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Solving fractional Bagley-Torvik equation by fractional order Fibonacci wavelet arising in fluid mechanics

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ABSTRACT

This study introduces a new fractional order Fibonacci wavelet technique proposed for solving the fractional Bagley-Torvik equation (BTE), along with the block pulse functions. To convert the specified initial and boundary value problems into algebraic equations, the Riemann–Liouville (R-L) fractional integral operator is defined, and the operational matrices of fractional integrals (OMFI) are built. This numerical scheme's performance is evaluated and examined on particular problems to show its proficiency and effectiveness, and other methods that are accessible in the current literature are compared. The numerical results demonstrate that the approach produces extremely precise results and is computationally more decisive than previous methods.

1. Introduction

Over the past decades, scientists, geologists, technicians, mathematicians, and statisticians have been paying close attention to fractional calculus. It has been noted that fractional derivatives can also be used to simulate a variety of transdisciplinary issues. Fractional type differential equations [37–43,46,50] have sparked lots of interest because of their capacity to simulate complicated processes, such as continuum and statistical mechanics, viscoelastic materials, and solid mechanics [48,49]. The dynamics of a viscoelastic rod with the fractional derivative type of dissipation under time-dependent loading have also been studied by Atanackovic and Stankovic [24]. Khan [25] used fractional calculus to investigate the fluxes in an Oldroyd-B fluid. He calculated the velocity field for an incompressible generalized Oldroyd-B fluid by a fractional derivative framework inside an infinite edge using Laplace and Fourier sine transforms. Khan and Wang [26] employed the fractional calculus technique in a non-Newtonian fluid and a fundamental normative framework in a generalized second-grade fluid to get accurate analytic solutions for the flow of fluid between two side walls that are perpendicular to the plate. The authors of [23] are the ones who initially suggested the BTE, which excels in its capacity as a simulation of the motion of a structural element arising in a Newtonian fluid.

Bagley and Torvik [23], have investigated the movement of a submerged plate that has been bound in a Newtonian fluid and a gas in a fluid, respectively, and established one of the earliest problems of this sort. The frequency-dependent damping materials have been effectively modelled using fractional order(order 1/2

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or 3/2) derivatives. BTE has been solved by several authors, both numerically and analytically. The BTE is defined as below:

$$\Lambda\mathcal{D}^2 z(\xi) + \beta\mathcal{D}^{3/2} z(\xi) + \gamma z(\xi) = h(\xi) \quad (\xi \in [0, U]), \tag{1}$$

with conditions

$$z(0) = z_0 \quad and \quad z(U) = z_1, \tag{2}$$

where the displacement of the plate is defined by the function $z(\xi)$. The constant coefficients (Λ, β, γ) are dependent upon the viscosity, fluid density, stiffness of the spring, and area of the plate. The external force denoted by $h(\xi)$ is a known function, the constant input represented by U lies within the interval $[0, U]$, the constants z_0 and z_1 are with $\Lambda \neq 0$.

The BTE has been proposed in several areas of physics and applied mathematics and carried out in this manner. For example, the fractional Taylor expansion method [4], the computational intelligence algorithms [13], the generalized Taylor collocation method [7], the Adomian decomposition method [21], the fractional iteration method [5], variational iteration method [20]. By matrix method and successive approximation, the problem (1) has been meticulously examined by Podlubny [11]. Additionally, in order to solve the BTE(1) some methods have been developed based on the orthogonal basis function. For instance, Chebyshev and Laguerre functions, the sinc and Bessels functions, Jacobi and Legendre polynomials, and the block-pulse functions. The wavelet functions are a recent addition to these orthogonal functions. Wavelets are mathematical operations that can find information throughout the whole computational domain at various sizes and locations. These functions are capable of providing bases, which are created by enlarging and translating the mother wavelet, a fixed function. Currently, the OMFI for the Bernoulli wavelets [1,3,10,14], the Spline wavelets [22], the Chebyshev wavelets [15,19], the Legendre wavelet [8,47], the Haar wavelets [6,33–36] and many more [2,9,12] been developed to resolve several types of fractional order differential equations.

Recently, the author [28], defined the transformation $\xi = x^\alpha$ $(\alpha > 0)$ for shifted Legendre polynomials to obtain better findings for the solution of fractional order differential equations. The author [29] used the same transformation for the generalized Laguerre polynomials to approximate fractional differential equations. Moreover, the papers [30,31,14] used Bernstein polynomials, Legendre functions, and Bernoulli wavelets to obtain numerical solutions to fractional differential equations. This motivates us to extend the Fibonacci wavelet [32] to fractional order Fibonacci wavelet by using the transformation $\xi = x^\alpha$ $(\alpha > 0)$ to obtain the numerical solutions of the fractional BTE.

