mathematics

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Ulam Stability Results of Functional Equations in Modular Spaces and 2-Banach Spaces

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Abstract: In this work, we investigate the refined stability of the additive, quartic, and quintic functional equations in modular spaces with and without the Δ_2-condition using the direct method (Hyers method). We also examine Ulam stability in 2-Banach space using the direct method. Additionally, using a suitable counterexample, we eventually demonstrate that the stability of these equations fails in a certain case.

Keywords: quartic and quintic functional equations; modular spaces; 2-Banach spaces; refined stability

MSC: 39B52; 39B72; 39B82

Citation: Tamilvanan, K.; Alkhaldi, A.H.; Jakhar, J.; Chugh, R.; Jakhar, J.; Rassias, J.M. Ulam Stability Results of Functional Equations in Modular Spaces and 2-Banach Spaces. *Mathematics* **2023**, *11*, 371. https://doi.org/10.3390/math11020371

Academic Editor: Roman Ger

Received: 1 December 2022
Revised: 30 December 2022
Accepted: 4 January 2023
Published: 10 January 2023

1. Introduction

Functional equations play a crucial role in the study of stability problems in several frameworks. Ulam was the first who questioned the stability of group homomorphisms, and this established the foundation for work on stability problems. In the case that an equation admits a unique solution, we say that the equation is stable. For the Cauchy functional equation,

$$\phi(u_1 + u_2) = \phi(u_1) + \phi(u_2),$$

Ulam [1] formulated such a problem. Using Banach spaces, Hyers [2] solved this stability problem by considering Cauchy's functional equation. Hyers' work was expanded upon by Aoki [3] by assuming an unbounded Cauchy difference. Rassias [4] presented work on additive mapping, and these kinds of results are further presented by Găvruţa [5].

In 1950, Nakano [6] studied the theory of modular linear spaces. Since then, this theory has been thoroughly established by many authors, e.g., Amemiya [7], Koshi [8], Luxemburg [9], Mazur [10], Musielak [11], Orlicz [12], and Turpin [13]. Orlicz spaces [14] and the idea of interpolation [11,14] are both extensively applicable to the theory of modular spaces.

On the other hand, several mathematicians, using the fixed-point theorem of quasi-contraction mappings, investigated stability in modular spaces but without using the Δ_2-condition, as was proposed by Khamsi [15]. More recently, Sadeghi [16] established the stability results of functional equations with both the Fatou property and the Δ_2-condition together in modular spaces.

Firstly, we recall some definitions, notations, and usual properties of the theory of given spaces.

Definition 1 ([16,17]). *Let S be a linear space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A generalized functional $\rho : S \to [0, \infty)$ is called* modular *if for any given $u, v \in S$, the following conditions hold:*
(M_1) $\rho(u) = 0 \Leftrightarrow u = 0$,
(M_2) $\rho(\epsilon u) = \rho(u)$ *for any scalar* ϵ *with* $|\epsilon| = 1$,
(M_3) $\rho(\epsilon_1 u + \epsilon_2 v) \leq \rho(u) + \rho(v)$ *for any scalars* $\epsilon_1, \epsilon_2 \geq 0$ *with* $\epsilon_1 + \epsilon_2 = 1$.

If the condition (M_3) is replaced by
(M_3') $\rho(\epsilon_1 u + \epsilon_2 v) \leq \epsilon_1 \rho(u) + \epsilon_2 \rho(v)$ for any scalars $\epsilon_1, \epsilon_2 \geq 0$ with $\epsilon_1 + \epsilon_2 = 1$,
then ρ is called a convex modular. Furthermore, the vector space induced by a modular ρ,

$$S_\rho = \{u \in S : \rho(au) \to 0 \;\; as \;\; a \to 0\}$$

is a modular space.

Definition 2 ([16,17]). *Let S_ρ be a modular space and $\{u_\alpha\}$ be a sequence in S_ρ. Then,*

(1) *$\{u_\alpha\}$ is ρ-convergent to a point $u \in S_\rho$, and we write $u_\alpha \to u$ if $\rho(u_\alpha - u) \to 0 \;\; as \;\; \alpha \to \infty$.*
(2) *$\{u_\alpha\}$ is said to be ρ-Cauchy if for any $\varepsilon > 0$ one has $\rho(u_\alpha - u_\beta) < \varepsilon$ for sufficiently large $\alpha, \beta \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers.*
(3) *$K \subseteq S_\rho$ is called as a ρ-complete if any ρ-Cauchy sequence is ρ-convergent in $\mathbb{K}$.*

The modular ρ has the Fatou property if $\rho(u) \leq \lim\ \inf_{\alpha\to\infty} \rho(u_\alpha)$, whereas the sequence $\{u_\alpha\}$ is ρ-convergent to u in modular space S_ρ and conversely.

Proposition 1 ([18]). *In modular spaces,*

(1) *if $u_\alpha \to u$ and c is a constant vector, then $u_\alpha + c \to u + c$.*
(2) *if $u_\alpha \to u$ and $v_\alpha \to v$, then $\epsilon_1 u_\alpha + \epsilon_2 v_\alpha \to \epsilon_1 u + \epsilon_2 v$, where $\epsilon_1 + \epsilon_2 \leq 1$ and $\epsilon_1, \epsilon_2 \geq 0$.*

Remark 1. *Assume that ρ satisfies a Δ_2-condition with Δ_2-constant $k > 0$ and is convex. If $k < 2$, then $\rho(u) \leq k\rho(\frac{u}{2}) \leq \frac{k}{2}\rho(u)$, which indicates $\rho = 0$. So, we should have the Δ_2-constant $k \geq 2$ if ρ is convex modular.*

It should be noted that the convergence of a sequence $\{u_n\}$ to u does not imply that $\{\alpha u_n\}$ converges to αu if α is selected from the equivalent scalar field with $|\alpha| > 1$ in modular spaces. This is because the multiples of the convergent sequence $\{u_n\}$ naturally converges in modular spaces. Several mathematicians have investigated stability without using the Δ_2-condition using the fixed-point approach of quasi-contraction functions in modular spaces, which is the method introduced by Khamsi [15]. The stability findings of functional equations were established by Sadeghi [16] recently using the Fatou property and the Δ_2-condition in modular spaces.

In this paper, the refined stability of the additive functional equation, the quartic functional equation, and the quintic functional equation

$$\chi\left(\frac{v_1 - v_2}{n} + v_3\right) + \chi\left(\frac{v_2 - v_3}{n} + v_1\right) + \chi\left(\frac{v_3 - v_1}{n} + v_2\right) = \chi(v_1 + v_2 + v_3),$$

$$\chi(2v_1 + v_2) + \chi(2v_1 - v_2) = 4\chi(v_1 - v_2) + 4\chi(v_1 + v_2) + 24\chi(v_1) - 6\chi(v_2),$$

and

$$\chi(v_1 + 3v_2) - 5\chi(v_1 + 2v_2) + 10\chi(v_1 + v_2) - 10\chi(v_1) + 5\chi(v_1 - v_2) - \chi(v_1 - 2v_2) = 120\chi(v_2),$$

respectively, in modular spaces with and without the Δ_2-condition are investigated using the direct method. The Ulam stability in 2-Banach spaces is also investigated. Using a suitable counterexample, we eventually demonstrate that the stability of these equations fails in a certain case.

2. Stability Results in Modular Spaces

In this section, we use the direct method to investigate the stability results of additive functional equation, quartic functional equation, and quintic functional equation, which are improved forms of Wongkum ([19,20]) and Sadeghi [16]. Consider that W is a linear space and S_ρ is a complete convex modular space.

2.1. Stability Results of Additive Functional Equation

In 2013, Kim [21] investigated the stability of the additive functional equation in fuzzy Banach spaces. Motivated by the method and direction of research of Kim [21], an effort has been made here to investigate the stability of the additive functional equation

$$\chi\left(\frac{v_1 - v_2}{n} + v_3\right) + \chi\left(\frac{v_2 - v_3}{n} + v_1\right) + \chi\left(\frac{v_3 - v_1}{n} + v_2\right) = \chi(v_1 + v_2 + v_3), \tag{1}$$

where any integer $n > 0$, in modular spaces without Δ_2-conditions.

For notational handiness, we define a mapping $\chi : W \to S_\rho$ as

$$\Delta\chi(v_1, v_2, v_3) = \chi\left(\frac{v_1 - v_2}{n} + v_3\right) + \chi\left(\frac{v_2 - v_3}{n} + v_1\right) + \chi\left(\frac{v_3 - v_1}{n} + v_2\right) - \chi(v_1 + v_2 + v_3),$$

for all $v_1, v_2, v_3 \in W$.

Theorem 1. *If there exists a mapping $\phi : W^3 \to [0, \infty)$ such that*

$$\phi(v_1, v_2, v_3) = \sum_{\alpha=1}^{\infty} \frac{1}{3^\alpha} \varphi(3^{\alpha-1}v_1, 3^{\alpha-1}v_2, 3^{\alpha-1}v_3) < \infty \tag{2}$$

and a mapping $\chi : W \to S_\rho$ with $\chi(0) = 0$ and

$$\rho(\Delta\chi(v_1, v_2, v_3)) \leq \phi(v_1, v_2, v_3), \tag{3}$$

for all $v_1, v_2, v_3 \in W$, then there exists a unique additive mapping $Q : W \to S_\rho$ satisfying

$$\rho(\chi(v_1) - Q(v_1)) \leq \phi(v_1, v_1, v_1), \tag{4}$$

for all $v_1 \in W$.

