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A new parameter estimation method for the extended power Lindley distribution based on order statistics, with application

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Abstract

In this paper, we propose inference procedures for the estimation of parameters by using order statistics. First, we derive some new expressions for single and product moments of the order statistics from the extended power Lindley distribution. We then use these moments to obtain the best linear unbiased estimates (BLUEs) of the location and scale parameters based on Type-II right-censored samples. A real data set is analysed to illustrate the flexibility and importance of the model.
Key words: extended power Lindley distribution, order statistics, moments, best linear unbiased estimator.

1. Introduction

The Lindley distribution was proposed by Lindley (1958) in the context of fiducial and Bayesian statistics, which can be seen as a mixture of $exp(\xi)$ and $gamma(2,\xi)$ distributions. Later, Ghitany et al. (2008) studied the statistical and mathematical properties of this distribution and also showed that this distribution performs better than the well-known exponential distribution in many ways. The Lindley distribution has only one scale parameter and is capable of modelling data with monotonic increasing failure rates and due to this, the Lindley distribution does not provide enough flexibility for analysing different types of lifetime data. To increase the flexibility for modeling purposes it will be useful to consider further alternatives of this distribution. Also, Bouchahed and Zeghdoudi (2018) and Zeghdoudi et al. (2018) generalized the Lindley distribution. Recently, the three parameter extended power Lindley distribution was proposed by Alkarni (2015) for the flexibility of purpose. The extended power Lindley (EPL) distribution is specified by the following probability density function (pdf):

$$f(x;\tau,\xi,\delta)=\frac{\tau\xi^2}{\xi+\delta}(1+\delta x^\tau)\,x^{\tau-1}\,e^{-\xi x^\tau},\quad x>0;\quad \tau>0,\xi>0,\delta>0 \tag{1}$$

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and the cumulative distribution function (cdf) is

$$F(x;\tau,\xi,\delta) = 1 - \left(1 + \frac{\delta\xi}{\xi+\delta}x^{\tau}\right) e^{-\xi x^{\tau}}, \quad x > 0; \quad \tau > 0, \xi > 0, \delta > 0. \tag{2}$$

For $\delta = 1$ and $\delta = 1,\ \ \tau = 1$, the EPL distribution reduces to the power Lindley (PL) and Lindley distributions respectively.

Let us consider the pdf of the scale-parameter EPL distribution as

$$f(x;\tau,\delta,\xi,\sigma) = \frac{\tau\xi^2}{\sigma(\xi+\delta)} \left[1 + \delta\left(\frac{x}{\sigma}\right)^{\tau}\right] \left(\frac{x}{\sigma}\right)^{\tau-1} e^{-\xi\left(\frac{x}{\sigma}\right)^{\tau}}, \ x > 0; \ \tau > 0, \delta > 0, \xi > 0, \sigma > 0, \tag{3}$$

and the pdf of the location-scale of the EPL distribution is

$$\begin{aligned} f(x;\tau,\delta,\xi,\sigma) &= \frac{\tau\xi^2}{\sigma(\xi+\delta)} \left[1 + \delta\left(\frac{x-\mu}{\sigma}\right)^{\tau}\right] \left(\frac{x-\mu}{\sigma}\right)^{\tau-1} \\ &\times \ e^{-\xi\left(\frac{x-\mu}{\sigma}\right)^{\tau}}, \ x > \mu; \ \tau > 0, \delta > 0, \xi > 0, \sigma > 0, \mu > 0, \end{aligned} \tag{4}$$

Order statistics play an important role in a wide range of theoretical and practical problems such as estimation of the problems, reliability analysis, quality control and strength of materials, for example, Balakrishnan and Cohan (1991), Sanmel and Thomas (1997), Balakrishnan and Chandramouleeswaran, (1996), Sultan et al. (2000), Sultan and Balakrishnan (2000), Mahmoud et al. (2005), Jabeen et al. (2013), Sultan and AL-Thubyani (2016), Kumar et al. (2017), Kumar and Dey (2017a, 2017b), Ahsanullah and Alzaatreh (2018), Kumar and Goyal (2019a, 2019b), Kumar et al. (2020a, 2020b) and the references cited therein.

The rest of the paper is organized as follows. In Section 2, we introduce some lemmas on the EPL distribution and we discuss some expressions for single and double moments of order statistics. We use these moments to obtain the BLUEs for μ and σ in Section 3. In Section 4, we performed a numerical study using R software and we see that these expressions provide precise numerical evaluations. A real data application is presented in Section 5. Finally, we conclude the paper in Section 6.

2. Moments of Order Statistics

In this section, we provide and prove some lemmas and some new expressions for single and product moments of order statistics for the given a random sample $X_1, X_2, \ldots, X_n$ from the EPL distribution. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ denote the order statistics corresponding to $X_1, X_2, \ldots, X_n$. The pdf of the rth order statistic is given by Arnold et al. (2003) and David and Nagaraja (2003) as

$$f_{X_{(r:n)}}(x) = C_{r:n}\, F^{r-1}(x)\, [1 - F(x)]^{n-r} f(x) \tag{5}$$

for $x > 0$. The joint pdf of the rth and sth order statistics is given by Arnold et al. (2003) and David and Nagaraja (2003) as

$$f_{X_{(r:n)},X_{(s:n)}}(x,y) \quad = \quad C_{r,s:n}\, F^{r-1}(x)\,[F(y)-F(x)]^{s-1-r}\,[1-F(x)]^{n-s}\, f(x)f(y) \qquad (6)$$

for $0 < x < y$, where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$ and $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

2.1. Some Lemmas

Here, we provide and prove two lemmas.

Lemma 1 *Let $f(x)$ and $F(x)$ be given by (1) and (2), respectively. For $a > 0$, $b > 0$ and $p > 0$, let*

$$K(a,b,p) = \int_0^\infty x^p F^a(x)[1-F(x)]^b f(x)dx.$$

Then,

$$\begin{aligned} K(a,b,p) \quad &= \quad \frac{\xi^2}{(\xi+\delta)^{b+1}} \sum_{i_1=0}^{a}\sum_{i_2=0}^{i_1+b}\sum_{i_3=0}^{i_2+1}(-1)^{i_1}\frac{\xi^{i_2}\delta^{b+i_1-i_2+i_3}}{(\xi+\delta)^{i_1}}\binom{a}{i_1}\binom{i_1+b}{i_2}\binom{i_2+1}{i_3} \\ &\times \quad \frac{\Gamma\left(\frac{p+\tau(i_3+1)}{\tau}\right)}{[\xi(i_1+b+1)]^{\frac{p+\tau(i_3+1)}{\tau}}}. \end{aligned}$$

Proof. By using binomial expansion in (5) and then from (1) and (2), we get

$$\begin{aligned} K(a,b,p) \quad &= \quad \sum_{i_1=0}^{a}(-1)^{i_1}\binom{a}{i_1}\int_0^\infty x^p[1-F(x)]^{i_1+b}f(x)dx. \\ &= \quad \frac{\tau\xi^2}{(\xi+\delta)}\sum_{i_1=0}^{a}(-1)^{i_1}\binom{a}{i_1}\int_0^\infty x^{p+\tau-1}(1+\delta x^\tau)e^{-\xi(i_1+b+1)x^\tau} \\ &\times \quad \left[\left(1+\frac{\delta\xi}{\xi+\delta}x^\tau\right)\right]^{i_1+b}dx \\ &= \quad \frac{\tau\xi^2}{(\xi+\delta)^{b+1}} \sum_{i_1=0}^{a}\sum_{i_2=0}^{i_1+b}\sum_{i_3=0}^{i_2+1}(-1)^{i_1}\frac{\xi^{i_2}\delta^{b+i_1-i_2+i_3}}{(\xi+\delta)^{i_1}}\binom{a}{i_1}\binom{i_1+b}{i_2}\binom{i_2+1}{i_3} \\ &\times \quad \int_0^\infty x^{p+\tau(i_3+1)-1}\, e^{-\xi(i_1+b+1)x^\tau}dx \\ &= \quad \frac{\xi^2}{(\xi+\delta)^{b+1}} \sum_{i_1=0}^{a}\sum_{i_2=0}^{i_1+b}\sum_{i_3=0}^{i_2+1}(-1)^{i_1}\frac{\xi^{i_2}\delta^{b+i_1-i_2+i_3}}{(\xi+\delta)^{i_1}}\binom{a}{i_1}\binom{i_1+b}{i_2}\binom{i_2+1}{i_3} \\ &\times \quad \frac{1}{[\xi(i_1+b+1)]^{\frac{p+\tau(i_3+1)}{\tau}}}\int_0^\infty z^{\frac{p+\tau i_3}{\tau}}\, e^{-z}dz, \end{aligned}$$

where $z=\xi(i_1+b+1)x^{\tau}$.