Fibonacci polynomials [44], which are used to generate the Fibonacci wavelets, are a relatively new addition to the wavelet family category, these Fibonacci polynomials contain fewer terms than the shifted Legendre polynomials, which speed up the computation and reduce the likelihood of creating an error. The operational matrix of integration in Fibonacci polynomials has less error compared to the Legendre polynomials. Individual term coefficients in Fibonacci polynomials are smaller, corresponding to the Legendre polynomials, which provide a lower computational error of Fibonacci polynomials. These special types of Fibonacci wavelets are based on nonorthogonal functions. Though, the operational matrix of integration is constructed. Inspired by the superior characteristics of the Fibonacci wavelet over other existing wavelets, many researchers have studied the Fibonacci wavelets. For approximating smooth and piecewise smooth functions, we suggest that Fibonacci wavelets are suitable. This polynomial based wavelet method has been used to resolve Stratonovich Volterra integral equations [16], fractional optimal control problems [17], delay problems [18], bioheat transfer equations [45], Fredholm integral equations [51]. In retrospect, the pleasant properties of the Fibonacci wavelets led us to solve the fractional type equations such as BTE by the fractional Fibonacci wavelet approach using the block pulse function. For the Fibonacci wavelet, Chen and Haiso [19] technique is taken into account to build the OMFI.

The goal of this study is to develop a new fractional Fibonacci wavelet approach based on an operational matrix of integration (OMI) for solving the BTE arising in fluid mechanics. Firstly, the unknown function $z(\xi)$ is approximated by the linear combination of the fractional Fibonacci wavelet and then its fractional derivatives $\mathcal{D}^\alpha z(\xi)$ by truncating at optimal levels. Further, the OMI of fractional Fibonacci wavelets for the given problem is introduced and then transformed into a system of algebraic equations. Furthermore, two cases of fractional BTE are discussed with valid examples, and to show the efficiency and novelty of the present scheme, a comparison is made with the existing literature. Presently, there is no such methodology in the literature as our presented technique. The proposed methodology has not shown any significant drawbacks, but this present scheme only works in a limited domain.

This paper is designed as: Section 2 contains the preliminaries of the fractional calculus, wavelets, and Fibonacci wavelet, which are used in further sections. Section 3 contains the block pulse functions and the construction of the fractional Fibonacci wavelet of fractional order integration. The description of the method to solve the fractional BTE for particular cases by a fractional Fibonacci wavelet is given in Section 4. By demonstrating the four test problems, Section 5 illustrates the precision of the currently suggested technique. Finally, Section 6 gives a brief conclusion.

2. Preliminaries

Several definitions exist in the literature to calculate fractional derivatives and integrations. In this paper, Caputo fractional derivatives and R-L fractional integration are used for calculations. When the starting specifications are considered in the form of the field variables and their integer order, as is the case for the bulk of physical processes, the Liouville-Caputo technique has the advantage of being more appropriate for initial-value problems.

Definition 2.1. A real function $h(\xi), \xi > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there is a real number $\lambda(\lambda > \mu)$ such that $h(\xi) = \xi^\lambda h_0(\xi)$, where $h_0(\xi) \in C[0, \infty)$, and $h(\xi) \in C_\mu^m$ if $h^{(m)}(\xi) \in C_\mu, m \in \mathbb{N}$.[16].

Definition 2.2. The R-L fractional integral operator J^α of α order$(\alpha \geqslant 0)$ for a function $h(\xi) \in C_\gamma(\gamma \geqslant -1)$ is defined as [16]

$$(J^\alpha h)(\xi) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^\xi (\xi - t)^{\alpha-1} h(t) dt, & \alpha > 0, \\ h(\xi), & \alpha = 0. \end{cases}$$

Definition 2.3. The Caputo fractional derivative operator D^α of α order$(\alpha \geqslant 0)$ for a function $h(\xi) \in C_1^m$ is defined as[16]

$$(D^\alpha h)(\xi) = \begin{cases} h^{(m)}(\xi), & \alpha = m \in \mathbb{N}, \\ \frac{1}{\Gamma(m-\alpha)} \int_0^\xi \frac{h^{(m)}(t)}{(\xi - t)^{\alpha+1-m}} dt, & m - 1 < \alpha < m. \end{cases}$$

Here, R–L fractional integral operator and the Caputo fractional derivative operator have relation shown below:

- $\mathcal{D}^\alpha J^\alpha h(\xi) = h(\xi)$,
- $\mathcal{D}^\beta J^\alpha h(\xi) = J^{\alpha-\beta} h(\xi), \quad \alpha > \beta$,
- $J^\alpha D^\alpha h(\xi) = h(\xi) - \sum_{k=0}^{m-1} h^k(0^+) \frac{\xi^k}{k!}, \quad \mu > 0, \quad m - 1 < \alpha \leqslant m.$

3. Fractional order Fibonacci wavelet

In order to define the fractional order Fibonacci wavelet, we covered the fundamental ideas of wavelets and the Fibonacci wavelet in this part. Additionally, the construction of the OMFI of the fractional Fibonacci wavelet with the block-pulse function is explained.

3.1. Wavelets and Fibonacci wavelet

A signal function known as the mother wavelet is translated and dilated to produce a family of functions known as wavelets. When the dilation parameters u, v fluctuate continuously, the given equation below is defined for the continuous wavelet family. [18]

$$\psi_{u,v}(\xi) = |u|^{-\frac{1}{2}}\psi\left(\frac{\xi - v}{u}\right), \quad u, v \in R, u \neq 0,$$

If we choose $u = u_0^{-p}$, $v = \eta u_0^{-p} v_0, u_0 > 1, v_0 > 1$, and $\eta, p \in \mathbb{Z}^+$, and discrete wavelets family is introduced by

$$\psi_{p,\eta}(\xi) = |u_0|^{\frac{p}{2}}\psi(u_0^p\xi - \eta v_0),$$

in which wavelet basis $\psi_{p,\eta}(\xi)$ is in $L^2(R)$.