Proof. Setting $v_1 = v_2 = v_3$ in (3) and letting $\Phi(v_1) = \phi(v_1, v_1, v_1)$, we obtain

$$\rho(3\chi(v_1) - \chi(3v_1)) \leq \phi(v_1, v_1, v_1) = \Phi(v_1). \tag{5}$$

Hence,

$$\rho\left(\chi(v_1) - \frac{1}{3}\chi(3v_1)\right) \leq \frac{1}{3}\Phi(v_1). \tag{6}$$

Then, by mathematical induction, we have

$$\rho\left(\chi(v_1) - \frac{\chi(3^\alpha v_1)}{3^\alpha}\right) \leq \sum_{i=1}^{\alpha} \frac{1}{3^i}\Phi(3^{i-1}v_1) \tag{7}$$

for all $v_1 \in W$ and all natural numbers α. Certainly, the case $\alpha = 1$ arises from (6). If the inequality (7) holds for $\alpha \in \mathbb{N}$, then we obtain

$$\begin{aligned}
\rho\left(\frac{\chi(3^{\alpha+1}v_1)}{3^{\alpha+1}}-\chi(v_1)\right) &= \rho\left(\frac{1}{3}\left(\chi(3v_1)-\frac{\chi(3^{\alpha}3v_1)}{3^{\alpha}}\right)+\frac{1}{3}(3\chi(v_1)-\chi(3v_1))\right)\\
&\leq \frac{1}{3}\rho\left(\frac{\chi(3^{\alpha}3v_1)}{3^{\alpha}}-\chi(3v_1)\right)+\frac{1}{3}\rho(\chi(3v_1)-3\chi(v_1))\\
&\leq \frac{1}{3}\sum_{i=1}^{\alpha}\frac{1}{3^i}\Phi(3^iv_1)+\frac{1}{3}\Phi(v_1)\\
&= \sum_{i=1}^{\alpha}\frac{1}{3^{i+1}}\Phi(3^iv_1)+\frac{1}{3}\Phi(v_1)\\
&= \sum_{i=1}^{\alpha+1}\frac{1}{3^i}\Phi(3^{i-1}v_1).
\end{aligned}$$

Hence, the inequality (7) holds for every $\alpha \in \mathbb{N}$. Let β and γ be natural numbers with $\gamma > \beta$. By (7), we have

$$\begin{aligned}
\rho\left(\frac{\chi(3^{\gamma}v_1)}{3^{\gamma}}-\frac{\chi(3^{\beta}v_1)}{3^{\beta}}\right) &= \rho\left(\frac{1}{3^{\beta}}\left(\frac{\chi(3^{\gamma-\beta}3^{\beta}v_1)}{3^{\gamma-\beta}}-\chi(3^{\beta}v_1)\right)\right)\\
&\leq \frac{1}{3^{\beta}}\sum_{i=1}^{\gamma-\beta}\frac{\Phi(3^{i-1}3^{\beta}v_1)}{3^i}\\
&= \sum_{i=1}^{\gamma-\beta}\frac{\Phi(3^{\beta+i-1}v_1)}{3^{\beta+i}}\\
&= \sum_{\alpha=\beta+1}^{\gamma}\frac{\Phi(3^{\alpha-1}v_1)}{3^{\alpha}}.
\end{aligned} \tag{8}$$

From (2) and (8), it is implied that the sequence $\left\{\frac{\chi(3^{\gamma}v_1)}{3^{\gamma}}\right\}$ is a ρ-Cauchy sequence in S_ρ. The ρ-completeness of S_ρ confers its ρ-convergence. Now, we can define a mapping $Q : W \to S_\rho$ by

$$Q(v_1) := \lim_{\gamma\to\infty}\frac{\chi(3^{\gamma}v_1)}{3^{\gamma}}, \quad v_1 \in W. \tag{9}$$

Thus,

$$\begin{aligned}
\rho\left(\frac{3Q(v_1)-Q(3v_1)}{3^3}\right) &= \rho\left(\frac{1}{3^3}\left(\frac{\chi(3^{\gamma+1}v_1)}{3^{\gamma}}-Q(3v_1)\right)+\frac{1}{3}\left(\frac{1}{3}Q(v_1)-\frac{1}{3}\frac{\chi(3^{\gamma+1}v_1)}{3^{\gamma+1}}\right)\right)\\
&\leq \frac{1}{3^3}\rho\left(Q(3v_1)-\frac{\chi(3^{\gamma+1}v_1)}{3^{\gamma}}\right)+\frac{1}{9}\rho\left(\frac{\chi(3^{\gamma+1}v_1)}{3^{\gamma+1}}-Q(v_1)\right)
\end{aligned} \tag{10}$$

for all $v_1 \in W$. Then, by (9), the right-hand side of (10) tends toward 0 as $\gamma \to \infty$. Thus, we obtain that

$$Q(3v_1) = 3Q(v_1) \tag{11}$$

for all $v_1 \in W$. We observe that for all $\gamma \in \mathbb{N}$, by (11) we have

$$\begin{aligned}
rl\rho(\chi(v_1)-Q(v_1)) &=\rho\left(\sum_{\alpha=1}^{\gamma}\frac{3\chi(3^{\alpha-1}v_1)-\chi(3^{\alpha}v_1)}{3^{\alpha}}+\left(\frac{\chi(3^{\gamma}v_1)}{3^{\gamma}}-Q(v_1)\right)\right)\\
&=\rho\left(\sum_{\alpha=1}^{\gamma}\frac{3\chi(3^{\alpha-1}v_1)-\chi(3^{\alpha}v_1)}{3^{\alpha}}+\frac{1}{3}\left(\frac{\chi(3^{\gamma-1}3v_1)}{3^{\gamma-1}}-Q(3v_1)\right)\right).
\end{aligned} \tag{12}$$

Because $\sum_{\alpha=1}^{\gamma} \frac{1}{3^{\alpha}} + \frac{1}{3} < 1$, from inequalities (5) and (12) we obtain that

$$\rho(\chi(v_1) - Q(v_1)) \leq \sum_{\alpha=1}^{\gamma} \frac{1}{3^{\alpha}} \rho\Big(3\chi(3^{\alpha-1}v_1) - \chi(3^{\alpha}v_1)\Big) + \frac{1}{3}\rho\left(\frac{\chi(3^{\gamma-1}3v_1)}{3^{\gamma-1}} - Q(3v_1)\right)$$

$$\leq \sum_{\alpha=1}^{\gamma} \Phi(3^{\alpha-1}v_1) + \frac{1}{3}\rho\left(\frac{\chi(3^{\gamma-1}3v_1)}{3^{\gamma-1}} - Q(3v_1)\right) \tag{13}$$

$$= \sum_{\alpha=1}^{\gamma} \frac{1}{3^{\alpha}} \varphi(3^{\alpha-1}v_1, 3^{\alpha-1}v_1, 3^{\alpha-1}v_1) + \frac{1}{3}\rho\left(\frac{\chi(3^{\gamma-1}3v_1)}{3^{\gamma-1}} - Q(3v_1)\right).$$

Taking the limit $\gamma \to \infty$ in (13), we obtain

$$\rho(\chi(v_1) - Q(v_1)) \leq \Phi(v_1, v_1, v_1),$$

for all $v_1 \in W$. Thus, we obtain (4). Now, we want to prove that the function Q is additive. We observe that

$$\begin{aligned} rcl\rho\left(\frac{1}{3^i}\Delta\chi\Big(3^i v_1, 3^i v_2, 3^i v_3\Big)\right) &\leq \frac{1}{3^i}\rho\left(\Delta\chi\Big(3^i v_1, 3^i v_2, 3^i v_3\Big)\right) \\ &\leq \frac{1}{3^i}\varphi(3^i v_1, 3^i v_2, 3^i v_3) \to 0 \ \ as \ \ i \to \infty, \end{aligned} \tag{14}$$

for all $v_1, v_2, v_3 \in W$. By the inequality (14), we have

$$\rho(\Delta Q(v_1, v_2, v_3)) \to \quad 0 \ \ as \ \ i \to \infty.$$

Hence, we have

$$\Delta Q(v_1, v_2, v_3) = 0.$$

Thus, it proves that the function Q is additive. Next, we want to prove that the mapping Q is unique. Suppose that Q_1 and Q_2 are additive mappings which satisfy (4). Then,

$$\begin{aligned} \rho\left(\frac{Q_1(v_1) - Q_2(v_1)}{2}\right) &= \rho\left(\frac{1}{2}\left(\frac{Q_1(3^{\alpha}v_1)}{3^{\alpha}} - \frac{\chi(3^{\alpha}v_1)}{3^{\alpha}}\right) + \frac{1}{2}\left(\frac{\chi(3^{\alpha}v_1)}{3^{\alpha}} - \frac{Q_2(3^{\alpha}v_1)}{3^{\alpha}}\right)\right) \\ &\leq \frac{1}{2}\rho\left(\frac{Q_1(3^{\alpha}v_1)}{3^{\alpha}} - \frac{\chi(3^{\alpha}v_1)}{3^{\alpha}}\right) + \frac{1}{2}\rho\left(\frac{\chi(3^{\alpha}v_1)}{3^{\alpha}} - \frac{Q_2(3^{\alpha}v_1)}{3^{\alpha}}\right) \\ &\leq \frac{1}{2}\frac{1}{3^{\alpha}}\Big(\rho(Q_1(3^{\alpha}v_1) - \chi(3^{\alpha}v_1)) + \rho(Q_2(3^{\alpha}v_1) - \chi(3^{\alpha}v_1))\Big) \\ &\leq \frac{1}{3^{\alpha}}\Phi(3^{\alpha}v_1, 3^{\alpha}v_1, 3^{\alpha}v_1) \\ &\leq \sum_{k=\alpha+1}^{\infty} \frac{1}{3^k}\varphi(3^{k-1}v_1, 3^{k-1}v_1, 3^{k-1}v_1) \\ &\to 0 \ \ as \ \ \alpha \to \infty. \end{aligned}$$

This implies that $Q_1 = Q_2$. Hence, the proof of the theorem is now completed. □

Corollary 1. *If there exists a mapping $\chi : W \to S_{\rho}$ such that $\chi(0) = 0$ and*

$$\rho(\Delta\chi(v_1, v_2, v_3)) \leq \varepsilon, \tag{15}$$

for all $v_1, v_2, v_3 \in W$, then there exists a unique additive mapping $Q : W \to S\rho$ satisfying

$$\rho(\chi(v_1) - Q(v_1)) \leq \frac{\varepsilon}{2}$$

for all $v_1 \in W$.

Corollary 2. *If there exists a mapping* $\chi : W \to S_\rho$ *such that* $\chi(0) = 0$ *and*

$$\rho(\Delta\chi(v_1, v_2, v_3)) \leq \theta(\|v_1\|^q + \|v_2\|^q + \|v_3\|^q),$$

for all $v_1, v_2, v_3 \in W$, *and* $\theta > 0$ *and* $0 < q < 1$, *then there exists a unique additive mapping* $Q : W \to S\rho$ *satisfying*

$$\rho(\chi(v_1) - Q(v_1)) \leq \frac{3\theta}{3 - 3^q}\|v_1\|^q$$

for all $v_1 \in W$.

The following theorem gives another stability of Theorem 1 in modular spaces with the Δ_2-condition.

Theorem 2. *Suppose that* S *is a linear space and* S_ρ *satisfies the* Δ_2*-condition with the function* $\phi : W^3 \to [0, \infty)$ *for which there exists a mapping* $\chi : W \to S_\rho$ *such that*

$$\rho(\Delta\chi(v_1, v_2, v_3)) \leq \phi(v_1, v_2, v_3),$$

and

$$\lim_{\alpha\to\infty} k^\alpha \phi\left(\frac{v_1}{3^\alpha}, \frac{v_2}{3^\alpha}, \frac{v_3}{3^\alpha}\right) = 0 \quad and \quad \sum_{i=1}^{\infty}\left(\frac{k^2}{3}\right)^i \phi\left(\frac{v_1}{3^\alpha}, \frac{v_1}{3^\alpha}, \frac{v_1}{3^\alpha}\right) < \infty$$

for all $v_1, v_2 \in W$. *Then, there exists a unique additive mapping* $Q : W \to S_\rho$, *defined as*

$$Q(v_1) = \lim_{\alpha\to\infty} 3^\alpha \chi\left(\frac{v_1}{3^\alpha}\right)$$

and

$$\rho(\chi(v_1) - Q(v_1)) \leq \frac{\tau}{3k}\sum_{i=1}^{\infty}\left(\frac{k^2}{3}\right)^i \phi\left(\frac{v_1}{2^i}, \frac{v_1}{2^i}, \frac{v_1}{2^i}\right)$$

for all $v_1 \in W$.