Lemma 2 *Let $f(x)$ and $F(x)$ be given by (1) and (2), respectively. For $a>0$, $b>0$, c, $p>0$ and $q>0$, let*

$$L(a,b,c,p,q) = \int_0^{\infty}\int_x^{\infty} x^p y^q [F(x)]^a [F(y)-F(x)]^b [1-F(y)]^c f(x)f(y)dydx.$$

Then,

$$\begin{aligned}
L(a,b,c,p,q) &= \frac{\xi^4}{(\xi+\delta)^{b+c+2}} \sum_{i_1=0}^{a}\sum_{i_2=0}^{b}\sum_{i_3=0}^{i_1+b-i_2}\sum_{i_4=0}^{i_2+c}\sum_{i_5=0}^{i_3+1}\sum_{i_6=0}^{i_4+1}\sum_{i_7=0}^{\frac{q+\tau i_6}{\tau}} (-1)^{i_1+i_2}\binom{a}{i_1}\binom{b}{i_2} \\
&\times \binom{i_1+b-i_2}{i_3}\binom{i_2+c}{i_4}\binom{i_3+1}{i_5}\binom{i_4+1}{i_6}\frac{\xi^{i_3+i_4}\delta^{i_1+b-i_3-i_4+i_5+i_6}}{(\xi+\delta)^{i_1}} \\
&\times \frac{\left(\frac{q+\tau i_6}{\tau}\right)!}{[\xi(i_2+c+1)]^{\frac{q+\tau(i_6+1)}{\tau}}}\frac{[\xi(i_2+c+1)]^{i_7}}{i_7!}\frac{\Gamma\left(\frac{p+\tau(i_5+i_7+1)}{\tau}\right)}{[\xi(i_1+b+c+2)]^{\frac{p+\tau(i_5+i_7+1)}{\tau}}}.
\end{aligned}$$

Proof. By using binomial expansion in (6) and then from (1) and (2), we get

$$\begin{aligned}
L(a,b,c,p,q) &= \sum_{i_1=0}^{a}\sum_{i_2=0}^{b}(-1)^{i_1+i_2}\binom{a}{i_1}\binom{b}{i_2} \\
&\times \int_0^{\infty}\int_x^{\infty} x^p y^q [1-F(x)]^{i_1+b-i_2}[1-F(y)]^{i_2+c} f(x)f(y)dydx \\
&= \frac{\tau^2\xi^4}{(\xi+\delta)^2}\sum_{i_1=0}^{a}\sum_{i_2=0}^{b}(-1)^{i_1+i_2}\binom{a}{i_1}\binom{b}{i_2}\int_0^{\infty}\int_x^{\infty} x^{p+\tau-1}y^{q+\tau-1} \\
&\times (1+\delta x^{\tau})(1+\delta y^{\tau})e^{-\xi(i_1+b-i_2+1)x^{\tau}}e^{-\xi(i_2+c+1)y^{\tau}} \\
&\times \left[\left(1+\frac{\xi\delta}{\xi+\delta}x^{\tau}\right)\right]^{i_1+b-i_2}\left[\left(1+\frac{\xi\delta}{\xi+\delta}y^{\tau}\right)\right]^{i_2+c} dxdy \\
&= \frac{\tau^2\xi^4}{(\xi+\delta)^{b+c+2}}\sum_{i_1=0}^{a}\sum_{i_2=0}^{b}\sum_{i_3=0}^{i_1+b-i_2}\sum_{i_4=0}^{i_2+c}\sum_{i_5=0}^{i_3+1}\sum_{i_6=0}^{i_4+1}(-1)^{i_1+i_2} \\
&\times \frac{\xi^{i_3+i_4}\delta^{i_1+b-i_3-i_4+i_5+i_6}}{(\xi+\delta)^{i_1}}\binom{a}{i_1}\binom{b}{i_2}\binom{i_1+b-i_2}{i_3}\binom{i_2+c}{i_4}\binom{i_3+1}{i_5} \\
&\times \binom{i_4+1}{i_6}\frac{1}{[\xi(i_2+c+1)]^{\frac{q+\tau(i_6+1)}{\tau}}}\int_0^{\infty} x^{p+\tau(i_5+1)-1}e^{-\xi(i_1+b-i_2+1)x^{\tau}} \\
&\times \Gamma\left(\frac{q+\tau(i_6+1)}{\tau},\ \xi(i_2+c+1)x^{\tau}\right)dx,
\end{aligned}$$

By using the relation

$$\Gamma(p,\ y) = (p-1)!\, e^{-y}\sum_{l=0}^{p-1}\frac{y^l}{l!}.$$

$$
\begin{aligned}
L(a,b,c,p,q) &= \frac{\tau^2 \xi^4}{(\xi+\delta)^{b+c+2}} \sum_{i_1=0}^{a} \sum_{i_2=0}^{b} \sum_{i_3=0}^{i_1+b-i_2} \sum_{i_4=0}^{i_2+c} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} \sum_{i_7=0}^{\frac{q+\tau i_6}{\tau}} (-1)^{i_1+i_2} \binom{a}{i_1} \binom{b}{i_2} \\
&\times \frac{\xi^{i_3+i_4} \delta^{i_1+b-i_3-i_4+i_5+6}}{(\xi+\delta)^{i_1}} \binom{i_1+b-i_2}{i_3} \binom{i_2+c}{i_4} \binom{i_3+1}{i_5} \binom{i_4+1}{i_6} \\
&\times \frac{\left(\frac{q+\tau i_6}{\tau}\right)!}{[\xi(i_2+c+1)]^{\frac{q+\tau(i_6+1)}{\tau}}} \frac{[\xi(i_2+c+1)]^{i_7}}{i_7!} \\
&\times \int_0^\infty x^{p+\tau(i_5+i_7+1)-1} e^{-\xi(i_1+b+c+2)x^\tau} dx \\
&= \frac{\xi^4}{(\xi+\delta)^{b+c+2}} \sum_{i_1=0}^{a} \sum_{i_2=0}^{b} \sum_{i_3=0}^{i_1+b-i_2} \sum_{i_4=0}^{i_2+c} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} \sum_{i_7=0}^{\frac{q+\tau i_6}{\tau}} (-1)^{i_1+i_2} \binom{a}{i_1} \binom{b}{i_2} \\
&\times \frac{\xi^{i_3+i_4} \delta^{i_1+b-i_3-i_4+i_5+i_6}}{(\xi+\delta)^{i_1}} \binom{i_1+b-i_2}{i_3} \binom{i_2+c}{i_4} \binom{i_3+1}{i_5} \binom{i_4+1}{i_6} \\
&\times \frac{\left(\frac{q+\tau i_6}{\tau}\right)!}{[\xi(i_2+c+1)]^{\frac{q+\tau(i_6+1)}{\tau}}} \frac{[\xi(i_2+c+1)]^{i_7}}{i_7!} \\
&\times \frac{1}{[\xi(i_1+b+c+2)]^{\frac{p+\tau(i_5+i_7+1)}{\tau}}} \int_0^\infty z^{\frac{p+\tau(i_5+i_7)}{\tau}} e^{-z} dz,
\end{aligned}
$$

where $z = \xi(i_1+b+c+2)x^\tau$.

2.2. Single moments

Here, we present expressions for the single moments of rth order statistics, $E\left(X_{r:n}^{(p)}\right) = \mu_{r:n}^{(p)}$ from the EPL distribution because these are very important to calculate the variance and draw the inferential techniques.

