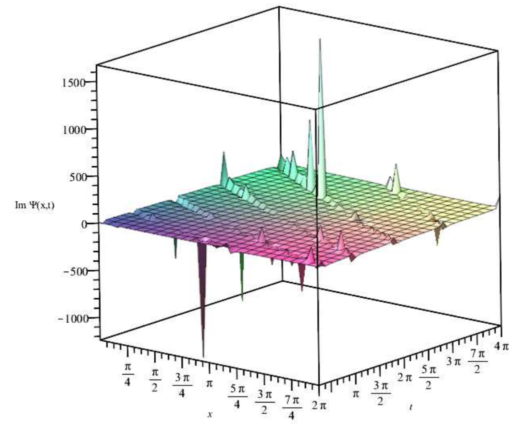
(b)

Fig. 3: Plots of (57) with the parametric values $\varUpsilon = 0.75$, $a = 0.65$, $b = 0.55$, $m = 0.2$, $\alpha = 0.65$, $\varLambda = 0.45$, $\sigma = 0.3$, $\varOmega = 0.5$, $\theta = 0$, $\kappa = 1$.

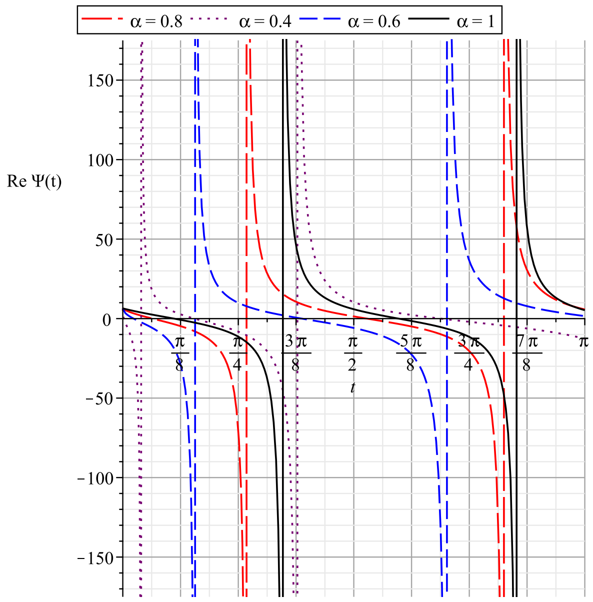
3D plots for the real and imaginary parts of the solution $\Psi(x,t)$ in (80) are shown in Fig.4a, 4b show that the solution exhibits periodic lump-type breather soliton for these parametric values. As the value of parameter α increases, an increase in amplitude of the wave function and associated phase shifts in observed in Fig. 4c, 4d.



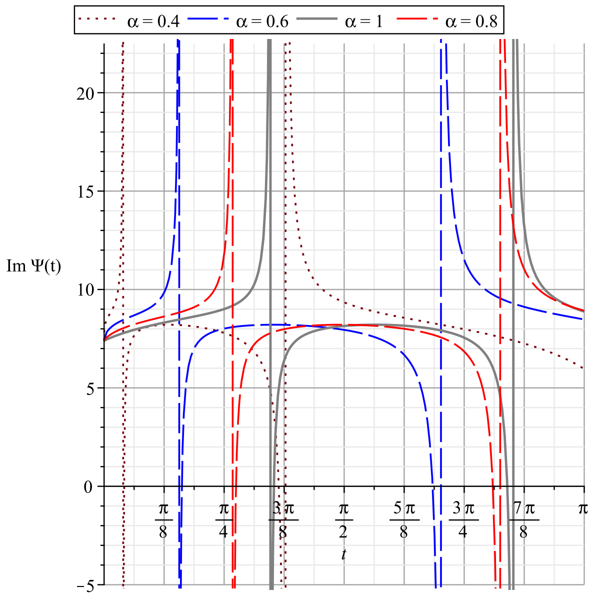
(a)



(b)