Proof. Because ρ fulfills the Δ_2-condition with τ, the inequality (3) implies

$$\rho(\Delta\chi(v_1, v_2, v_3)) \leq \tau\phi(v_1, v_2, v_3),$$

for all $v_1, v_2, v_3 \in W$. Then the conclusion directly follows from the proof of the Theorem 3. □

2.2. Stability Results of Quartic Functional Equation

In this subsection, we investigate the refined stability and the Ulam stability of the quartic functional equation

$$\chi(2v_1 + v_2) + \chi(2v_1 - v_2) = 4\chi(v_1 - v_2) + 4\chi(v_1 + v_2) + 24\chi(v_1) - 6\chi(v_2), \tag{16}$$

in modular spaces S_ρ, without using the Fatou property.

For notational simplicity, we can define a mapping $\chi : W \to S_\rho$ by

$$\Delta\chi(v_1, v_2) = \chi(2v_1 + v_2) + 6\chi(v_2) + \chi(2v_1 - v_2) - 4\chi(v_1 + v_2) - 24\chi(v_1) - 4\chi(v_1 - v_2),$$

for all $v_1, v_2 \in W$.

Theorem 3. *Suppose that* S *is a linear space and* S_ρ *satisfies the* Δ_2*-condition with a mapping* $\phi : W \times W \to [0, \infty)$, *for which there exists a mapping* $\chi : W \to S_\rho$ *such that*

$$\rho(\Delta\chi(v_1, v_2)) \leq \phi(v_1, v_2), \tag{17}$$

and

$$\lim_{\alpha\to\infty} k^{4\alpha}\phi\left(\frac{v_1}{2^\alpha}, \frac{v_2}{2^\alpha}\right) = 0 \quad and \quad \sum_{i=1}^{\infty}\left(\frac{k^5}{2}\right)^i \phi\left(\frac{v_1}{2^\alpha}, 0\right) < \infty$$

for all $v_1, v_2 \in W$. Then, there exists a unique quartic mapping $Q : W \to S_\rho$, defined as

$$Q(v_1) = \lim_{\alpha\to\infty} 2^{4\alpha}\chi\left(\frac{v_1}{2^\alpha}\right)$$

and

$$\rho(\chi(v_1) - Q(v_1)) \leq \frac{1}{2k}\sum_{i=1}^{\infty}\left(\frac{k^5}{2}\right)^i \phi\left(\frac{v_1}{2^i}, 0\right) \tag{18}$$

for all $v_1 \in W$.

Proof. Let us consider $\chi(0) = 0$ in view of $\phi(0, 0) = 0$ along the convergence of

$$\sum_{i=1}^{\infty}\left(\frac{k^5}{2}\right)^i \phi(0, 0) < \infty.$$

We suppose $v_2 = 0$ in (17) to have

$$\rho(2\chi(2v_1) - 32\chi(v_1)) \leq \phi(v_1, 0)$$

for all $v_1 \in W$. Because $\sum_{i=1}^{\infty} \frac{1}{2^i} \leq 1$, by the Δ_2-condition of ρ, the next functional inequality can be shown as

$$\begin{aligned} \rho\left(\chi(v_1) - 2^{4\alpha}\chi\left(\frac{v_1}{2^\alpha}\right)\right) &= \rho\left(\sum_{i=1}^{\alpha}\frac{1}{2^i}\left(2^{5i-4}\chi\left(\frac{v_1}{2^{i-1}}\right) - 2^{5i}\chi\left(\frac{v_1}{2^i}\right)\right)\right) \\ &\leq \frac{1}{k^4}\sum_{i=1}^{\alpha}\left(\frac{k^5}{2}\right)^i \phi\left(\frac{v_1}{2^i}, 0\right), \ \forall\, v_1 \in W. \end{aligned} \tag{19}$$

Now, replacing v_1 with $2^{-\beta}v_1$ in (19), we obtain the conclusion that the series of (17) converges, and

$$\begin{aligned} \rho\left(2^{4\beta}\chi\left(\frac{v_1}{2^\beta}\right) - 2^{4(\beta+\alpha)}\chi\left(\frac{v_1}{2^{\beta+\alpha}}\right)\right) &\leq k^{4\beta}\rho\left(\chi\left(\frac{v_1}{2^\beta}\right) - 2^{4\alpha}\chi\left(\frac{v_1}{2^{\beta+\alpha}}\right)\right) \\ &\leq k^{4\beta-4}\sum_{i=1}^{\alpha}\left(\frac{k^5}{2}\right)^i \phi\left(\frac{v_1}{2^{i+\beta}}, \frac{v_1}{2^{i+\beta}}\right) \\ &\leq \frac{2^\beta}{k^{\beta+4}}\sum_{i=\beta+1}^{\alpha+\beta}\left(\frac{k^5}{2}\right)^i \phi\left(\frac{v_1}{2^i}, 0\right) \end{aligned}$$

for all $v_1 \in W$, which tends to 0 as $\beta \to \infty$ because $\frac{2}{k} \leq 1$. Because the space S_ρ is ρ-complete, for all $v_1 \in W$ the sequence $\left\{2^{4\alpha}\chi\left(\frac{v_1}{2^\alpha}\right)\right\}$ is a ρ-Cauchy sequence, and it is ρ-convergent in S_ρ. Then, we can define a mapping $Q : W \to S_\rho$ as

$$\begin{aligned} \rho\left(\lim_{\alpha\to\infty} 2^{4\alpha}\chi\left(\frac{v_1}{2^\alpha}\right)\right) &= Q(v_1), \\ i.e., \quad \lim_{\alpha\to\infty}\rho\left(2^{4\alpha}\chi\left(\frac{v_1}{2^\alpha}\right) - Q(v_1)\right) &= 0 \end{aligned}$$

for all $v_1 \in W$. Therefore, we obtained the inequality without using the Fatou property from the Δ_2-condition that

$$\rho(\chi(v_1) - Q(v_1)) \leq \frac{1}{2}\rho\Big(2\chi(v_1) - 2(2^{4\alpha})\chi\Big(\frac{v_1}{2^\alpha}\Big)\Big) + \frac{1}{2}\rho\Big(2(2^{4\alpha})\chi\Big(\frac{v_1}{2^\alpha}\Big) - 2Q(v_1)\Big)$$

$$\leq \frac{k}{2}\rho\Big(\chi(v_1) - 2^{4\alpha}\chi\Big(\frac{v_1}{2^\alpha}\Big)\Big) + \frac{k}{2}\rho\Big(2^{4\alpha}\chi\Big(\frac{v_1}{2^\alpha}\Big) - Q(v_1)\Big)$$

$$\leq \frac{1}{2k^3}\sum_{i=1}^{\alpha}\Big(\frac{k^5}{2}\Big)^i \phi\Big(\frac{v_1}{2^i}, 0\Big) + \frac{k}{2}\rho\Big(2^{4\alpha}\chi\Big(\frac{v_1}{2^\alpha}\Big) - Q(v_1)\Big)$$

holds for all $v_1 \in W$ and all natural numbers $\alpha > 1$. Taking $\alpha \to \infty$, we obtain the estimate (18) of χ as Q. Replacing (v_1, v_2) with $(2^{-\alpha}v_1, 2^{-\alpha}v_2)$ in (17), we see that

$$\Big(2^{4\alpha}\Delta\chi\Big(\frac{v_1}{2^\alpha}, \frac{v_2}{2^\alpha}\Big)\Big) \leq k^{4\alpha}\phi\Big(\frac{v_1}{2^\alpha}, \frac{v_2}{2^\alpha}\Big) \quad \to \quad 0 \quad \text{as} \quad \alpha \to \infty.$$

Therefore, it develops with the convexity of ρ that

$$\begin{aligned}
&\rho\Big(\frac{1}{41}Q(2v_1+v_2) + \frac{1}{41}Q(2v_1-v_2) + \frac{6}{41}Q(v_2) - \frac{4}{41}Q(v_1-v_2) - \frac{4}{41}Q(v_1+v_2) - \frac{24}{41}Q(v_1)\Big) \\
&\leq \frac{1}{41}\rho\Big(Q(2v_1+v_2) - 2^{4\alpha}\chi\Big(\frac{2v_1+v_2}{2^\alpha}\Big)\Big) + \frac{1}{41}\rho\Big(Q(2v_1-v_2) - 2^{4\alpha}\chi\Big(\frac{2v_1-v_2}{2^\alpha}\Big)\Big) \\
&+\frac{6}{41}\rho\Big(Q(v_2) - 6(2^{4\alpha})\chi\Big(\frac{v_2}{2^\alpha}\Big)\Big) + \frac{4}{41}\rho\Big(Q(v_1-v_2) - 4(2^{4\alpha})\chi\Big(\frac{v_1-v_2}{2^\alpha}\Big)\Big) \\
&+\frac{4}{41}\rho\Big(Q(v_1+v_2) - 4(2^{4\alpha})\chi\Big(\frac{v_1+v_2}{2^\alpha}\Big)\Big) + \frac{24}{41}\rho\Big(Q(v_1) - 24(2^{4\alpha})\chi\Big(\frac{v_1}{2^\alpha}\Big)\Big) \\
&+\frac{1}{41}\rho\Big(2^{4\alpha}\chi\Big(\frac{2v_1+v_2}{2^\alpha}\Big) + 2^{4\alpha}\chi\Big(\frac{2v_1-v_2}{2^\alpha}\Big) + 6(2^{4\alpha})\chi\Big(\frac{v_2}{2^\alpha}\Big) \\
&-4(2^{4\alpha})\chi\Big(\frac{v_1-v_2}{2^\alpha}\Big) - 4(2^{4\alpha})\chi\Big(\frac{v_1+v_2}{2^\alpha}\Big) - 24(2^{4\alpha})\chi\Big(\frac{v_1}{2^\alpha}\Big)\Big)
\end{aligned}$$

for all $v_1, v_2 \in W$ and all natural numbers $\alpha > 1$. Therefore, the function Q is quartic as $\alpha \to \infty$.