Theorem 1. *For the EPL distribution given in (1) and for,* $1 \leq r \leq n$

$$
\begin{aligned}
\mu_{r:n}^{(p)} &= \frac{\xi^2 C_{r:n}}{(\xi+\delta)^{n-r+1}} \sum_{i_1=0}^{r-1} \sum_{i_2=0}^{i_1+n-r} \sum_{i_3=0}^{i_2+1} (-1)^{i_1} \binom{r-1}{i_1} \binom{i_1+n-r}{i_2} \binom{i_2+1}{i_3} \\
&\times \frac{\xi^{i_2} \delta^{i_1+n-r-i_2+i_3}}{(\xi+\delta)^{i_1}} \frac{\Gamma\left(\frac{p+\tau(i_3+1)}{\tau}\right)}{[\xi(i_1+n-r+1)]^{\frac{p+\tau(i_3+1)}{\tau}}}.
\end{aligned} \tag{7}
$$

Proof. By using (5), we get

$$
\mu_{r:n}^{(p)} = C_{r:n} \int_0^\infty x^p \, F^{r-1}(x) \, [1-F(x)]^{n-r} f(x) dx. \tag{8}
$$

The result follows from applying lemma 1.

Special Cases

i For $\tau = 1$ and $\delta = 1$ in (7), we obtain relation for order statistic of the Lindley distribution

$$\mu_{r:n}^{(p)} = \frac{\xi^2\, C_{r:n}}{(\xi+1)^{n-r+1}} \sum_{i_1=0}^{r-1} \sum_{i_2=0}^{i_1+n-r} \sum_{i_3=0}^{i_2+1} (-1)^{i_1} \binom{r-1}{i_1} \binom{i_1+n-r}{i_2} \binom{i_2+1}{i_3} \times \frac{\xi^{i_2}}{(\xi+1)^{i_1}} \frac{\Gamma(p+i_3+1)}{[\xi(i_1+n-r+1)]^{p+i_3+1}},$$

as obtained by Sultan and AL-Thubyani (2016).

ii For $\delta = 1$ in (7), we obtain relation for order statistic of the power Lindley distribution

$$\mu_{r:n}^{(p)} = \frac{\xi^2\, C_{r:n}}{(\xi+1)^{n-r+1}} \sum_{i_1=0}^{r-1} \sum_{i_2=0}^{i_1+n-r} \sum_{i_3=0}^{i_2+1} (-1)^{i_1} \binom{r-1}{i_1} \binom{i_1+n-r}{i_2} \binom{i_2+1}{i_3} \times \frac{\xi^{i_2}}{(\xi+1)^{i_1}} \frac{\Gamma\frac{(p+\tau(i_3+1))}{\tau}}{[\xi(i_1+n-r+1)]^{\frac{p+\tau(i_3+1)}{\tau}}},$$

as obtained by Kumar and Goyal (2019a).

iii If $p = n = r = 1$ in (7), we unordered mean of the extended power Lindley distribution

$$\mu_{1:1}^{(1)} = \frac{\xi^2}{(\xi+\delta)} \sum_{i_3=0}^{1} \frac{\delta^{i_3}\, \Gamma(\frac{\tau(i_3+1)+1}{\tau})}{\xi^{\frac{\tau(i_3+1)+1}{\tau}}} = E(X),$$

as obtained by Alkarni (2015).

In particular, the first moment (mean) of the r-th order statistic is

$$\mu_{r:n} = \mu_{r:n}^{(1)} = \vartheta(\tau, \xi, \delta, r, n, 1),$$

where

$$\vartheta(\tau, \xi, \delta, r, n, k) = \frac{\xi^2\, C_{r:n}}{(\xi+\delta)^{n-r+1}} \sum_{i_1=0}^{r-1} \sum_{i_2=0}^{i_1+n-r} \sum_{i_3=0}^{i_2+1} (-1)^{i_1} \binom{r-1}{i_1} \binom{i_1+n-r}{i_2} \binom{i_2+1}{i_3} \times \frac{\xi^{i_2} \delta^{i_1+n-r-i_2+i_3}}{(\xi+\delta)^{i_1}} \frac{\Gamma(\frac{k+\tau(i_3+1)}{\tau})}{[\xi(i_1+n-r+1)]^{\frac{k+\tau(i_3+1)}{\tau}}}.$$

In addition, the variance of $X_{r:n}$ is found to be

$$\sigma_{r:n}^2 = \mu_{r:n}^{(2)} - [\mu_{r:n}^{(1)}]^2 = \vartheta(\tau, \xi, \delta, r, n, 2) - [\vartheta(\tau, \xi, \delta, r, n, 1)]^2.$$

2.3. Double moments

Here, we provide relation for the double moment of rth and sth order statistics, $E\left(X_{r,s:n}^{(p,q)}\right) = \mu_{r,s:n}^{(p,q)}$ from the EPL distribution, which are useful to calculate the variances, covariances, BLUEs and other inferential techniques.

Theorem 2. *For the EPL distribution given in (1) and for* $1 \leq r < s \leq n$

$$\begin{aligned}
\mu_{r,s:n}^{(p,q)} &= C_{r,s:n} \frac{\xi^4}{(\xi+\delta)^{n-r+1}} \sum_{i_1=0}^{r-1} \sum_{i_2=0}^{s-r-1} \sum_{i_3=0}^{i_1+s-r-i_2-1} \sum_{i_4=0}^{i_2+n-s} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} \sum_{i_7=0}^{\frac{q+\tau i_6}{\tau}} (-1)^{i_1+i_2} \\
&\times \frac{\xi^{i_3+i_4} \delta^{i_1+s-r-1-i_3-i_4+i_5+i_6}}{(\xi+\delta)^{i_1}} \binom{r-1}{i_1} \binom{s-r-1}{i_2} \binom{i_1+s-r-i_2-1}{i_3} \\
&\times \binom{i_2+n-s}{i_4} \binom{i_3+1}{i_5} \binom{i_4+1}{i_6} \frac{\left(\frac{q+\tau i_6}{\tau}\right)!}{[\xi(i_2+n-s+1)]^{\frac{q+\tau(i_6+1)}{\tau}}} \\
&\times \frac{[\xi(i_2+n-s+1)]^{i_7}}{i_7!} \frac{\Gamma\left(\frac{p+\tau(i_5+i_7+1)}{\tau}\right)}{[\xi(i_1+n-r+1)]^{\frac{p+\tau(i_5+i_7+1)}{\tau}}}.
\end{aligned}$$

Proof. By using (6), we obtain

$$\mu_{r,s:n}^{(p,q)} = C_{r,s:n} \int_0^\infty \int_x^\infty x^p\, y^q\, F^{r-1}(x) \left[F(y) - F(x)\right]^{s-1-r} \left[1 - F(y)\right]^{n-s} f(x) f(y) dx dy \qquad (9)$$

The result follows from applying lemma 2.

Special Cases

i For $\tau = 1$ and $\delta = 1$ in Theorem 2, we obtain the relation for order statistic of the Lindley distribution

$$\begin{aligned}
\mu_{r,s:n}^{(p,q)} &= \frac{\xi^4 C_{r,s:n}}{(\xi+1)^{n-r+1}} \sum_{i_1=0}^{r-1} \sum_{i_2=0}^{s-r-1} \sum_{i_3=0}^{i_1+s-r-i_2-1} \sum_{i_4=0}^{i_2+n-s} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} \sum_{i_7=0}^{q+i_6} (-1)^{i_1+i_2} \frac{\xi^{i_3+i_4}}{(\xi+1)^{i_1}} \\
&\times \binom{r-1}{i_1} \binom{s-r-1}{i_2} \binom{i_1+s-r-i_2-1}{i_3} \binom{i_2+n-s}{i_4} \binom{i_3+1}{i_5} \binom{i_4+1}{i_6} \\
&\times \frac{(q+i_6)!}{[\xi(i_2+n-s+1)]^{q+i_6+1}} \frac{[\xi(i_2+n-s+1)]^{i_7}}{i_7!} \frac{\Gamma(p+i_5+i_7+1)}{[\xi(i_1+n-r+1)]^{p+i_5+i_7+1}}.
\end{aligned}$$

as obtained by Sultan and AL-Thubyani (2016).

ii For $\delta = 1$ in Theorem 2, we get the explicit expression for order statistic of the power Lindley distribution

$$\begin{aligned}
\mu_{r,s:n}^{(p,q)} &= \frac{\xi^4 C_{r,s:n}}{(\xi+1)^{n-r+1}} \sum_{i_1=0}^{r-1} \sum_{i_2=0}^{s-r-1} \sum_{i_3=0}^{i_1+s-r-i_2-1} \sum_{i_4=0}^{i_2+n-s} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} \sum_{i_7=0}^{\frac{q+\tau i_6}{\tau}} (-1)^{i_1+i_2} \frac{\xi^{i_3+i_4}}{(\xi+1)^{i_1}} \\
&\times \binom{r-1}{i_1} \binom{s-r-1}{i_2} \binom{i_1+s-r-i_2-1}{i_3} \binom{i_2+n-s}{i_4} \binom{i_3+1}{i_5} \binom{i_4+1}{i_6} \\
&\times \frac{\left(\frac{q+\tau i_6}{\tau}\right)!}{[\xi(i_2+n-s+1)]^{\frac{q+\tau(i_6+1)}{\tau}}} \frac{[\xi(i_2+n-s+1)]^{i_7}}{i_7!} \frac{\Gamma\left(\frac{p+\tau(i_5+i_7+1)}{\tau}\right)}{[\xi(i_1+n-r+1)]^{\frac{p+\tau(i_5+i_7+1)}{\tau}}}.
\end{aligned}$$

as obtained by Kumar and Goyal (2019a).