For any $\xi \in \mathbb{R}^+$, the recurrence relation defines the Fibonacci polynomials as follows:

$$\widetilde{W}_{s+2}(\xi) = \xi\widetilde{W}_{s+1}(\xi) + \widetilde{W}_s(\xi) \quad (s \geqslant 0),$$

with initial conditions(ICs) $\widetilde{W}_0(\xi) = 0, \widetilde{W}_1(\xi) = 1$ [18].

Following generic formula is used to define the Fibonacci polynomials

$$\widetilde{W}_s(\xi) = \begin{cases} 1, & s = 0, \\ \xi, & s = 1, \\ \xi\widetilde{W}_{s-1}(\xi) + \widetilde{W}_{s-2}(\xi), & s > 1. \end{cases}$$

and further the formula in closed form represented as:

$$\widetilde{W}_{s-1}(\xi) = \frac{\rho^s - \chi^s}{\rho - \chi} \quad (s \geqslant 1),$$

where ρ , χ represent the roots of the recursion's partner polynomial, $y^2 - \xi y - 1$.

Additionally, the Fibonacci polynomials power-form representation appears as follow [18]:

$$\widetilde{W}_s(\xi) = \sum_{i=0}^{\lfloor s/2 \rfloor}\binom{s-i}{i}\xi^{s-2i} \quad (s \geqslant 0),$$

where $\lfloor\cdot\rfloor$ stands the well recognised floor function.

The Fibonacci polynomial has the following properties:

$$\int_0^\xi \widetilde{W}_s(t)dt = \frac{1}{s+1}\left[\widetilde{W}_{s+1}(\xi) + \widetilde{W}_{s-1}(\xi) - \widetilde{W}_{s+1}(0) + \widetilde{W}_{s-1}(0)\right],$$

$$\int_0^1 \widetilde{W}_s(\xi)\widetilde{W}_r(\xi)d\xi = \sum_{i=0}^{\lfloor s/2 \rfloor}\sum_{j=0}^{\lfloor r/2 \rfloor}\binom{s-i}{i}\binom{r-j}{j}\frac{1}{s+r-2i-2j+1}. \tag{3}$$

The following is how we define Fibonacci wavelets:

$$\psi_{r,s}(\xi) = \begin{cases} 2^{\frac{k-1}{2}}\widehat{W}_s\left(2^{k-1}\beta - r + 1\right), & \frac{r-1}{2^{k-1}} \leqslant \xi < \frac{r}{2^{k-1}}, \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

with

$$\widehat{W}_s(\xi) = \frac{1}{\sqrt{\omega_s}}\widetilde{W}_s(\xi),$$

and

$$\omega_s = \int_0^1 \widetilde{W}_s^2(\xi)d\xi.$$

where $\omega_s, s = 0, 1, \ldots, M-1$, are obtained using property given by (3), the order of Fibonacci polynomials is represent by $r, r = 1, 2, \ldots, 2^{k-1}$ where $k \in \mathbb{Z}^+$.

3.2. Fractional order Fibonacci wavelets

Fractional order Fibonacci polynomial is defined as:

$$\widetilde{W}_s^\alpha(\xi) = \begin{cases} 1, & s = 0, \\ \xi^\alpha, & s = 1, \\ \xi^\alpha\widetilde{W}_{s-1}^\alpha(\xi) + \widetilde{W}_{s-2}^\alpha(\xi), & s > 1. \end{cases}$$

The Fractional Fibonacci polynomials power-form representation appears as follow:

$$\widetilde{W}_s^\alpha(\xi) = \sum_{i=0}^{\lfloor s/2 \rfloor}\binom{s-i}{i}\xi^{\alpha(s-2i)} \quad (s \geqslant 0),$$

where $\lfloor\cdot\rfloor$ stands the well recognised floor function.

Fractional order Fibonacci polynomial for n = 3 are.

$\widetilde{W}_0^\alpha(\xi) = 1$, $\widetilde{W}_1^\alpha(\xi) = \xi^\alpha$, $\widetilde{W}_2^\alpha(\xi) = \xi^{2\alpha} + 1$, $\widetilde{W}_3^\alpha(\xi) = \xi^{3\alpha} + 2\xi^\alpha$.

Using properties, we obtain

$$\int_0^1 \widetilde{W}_s^\alpha(\xi)\widetilde{W}_r^\alpha(\xi)\xi^{\alpha-1}d\xi = \sum_{i=0}^{\lfloor s/2 \rfloor}\sum_{j=0}^{\lfloor r/2 \rfloor}\binom{s-i}{i}\binom{r-j}{j}\frac{1}{\alpha(s+r-2i-2j+1)}. \tag{5}$$

By changing the variable ξ to ξ^α, $(\alpha > 0)$ on the Fibonacci wavelet, we build fractional order Fibonacci wavelet which is a new class of fractional functions, let the fractional Fibonacci wavelets $\psi_{r,s}(\xi^\alpha)$ be denoted by $\psi_{r,s}^\alpha(\xi)$.