To show the uniqueness of the function Q, we prove that there is a quartic mapping $Q' : W \to S_\rho$ satisfying

$$\rho(\chi(v_1) - Q'(v_1)) \leq \frac{1}{2k}\sum_{i=1}^{\alpha}\Big(\frac{k^5}{2}\Big)^i \phi\Big(\frac{v_1}{2^i}, 0\Big)$$

for all $v_1 \in W$. Then, we see from the equality $Q(2^{-\alpha}v_1) = 2^{-4\alpha}Q(v_1)$ and $Q'(2^{-\alpha}v_1) = 2^{-4\alpha}Q'(v_1)$ that

$$\begin{aligned}
&\rho(Q(v_1) - Q'(v_1)) \\
\leq\ & \frac{1}{2}\rho\Big(2(2^{4\alpha})Q\Big(\frac{v_1}{2^\alpha}\Big) - 2(2^{4\alpha})\chi\Big(\frac{v_1}{2^\alpha}\Big)\Big) + \frac{1}{2}\rho\Big(2(2^{4\alpha})\chi\Big(\frac{v_1}{2^\alpha}\Big) - 2(2^{4\alpha})Q'\Big(\frac{v_1}{2^\alpha}\Big)\Big) \\
\leq\ & \frac{k^{4\alpha+1}}{2}\rho\Big(Q\Big(\frac{v_1}{2^\alpha}\Big) - \chi\Big(\frac{v_1}{2^\alpha}\Big)\Big) + \frac{k^{4\alpha+1}}{2}\rho\Big(\chi\Big(\frac{v_1}{2^\alpha}\Big) - Q'\Big(\frac{v_1}{2^\alpha}\Big)\Big) \\
\leq\ & \frac{k^{4\alpha}}{2}\sum_{i=1}^{\alpha}\Big(\frac{k^5}{2}\Big)^i \phi\Big(\frac{v_1}{2^{\alpha+i}}, 0\Big) \\
\leq\ & \frac{2^{\alpha-1}}{k^{\alpha+2}}\sum_{i=1}^{\alpha}\Big(\frac{k^5}{2}\Big)^i \phi\Big(\frac{v_1}{2^i}, 0\Big)
\end{aligned}$$

for all $v_1 \in W$ and all sufficiently large natural numbers α. Taking the limit as $\alpha \to \infty$, we obtain our required result. $\square$

Corollary 3. *Suppose that S_ρ satisfies the Δ_2-condition and let W be a normed space with a norm $\|.\|$. If there is a real number $\theta > 0$, $q > log_2 \frac{k^5}{2}$ and a mapping $\chi : W \to S_\rho$ such that*

$$\rho(\Delta\chi(v_1, v_2)) \leq \theta(\|v_1\|^q + \|v_2\|^q)$$

for all $v_1, v_2 \in W$, then there exists a unique quartic mapping $Q : W \to S_\rho$ satisfying

$$(\chi(v_1) - Q(v_1)) \leq \frac{k^4\theta}{2^{q+1} - k^5},$$

for all $v_1 \in W$.

The following theorem gives an alternative stability of Theorem 3 in modular spaces without using the Fatou property or the Δ_2-condition.

Theorem 4. *Let there exist a mapping $\chi : W \to S_\rho$ that satisfies (17), and suppose a mapping $\phi : W^2 \to [0, \infty)$ exists such that*

$$\lim_{\alpha\to\infty} \frac{\phi(2^\alpha v_1, 2^\alpha v_2)}{2^{4\alpha}} = 0, \qquad \sum_{i=1}^{\infty} \frac{\phi(2^i v_1, 0)}{2^{4i}} < \infty$$

for all $v_1, v_2 \in W$. Then, there exists a unique quartic mapping $Q : W \to S_\rho$ satisfying

$$\rho\left(\chi(v_1) - \frac{1}{5}\chi(0) - Q(v_1)\right) \leq \frac{1}{2^4} \sum \frac{\phi(2^i v_1, 0)}{2^{4i}} \tag{20}$$

for all $v_1, v_2 \in W$.

Proof. Taking $v_2 = 0$ in (17), one has

$$\rho(2\chi(2v_1) - 32\chi(v_1)) \leq \phi(v_1, 0),$$

where $\chi'(v_1) = \chi(v_1) - \dfrac{\chi(0)}{5}$. Then, we obtain the following inequalities and also a convexity of ρ, $\sum \dfrac{1}{2^{4(i+1)}} \leq 1$

$$\begin{aligned}
\left(\chi'(v_1) - \frac{\chi'(2^\alpha v_1)}{2^{4\alpha}}\right) &\leq \rho\left(\sum_{0\leq i\leq \alpha-1}\left(\frac{2^4\chi'(2^i v_1) - \chi'(2^{i+1}v_1)}{2^{4(1+i)}}\right)\right) \\
&\leq \sum_{0\leq i\leq \alpha-1} \frac{\rho(2^4\chi'(2^i v_1) - \chi'(2^{i+1}v_1))}{2^{4(1+i)}} \\
&\leq \frac{1}{2^4}\sum_{0\leq i\leq \alpha-1} \frac{\phi(2^i v_1, 0)}{2^{4i}}
\end{aligned}$$

for all $v_1 \in W$ and all $\alpha \in \mathbb{N}$. Then, one has a ρ-Cauchy sequence $\left\{\dfrac{\chi'(2^\alpha v_1)}{2^{4\alpha}}\right\}$ and the mapping $Q : W \to S_\rho$ is defined as

$$\begin{aligned}
\rho\left(\lim_{\alpha\to\infty} \frac{\chi'(2^\alpha v_1)}{2^{4\alpha}}\right) &= Q(v_1) \\
\textit{i.e.,} \quad \lim_{\alpha\to\infty} \rho\left(\frac{\chi'(2^\alpha v_1)}{2^{4\alpha}} - Q(v_1)\right) &= 0
\end{aligned}$$

for all $v_1 \in W$, without using the Δ_2-condition and the Fatou property. Moreover, it is obvious that the function Q satisfies the quartic functional equation in the proof that follows using the ideas from Theorem 3.

Now, we prove that (20) holds with utilization of the Fatou property and the Δ_2-condition. By using the convexity of ρ and

$$\sum_{0\leq i\leq \alpha-1} \frac{1}{2^{4(i+1)}} + \frac{1}{2^4} \leq 1,$$

we obtain the below inequality:

$$\begin{aligned} \rho(\chi'(v_1) - Q(v_1)) &= \rho\left(\sum_{i=1}^{\alpha-1}\left(\frac{2^4\chi'(2^i v_1) - \chi'(2^{i+1}v_1)}{2^{4(i+1)}}\right) + \frac{\chi'(2^\alpha v_1)}{2^{4\alpha}} - \frac{Q(2v_1)}{2^4}\right) \\ &\leq \sum_{i=0}^{\alpha-1} \frac{1}{2^{4(i+1)}}\rho(2^4\chi'(2^i v_1) - \chi'(2^{i+1}v_1)) + \frac{1}{2^4}\rho\left(\frac{\chi'(2^{\alpha-1}2v_1)}{2^{4(\alpha-1)}} - Q(2v_1)\right) \\ &\leq \frac{1}{2^4}\sum_{i=0}^{\alpha-1} \frac{1}{2^{4i}}\phi(2^i v_1, 0) + \frac{1}{2^4}\left(\frac{\chi'(2^{\alpha-1}2v_1)}{2^{4(\alpha-1)}} - Q(2v_1)\right) \end{aligned}$$

for all $v_1 \in W$ and all natural numbers $\alpha > 1$. Taking the limit as $\alpha \to \infty$, we obtain our needed result. □

Corollary 4. *Let there exist a mapping $\phi : W^2 \to [0, \infty)$ such that*

$$\lim_{\alpha\to\infty} \frac{\phi(2^\alpha v_1, 2^\alpha v_2)}{2^{4\alpha}} = 0, \quad \phi(2v_1, 0) \leq 2^4 L\phi(v_1, 0)$$

for all $v_1, v_2 \in W$ and for some $L \in (0, 1)$. If there exists a mapping $\chi : W \to S_\rho$ satisfies (17), then there exists a unique quartic mapping $Q : W \to S_\rho$ satisfying

$$\rho\left(\chi(v_1) - \frac{1}{5}\chi(0) - Q(v_1)\right) \leq \frac{1}{2^4(1-L)}\phi(v_1, 0)$$

for all $v_1 \in W$.

Corollary 5. *Suppose that W is a normed linear space with the norm $\|\cdot\|$. If there exists a real number $\theta > 0$, $\epsilon > 0$, $q \in (-\infty, 2)$ and a mapping $\chi : W \to S_\rho$ such that*

$$\rho(\Delta\chi(v_1, v_2)) \leq \theta(\|v_1\|^q + \|v_2\|^q) + \epsilon$$

for all $v_1, v_2 \in W$, then there exists a unique quartic mapping $Q : W \to S_\rho$ satisfying

$$\rho\left(\chi(v_1) - \frac{1}{5}\chi(0) - Q(v_1)\right) \leq \frac{2\theta}{2^4 - 2^q}\|v_1\|^q + \frac{\epsilon}{3}$$

for all $v_1 \in W$, where $v_1 \neq 0$ if $q < 0$.

2.3. Stability Results of Quintic Functional Equation

In this subsection, we investigate the refined stability and the Ulam stability of the quintic functional equation

$$\chi(v_1+3v_2) - 5\chi(v_1+2v_2) + 10\chi(v_1+v_2) - 10\chi(v_1) + 5\chi(v_1-v_2) - \chi(v_1-2v_2) - 120\chi(v_2) = 0 \quad (21)$$

in modular spaces S_ρ without using the Fatou property.

Theorem 5. *Let S be a linear space, and suppose that S_ρ satisfies the Δ_2-condition with the mapping $\phi : W^2 \to [0, \infty)$ for which a mapping $\chi : W \to S_\rho$ exists such that*

$$\left.\begin{array}{r}\rho(\chi(v_1+3v_2)-5\chi(v_1+2v_2)+10\chi(v_1+v_2)\\ -10\chi(v_1)+5\chi(v_1-v_2)-\chi(v_1-2v_2)-120\chi(v_2))\end{array}\right\}\le\phi(v_1,v_2), \tag{22}$$

and

$$\lim_{\alpha\to\infty}k^{5\alpha}\phi\left(\frac{v_1}{2^\alpha},\frac{v_2}{2^\alpha}\right)=0 \quad and \quad \sum_{i=1}^{\infty}\left(\frac{k^6}{2}\right)\phi\left(\frac{v_1}{2^i},\frac{v_1}{2^i}\right)<\infty$$

for all $v_1, v_2 \in W$, then there exists a unique quintic mapping $Q : W \to S_\rho$, defined as

$$Q(v_1)=\lim_{\alpha\to\infty}2^{5\alpha}\chi\left(\frac{v_1}{2^\alpha}\right)$$

and

$$\rho(\chi(v_1)-Q(v_1))\le\frac{1}{2k}\sum_{i=1}^{\infty}\left(\frac{k^6}{2}\right)^i\phi\left(\frac{v_1}{2^i},\frac{v_1}{2^i}\right) \tag{23}$$

for all $v_1 \in W$.