For simplicity, we denote the (1, 1)-th moment of $X_{r:n}$ and $X_{s:n}$, which are also called the simple product moment of these order statistics, by $\mu_{r,s:n}$. The simple product moments are used for evaluating the covariances, in other words

$$\sigma_{r,s:n} = Cov(X_{r:n}, X_{s:n}) = \mu_{r,s:n} - \mu_{r:n}\mu_{s:n}.$$

3. Location and scale parameter estimation based on Type-II right-censored sample

In this section we have obtained the BLUEs of the location and scale parameters to approximate population parameters. Let $Y_{1:n} \leq Y_{2:n} \leq \cdots \leq Y_{n-c:n}$, $c = 0, 1, \ldots, n-1$ denote Type-II right-censored sample from the location-scale parameter extended power Lindley distribution in Equation (1). Let us denote $Z_{r:n} = (Y_{r:n} - \mu)/\sigma$, $E(Z_{r:n}) = \mu_{r;n}^{(1)}$, $1 \leq r \leq (n-c)$, and $Cov(Z_{r:n}, Z_{s:n}) = \sigma_{r,s:n} = \mu_{r,s:n}^{(1,1)} - \mu_{r:n}^{(1)} \mu_{s:n}^{(1)}$, $1 \leq r < s \leq (n-c)$. We shall use the following notations:

$$\mathbf{Y} = (Y_{1:n}, Y_{2:n}, \ldots, Y_{(n-c):n})^T,$$

$$\mu = (\mu_{1:n}, \mu_{2:n}, \ldots, \mu_{(n-c):n})^T,$$

$$\mathbf{1} = \underbrace{(1,\ 1,\ \ldots,\ 1)^T}_{n-c},$$

and

$$\sum = ((\sigma_{r,s})),\ \ 1 \leq r,\ \ s \leq n-c.$$

The BLUEs of μ and σ are given by Arnold *et al*. (2003)

$$\mu^* = \left\{ \frac{\boldsymbol{\mu}^{\mathbf{T}} \Sigma^{-1} \boldsymbol{\mu} \mathbf{1}^{\mathbf{T}} \Sigma^{-1} - \boldsymbol{\mu}^{\mathbf{T}} \Sigma^{-1} \mathbf{1} \boldsymbol{\mu}^{\mathbf{T}} \Sigma^{-1}}{(\boldsymbol{\mu}^{\mathbf{T}} \Sigma^{-1} \boldsymbol{\mu})(\mathbf{1}^{\mathbf{T}} \Sigma^{-1} \mathbf{1}) - (\boldsymbol{\mu}^{\mathbf{T}} \Sigma^{-1} \mathbf{1})^{\mathbf{2}}} \right\} Y_{r:n} = \sum_{r=1}^{n-c} a_r Y_{r:n}, \tag{10}$$

$$\sigma^* = \left\{ \frac{\mathbf{1}^{\mathbf{T}} \Sigma^{-1} \mathbf{1} \boldsymbol{\mu}^{\mathbf{T}} \Sigma^{-1} - \mathbf{1}^{\mathbf{T}} \Sigma^{-1} \boldsymbol{\mu} \mathbf{1}^{\mathbf{T}} \Sigma^{-1}}{(\boldsymbol{\mu}^{\mathbf{T}} \Sigma^{-1} \boldsymbol{\mu})(\mathbf{1}^{\mathbf{T}} \Sigma^{-1} \mathbf{1}) - (\boldsymbol{\mu}^{\mathbf{T}} \Sigma^{-1} \mathbf{1})^{\mathbf{2}}} \right\} Y_{r:n} = \sum_{r=1}^{n-c} b_r Y_{r:n}. \tag{11}$$

Furthermore, the variances and covariance of these BLUEs are given by Arnold *et al*. (2003)

$$Var(\mu^*) = \sigma^2 \left\{ \frac{\boldsymbol{\mu}^{\mathbf{T}} \Sigma^{-1} \boldsymbol{\mu}}{(\boldsymbol{\mu}^{\mathbf{T}} \Sigma^{-1} \boldsymbol{\mu})(\mathbf{1}^{\mathbf{T}} \Sigma^{-1} \mathbf{1}) - (\boldsymbol{\mu}^{\mathbf{T}} \Sigma^{-1} \mathbf{1})^{\mathbf{2}}} \right\} \sigma^2 V_1, \tag{12}$$

$$Var(\sigma^*) = \sigma^2 \left\{ \frac{\mathbf{1}^{\mathbf{T}} \Sigma^{-1} \mathbf{1}}{(\boldsymbol{\mu}^{\mathbf{T}} \Sigma^{-1} \boldsymbol{\mu})(\mathbf{1}^{\mathbf{T}} \Sigma^{-1} \mathbf{1}) - (\boldsymbol{\mu}^{\mathbf{T}} \Sigma^{-1} \mathbf{1})^{\mathbf{2}}} \right\} \sigma^2 V_2, \tag{13}$$

and

$$Cov(\mu^*,\ \sigma^*) = \sigma^2 \left\{ \frac{-\boldsymbol{\mu}^{\mathbf{T}} \Sigma^{-1} \mathbf{1}}{(\boldsymbol{\mu}^{\mathbf{T}} \Sigma^{-1} \boldsymbol{\mu})(\mathbf{1}^{\mathbf{T}} \Sigma^{-1} \mathbf{1}) - (\boldsymbol{\mu}^{\mathbf{T}} \Sigma^{-1} \mathbf{1})^{\mathbf{2}}} \right\} \sigma^2 V_3. \tag{14}$$

4. Numerical Results

The relations obtained in the preceding sections allow us to evaluate the expected values, second moments, variances, product moment and covariances of order statistics from samples of sizes up to 10 for various values of the parameters. The relation in (7) can be used to compute the expected values, second order moments and variances of all order statistics for sample sizes $n = 1,2,3,\ldots,10$. In Tables 1, we have presented expected values, second order moments, variances, skewness and kurtosis of the rth order statistics from the EPL distribution for $n = 1,2,3,\ldots,10$ and $\tau = 2$, $\delta = 0.5$ and $\xi = 5$. One can see that the means and variances are decreasing both with respect to n and ξ. In Table 2 we have reported the double moments and covariances of the rth and sth order statistics from the EPL distribution for $n = 1,2,3,\ldots,10$ and $\tau = 2$, $\delta = 0.5$ and $\xi = 5,10$. From Table 3, one can observe that product moments are decreasing with respect to n.

The coefficients of the BLUEs for Type-II right-censored samples of sample sizes $n = 7,\ 10,\ \tau = 2,\ \delta = 0.5$ and $\xi = 5$ and different censoring cases $c = 0(1)([n/2]-1)$ are displayed in Tables 3 and 4. Also, the variances and covariances of the BLUEs are provided in Table 5.