The following equation define the fractional order Fibonacci wavelet

$$\psi_{r,s}^\alpha(\xi) = \begin{cases} \frac{2^{\frac{k-1}{2}}}{\sqrt{\omega_s}}\widehat{W}_s\left(2^{k-1}\xi^\alpha - r + 1\right), & \frac{r-1}{2^{k-1}} \leqslant \xi^\alpha < \frac{r}{2^{k-1}}, \\ 0, & \text{otherwise}. \end{cases} \tag{6}$$

The coefficient $\frac{1}{\sqrt{\omega_s}}$ in (6) is a normalised factor that may be calculated using (5)

$$\omega_s = \int_0^1 \widetilde{W}_s^{2\alpha}(\xi)\xi^{\alpha-1}d\xi,$$

3.3. Function approximation

In form of the fractional order Fibonacci wavelet, any function $h \in L^2[0,1)$ may be extended as

$$h(\xi) \approx \sum_{r=1}^{2^{k-1}}\sum_{s=0}^{M-1} q_{r,s}\psi_{r,s}^\alpha(\xi), \tag{7}$$

where

$$q_{r,s} = \left\langle h, \psi_{r,s}^\alpha\right\rangle = \int_0^1 h(\xi)\psi_{r,s}^\alpha(\xi)\xi^{\alpha-1}d\xi,$$

are the coefficients of Fibonacci wavelet. The matrix equivalent of (7) is written as follows:

$$h(\xi) = Q^T\Psi^\alpha(\xi), \tag{8}$$

where Q is the row vector defined below:

$$Q = \left[q_{1,0}, q_{1,1}, \ldots, q_{1,M-1}, q_{2,0}, q_{2,1}, \cdots, q_{2,M-1}, \ldots, q_{2^{k-1},0}, q_{2^{k-1},1}, \ldots, q_{2^{k-1},M-1}\right]^T. \quad (9)$$

The matrix $\Psi^\alpha(\xi)$ in (8) is of order $1 \times 2^{k-1}M$ Fibonacci wavelet matrix and is given by

$$\Psi^\alpha(\xi) = \left[\psi^\alpha_{1,0}, \psi^\alpha_{1,1}, \cdots, \psi^\alpha_{1,M-1}, \psi^\alpha_{2,0}, \psi^\alpha_{2,1}, \ldots, \psi^\alpha_{2,M-1}, \ldots, \psi^\alpha_{2^{k-1},0}, \psi^\alpha_{2^{k-1},1}, \ldots, \psi^\alpha_{2^{k-1},M-1}\right]^T. \quad (10)$$

By defining the collocation points as:

$$\xi_i = \frac{2i-1}{2^kM} \quad \left(1 \leqslant i \leqslant 2^{k-1}M\right),$$

The following Fibonacci wavelet matrix Ψ^α is defined

$$\Psi^\alpha = \left[\Psi^\alpha\left(\frac{1}{2^kM}\right), \Psi^\alpha\left(\frac{3}{2^kM}\right), \ldots, \Psi^\alpha\left(\frac{2^kM-1}{2^kM}\right)\right]^T_{2^{k-1}M \times 2^{k-1}M}. \quad (11)$$

In particular, when α=1, k = 2 and M = 3, the matrix obtained is:

$$\Psi^1_{6\times 6} = \begin{pmatrix} 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 \\ 0.4082 & 1.2247 & 2.0412 & 0 & 0 & 0 \\ 1.0638 & 1.2939 & 1.7539 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & 0.4082 & 1.2247 & 2.0412 \\ 0 & 0 & 0 & 1.0638 & 1.2939 & 1.7539 \end{pmatrix}. \quad (12)$$

3.4. Fractional Fibonacci wavelets OMFI

In terms of block-pulse function, any function $h \in L^2[0,1)$, can be extended as

$$h(\xi) \approx \sum_{i=0}^{\hat{m}-1} h_i b_i(\xi) = \mathbf{h}^T\mathbf{B}(x),$$

here the block-pulse function coefficients are denoted by h_i. Now, block-pulse functions could be extended into $\hat{m}$-set in terms of fractional order Fibonacci wavelet as

$$\mathbf{\Psi}^\alpha(\xi) = \mathbf{\Psi}^\alpha_{\hat{m}\times\hat{m}}\mathbf{B}(\xi), \quad \left(\hat{m} = 2^{k-1}M\right),$$

The fractional integral of the vector of the block-pulse function may be expressed as

$$\left(\mathbf{J}^\delta\mathbf{B}\right)(\xi) = \mathbf{F}^\delta_{\hat{m}\times\hat{m}}\mathbf{B}(\xi),$$

as $\mathbf{F}^\delta_{\hat{m}\times\hat{m}}$ is defined in[27] we obtain

$$\mathbf{P}^{\alpha,\delta}_{\hat{m}\times\hat{m}} = \Psi^\alpha_{\hat{m}\times\hat{m}}\mathbf{F}^\delta\left(\Psi^\alpha_{\hat{m}\times\hat{m}}\right)^{-1}.$$

The fractional order Fibonacci wavelet OMI $\mathbf{P}^{\alpha,\delta}_{\hat{m}\times\hat{m}}$ of fractional order δ is developed for various α and then these are utilized to solve differential equations.