Proof. Initially, assume that $\chi(0)=0$ in view of $\phi(0,0)=0$ along the convergence of

$$\sum_{i=1}^{\infty}\left(\frac{k^6}{2}\right)\phi(0,0)<\infty.$$

We take $v_1=v_2$ in (22) to have

$$\rho(\chi(4v_1)-5\chi(3v_1)+10\chi(2v_1)-10\chi(v_1)-\chi(-v_1)-120\chi(v_1))\le\phi(v_1,v_1).$$

By the Δ_2-condition of ρ and $\sum_{i=1}^{\alpha}\frac{1}{2^i}\le 1$, the next functional inequality can be shown as

$$\begin{aligned}\rho\left(\chi(v_1)-2^{5\alpha}\chi\left(\frac{v_1}{2^\alpha}\right)\right) &= \rho\left(\sum_{i=1}^{\alpha}\frac{1}{2^i}\left(2^{6i-5}\chi\left(\frac{v_1}{2^{i-1}}\right)-2^{6i}\chi\left(\frac{v_1}{2^i}\right)\right)\right)\\ &\le \frac{1}{k^5}\sum_{i=1}^{\alpha}\left(\frac{k^6}{2}\right)^i\phi\left(\frac{v_1}{2^i},\frac{v_1}{2^i}\right)\end{aligned} \tag{24}$$

for all $v_1 \in W$. Now, replacing v_1 by $2^{-\beta}v_1$ in (24), we obtain the series of (22) converges and

$$\begin{aligned}\rho\left(2^{5\beta}\chi\left(\frac{v_1}{2^\beta}\right)-2^{5(\beta+\alpha)}\chi\left(\frac{v_1}{2^{\beta+\alpha}}\right)\right) &\le k^{5\beta}\rho\left(\chi\left(\frac{v_1}{2^\beta}\right)-2^{5\alpha}\chi\left(\frac{v_1}{2^{\beta+\alpha}}\right)\right)\\ &\le k^{5\beta-5}\sum_{i=1}^{\alpha}\left(\frac{k^6}{2}\right)^i\phi\left(\frac{v_1}{2^{i+\beta}},\frac{v_1}{2^{i+\beta}}\right)\\ &\le \frac{2^\beta}{k^{\beta+4}}\sum_{i=\beta+1}^{\alpha+\beta}\left(\frac{k^6}{2}\right)^i\phi\left(\frac{v_1}{2^i},\frac{v_1}{2^i}\right),\end{aligned}$$

for all $v_1 \in W$, which tends to 0 as $\beta\to\infty$ because $\frac{2}{k}\le 1$. Because the space S_ρ is ρ-complete, the sequence $\left\{2^{5\alpha}\chi\left(\frac{v_1}{2^\alpha}\right)\right\}$ is a ρ-Cauchy sequence for every $v_1 \in W$, and it is ρ-convergent in S_ρ. Hence, we can define a mapping $Q : W \to S_\rho$ by

$$Q(v_1)=\rho\left(\lim_{\alpha\to\infty}2^{5\alpha}\chi\left(\frac{v_1}{2^\alpha}\right)\right), \quad i.e., \quad \lim_{\alpha\to\infty}\rho\left(2^{5\alpha}\chi\left(\frac{v_1}{2^\alpha}\right)-Q(v_1)\right)=0$$

for all $v_1 \in W$. Therefore, without using the Fatou property from the Δ_2-condition, we have obtained that the inequality that

$$\rho(\chi(v_1)-Q(v_1))\le\frac{1}{2}\rho\left(2\chi(v_1)-2(2^{5\alpha})\chi\left(\frac{v_1}{2^\alpha}\right)\right)+\frac{1}{2}\rho\left(2(2^{5\alpha})\chi\left(\frac{v_1}{2^\alpha}\right)-2Q(v_1)\right)$$

$$\leq \frac{k}{2}\rho\Big(\chi(v_1) - 2^{5\alpha}\chi\Big(\frac{v_1}{2^\alpha}\Big)\Big) + \frac{k}{2}\rho\Big(2^{5\alpha}\chi\Big(\frac{v_1}{2^\alpha}\Big) - Q(v_1)\Big)$$
$$\leq \frac{1}{2k}\sum_{i=1}^{\alpha}\Big(\frac{k^6}{2}\Big)^i \phi\Big(\frac{v_1}{2^i}, \frac{v_1}{2^i}\Big) + \frac{k}{2}\rho\Big(2^{5\alpha}\chi\Big(\frac{v_1}{2^\alpha}\Big) - Q(v_1)\Big)$$

holds for all $v_1 \in W$ and all natural numbers $\alpha > 1$. Taking the limit $\alpha \to \infty$, we obtain the estimate (23) of χ by Q. Replacing (v_1, v_2) with $(2^{-\alpha}v_1, 2^{-\alpha}v_2)$ in (22), we see that

$$\rho\Big(2^{5\alpha}\chi\Big(\frac{v_1+3v_2}{2^\alpha}\Big) - 5(2^{5\alpha})\chi\Big(\frac{v_1+2v_2}{2^\alpha}\Big) + 10(2^{5\alpha})\chi\Big(\frac{v_1+v_2}{2^\alpha}\Big) - 10(2^{5\alpha})\chi\Big(\frac{v_1}{2^\alpha}\Big)$$
$$+5(2^{5\alpha})\chi\Big(\frac{v_1-v_2}{2^\alpha}\Big) - 2^{5\alpha}\chi\Big(\frac{v_1-2v_2}{2^\alpha}\Big) - 120(2^{5\alpha})\chi\Big(\frac{v_2}{2^\alpha}\Big)\Big) \leq k^{5\alpha}\phi\Big(\frac{v_1}{2^\alpha}, \frac{v_2}{2^\alpha}\Big),$$

which approaches 0 as $\alpha \to \infty$ for all v_1, v_2 in W. Thus, it develops from the convexity of ρ that

$$\begin{aligned}
&\rho\Big(\frac{1}{152}Q(v_1+3v_2) - \frac{5}{152}Q(v_1+2v_2) + \frac{10}{152}Q(v_1+v_2) - \frac{10}{152}Q(v_1) + \frac{5}{152}Q(v_1-v_2)\\
&\qquad - \frac{1}{152}Q(v_1-2v_2)) - \frac{120}{152}Q(v_2)\Big)\\
\leq\ & \frac{1}{152}\rho\Big(Q(v_1+3v_2) - 2^{5\alpha}\chi\Big(\frac{v_1+3v_2}{2^\alpha}\Big)\Big)\\
&+\frac{5}{152}\rho\Big(Q(v_1+2v_2) - 5(2^{5\alpha})\chi\Big(\frac{v_1+2v_2}{2^\alpha}\Big)\Big) + \frac{10}{152}\rho\Big(Q(v_1+v_2) - 10(2^{5\alpha})\chi\Big(\frac{v_1+v_2}{2^\alpha}\Big)\Big)\\
&+\frac{10}{152}\rho\Big(Q(v_1) - 10(2^{5\alpha})\chi\Big(\frac{v_1}{2^\alpha}\Big)\Big) + \frac{5}{152}\rho\Big(Q(v_1-v_2) - 5(2^{5\alpha})\chi\Big(\frac{v_1-v_2}{2^\alpha}\Big)\Big)\\
&+\frac{1}{152}\rho\Big(Q(v_1-2v_2) - 2^{5\alpha}\chi\Big(\frac{v_1-2v_2}{2^\alpha}\Big)\Big) + \frac{120}{152}\rho\Big(Q(v_2) - 2^{5\alpha}\chi\Big(\frac{v_2}{2^\alpha}\Big)\Big)\\
&+\frac{1}{152}\rho\Big(2^{5\alpha}\chi\Big(\frac{v_1+3v_2}{2^\alpha}\Big) - 5(2^{5\alpha})\chi\Big(\frac{v_1+2v_2}{2^\alpha}\Big) + 10(2^{5\alpha})\chi\Big(\frac{v_1+v_2}{2^\alpha}\Big) - 10(2^{5\alpha})\chi\Big(\frac{v_1}{2^\alpha}\Big)\\
&+5(2^{5\alpha})\chi\Big(\frac{v_1-v_2}{2^\alpha}\Big) - 2^{5\alpha}\chi\Big(\frac{v_1-2v_2}{2^\alpha}\Big) - 120(2^{5\alpha})\chi\Big(\frac{v_2}{2^\alpha}\Big)\Big)
\end{aligned}$$

for all $v_1, v_2 \in W$ and all natural numbers $\alpha > 1$. Therefore, the mapping Q is quintic as $\alpha \to \infty$.

To prove the uniqueness of the function Q, assume that there is another quintic mapping $Q' : W \to S_\rho$ satisfying

$$\rho(\chi(v_1) - Q'(v_1)) \leq \frac{1}{2k}\sum_{i=1}^{\infty}\Big(\frac{k^6}{2}\Big)^i \phi\Big(\frac{v_1}{2^i}, \frac{v_1}{2^i}\Big) \quad \forall\, v_1 \in W.$$

Then, we see from the equality $Q(2^{-\alpha}v_1) = 2^{-5\alpha}Q(v_1)$ and $Q'(2^{-\alpha}v_1) = 2^{-5\alpha}Q'(v_1)$ that

$$\begin{aligned}
\rho(Q(v_1) - Q'(v_1)) \ \leq\ & \frac{1}{2}\rho\Big(2(2^{5\alpha})Q\Big(\frac{v_1}{2^\alpha}\Big) - 2(2^{5\alpha})\chi\Big(\frac{v_1}{2^\alpha}\Big)\Big) + \frac{1}{2}\rho\Big(2(2^{5\alpha})\chi\Big(\frac{v_1}{2^\alpha}\Big) - 2(2^{5\alpha})Q'\Big(\frac{v_1}{2^\alpha}\Big)\Big)\\
\leq\ & \frac{k^{5\alpha+1}}{2}\rho\Big(Q\Big(\frac{v_1}{2^\alpha}\Big) - \chi\Big(\frac{v_1}{2^\alpha}\Big)\Big) + \frac{k^{5\alpha+1}}{2}\rho\Big(\chi\Big(\frac{v_1}{2^\alpha}\Big) - Q'\Big(\frac{v_1}{2^\alpha}\Big)\Big)\\
\leq\ & \frac{k^{5\alpha}}{2}\sum_{i=1}^{\infty}\Big(\frac{k^6}{2}\Big)^i \phi\Big(\frac{v_1}{2^{\alpha+i}}, \frac{v_1}{2^{\alpha+i}}\Big)\\
\leq\ & \frac{2^{\alpha-1}}{k^{\alpha+3}}\sum_{i=\alpha+1}^{\infty}\Big(\frac{k^6}{2}\Big)^i \phi\Big(\frac{v_1}{2^i}, \frac{v_1}{2^i}\Big)
\end{aligned}$$

for all $v_1 \in W$ and all sufficiently large natural numbers α. Taking the limit $\alpha \to \infty$, we obtain our needed result. $\square$

Corollary 6. *Suppose that S_ρ satisfies the Δ_2-condition, and let W be a normed space with a norm of $\|\cdot\|$. If there exists a real number $\theta > 0$, $q > log_2 \frac{k^6}{2}$ and a mapping $\chi : W \to S_\rho$ such that*

$$\rho(\chi(v_1 + 3v_2) - 5\chi(v_1 + 2v_2) + 10\chi(v_1 + v_2) - 10\chi(v_1) + 5\chi(v_1 - v_2) - \chi(v_1 - 2v_2) - 120\chi(v_2)) \leq \theta(\|v_1\|^q + \|v_2\|^q)$$

for all $v_1, v_2 \in W$, then there exists a unique quartic mapping $Q : W \to S_\rho$ satisfying

$$\rho(\chi(v_1) - Q(v_1)) \leq \frac{k^5\theta}{2^{q+1} - k^6}\|v_1\|^q$$

for all $v_1, v_2 \in W$.