Table 1: Moments, variances skewness and kurtosis of order statistic from EPL distribution $\tau = 2$, $\delta = 0.5$ and $\xi = 5$

r	n	$E(X)$	$E(X^2)$	$V(X)$	δ_1	δ_2	γ_1	γ_2
1	1	0.414348	0.218182	0.046498	0.379571	3.207079	0.616093	0.207079
	2	0.293422	0.109504	0.023408	0.387238	3.219388	0.622284	0.219388
	3	0.239709	0.073106	0.015646	0.389946	3.229529	0.624457	0.229529
	4	0.207652	0.054870	0.011751	0.391543	3.234472	0.625734	0.234472
	5	0.185762	0.043916	0.009408	0.392307	3.236857	0.626344	0.236857
	6	0.169597	0.036608	0.007845	0.394587	3.229235	0.628161	0.229235
	7	0.157029	0.031385	0.006727	0.395586	3.221588	0.628956	0.221588
	8	0.146897	0.027467	0.005888	0.395741	3.229525	0.629079	0.229525
	9	0.138503	0.024418	0.005235	0.394610	3.235330	0.628180	0.235330
	10	0.131400	0.021978	0.004712	0.396862	3.226521	0.629970	0.226521
2	2	0.535273	0.326860	0.040343	0.241593	3.215795	0.491521	0.215795
	3	0.400850	0.182301	0.021620	0.189444	3.115636	0.435251	0.115636
	4	0.335877	0.127814	0.015001	0.174957	3.088871	0.418279	0.088871
	5	0.295213	0.098686	0.011535	0.170039	3.070969	0.412358	0.070969
	6	0.266589	0.080456	0.009386	0.167111	3.066828	0.408792	0.066828
	7	0.245001	0.067944	0.007919	0.165394	3.065728	0.406687	0.065728
	8	0.227956	0.058815	0.006851	0.164916	3.061962	0.406099	0.061962
	9	0.214051	0.051856	0.006038	0.164358	3.069852	0.405411	0.069852
	10	0.202423	0.046373	0.005398	0.164208	3.048848	0.405226	0.048848
3	3	0.602485	0.399139	0.036151	0.225634	3.273408	0.475009	0.273408
	4	0.465823	0.236788	0.019797	0.145361	3.125628	0.381262	0.125628
	5	0.396873	0.171505	0.013997	0.123096	3.072745	0.350850	0.072745
	6	0.352462	0.135146	0.010917	0.112742	3.054180	0.335770	0.054180
	7	0.320559	0.111736	0.008978	0.108337	3.051939	0.329145	0.051939
	8	0.296135	0.095332	0.007636	0.105760	3.035578	0.325207	0.035578
	9	0.276626	0.083171	0.006649	0.105379	3.008423	0.324621	0.008423
	10	0.260564	0.073785	0.005891	0.104265	3.025598	0.322900	0.025598
4	4	0.648039	0.453256	0.033301	0.229320	3.321976	0.478874	0.321976
	5	0.511790	0.280311	0.018382	0.130464	3.147984	0.361198	0.147984
	6	0.441284	0.207865	0.013133	0.102281	3.090069	0.319815	0.090069
	7	0.394999	0.166359	0.010335	0.090404	3.056551	0.300673	0.056551
	8	0.361267	0.139076	0.008562	0.084264	3.040724	0.290282	0.040724
	9	0.335151	0.119652	0.007326	0.081505	3.010495	0.285491	0.010495
	10	0.314105	0.105073	0.006411	0.078213	3.023921	0.279667	0.023921
5	5	0.682101	0.496492	0.031230	0.237912	3.362265	0.487762	0.362265
	6	0.547043	0.316534	0.017278	0.124926	3.165975	0.353448	0.165975
	7	0.475998	0.238994	0.012420	0.092794	3.093716	0.304621	0.093716
	8	0.428730	0.193642	0.009833	0.078295	3.075215	0.279812	0.075215
	9	0.393912	0.163355	0.008188	0.071074	3.053238	0.266597	0.053238
	10	0.366721	0.141522	0.007038	0.067297	3.019226	0.259417	0.019226
6	6	0.709112	0.532483	0.029643	0.247927	3.393989	0.497922	0.393989
	7	0.575461	0.347549	0.016394	0.123722	3.179939	0.351742	0.179939
	8	0.504359	0.266206	0.011828	0.087950	3.098604	0.296564	0.098604
	9	0.456584	0.217871	0.009402	0.072183	3.067347	0.268669	0.067347
	10	0.421103	0.185187	0.007859	0.063867	3.048346	0.252719	0.048346
7	7	0.731387	0.563306	0.028379	0.257507	3.423281	0.507451	0.423281
	8	0.599162	0.374664	0.015668	0.123781	3.197735	0.351825	0.197735
	9	0.528246	0.290374	0.011330	0.081133	3.116019	0.284838	0.116019
	10	0.480238	0.239660	0.009031	0.067515	3.077528	0.259836	0.077528
8	8	0.750277	0.590255	0.027339	0.267491	3.444852	0.517195	0.444852
	9	0.619424	0.398746	0.015060	0.125692	3.209343	0.354531	0.209343
	10	0.548821	0.312108	0.010904	0.083717	3.116898	0.289339	0.116898
9	9	0.766633	0.614193	0.026467	0.276388	3.468726	0.525726	0.468726
	10	0.637074	0.420406	0.014543	0.127287	3.217234	0.356773	0.217234
10	10	0.781029	0.635725	0.025719	0.285348	3.482843	0.534180	0.482843

Table 2: Covariances of order statistics

			$\tau=2,\xi=5,\delta=0.5$				$\tau=2,\xi=10,\delta=0.5$			
s	r	n	$\mu_{r,s:n}$	$E(X_{r:n})$	$E(X_{s:n})$	$\sigma_{r,s:n}$	$\mu_{r,s:n}$	$E(X_{r:n})$	$\mu_{r,s:n}$	$\sigma_{r,s:n}$
2	1	2	0.125100	0.293422	0.535273	-0.03196	0.059192	0.202969	0.370875	-0.01608
		3	0.073220	0.239709	0.400850	-0.02287	0.034402	0.165748	0.277412	-0.01158
		4	0.052133	0.207652	0.335877	-0.01761	0.024407	0.143552	0.232333	-0.00895
		5	0.040552	0.185762	0.295213	-0.01429	0.018943	0.128403	0.204150	-0.00727
		6	0.033203	0.169597	0.266589	-0.01201	0.015488	0.117219	0.184323	-0.00612
		7	0.028119	0.157029	0.245001	-0.01035	0.013102	0.108526	0.169376	-0.00528
		8	0.024389	0.146897	0.227956	-0.00910	0.011355	0.101518	0.157579	-0.00464
		9	0.021534	0.138503	0.214051	-0.00811	0.010020	0.095714	0.147956	-0.00414
		10	0.019279	0.131400	0.202423	-0.00732	0.008966	0.090803	0.139911	-0.00374
3	1	3	0.146966	0.239709	0.602485	0.002545	0.066242	0.165748	0.417607	-0.00298
		4	0.100791	0.207652	0.465823	0.004062	0.044251	0.143552	0.322490	-0.00204
		5	0.079630	0.185762	0.396873	0.005906	0.034146	0.128403	0.274609	-0.00112
		6	0.067164	0.169597	0.352462	0.007387	0.028186	0.117219	0.243803	-0.00039
		7	0.058882	0.157029	0.320559	0.008545	0.024224	0.108526	0.221690	0.000165
		8	0.052962	0.146897	0.296135	0.009461	0.021392	0.101518	0.204769	0.000604
		9	0.048512	0.138503	0.276626	0.010199	0.019262	0.095714	0.191258	0.000956
		10	0.045043	0.131400	0.260564	0.010805	0.017601	0.090803	0.180137	0.001244
3	2	3	0.155112	0.400850	0.602485	-0.08639	0.076932	0.277412	0.417607	-0.03892
		4	0.087823	0.335877	0.465823	-0.06864	0.044543	0.232333	0.322490	-0.03038
		5	0.059381	0.295213	0.396873	-0.05778	0.031058	0.204150	0.274609	-0.02500
		6	0.043331	0.266589	0.352462	-0.05063	0.023522	0.184323	0.243803	-0.02142
		7	0.032948	0.245001	0.320559	-0.04559	0.018679	0.169376	0.221690	-0.01887
		8	0.025656	0.227956	0.296135	-0.04185	0.015295	0.157579	0.204769	-0.01697
		9	0.020245	0.214051	0.276626	-0.03897	0.012794	0.147956	0.191258	-0.01550
		10	0.016067	0.202423	0.260564	-0.03668	0.010868	0.139911	0.180137	-0.01434
4	1	4	0.157123	0.207652	0.648039	0.022556	0.067875	0.143552	0.449313	0.003375
		5	0.116281	0.185762	0.511790	0.021210	0.048633	0.128403	0.354410	0.003126
		6	0.096719	0.169597	0.441284	0.021879	0.039214	0.117219	0.305414	0.003413
		7	0.084955	0.157029	0.394999	0.022929	0.033425	0.108526	0.273287	0.003766
		8	0.077118	0.146897	0.361267	0.024049	0.029472	0.101518	0.249893	0.004104
		9	0.071578	0.138503	0.335151	0.025159	0.026598	0.095714	0.231790	0.004413
		10	0.067511	0.131400	0.314105	0.026238	0.024416	0.090803	0.217207	0.004693
	2	4	0.172829	0.335877	0.648039	-0.04483	0.084966	0.232333	0.449313	-0.01942
		5	0.112133	0.295213	0.511790	-0.03895	0.055696	0.204150	0.354410	-0.01666
		6	0.082851	0.266589	0.441284	-0.03479	0.041888	0.184323	0.305414	-0.01441
		7	0.064709	0.245001	0.394999	-0.03207	0.033554	0.169376	0.273287	-0.01273
		8	0.052012	0.227956	0.361267	-0.03034	0.027892	0.157579	0.249893	-0.01149
		9	0.042428	0.214051	0.335151	-0.02931	0.023756	0.147956	0.231790	-0.01054
		10	0.034802	0.202423	0.314105	-0.02878	0.020579	0.139911	0.217207	-0.00981
	3	4	0.179898	0.465823	0.648039	-0.12197	0.089110	0.322490	0.449313	-0.05579
		5	0.104111	0.396873	0.511790	-0.09900	0.052452	0.274609	0.354410	-0.04487
		6	0.071720	0.352462	0.441284	-0.08382	0.036947	0.243803	0.305414	-0.03751
		7	0.053410	0.320559	0.394999	-0.07321	0.028192	0.221690	0.273287	-0.03239
		8	0.041644	0.296135	0.361267	-0.06534	0.022532	0.204769	0.249893	-0.02864
		9	0.033502	0.276626	0.335151	-0.05921	0.018569	0.191258	0.231790	-0.02576
		10	0.027592	0.260564	0.314105	-0.05425	0.015641	0.180137	0.217207	-0.02349
5	1	5	0.165311	0.185762	0.682101	0.038602	0.068527	0.128403	0.473039	0.007787
		6	0.128166	0.169597	0.547043	0.035389	0.051311	0.117219	0.378909	0.006896
		7	0.110052	0.157029	0.475998	0.035306	0.042581	0.108526	0.329508	0.006820
		8	0.099086	0.146897	0.428730	0.036107	0.037070	0.101518	0.296682	0.006952
		9	0.091832	0.138503	0.393912	0.037274	0.033237	0.095714	0.272522	0.007153
		10	0.086817	0.131400	0.366721	0.038630	0.030414	0.090803	0.253665	0.007380
	2	5	0.173400	0.295213	0.682101	-0.02796	0.085162	0.204150	0.473039	-0.01141
		6	0.119750	0.266589	0.547043	-0.02609	0.059439	0.184323	0.378909	-0.01040
		7	0.092013	0.245001	0.475998	-0.02461	0.046480	0.169376	0.329508	-0.00933
		8	0.073911	0.227956	0.428730	-0.02382	0.038296	0.157579	0.296682	-0.00846