Particularly, with α=1, k = 2, M = 3, δ=0.5, the fractional order integration of the Fibonacci wavelet operational matrix $P^{1,0.5}_{6\times6}$ is given by

$$\Psi^{1,0.5}_{6\times 6} = \begin{pmatrix} 0.2742 & -0.4953 & 0.3151 & -0.1488 & 0.4353 & -0.1723 \\ 0.4567 & -0.9543 & 0.5916 & -0.2028 & 0.5929 & -0.2339 \\ 0.7332 & -1.6013 & 0.9622 & -0.2777 & 0.8096 & -0.3211 \\ 0 & 0 & 0 & 0.2742 & -0.4953 & 0.3151 \\ 0 & 0 & 0 & 0.4567 & -0.9543 & 0.5916 \\ 0 & 0 & 0 & 0.7332 & -1.6013 & 0.9621 \end{pmatrix}. \quad (13)$$

4. Description of Fractional Fibonacci Wavelet Scheme

Here, the fractional BTE problem is solve by using fractional Fibonacci wavelet OMI to demonstrate the proposed wavelet effectiveness.

Here, we discussed the following fractional BTE problem.

Problem 1. Initial value problems

$$\begin{cases} \Lambda D^2 z(\xi) + \beta D^{\frac{3}{2}} z(\xi) + \gamma z(\xi) = h(\xi), \quad 0 \leqslant \xi \leqslant 1, \\ z(0) = z_0, \quad z'(0) = z'_0. \end{cases} \quad (14)$$

Problem 2. Boundary value problems

$$\begin{cases} \Lambda D^2 z(\xi) + \beta D^{\frac{3}{2}} z(\xi) + \gamma z(\xi) = h(\xi), \quad 0 \leqslant \xi \leqslant 1, \\ z(0) = z_0, \quad z(1) = z_1. \end{cases} \quad (15)$$

The fractional order Fibonacci wavelets is used for solving Problem 1 and Problem 2. For the given problem, the highest-order term $\mathcal{D}^2 y(t)$ is extended in Fibonacci wavelet terms $\psi^\alpha_{r,s}(t)$ shown by (6). Thus,

$$\mathcal{D}^2 z(\xi) = \sum_{r=0}^{M-1}\sum_{s=1}^{2^{k-1}} q_{r,s}\psi^\alpha_{r,s}(\xi) = Q^T\Psi^\alpha(\xi). \quad (16)$$

Similarly, the following terms are expressed as below:

$$\mathcal{D}z(\xi) = \sum_{r=0}^{M-1}\sum_{s=1}^{2^{k-1}} q_{r,s}\left(J^1\psi^{\alpha(\xi)}_{r,s}\right) + z\prime_0 = Q^TP^{1,\alpha}\Psi^\alpha(\xi) + z\prime_0, \quad (17)$$

$$z(\xi) = \sum_{r=0}^{M-1}\sum_{s=1}^{2^{k-1}} q_{r,s}\left(J^2\psi^\alpha_{r,s}(\xi)\right) + z\prime_0\xi + z_0$$
$$= Q^TP^{2,\alpha}\Psi^\alpha(\xi) + z\prime_0\xi + z_0, \quad (18)$$

$$\mathcal{D}^{3/2} z(\xi) = \sum_{r=0}^{M-1}\sum_{s=1}^{2^{k-1}} q_{r,s}\left(J^{1/2}\psi^\alpha_{r,s}(\xi)\right) = Q^TP^{1/2,\alpha}\Psi^\alpha(\xi), \quad (19)$$

Now, substituting the Eqs. (16), (18), and (19) into Eq. (14), we now get a system of algebraic equations given below:

$$Q^T\left[\Lambda\Psi^\alpha(\xi) + \beta P^{1/2,\alpha}\Psi^\alpha(\xi) + \gamma P^{2,\alpha}\Psi^\alpha(\xi)\right]$$
$$= h(\xi) - \gamma[z\prime_0\xi + z_0]. \quad (20)$$

By collocating the system of algebraic Eqs. (20), we take:

$$\xi_i = \frac{2i-1}{2^kM} \quad \left(i = 1, 2, \ldots, 2^{k-1}M\right).$$

we get the unknown vector Q. Furthermore, the required Fibonacci wavelet solution for the Eq. (14) is given by:

$$Z = Q^TP^{2,\alpha}_{m\times m}\Psi^\alpha_{m\times m} + z\prime_0 I_{m\times m} + z_0 I_{m\times m}. \quad (21)$$

For Problem 2, we obtain: $z\prime_0 = z_1 - \left(Q^TP^{2,\alpha}\Psi^\alpha(1) + z_0\right)$.

5. Convergence analysis

Theorem 1. Any square integrable function $h \in L^2[0,1]$ can be expanded as an infinite series of fractional-order Fibonacci wavelets, and the series converges uniformly to h, i.e.,:

$$h(\xi) = \sum_{r=1}^{\infty}\sum_{s=0}^{\infty} q_{rs}\psi^\alpha_{rs}(\xi),$$

Next, truncate the fractional order Fibonacci wavelet series approximates a system's solution, hence the following is the error function $E(x)$ for $h(\xi)$:

$$E(\xi) = \left| h(\xi) - \sum_{r=1}^{2^{k-1}} \sum_{s=0}^{M-1} q_{rs} \psi_{rs}^{\alpha}(\xi) \right|,$$

where setting $\xi = \xi_j \in [0,1], \xi_j$ can be determined which is the absolute error value. The next theorem provides the approximate solution's error bound obtained by utilising fractional Fibonacci wavelet series.