The following theorem gives an alternative stability of Theorem 5 in modular spaces without using the Δ_2-condition and the Fatou property.

Theorem 6. *If there exists a mapping $\chi : W \to S_\rho$ such that*

$$\left.\begin{array}{r}\rho(\chi(v_1 + 3v_2) - 5\chi(v_1 + 2v_2) + 10\chi(v_1 + v_2) \\ -10\chi(v_1) + 5\chi(v_1 - v_2) - \chi(v_1 - 2v_2) - 120\chi(v_2))\end{array}\right\} \leq \phi(v_1, v_2), \tag{25}$$

and a mapping $\phi : W^2 \to [0, \infty)$ satisfies

$$\lim_{\alpha\to\infty} \frac{\phi(2^\alpha v_1, 2^\alpha v_2)}{2^{5\alpha}} = 0, \qquad \sum \frac{\phi(2^i v_1, 2^i v_1)}{2^{5i}} < \infty$$

for all $v_1, v_2 \in W$, then there exists a unique quintic mapping $Q : W \to S_\rho$ satisfying

$$\rho(\chi(v_1) - \frac{1}{6}\chi(0) - Q(v_1)) \leq \frac{1}{2^5} \sum \frac{\phi(2^i v_1, 2^i v_1)}{2^{5i}} \tag{26}$$

for all $v_1 \in W$.

Proof. Taking $v_1 = v_2$ in (25), one has

$$\begin{aligned}&\rho(\chi(4v_1) - 5\chi(3v_1) + 10\chi(2v_1) - 10\chi(v_1) + 5\chi(0) - \chi(-v_1) - 120\chi(v_1)) \\ &\quad = \rho(\chi'(4v_1) - 5\chi'(3v_1) + 10\chi'(2v_1) - 10\chi'(v_1) + \chi'(v_1) - 120\chi'(v_1)) \\ &\quad \leq \phi(v_1, v_1),\end{aligned}$$

where $\chi'(v_1) = \chi(v_1) - \frac{g(0)}{6}$, and then we obtain the convexity of ρ and $\sum_{i=0}^{\alpha-1} \frac{1}{2^{5(i+1)}} \leq 1$

$$\begin{aligned}\left(\chi'(v_1) - \frac{\chi'(2^\alpha v_1)}{2^{5\alpha}}\right) &\leq \rho\left(\sum_{i=0}^{\alpha-1}\left(\frac{2^5\chi'(2^i v_1) - \chi'(2^{i+1}v_1)}{2^{5(i+1)}}\right)\right) \\ &\leq \sum_{i=0}^{\alpha-1} \frac{\rho(2^5\chi'(2^i v_1) - \chi'(2^{i+1}v_1))}{2^{5(i+1)}} \\ &\leq \frac{1}{2^5}\sum_{i=0}^{\alpha-1} \frac{\phi(2^i(v_1), 2^i(v_1))}{2^{5i}}\end{aligned}$$

for all $v_1 \in W$ and all natural numbers α. Then, one has a ρ-Cauchy sequence $\left\{\frac{\chi'(2^\alpha v_1)}{2^{5\alpha}}\right\}$ and the mapping $Q : W \to S_\rho$ is defined as

$$\rho\left(\lim_{\alpha\to\infty}\frac{\chi'(2^\alpha v_1)}{2^{5\alpha}}\right)=Q(v_1),\quad i.e.,\quad \lim_{\alpha\to\infty}\rho\left(\frac{\chi'(2^\alpha v_1)}{2^{5\alpha}}-Q(v_1)\right)=0$$

for all $v_1 \in W$ without using the Fatou property and the Δ_2-condition. Moreover, it is obvious that the function Q satisfies the quintic functional equation in the proof that follows using the ideas from Theorem 5.

Now, we need to prove that (26) holds without using the Fatou property and the Δ_2-condition. By using the convexity of ρ and

$$\sum_{i=0}^{\alpha-1}\frac{1}{2^{5(i+1)}}+\frac{1}{2^5}\le 1,$$

we obtain the next inequality

$$\begin{aligned}\rho(\chi'(v_1)-Q(v_1)) &= \rho\left(\sum_{i=1}^{\alpha-1}\left(\frac{2^5\chi'(2^iv_1)-\chi'(2^{i+1}v_1)}{2^{5(i+1)}}\right)+\frac{\chi'(2^\alpha v_1)}{2^{5\alpha}}-\frac{Q(2v_1)}{2^5}\right)\\ &\le \sum_{i=0}^{\alpha-1}\frac{1}{2^{5(i+1)}}\rho(2^5\chi'(2^iv_1)-\chi'(2^{i+1}v_1))+\frac{1}{2^5}\rho\left(\frac{\chi'(2^{\alpha-1}2v_1)}{2^{5(\alpha-1)}}-Q(2v_1)\right)\\ &\le \frac{1}{2^5}\sum_{i=0}^{\alpha-1}\frac{1}{2^{5i}}\phi(2^iv_1,2^iv_1)+\frac{1}{2^5}\left(\frac{\chi'(2^{\alpha-1}2v_1)}{2^{5(\alpha-1)}}-Q(2v_1)\right)\end{aligned}$$

for all $v_1 \in W$ and all natural numbers $\alpha > 1$. Taking $\alpha \to \infty$, we obtain our needed result. □

Corollary 7. *Let there exist a mapping $\phi : W^2 \to [0,\infty)$ such that*

$$\lim_{\alpha\to\infty}\frac{\phi(2^\alpha v_1,2^\alpha v_2)}{2^{5\alpha}}=0,\ \ \phi(2v_1,2v_1)\le 2^5L\phi(v_1,v_1)$$

for all $v_1, v_2 \in W$ and for some $L \in (0,1)$. If there exists a mapping $\chi : W \to S_\rho$ such that

$$\left.\begin{array}{r}\rho(\chi(v_1+3v_2)-5\chi(v_1+2v_2)+10\chi(v_1+v_2)\\ -10\chi(v_1)+5\chi(v_1-v_2)-\chi(v_1-2v_2)-120\chi(v_2))\end{array}\right\}\le\phi(v_1,v_2),$$

for all $v_1, v_2 \in W$, then there exists a unique quintic mapping $Q : W \to S_\rho$ satisfying

$$\rho\left(\chi(v_1)-\frac{1}{6}\chi(0)-Q(v_1)\right)\le\frac{1}{2^5(1-L)}\phi(v_1,v_1),$$

for all $v_1 \in W$.

Corollary 8. *Suppose that W is a normed space with the norm $\|\cdots\|$. If there exists a real number $\theta > 0$, $\epsilon > 0$, $q \in (-\infty, 2)$ and a mapping $\chi : W \to S_\rho$ such that*

$$\left.\begin{array}{r}\rho(\chi(v_1+3v_2)-5\chi(v_1+2v_2)+10\chi(v_1+v_2)\\ -10\chi(v_1)+5\chi(v_1-v_2)-\chi(v_1-2v_2)\\ -120\chi(v_2))\end{array}\right\}\le\theta(\|v_1\|^q+\|v_2\|^q)+\epsilon,$$

for all $v_1, v_2 \in W$, then there exists a unique quintic mapping $Q : W \to S_\rho$ satisfying

$$\rho\left(\chi(v_1)-\frac{1}{6}\chi(0)-Q(v_1)\right)\le\frac{2\theta}{2^5-2^q}\|v_1\|^q+\frac{\epsilon}{3}$$

for all $v_1 \in W$.

3. Stability Results in 2-Banach Spaces

Gahler [22,23] developed the concept of linear 2-normed spaces in the 1960s.

Definition 3. *Let W over $\mathbb{R}$ be a linear space with $\dim W > 1$ and a mapping $\|\cdot,\cdot\| : W^2 \to \mathbb{R}$ such that*

(a) $\|p,a\| = 0$ if and only if p and a are linearly dependent.
(b) $\|p,a\| = \|a,p\|$,
(c) $\|\tau p,a\| = |\tau|\,\|p,a\|$,
(d) $\|p,a+b\| \leq \|p,a\| + \|p,b\|$

for all $p,a,b \in W$ and $\tau \in \mathbb{R}$.

Then, the function $\|\cdot,\cdot\|$ is defined as a 2-norm on W, and the pair $(W,\|\cdot,\cdot\|)$ is defined as a linear 2-normed space. A typical example of a 2-normed space is $\mathbb{R}^2$ equipped with a 2-norm defined as $|p,q|$ = the area of the triangle with the vertices 0, p, and q.

A classical illustration of a 2-normed space is $\mathbb{R}^2$ with the 2-norm defined as $|p,a| =$ the area of the triangle with the vertices 0, p, and a.

Because of (d), it is evident that

$$\|p+a,b\| \leq \|p,b\| + \|a,b\| \quad \text{and} \quad \big|\, \|p,b\| - \|a,b\| \,\big| \leq \|p-a,b\|.$$

Thus, $p \to \|p,a\|$ are continuous mappings of W into $\mathbb{R}$ for any fixed $a \in W$.

Definition 4. *If there exists a,b in a linear 2-normed space that W satisfy the condition that a and b are linearly independent, then the sequence $\{p_j\}$ in W is known as a Cauchy sequence.*

$$i.e., \quad \lim_{i,j\to\infty} \|p_i - p_j, a\| = 0$$

and

$$\lim_{i,j\to\infty} \|p_i - p_j, b\| = 0.$$

Definition 5. *A sequence $\{p_j\}$ in a linear 2-normed space W is called as a convergent if there exists an element $p \in W$ such that*

$$\lim_{i,j\to\infty} \|p_j - p, a\| = 0,$$

for every $a \in W$.

If $\{p_j\}$ converges to p, then we denote $p_j \to p$ as $j \to \infty$, and say that p is the limit point of $\{p_j\}$. We also write in this case,

$$\lim_{j\to\infty} p_j = p.$$

Definition 6. *Every Cauchy sequence is convergent in a 2-Banach space, which is a linear 2-normed space.*

Lemma 1 ([24])**.** *Let $(W,\|\cdot,\cdot\|)$ be a linear 2-normed space. If $p \in W$ and $\|p,q\| = 0$ for all $q \in W$, then $p = 0$.*

Lemma 2 ([24])**.** *For a convergent sequence $\{p_j\}$ in a linear 2-normed space W,*

$$\lim_{j\to\infty} \|p_j, a\| = \|\lim_{j\to\infty} p_j, a\|$$

for all $a \in W$.

Choonkil Park studied approximate additive mappings, approximate Jensen mappings, and approximate quadratic mappings in 2-Banach spaces in their paper [24]. In [25], Choonkil Park examined the superstability of the Cauchy functional inequality and the Cauchy–Jensen functional inequality in 2-Banach spaces under certain conditions.

In this section, we consider W to be a normed linear space and S to be a 2-Banach space.