Table 2: Continued.

			$\tau=2,\xi=5,\delta=0.5$				$\tau=2,\xi=10,\delta=0.5$			
s	r	n	$\mu_{r,s:n}$	$E(X_{r:n})$	$E(X_{s:n})$	$\sigma_{r,s:n}$	$\mu_{r,s:n}$	$E(X_{r:n})$	$\mu_{r,s:n}$	$\sigma_{r,s:n}$
		9	0.060674	0.214051	0.393912	-0.02364	0.032550	0.147956	0.272522	-0.00777
		10	0.050264	0.202423	0.366721	-0.02397	0.028244	0.139911	0.253665	-0.00725
	3	5	0.202606	0.396873	0.682101	-0.06810	0.099405	0.274609	0.473039	-0.03050
		6	0.133799	0.352462	0.547043	-0.05901	0.065762	0.243803	0.378909	-0.02662
		7	0.100903	0.320559	0.475998	-0.05168	0.049799	0.221690	0.329508	-0.02325
		8	0.080899	0.296135	0.428730	-0.04606	0.040135	0.204769	0.296682	-0.02062
		9	0.067311	0.276626	0.393912	-0.04166	0.033574	0.191258	0.272522	-0.01855
		10	0.057471	0.260564	0.366721	-0.03808	0.028798	0.180137	0.253665	-0.01690
	4	5	0.197590	0.511790	0.682101	-0.15150	0.097898	0.354410	0.473039	-0.06975
		6	0.115810	0.441284	0.547043	-0.12559	0.058351	0.305414	0.378909	-0.05737
		7	0.080302	0.394999	0.475998	-0.10772	0.041419	0.273287	0.329508	-0.04863
		8	0.059934	0.361267	0.428730	-0.09495	0.031757	0.249893	0.296682	-0.04238
		9	0.046659	0.335151	0.393912	-0.08536	0.025457	0.231790	0.272522	-0.03771
		10	0.037334	0.314105	0.366721	-0.07786	0.021014	0.217207	0.253665	-0.03408
6	1	6	0.173540	0.169597	0.709112	0.053277	0.069056	0.117219	0.491865	0.011400
		7	0.138937	0.157029	0.575461	0.048573	0.053325	0.108526	0.398669	0.010059
		8	0.122004	0.146897	0.504359	0.047915	0.045185	0.101518	0.349204	0.009735
		9	0.111775	0.138503	0.456584	0.048537	0.039958	0.095714	0.316010	0.009712
		10	0.105094	0.131400	0.421103	0.049761	0.036274	0.090803	0.291379	0.009816
	2	6	0.169539	0.266589	0.709112	-0.01950	0.083624	0.184323	0.491865	-0.00704
		7	0.121311	0.245001	0.575461	-0.01968	0.060716	0.169376	0.398669	-0.00681
		8	0.095139	0.227956	0.504359	-0.01983	0.048693	0.157579	0.349204	-0.00633
		9	0.077345	0.214051	0.456584	-0.02039	0.040853	0.147956	0.316010	-0.00590
		10	0.063859	0.202423	0.421103	-0.02138	0.035208	0.139911	0.291379	-0.00556
	3	6	0.206018	0.352462	0.709112	-0.04392	0.100332	0.243803	0.491865	-0.01959
		7	0.145146	0.320559	0.575461	-0.03932	0.070483	0.221690	0.398669	-0.01790
		8	0.114479	0.296135	0.504359	-0.03488	0.055437	0.204769	0.349204	-0.01607
		9	0.095232	0.276626	0.456584	-0.03107	0.045952	0.191258	0.316010	-0.01449
		10	0.081919	0.260564	0.421103	-0.02781	0.039327	0.180137	0.291379	-0.01316
	4	6	0.223267	0.441284	0.709112	-0.08965	0.110002	0.305414	0.491865	-0.04022
		7	0.148164	0.394999	0.575461	-0.07914	0.073256	0.273287	0.398669	-0.03569
		8	0.111855	0.361267	0.504359	-0.07035	0.055703	0.249893	0.349204	-0.03156
		9	0.089461	0.335151	0.456584	-0.06356	0.045006	0.231790	0.316010	-0.02824
		10	0.073972	0.314105	0.421103	-0.05830	0.037697	0.217207	0.291379	-0.02559
	5	6	0.211616	0.547043	0.709112	-0.17630	0.104759	0.378909	0.491865	-0.08161
		7	0.125475	0.475998	0.575461	-0.14844	0.063092	0.329508	0.398669	-0.06827
		8	0.087690	0.428730	0.504359	-0.12854	0.045094	0.296682	0.349204	-0.05851
		9	0.065827	0.393912	0.456584	-0.11403	0.034744	0.272522	0.316010	-0.05138
		10	0.051487	0.366721	0.421103	-0.10294	0.027951	0.253665	0.291379	-0.04596
7	1	7	0.182405	0.157029	0.731387	0.067556	0.069687	0.108526	0.507398	0.014621
		8	0.149550	0.146897	0.599162	0.061535	0.055063	0.101518	0.415157	0.012917
		9	0.133567	0.138503	0.528246	0.060403	0.047410	0.095714	0.365801	0.012398
		10	0.124002	0.131400	0.480238	0.060899	0.042442	0.090803	0.332431	0.012256
	2	7	0.163819	0.245001	0.731387	-0.01537	0.081635	0.169376	0.507398	-0.00431
		8	0.119737	0.227956	0.599162	-0.01685	0.060918	0.157579	0.415157	-0.00450
		9	0.094876	0.214051	0.528246	-0.01820	0.049741	0.147956	0.365801	-0.00438
		10	0.077366	0.202423	0.480238	-0.01984	0.042276	0.139911	0.332431	-0.00423
	3	7	0.205092	0.320559	0.731387	-0.02936	0.099057	0.221690	0.507398	-0.01343
		8	0.150629	0.296135	0.599162	-0.02680	0.072334	0.204769	0.415157	-0.01268
		9	0.122355	0.276626	0.528246	-0.02377	0.058356	0.191258	0.365801	-0.01161
		10	0.104276	0.260564	0.480238	-0.02086	0.049294	0.180137	0.332431	-0.01059
	4	7	0.227234	0.320559	0.731387	-0.00722	0.111556	0.273287	0.507398	-0.02711
		8	0.160109	0.296135	0.599162	-0.01732	0.078663	0.249893	0.415157	-0.02508
		9	0.125850	0.276626	0.528246	-0.02028	0.062003	0.231790	0.365801	-0.02279
		10	0.103939	0.260564	0.480238	-0.02119	0.051445	0.217207	0.332431	-0.02076