Theorem 2. Suppose $z \in Q^{M\alpha}[0,1)$ and $Y_{M\alpha} = \text{span}\{\psi_0^{\alpha}(\xi), \psi_1^{\alpha}(\xi), \ldots, \psi_{M-1}^{\alpha}(\xi)\}$. If $z_M(\xi) = A^T \widehat{F}^{\alpha}(\xi)$ is the best approximation of $z(\xi)$ from Y_M on $\left[\frac{r-1}{2^{k-1}}, \frac{r}{2^{k-1}}\right]$, by using Fibonacci wavelet on interval $[0,1)$ the error bound of the approximate solution $z^*(\xi)$ would be obtained in the following form:

$$\|e(\xi)\|_2 = \|z - z^*\|_2 \leqslant \frac{R}{M\alpha!\sqrt{2M\alpha+1}},$$

Proof. Define the function,

$$\hat{z}(\xi) = \sum_{r=0}^{M-1} \frac{z^r(0)\xi^r}{r!},$$

From the expansion of Taylor series, we obtain

$$|z(\xi) - \hat{z}(\xi)| \leqslant \frac{\xi^{M\alpha} sup_{t \in I_{k,s}} |z^{M\alpha}(t)|}{M\alpha!},$$

As, $z_M(\xi) = A^T \widehat{F}(\xi)$ is the close approximation of $z(\xi)$ from Y_M on the interval $\left[\frac{s-1}{2^{k-1}}, \frac{s}{2^{k-1}}\right]$ and $\sum_{s=0}^{M-1} \frac{z^s(0)\xi^s}{s!} \in Y_M$. Therefore,

$$||z - z^*||^2_{L^2[0,1]}, = \left| |z - Q^T \Psi^{\alpha}| \right|^2_{L^2[0,1]},$$
$$= \sum_{s=1}^{2^{k-1}} \left| |z - A^T \hat{F}^{\alpha}| \right|^2_{I_{s,k}},$$
$$\leqslant \sum_{s=1}^{2^{k-1}} \|z - \hat{z}\|^2_{I_{s,k}} \leqslant \sum_{s=1}^{2^{k-1}} \int_{I_{s,k}} [z(\xi) - \hat{z}(\xi)]^2 d\xi,$$
$$\leqslant \sum_{s=1}^{2^{k-1}} \int_{I_{s,k}} \left(\frac{\xi^{M\alpha} \sup_{t \in I_{s,k}} |z^{M\alpha}(t)|}{M\alpha!} \right)^2 d\xi,$$
$$\leqslant \int_0^1 \left(\frac{\xi^{M\alpha} \sup_{t \in [0,1)} |z^{M\alpha}(t)|}{M\alpha!} \right)^2 d\xi,$$
$$= \frac{R^2}{(M\alpha!)^2 (2M\alpha+1)}.$$

Where $\widehat{F}^{\alpha}(\xi) = \left[\widetilde{F}_0^{\alpha}(\xi), \ldots, \widetilde{F}_{M-1}^{\alpha}(\xi)\right]^T$ and $R = \sup_{t \in [0,1)} |z^{M\alpha}(t)|$. Consequently, as $M \to \infty$ the $\|e(\xi)\| \to 0$ and M defined as maximal level of resolution.

6. Numerical examples

In this part, we use the Fibonacci wavelet OMFI to solve the fractional BTE that results from simulating the motion of a rigid plate submerged in a Newtonian fluid. To demonstrate the effectiveness and application of the suggested strategy, a few particular examples of Eq. (1) are taken into consideration with initial conditions (ICs) and boundary conditions (BCs).These conditions are taken into account either because analytical solutions are known for them or because other numerical techniques have also been used to resolve them. This makes it possible to compare the outcomes of the present approach with those of other approaches or the analytical solution. The MATLAB (R2020a) software is used to calculate all findings.

Example 1. Considering the Eq. (1) having the conditions:

Table 1
Analytical and approximate solution at k = 2, M = 4.

ξ	Analytical solution	Approximate solution	Absolute error
0.0625	1.0625	1.0625	0.0000
0.1875	1.1875	1.1875	0.0000
0.3125	1.3125	1.3125	0.0000
0.4375	1.4375	1.4375	0.0000
0.5625	1.5625	1.5625	0.0000
0.6875	1.6875	1.6875	0.0000
0.8125	1.8125	1.8125	0.0000
0.9375	1.9375	1.9375	0.0000

$$\Lambda = \beta = \gamma = 1, \quad h(\xi) = 1 + \xi \quad (0 \leqslant \xi \leqslant 1), \tag{22}$$

with ICs

$$z(0) = 1, \quad z'(0) = 1.$$

Analytical solution of (22) is $z(\xi) = 1 + \xi$. The proposed scheme discussed in Section 4 is used to solve the problem using a fractional Fibonacci wavelet. Table 1 presents the numerical values obtained at k = 2, M = 4, $\alpha = 1$. And Fig. 1 plots compares the Chebyshev wavelet of second kind[15], analytical solution and current method at $\alpha = 1$, k = 2 and M = 4. The outcomes make it abundantly evident that the current technique solution and the analytical solution are in excellent accord.