3.1. Stability Results of Cauchy Additive Functional Equation

Theorem 7. *Let there exist a mapping $\phi : W^3 \times S \to [0, +\infty)$ exist such that*

$$\lim_{i\to\infty} \frac{1}{3^i}\phi\left(3^i v_1, 3^i v_2, 3^i v_3, s\right) = 0 \tag{27}$$

for all $v_1, v_2, v_3 \in W$ and all $s \in S$. If there exists a mapping $\chi : W \to S$ exists with $\chi(0) = 0$ such that

$$\left\|D\chi(v_1, v_2, v_3), s\right\| \le \phi(v_1, v_2, v_3, s), \tag{28}$$

and

$$\hat{\phi}(v_1, s) =: \sum_{j=0}^{\infty} \frac{1}{3^j}\phi(3^j v_1, 3^j v_1, 3^j v_1, s) < \infty \tag{29}$$

exist for all $v_1, v_2, v_3 \in W, s \in S$, then there exists a unique additive mapping $A : W \to S$ satisfying

$$\left\|\chi(v_1) - A(v_1), s\right\| \le \hat{\phi}(v_1, s) \tag{30}$$

for all $v_1 \in W$ and all $s \in S$.

Proof. Setting $v_1 = v_2 = v_3$ in (28), we have

$$\left\|\chi(3v_1) - 3\chi(v_1), s\right\| \le \phi(v_1, v_1, v_1, s), \tag{31}$$

for all $v_1 \in W$ and all $s \in S$. Replacing v_1 with $3^i v_1$ in (31), we have

$$\left\|\frac{1}{3^{(i+1)}}\chi(3^{i+1}v_1) - \frac{1}{3^i}\chi(3^i v_1), s\right\| \le \frac{1}{3^{i+1}}\phi(3^i v_1, 3^i v_1, 3^i v_1, s), \tag{32}$$

for all $v_1 \in W,\ s \in S$, and all $i > 0$. Thus,

$$\begin{aligned} \left\|\frac{1}{3^{i+1}}\chi(3^{i+1}v_1) - \frac{1}{3^m}\chi(3^m v_1), s\right\| &\le \sum_{j=m}^{i}\left\|\frac{1}{3^{j+1}}\chi(3^{j+1}v_1) - \frac{1}{3^j}\chi(3^j v_1), s\right\| \\ &\le \frac{1}{3}\sum_{j=m}^{i}\frac{1}{3^j}\phi(3^j v_1, 3^j v_1, 3^j v_1, s) \end{aligned} \tag{33}$$

for all $v_1 \in W,\ s \in S$ and all integers $i > 0$ and $m > 0$ with $m \le i$.

Thus, from (28) and (33) we conclude that the sequence $\{\frac{\chi(3^i v_1)}{3^i}\}$ is a Cauchy sequence in S for all $v_1 \in W$. We know that S is complete, which implies that the sequence $\{\frac{\chi(3^i v_1)}{3^i}\}$ converges in S for all $v_1 \in W$. So, we can define a mapping $A : W \to S$ by

$$A(v_1) := \lim_{i\to\infty}\frac{1}{3^i}\chi(3^i v_1) \tag{34}$$

for all $v_1 \in W$. Therefore,

$$\lim_{i\to\infty}\left\|\frac{1}{3^i}\chi(3^i v_1) - A(v_1), s\right\| = 0$$

for all $v_1 \in W$ and all $s \in S$. Replacing $m = 0$ and taking the limit $i \to \infty$ in (33), we obtain (30).

Now, we want to prove that the function A is additive. From inequalities (27), (28), (34), and Lemma 2 we can obtain that

$$\begin{aligned} \left\| D\chi(v_1, v_2, v_3), s \right\| &= \lim_{i\to\infty} \left\| D\chi(3^i v_1, 3^i v_2, 3^i v_3), s \right\| \\ &\leq \lim_{i\to\infty} \frac{1}{3^i} \phi(3^i v_1, 3^i v_2, 3^i v_3, s) = 0. \end{aligned}$$

By Lemma 1,

$$DA(v_1, v_2, v_3) = 0$$

for all $v_1, v_2, v_3 \in W$. Thus, the function A is additive.

To prove the uniqueness of the function A, we consider another additive mapping $A' : W \to S$ satisfying (30). Then,

$$\begin{aligned} \left\| A(v_1) - A'(v_1), s \right\| &= \lim_{i\to\infty} \frac{1}{3^i} \left\| A(3^i v_1) - \chi(3^i v_1) + \chi(3^i v_1) - A'(3^i v_1), s \right\| \\ &\leq \lim_{i\to\infty} \frac{1}{3^i} \hat{\phi}(3^i v_1, s) = 0 \end{aligned}$$

for all $v_1 \in W$ and all $s \in S$. By Lemma 1, $A'(v_1) - A(v_1) = 0$ for all $v_1 \in W$, which implies $A' = A$. □

Remark 2. *A Theorem analogous to 7 can be formulated, in which the sequence*

$$A(v_1) := \lim_{i\to\infty} 3^i \chi\left(\frac{v_1}{3^i}\right)$$

is defined with appropriate assumptions for ϕ.

Corollary 9. *Let there exist a mapping $\tau : [0, \infty) \to [0, \infty)$ exist such that $\tau(0) = 0$ and*

(i) $\tau(rs) \leq \tau(r)\tau(s)$.

(ii) $\tau(r) < r$ *for all* $r > 1$.

If a mapping $\chi : W \to S$ exists with $\chi(0) = 0$ and

$$\left\| D\chi(v_1, v_2, v_3), s \right\| \leq \tau(\|v_1\| + \|v_2\| + \|v_3\|) + \tau(\|s\|) \tag{35}$$

for all $v_1, v_2, v_3 \in W$ and all $s \in S$, then there exists a unique additive mapping $A : W \to S$ satisfying

$$\left\| \chi(v_1) - A(v_1), s \right\| \leq \left[\frac{3\tau(\|v_1\|)}{3 - \tau(3)} + \tau(\|s\|) \right] \tag{36}$$

for all $v_1 \in W$ and all $s \in S$.

Proof. Let

$$\varphi(v_1, v_2, v_3, s) = \tau(\|v_1\| + \|v_2\| + \|v_3\|) + \tau(\|s\|)$$

for all $v_1, v_2, v_3 \in W$ and all $s \in S$. From condition (i), we have

$$\tau(3^i) \leq (\tau(3))^i$$

and

$$\phi(3^i v_1, 3^i v_2, 3^i v_3, s) \leq (\tau(3))^i (\tau(\|v_1\| + \|v_2\| + \|v_3\|)) + \tau(\|s\|).$$

By using Theorem 7, we reach (36). □

Corollary 10. *Let there exist an $a \in \mathbb{R}^+$ with $a < 1$ and a homogeneous mapping $G : [0, \infty) \times [0, \infty) \to [0, \infty)$ with degree a. If there exists a mapping $\chi : W \to S$ with $\chi(0) = 0$ and*

$$\left\| D\chi(v_1, v_2, v_3), s \right\| \leq G(\|v_1\|, \|v_2\|, \|v_3\|) + \|s\|$$

for all $v_1, v_2, v_3 \in W$ and all $s \in S$, then there exists a unique additive mapping $A : W \to S$ satisfying

$$\left\| \chi(v_1) - A(v_1), s \right\| \leq \frac{G(\|v_1\|, \|v_2\|, \|v_3\|) + \|s\|}{3 - a} \tag{37}$$

for all $v_1 \in W$ and all $s \in S$.

Corollary 11. *Let there exist a homogeneous mapping $G : [0, \infty) \times [0, \infty) \to [0, \infty)$ with degree q. If there exists a mapping $\chi : W \to S$ with $\chi(0) = 0$ and*

$$\left\| D\chi(v_1, v_2, v_3), s \right\| \leq G(\|v_1\|, \|v_2\|, \|v_3\|)\|s\|$$

for all $v_1, v_2, v_3 \in W$ and all $s \in S$. Then, there exists a unique additive mapping $A : W \to S$ satisfying

$$\left\| \chi(v_1) - A(v_1), s \right\| \leq \frac{G(\|v_1\|, \|v_1\|, \|v_1\|)\|s\|}{3 - 3^q} \tag{38}$$

for all $v_1 \in W$ and all $s \in S$, where $q \in \mathbb{R}^+$ with $q < 1$.

Corollary 12. *If there exists a mapping $\chi : W \to S$ with $\chi(0) = 0$ and*

$$\left\| D\chi(v_1, v_2, v_3), s \right\| \leq \|v_1\|^p + \|v_2\|^p + \|v_3\|^p + \|s\|$$

for all $v_1, v_2, v_3 \in W$ and all $s \in S$, then there exists a unique additive mapping $A : W \to S$ satisfying

$$\left\| \chi(v_1) - A(v_1), s \right\| \leq \frac{2\|v_1\|^p + \|s\|}{3 - p}$$

for all $v_1 \in W$ and all $s \in S$, where $p \in \mathbb{R}^+$ with $p < 1$.

3.2. Stability of Quartic Functional Equation

For notational simplicity, we can define a mapping $\chi : W \to S$ by

$$D\chi(v_1, v_2) = \chi(2v_1 + v_2) + 6\chi(v_2) + \chi(2v_1 - v_2) - 4\chi(v_1 + v_2) - 24\chi(v_1) - 4\chi(v_1 - v_2),$$

for all $v_1, v_2 \in W$.

Theorem 8. *Let there exist a mapping $\phi : W^2 \times S \to [0, +\infty)$ such that*

$$\lim_{i \to \infty} \frac{1}{2^{4i}} \phi\left(2^i v_1, 2^i v_2, s\right) = 0 \tag{39}$$

for all $v_1, v_2 \in W$ and all $s \in S$. Suppose that there exists a mapping $\chi : W \to S$ with $\chi(0) = 0$ such that

$$\left\| D\chi(v_1, v_2), s \right\| \leq \phi(v_1, v_2, s) \tag{40}$$

and

$$\hat{\phi}(v_1, s) =: \frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{2^{4j}} \phi(2^j v_1, 0, s) < \infty \tag{41}$$

exists for all $v_1 \in W$ and all $s \in S$. Then, there exists a unique quartic mapping $Q_4 : W \to S$ that satisfies

$$\left\| \chi(v_1) - Q_4(v_1), s \right\| \leq \hat{\phi}(v_1, s), \tag{42}$$

for all $v_1 \in W$ and all $s \in S$.