Table 2: Continued.

			$\tau=2,\xi=5,\delta=0.5$				$\tau=2,\xi=10,\delta=0.5$			
s	r	n	$\mu_{r,s:n}$	$E(X_{r:n})$	$E(X_{s:n})$	$\sigma_{r,s:n}$	$\mu_{r,s:n}$	$E(X_{r:n})$	$\mu_{r,s:n}$	$\sigma_{r,s:n}$
	5	7	0.240059	0.475998	0.731387	-0.10808	0.118411	0.329508	0.507398	-0.04878
		8	0.160346	0.428730	0.599162	-0.09653	0.079330	0.296682	0.415157	-0.04384
		9	0.121653	0.393912	0.528246	-0.08643	0.060576	0.272522	0.365801	-0.03911
		10	0.097711	0.366721	0.480238	-0.07840	0.049095	0.253665	0.332431	-0.03523
	6	7	0.223156	0.575461	0.731387	-0.19773	0.110362	0.398669	0.507398	-0.09192
		8	0.133614	0.504359	0.599162	-0.16858	0.067043	0.349204	0.415157	-0.07793
		9	0.094015	0.456584	0.528246	-0.14717	0.048208	0.316010	0.365801	-0.06739
		10	0.070923	0.421103	0.480238	-0.13131	0.037307	0.291379	0.332431	-0.05956
8	1	8	0.192134	0.146897	0.750277	0.081920	0.070480	0.101518	0.520575	0.017632
		9	0.160463	0.138503	0.619424	0.074671	0.056695	0.095714	0.429259	0.015609
		10	0.145243	0.131400	0.548821	0.073128	0.049442	0.090803	0.380103	0.014928
	2	8	0.156885	0.227956	0.750277	-0.01415	0.079554	0.157579	0.520575	-0.00248
		9	0.116038	0.214051	0.619424	-0.01655	0.060572	0.147956	0.429259	-0.00294
		10	0.092204	0.202423	0.548821	-0.01889	0.050124	0.139911	0.380103	-0.00306
	3	8	0.203197	0.296135	0.750277	-0.01899	0.097193	0.204769	0.520575	-0.00940
		9	0.153848	0.276626	0.619424	-0.01750	0.072961	0.191258	0.429259	-0.00914
		10	0.127820	0.260564	0.548821	-0.01518	0.059979	0.180137	0.380103	-0.00849
	4	8	0.225773	0.361267	0.750277	-0.04528	0.110449	0.249893	0.520575	-0.01964
		9	0.165186	0.335151	0.619424	-0.04241	0.080817	0.231790	0.429259	-0.01868
		10	0.133147	0.314105	0.548821	-0.03924	0.065255	0.217207	0.380103	-0.01731
	5	8	0.245110	0.428730	0.750277	-0.07656	0.120610	0.296682	0.520575	-0.03384
		9	0.173431	0.393912	0.619424	-0.07057	0.085377	0.272522	0.429259	-0.03161
		10	0.136810	0.366721	0.548821	-0.06445	0.067482	0.253665	0.380103	-0.02894
	6	8	0.253982	0.504359	0.750277	-0.12443	0.125347	0.349204	0.520575	-0.05644
		9	0.170589	0.456584	0.619424	-0.11223	0.084412	0.316010	0.429259	-0.05124
		10	0.129956	0.421103	0.548821	-0.10115	0.064702	0.291379	0.380103	-0.04605
	7	8	0.232942	0.504359	0.750277	-0.14547	0.115084	0.415157	0.520575	-0.10104
		9	0.140652	0.456584	0.619424	-0.14217	0.070427	0.365801	0.429259	-0.08660
		10	0.099571	0.421103	0.548821	-0.13154	0.050909	0.332431	0.380103	-0.07545
9	1	9	0.202844	0.138503	0.766633	0.096663	0.071447	0.095714	0.531990	0.020528
		10	0.171939	0.131400	0.637074	0.088228	0.058309	0.090803	0.441548	0.018215
	2	9	0.148835	0.214051	0.766633	-0.01526	0.077490	0.147956	0.531990	-0.00122
		10	0.110573	0.202423	0.637074	-0.01839	0.059909	0.139911	0.441548	-0.00187
	3	9	0.201458	0.276626	0.766633	-0.01061	0.095239	0.191258	0.531990	-0.00651
		10	0.156264	0.260564	0.637074	-0.00973	0.073008	0.180137	0.441548	-0.00653
	4	9	0.222547	0.335151	0.766633	-0.03439	0.108528	0.231790	0.531990	-0.01478
		10	0.167173	0.314105	0.637074	-0.03294	0.081537	0.217207	0.441548	-0.01437
	5	9	0.244147	0.393912	0.766633	-0.05784	0.119779	0.272522	0.531990	-0.02520
		10	0.179218	0.366721	0.637074	-0.05441	0.087879	0.253665	0.441548	-0.02413
	6	9	0.260024	0.456584	0.766633	-0.09001	0.128154	0.316010	0.531990	-0.03996
		10	0.184607	0.421103	0.637074	-0.08367	0.091033	0.291379	0.441548	-0.03762
	7	9	0.265867	0.528246	0.766633	-0.13910	0.131234	0.365801	0.531990	-0.06337
		10	0.179461	0.480238	0.637074	-0.12649	0.088775	0.332431	0.441548	-0.05801
	8	9	0.241423	0.619424	0.766633	-0.23345	0.119156	0.429259	0.531990	-0.10921
		10	0.146845	0.548821	0.637074	-0.20279	0.073382	0.380103	0.441548	-0.09445
10	1	10	0.214615	0.131400	0.781029	0.111988	0.072587	0.090803	0.542039	0.023369
	2	10	0.139577	0.202423	0.781029	-0.01852	0.075465	0.139911	0.542039	-0.00037
	3	10	0.200369	0.260564	0.781029	-0.00314	0.093369	0.180137	0.542039	-0.00427
	4	10	0.218699	0.314105	0.781029	-0.02663	0.106389	0.217207	0.542039	-0.01135
	5	10	0.241199	0.366721	0.781029	-0.04522	0.117950	0.253665	0.542039	-0.01955
	6	10	0.259491	0.421103	0.781029	-0.06940	0.127622	0.291379	0.542039	-0.03032
	7	10	0.272860	0.480238	0.781029	-0.10222	0.134607	0.332431	0.542039	-0.04558
	8	10	0.276212	0.548821	0.781029	-0.15243	0.136336	0.380103	0.542039	-0.06969
	9	10	0.248896	0.637074	0.781029	-0.24868	0.122731	0.441548	0.542039	-0.11661

Table 3: Coefficients of the BLUEs of the location parameter for $\tau = 2$, $\delta = 0.5$.