Example 2. Considering the fractional BTE (1) having conditions

$$\Lambda = \beta = \gamma = 1, \quad h(\xi) = (\xi^3) + 6\xi + \frac{8}{\Gamma(1/2)\xi^{3/2}} \quad (0 \leqslant \xi \leqslant 1), \tag{23}$$

with ICs

$$z(0) = 0, \qquad z\prime(0) = 0.$$

Analytical solution of (23) is $z(\xi) = \xi^3$. The given problem is solved by using a fractional Fibonacci wavelet and its operational matrices of integration with initial conditions. Table 2 presents the comparison of the present method's approximate solution obtained by k = 3, M = 4 and k = 2, M = 4 at $\alpha = 1$ with analytical solution and approximate solution of [11]. And Fig. 2 plots the comparison of the analytical solution and the current approach at k = 2, M = 4 and $\alpha = 0.5$.

Example 3. Considering the fractional Eq. (1) with following conditions

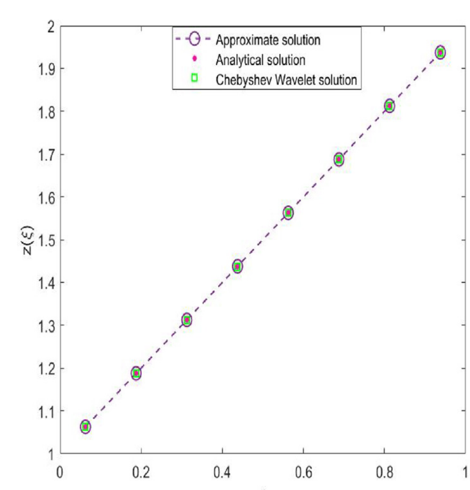

Fig. 1. Chebyshev wavelet of second kind, analytical solution and approximate solution.

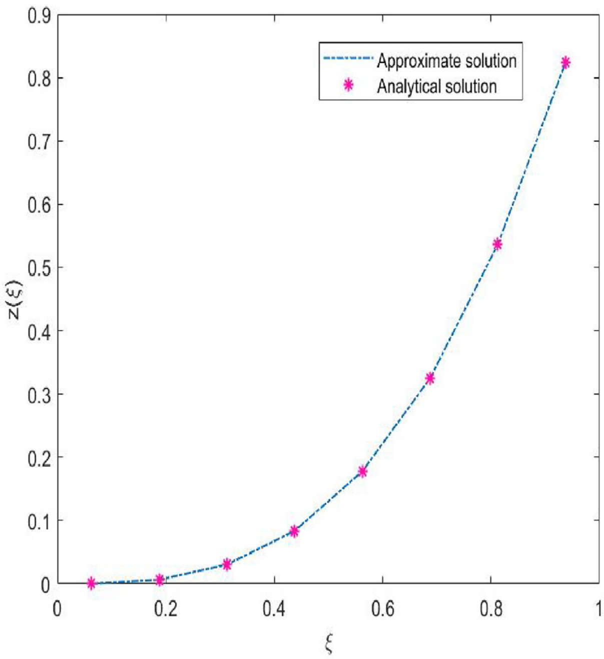

Fig. 2. Approximate and analytical solution.

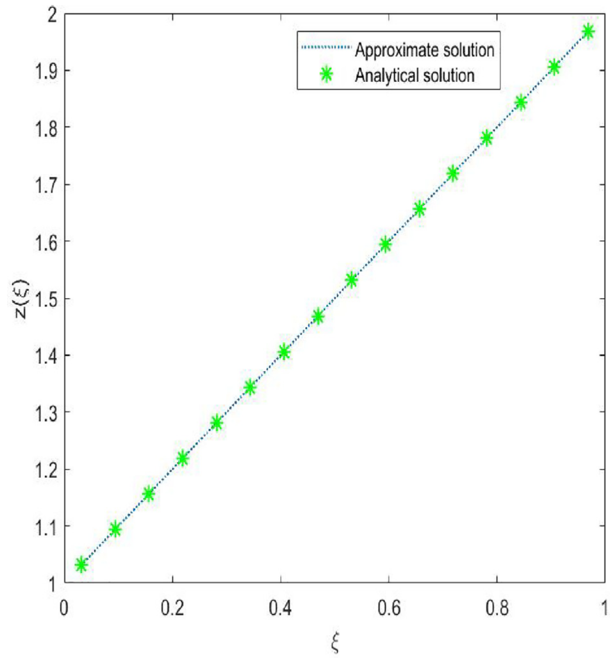

Fig. 3. Comparison of approximate and analytical solution.

Table 2
Approximate solutions, analytical solution at k = 3, M = 4 and k = 2, M = 4.