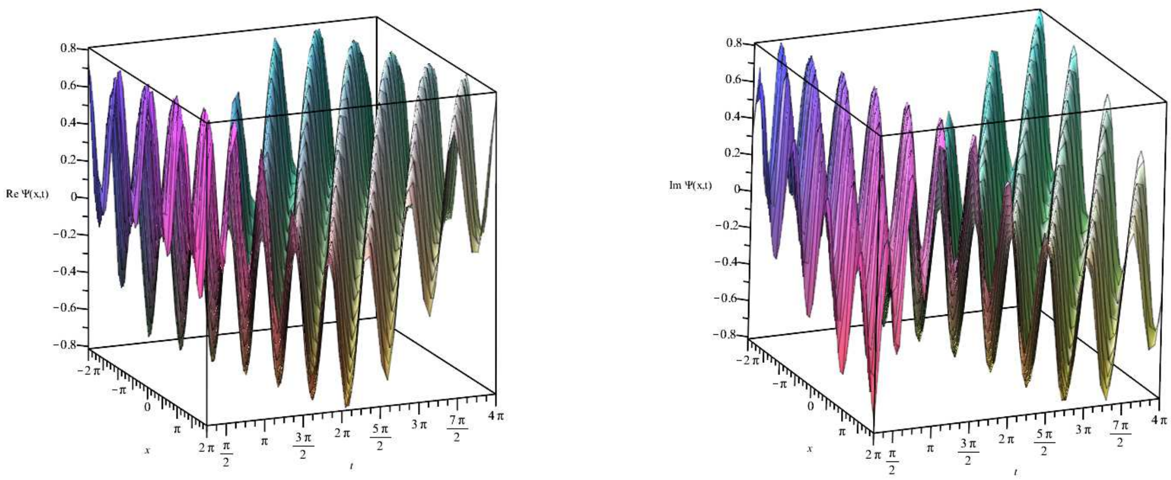
(c) 2D plot of 4a with x fixed at 1



(d) 2D plot of 4b with x fixed at 1

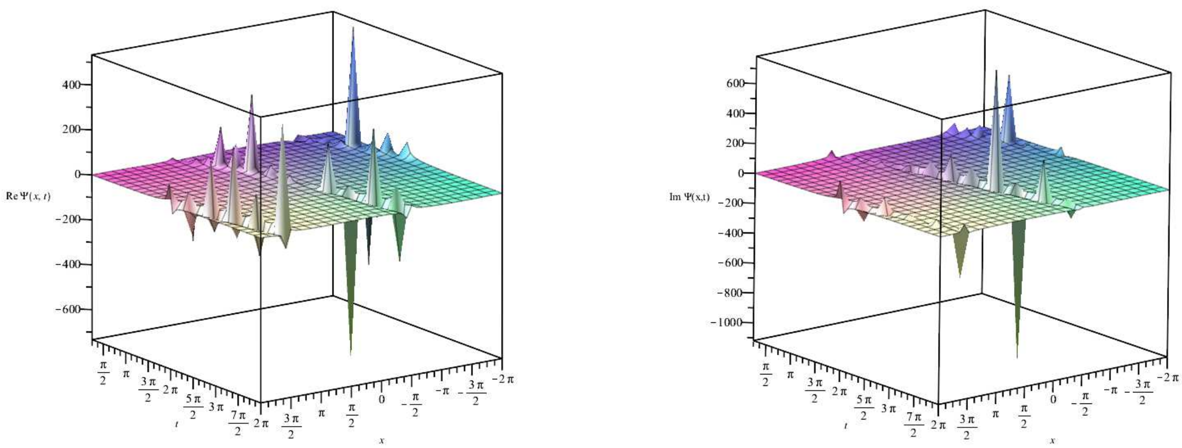
Fig. 4: Plots of (80) with the parametric values $\Upsilon = 0.75$, $a = 0.65$, $b = 0.55$, $\alpha = 0.8$, $m = 0.65$, $\Lambda = 0.45$, $\sigma = 0.3$, $\Omega = 0.5$, $\theta = 0$, $\kappa = 1$.

3D plots shown in Fig.5 is the representation of $\Psi(x,t)$ given in (18) and the graph shows that in this case the solution is periodic in nature and 3D plots of (87) are given in Fig.6.



(a) (b)

Fig. 5: Plots of (18) with the parametric values $\varUpsilon = 0.75$, $a = 0.65$, $b = 0.55$, $m = 0.2$, $\alpha = 0.8$, $\varLambda = 0.45$, $\sigma = 0.3$, $\varOmega = 0.5$, $\theta = 0$, $\kappa = 1$.



(a) (b)

Fig. 6: Plots of (87) with the parametric values $\varUpsilon = 0.75$, $a = 0.65$, $b = 0.55$, $m = 0.65$, $\alpha = 0.8$, $\varLambda = 0.45$, $\sigma = 0.3$, $\varOmega = 0.5$, $\theta = 0$, $\kappa = 1$.

A similar analysis of all the other solutions obtained in sec. 4 can also be done.

6 Conclusion

Wave solutions with rich physical structures of complex nonlinear Radhakrishnan-Kundu-Lakshmanan equation with conformable fractional time derivative have been obtained by the authors by utilizing to the GJEF method. The solutions obtained in this study can have potential applications in the explanation of physical interpretation of the studied nonlinear model in the field of nonlinear optics and fluid dynamics. By restricting the modulus of Jacobi elliptic functions, we obtain solitary wave solutions, bright-dark wave optical solitons, singular solitons, etc. The dependence of the behaviour of the solutions on the fractional derivative has also been analyzed in the present study. From the computations in this study, it becomes apparent that the GJEF method provides us with a powerful tool that can be used in the construction of optical soliton solutions of conformable fractional nonlinear models found in mathematical science.

Declarations

Acknowledgements This research has received funding from the Council of Scientific and Industrial Research, New Delhi, India (CSIR file no. 09/1152(0017)/2019 -EMR-1). The support of the CSIR is greatly acknowledged.
Conflicts of interest The authors declare that they have no conflict of interest concerning the publication of this manuscript.
Author Contributions All the authors have contributed equally in the preparation of this manuscript.
Data Availability Not Applicable.

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