ξ	n	c	$a_i, i = 1,2,3,\ldots,(n-c)$						
5	4	0	0.60056	0.52054	-0.0249	-0.0962			
		1	1.14951	-0.0022	-0.1473				
		2	1.46179	-0.4618					
	8	0	0.94978	-0.0079	-0.0355	0.11756	0.04068	-0.0357	-0.0211
			-0.0079						
		1	0.99500	-0.0542	-0.0168	0.13125	0.02120	-0.0520	-0.0245
		2	1.10008	-0.1932	0.02847	0.16721	-0.0256	-0.0769	
		3	-0.1442	3.22795	-1.6960	-1.0836	0.69588		
		4	1.22957	-5E-05	-0.2371	0.00762			
		5	1.20093	0.03644	-0.2374				
		6	1.55875	-0.5587					
	10	0	0.84633	0.46665	-0.1943	0.02331	0.04968	-0.1587	-0.0944
			0.02940	0.02612	0.00600				
		1	0.87109	0.40252	-0.1787	0.02622	0.03714	-0.1427	-0.0747
			0.03345	0.02564					
		2	0.92580	0.23990	-0.1220	0.04079	0.00905	-0.0958	-0.0266
			0.02881						
		3	1.14784	-0.0594	-0.0458	0.05472	-0.0716	-0.0521	0.02645
		4	0.97357	0.11809	-0.0719	0.05459	-0.0127	-0.0617	
		5	1.28460	-0.3778	0.12364	0.10493	-0.1354		
		6	1.19440	0.06579	-0.2574	-0.0028			

Table 4: Coefficients of the BLUEs of the scale parameter for $\tau = 2$, $\delta = 0.5$.

ξ	n	c	$b_i, i = 1,2,3,\ldots,(n-c)$						
5	4	0	-0.19072	0.15597	0.04744	-0.01269			
		1	-0.11835	0.08705	0.03130				
		2	-0.18472	0.18472					
	8	0	-0.11319	0.09364	0.00620	-0.02865	0.01231	0.02046	0.00619
			0.00305						
		1	-0.13061	0.11148	-0.00102	-0.03392	0.01981	0.02676	0.00750
		2	-0.16282	0.15410	-0.01488	-0.04495	0.03417	0.03438	
		3	0.39367	-1.37593	0.75633	0.51444	-0.28851		
		4	-0.17592	-0.03759	0.15149	0.06202			
		5	-0.40913	0.25959	0.14954				
		6	-0.63456	0.63456					
	10	0	-0.05656	-0.13237	0.10417	0.01245	-0.02274	0.06273	0.04769
			-0.00559	-0.00908	-0.00070				
		1	-0.05946	-0.12487	0.10235	0.01211	-0.02127	0.06086	0.04538
			-0.00606	-0.00902					
		2	-0.07871	-0.06765	0.08238	0.00698	-0.01139	0.04436	0.02846
			-0.00443						
		3	-0.11285	-0.02163	0.07068	0.00484	0.00101	0.03764	0.02030
		4	-0.24665	0.11466	0.05071	0.00474	0.04621	0.03032	
		5	-0.39958	0.35850	-0.04542	-0.02001	0.10651		
		6	-0.32861	0.00947	0.25437	0.06478			

Table 5: Variances and covariance of the BLUEs when $\tau = 0.5$, $\delta = 2$ and $\mu = 0$ and $\sigma = 1$.

ξ	n	c	$Var(\mu^*)$	$Var(\sigma^*)$	$Cov(\mu^*, \sigma^*)$
0.5	4	0	17.65257	1.410936	-0.566364
		1	14.02154	1.347831	-1.045046
		2	4.033713	0.896757	1.077511
	8	0	2.337863	0.846369	-0.520900
		1	2.167946	0.821161	-0.455453
		2	1.696970	0.776917	-0.311100
		3	20.75401	4.588457	-8.833814
		4	4.587047	1.809407	-2.130921
		5	4.594606	2.310717	-2.069361
		6	0.286717	0.600882	0.644635
	10	0	2.750427	1.010938	-1.077970
		1	2.628375	1.009270	-1.063700
		2	2.206944	0.957081	-0.915396
		3	1.727925	0.945757	-0.841744
		4	1.848615	1.016895	-0.749085
		5	0.544039	0.701471	-0.107607
		6	3.260254	2.383097	-2.244815
		7	3.261429	3.000961	-2.271760
		8	0.140384	0.562046	0.487219

5. Application with example

Here, we illustrate the applicability of the EPL distribution among a set of classical and recent models containing Pareto, generalized Lindley (GL) and type-II exponentiated log-logistic distribution (TIIELL), based on a set of goodness-of-fit statistics. We compare the goodness-of-fit of the models with the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) goodness-of-fit statistics. Further, we get the Kolmogorov-Smirnov (K-S) statistics with their corresponding p-value. In general, the smaller the values of these statistics and the larger value of the p-value, the better the fit to the data.

Description of the data is as follows: The data set refers to a study on the vinyl chloride from clean upgradient monitoring wells in mg/L, which was studied by Bhaumik et al. (2009). The data are:

```
5.1 1.2 1.3 0.6 0.5 2.4 0.5 1.1 8.0 0.8 0.4 0.6 0.9 0.4 2.0 0.5 5.3
3.2 2.7 2.9 2.5 2.3 1.0 0.2 0.1 0.1 1.8 0.9 2.0 4.0 6.8 1.2 0.4 0.2
```

In this data set, we select a random sample of size 10 and data are: 0.1, 1.1, 0.9, 2.3, 1.3, 2.5, 0.4, 2.0, 0.5, 3.2. By using the EPL distribution in Eq. (1) for the given sample, we have the maximum likelihood estimate of $\tau_{ML} = 1.2945$, $\delta_{ML} = 0.5514$ and $\xi_{ML} = 0.8099$. For this data set, a comparison of the EPL distribution with other classical and recent distributions is done as follows. We obtain the K-S, p-value, AIC and BIC statistics. The obtained results are reported in Table- 6. From this table, the smallest value of K-S, AIC and BIC and the largest value of p-value are obtained for the EPL distribution. Hence, we conclude

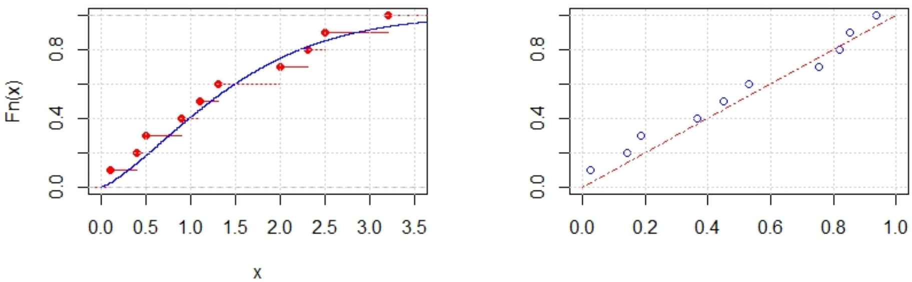

Figure 1: ECDF and Q-Q plot of the EPL distribution for the data set

that the EPL distribution provides the best fit among the compared distributions. Figure 1 shows Q-Q plot of the sample.

By using Tables 4 and 5, we have

$$\mu^* = \sum_{r=1}^{n} a_r X_{r:n} = 0.20081 \quad \text{and} \quad \sigma^* = \sum_{r=1}^{n} b_r X_{r:n} = 0.36501.$$

Table 6: The values of K-S, p- value, AIC and BIC statistics for some models fitted to the vinyl chloride data.

Distribution	K-S	p-value	AIC	BIC
Pareto	0.19982	0.1332	121.2704	122.7968
GL	0.11628	0.7474	116.2222	119.2747
TIIELL	0.10401	0.8556	116.1526	119.2053
EPL	0.08219	0.9757	116.0253	120.6044

6. Conclusion

In this paper, we have obtained the explicit expression for single and double moments of order statistics. Also, these results are reduced for special cases of order statistics of the power Lindley distribution and the Lindley distribution, which is obtained by Kumar and Goyal (2019a) and Sultan and AL-Thubyani (2016) respectively. The single and double moments of order statistics can be used to obtain BLUEs of the location and scale parameters. A simulation study is conducted for calculating BLUEs for the location and scale parameters based on Type-II right-censored samples. Finally, one real data set has been used to obtain the MLEs of the model parameters, BLUEs of the parameters and the EPL distribution is also compared with some existing distributions. We conclude that the EPL distribution provides the best fit among the compared distributions.

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