ξ	Approximate solution	Approximate solution	[11]	Analytical Solution
	k = 2,M = 4	k = 3, M = 4		
0.0	0.000000	0.000000	0.000000	0.000000
0.1	0.001000	0.001000	0.001000	0.001000
0.2	0.008000	0.008000	0.008000	0.008000
0.3	0.027000	0.027000	0.027000	0.027000
0.4	0.064000	0.064000	0.064000	0.064000
0.5	0.125000	0.125000	0.125000	0.125000
0.6	0.216000	0.216000	0.216000	0.216000
0.7	0.343000	0.343000	0.343000	0.343000
0.8	0.512000	0.512000	0.512000	0.512000
0.9	0.729000	0.729000	0.729000	0.729000

$$\Lambda = \beta = \gamma = 1, \quad h(\xi) = 1 + \xi \quad (0 \leqslant \xi \leqslant 1), \tag{24}$$

with the BCs

$z(0) = 1, \quad \text{and} \quad z(1) = 2.$

Analytical solution of Eq. (24) is $z(\xi) = 1 + \xi$. We acquire the numerical answers for Eq. (24) by using the suggested strategy that is discussed in Section 4. Fig. 3 compares the approximate solution and analytical solution for the parameters $\alpha = 0.5, k = 3$ and $M = 4$. Table 3 shows the present method's approximate values for $k = 3$ and $M = 4$ and the analytical values of the defined problem. Similarly, Table 4 for $k = 2, M = 3$. This shows that the level of Fibonacci polynomial degree increases, better results are obtained.

Problem 4. Finally, considering the Eq. (1) with following conditions

$$\Lambda = 0, \beta = \gamma = 1, \quad h(\xi) = \frac{2\sqrt{\xi}}{\Gamma(3/2)} + \xi(\xi - 1) \quad (0 \leqslant \xi \leqslant 1), \tag{25}$$

with the BCs

$z(0) = 0, \qquad z(1) = 0.$

Analytical solution of (25) is $z(\xi) = \xi^2 - \xi$. The proposed scheme discussed in Section 4 is used to solve the problem using a fractional

Table 3
Approximate solution, analytical solution and absolute error at k = 3, M = 4.

ξ	Analytical solution	Approximate solution	Absolute error
0.00	1.00000	1.00000	0.000000
0.10	1.10000	1.10000	0.000000
0.20	1.20000	1.20000	0.000000
0.30	1.30000	1.30000	0.000000
0.40	1.40000	1.40000	0.000000
0.50	1.50000	1.50000	0.000000
0.60	1.60000	1.60000	0.000000
0.70	1.70000	1.70000	0.000000
0.80	1.80000	1.80000	0.000000
0.90	1.90000	1.90000	0.000000
1.00	2.00000	2.00000	0.000000

Table 4
Absolute error, approximate solution and analytical solution at k = 2, M = 3.

ξ	Analytical solution	Approximate solution	Absolute error
0.0833	1.0833	1.0833	2.2204e − 16
0.2500	1.2500	1.2500	4.4409e − 16
0.4167	1.4167	1.4167	2.2204e − 16
0.5833	1.5833	1.5833	4.4409e − 16
0.7500	1.7500	1.7500	4.4409e − 16
0.9167	1.9167	1.9167	2.2204e − 16

Fibonacci wavelet. In Fig. 4, the analytical and present method's approximate solutions are compared at k = 2, M = 4 and $\alpha = 2$. The results make it abundantly evident that the current technique solutions are in excellent accord.

7. Conclusion

In this article, the fractional order Fibonacci wavelet based on the Fibonacci polynomial is used to solve the fractional BTE that is emerging in the study of fluid mechanics, along with utilising block-pulse functions. Firstly, the OMFI of a fractional Fibonacci wavelet is obtained. Further, by reducing the fractional BTE within

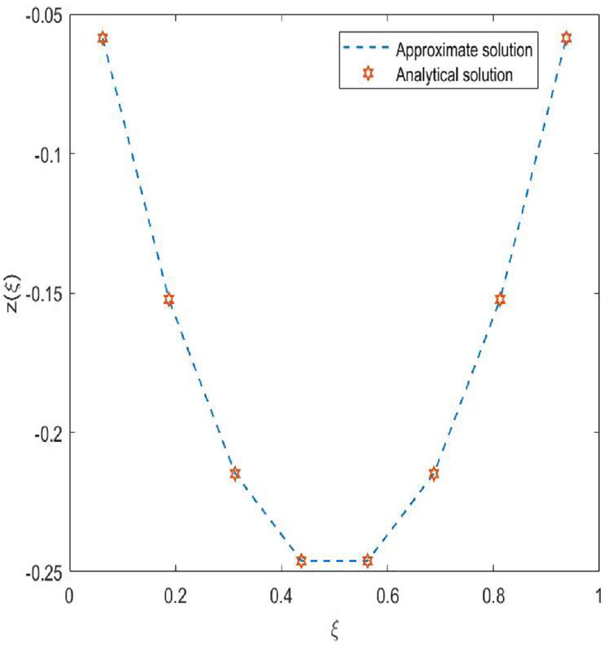

Fig. 4. Approximate and analytical solution comparison.

the algebraic equation and collocating them at defined points, we obtained the approximate solution. To illustrate the applicability of the suggested strategy, two cases are discussed. The figures are plotted to show the results of the current approach compared to the exact solutions and existing methods. The acquired findings demonstrate that a more accurate solution is produced as the level degree of the Fibonacci polynomial grows, as indicated in the tables. The fractional Fibonacci wavelet technique is widely applicable and may be used to address a variety of issues, including singular perturbation problems, optimization problems, and problems of nonlinear and linear systems with different conditions.

In the future, one can extend the methodology proposed in this paper for approximating the solution of fractional Riccati and fractional Pantograph differential equations and higher order fractional differential equations arising in various areas of biological and physical sciences.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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