Proof. Replacing (v_1, v_2) with $(v_1, 0)$ in (40), we obtain

$$\begin{aligned} \left\| 2\chi(2v_1) - 32\chi(v_1), s \right\| &\leq \phi(v_1, 0, s) \\ \left\| \frac{\chi(2v_1)}{2^4} - \chi(v_1), s \right\| &\leq \frac{1}{2(2^4)}\phi(v_1, 0, s) \end{aligned} \tag{43}$$

for all $v_1 \in W$ and all $s \in S$. Replacing v_1 with $2^i v_1$ in (43), we obtain

$$\left\| \frac{1}{2^{4(i+1)}}\chi(2^{i+1}v_1) - \frac{1}{2^{4i}}\chi(2^i v_1), s \right\| \leq \frac{1}{2(2^{4(i+1)})}\phi(2^i v_1, 0, s) \tag{44}$$

for all $v_1 \in W$, $s \in S$, and all integers $i > 0$. Hence,

$$\begin{aligned} \left\| \frac{1}{2^{4(i+1)}}\chi(2^{i+1}v_1) - \frac{1}{2^{4m}}\chi(2^m v_1), s \right\| &\leq \sum_{j=m}^{i} \left\| \frac{1}{2^{4(j+1)}}\chi(2^{j+1}v_1) - \frac{1}{2^{4j}}\chi(2^j v_1), s \right\| \\ &\leq \frac{1}{2}\sum_{j=m}^{i} \frac{1}{2^{4j}}\phi(2^j v_1, 0, s) \end{aligned} \tag{45}$$

for all $v_1 \in W$, $s \in S$ and all integers $i \geq m > 0$. Thus, it follows from inequalities (40) and (45) that the sequence $\{\frac{\chi(2^i v_1)}{2^{4i}}\}$ is Cauchy in S for every $v_1 \in W$. As S is complete, the sequence $\{\frac{\chi(2^i v_1)}{2^{4i}}\}$ converges in S for all $v_1 \in W$. Thus, we can define a mapping $Q_4 : W \to S$ by

$$Q_4(v_1) := \lim_{i\to\infty} \frac{\chi(2^i v_1)}{2^{4i}}, \tag{46}$$

for all $v_1 \in W$. Therefore,

$$\lim_{i\to\infty} \left\| \frac{\chi(2^i v_1)}{2^{4i}} - Q_4(v_1), s \right\| = 0$$

for all $v_1 \in W$ and all $s \in S$. Taking the limit as $i \to \infty$ and setting $m = 0$ in (45), we obtain (42). Next, we need to show that the function Q_4 is a quartic function. From the inequalities (39), (40), (46), and Lemma 2 we obtain that

$$\begin{aligned} \left\| D\chi(v_1, v_2, s) \right\| &= \lim_{i\to\infty} \left\| D\chi(2^i v_1, 2^i v_2, s \right\| \\ &\leq \lim_{i\to\infty} \frac{1}{2^{4i}}\phi(2^i v_1, 2^i v_2, s) = 0 \end{aligned}$$

for all $v_1, v_2 \in W$ and all $s \in S$. By Lemma 1,

$$DQ_4(v_1, v_2) = 0$$

for all $v_1, v_2 \in W$. Thus, the mapping $Q_4 : W \to S$ is quartic.

To verify the uniqueness of the function Q_4, consider that there exists another quartic mapping $Q_4' : W \to S$ satisfying (42). Then,

$$\begin{aligned} \left\| Q_4(v_1) - Q_4'(v_1), s \right\| &= \lim_{i\to\infty} \frac{1}{2^{4i}} \left\| Q_4(2^i v_1) - \chi(2^i v_1) + \chi(2^i v_1) - Q_4'(2^i v_1), s \right\| \\ &\leq \lim_{i\to\infty} \frac{1}{2^{4i}}\hat{\phi}(2^i v_1, s) = 0 \end{aligned}$$

for all $v_1 \in W$ and all $s \in S$. By Lemma 1, $Q_4(v_1) - Q_4'(v_1) = 0$ for all $v_1 \in W$. Therefore, $Q_4 = Q_4'$. □

Corollary 13. *Let there exist a mapping $\tau : [0, \infty) \to [0, \infty)$ such that $\tau(0) = 0$ and*

(i) $\tau(ab) \leq \tau(a)\tau(b)$.

(ii) $\tau(a) < a$ *for all* $a > 1$.

If there exists a mapping $\chi : W \to S$ with $\chi(0) = 0$ and

$$\left\| D\chi(v_1, v_2), s \right\| \leq \tau(\|v_1\| + \|v_2\|) + \tau(\|s\|) \tag{47}$$

for every $v_1, v_2, s \in W$, then there exists a unique quartic mapping $Q_4 : W \to S$ that satisfies

$$\left\| \chi(v_1) - Q_4(v_1), s \right\| \leq \left[\frac{\tau(\|v_1\|)}{2 - \tau(2)} + \tau(\|s\|) \right] \tag{48}$$

for all $v_1 \in W$ and all $s \in S$.

Proof. Let

$$\phi(v_1, v_2, s) = \tau(\|v_1\| + \|v_2\|) + \tau(\|s\|),$$

for all $v_1, v_2 \in W$ and $s \in S$. As a result of (i), that

$$\tau(2^i) \leq (\tau(2))^{4i}$$

and

$$\phi(2^i v_1, 2^i v_2, s) \leq (\tau(2))^{4i}(\tau(\|v_1\| + \|v_2\|)) + \tau(\|s\|).$$

By utilizing Theorem 8, we obtain (48). □

Corollary 14. *Let there exist a homogeneous mapping $G : [0, \infty) \times [0, \infty) \to [0, \infty)$ with degree q. If there exists a mapping $\chi : W \to S$ with $\chi(0) = 0$ and*

$$\left\| D\chi(v_1, v_2), s \right\| \leq G(\|v_1\|, \|v_2\|) + \|s\|,$$

for all $v_1, v_2 \in W$ and all $s \in S$, then there exists a unique quartic mapping $Q_4 : W \to S$ satisfying

$$\left\| \chi(v_1) - Q_4(v_1), s \right\| \leq \frac{G(\|v_1\|, \|v_2\|) + \|s\|}{2 - q} \tag{49}$$

for all $v_1 \in W$ and $s \in S$, where $q \in \mathbb{R}^+$ with $q < 1$.

Proof. Let

$$\phi(v_1, v_2, s) = G(\|v_1\|, \|v_2\|) + \|s\|,$$

for all $v_1, v_2 \in W$ and all $s \in S$. By using Theorem 8, we have (49). □

Corollary 15. *Let there exist a homogeneous mapping $G : [0, \infty) \times [0, \infty) \to [0, \infty)$ with degree q. If there exists a mapping $\chi : W \to S$ with $\chi(0) = 0$ and*

$$\left\| D\chi(v_1, v_2, s \right\| \leq G(\|v_1\|, \|v_2\|)\|s\|,$$

for all $v_1, v_2 \in W$ and all $s \in S$, then there exists a unique quartic mapping $Q_4 : W \to S$ satisfying

$$\left\| \chi(v_1) - Q_4(v_1), s \right\| \leq \frac{G(\|v_1\|, 0)\|s\|}{2 - 2^q} \tag{50}$$

for all $v_1 \in W$ and all $s \in S$, where $q \in \mathbb{R}^+$ with $q < 1$.

Corollary 16. *If there exists a mapping $\chi : W \to S$ with $\chi(0) = 0$ and*

$$\left\| D\chi(v_1, v_2), s \right\| \leq \|v_1\|^p + \|v_2\|^p + \|s\|,$$

for all $v_1, v_2 \in W$ and all $s \in S$, then there exists a unique quartic mapping $Q_4 : W \to S$ satisfying

$$\left\| \chi(v_1) - Q_4(v_1), s \right\| \leq \frac{\|v_1\|^p + \|s\|}{2 - p},$$

for all $v_1 \in W$ and all $s \in S$, where $p \in \mathbb{R}^+$ with $p < 1$.

4. Illustrative Examples

The functional Equations (1) and (16) are shown to be unstable in the singular condition using a relevant example. We provide the following counter-examples to Gajda's excellent example in [26], which illustrates the instability in Corollaries 2 and 3 of Equations (1) and (16), respectively, under the conditions $q \neq 1$ and $q \neq \log 2^{\frac{k^5}{2}}$, respectively.

Here, $\mathbb{R}$ denotes the real space and we can prove the below counter-examples as in [27,28].

Remark 3. *If a mapping $\chi : \mathbb{R} \to W$ satisfies the functional Equation (1), then the following conditions hold:*

(1) $\chi(n^c v_1) = n^c \chi(v_1)$, for all $v_1 \in \mathbb{R}$, $c \in \mathbb{Z}$ and $n \in \mathbb{Q}$.
(2) $\chi(v_1) = v_1 \chi(1)$, for all $v_1 \in \mathbb{R}$ if the mapping χ is continuous.

Example 1. *Let there exist a mapping $\chi : \mathbb{R} \to \mathbb{R}$ defined by*

$$\chi(v_1) = \sum_{i=0}^{\infty} \frac{\xi(3^i v_1)}{3^i}, \tag{51}$$

where

$$\xi(v_1) = \begin{cases} \zeta v_1, & -1 < v_1 < 1 \\ \zeta, & else. \end{cases}$$

Suppose that there exists a mapping $\chi : \mathbb{R} \to \mathbb{R}$ defined in (1) such that

$$|D\chi(v_1, v_2, w)| \leq 6\zeta(|v_1| + |v_2| + |w|), \tag{52}$$

for every $v_1, v_2, w \in \mathbb{R}$. We prove that there does not exist a additive mapping $A : \mathbb{R} \to \mathbb{R}$ which satisfies

$$|\chi(v_1) - A(v_1)| \leq \delta |v_1|, \tag{53}$$

for all $v_1 \in \mathbb{R}$, where δ and ζ are constants.

Remark 4. *If there exists a mapping $\chi : \mathbb{R} \to W$ satisfies the functional Equation (16), then the following assertions hold:*

(1) $\chi(n^{c/4} v_1) = n^c \chi(v_1)$, for all $v_1 \in \mathbb{R}$, $c \in \mathbb{Z}$ and $n \in \mathbb{Q}$.
(2) $\chi(v_1) = v_1^4 \chi(1)$, for all $v_1 \in \mathbb{R}$ if the mapping χ is continuous.

Example 2. *Let there exist a mapping $\chi : \mathbb{R} \to \mathbb{R}$ defined by*

$$\chi(v_1) = \sum_{i=0}^{\infty} \frac{\xi(2^i v_1)}{2^{4i}}, \tag{54}$$

where

$$\xi(v_1) = \begin{cases} \zeta v_1^4, & -1 < v_1 < 1 \\ \zeta, & else. \end{cases}$$

Suppose that there exists a mapping $\chi : \mathbb{R} \to \mathbb{R}$ *defined in* (16) *such that*

$$|D\chi(v_1, v_2)| \le 40\zeta\left(|v_1|^4 + |v_2|^4\right), \tag{55}$$

for all $v_1, v_2 \in \mathbb{R}$*. We prove that there does not exist a quartic mapping* $Q_4 : \mathbb{R} \to \mathbb{R}$ *that satisfies*

$$|\chi(v_1) - Q_4(v_1)| \le \delta|v_1|^4, \tag{56}$$

for all $v_1 \in \mathbb{R}$*, where* δ *and* ζ *are constants.*

5. Conclusions

Many mathematicians have obtained the stability results of various kinds of additive, quadratic, and cubic functional equations in various spaces. In our investigations, we investigated the stability results of additive, quartic, and quintic functional equations in the setting of modular space using Hyers' method without using the Δ_2-condition. In addition, an appropriate counter-example is provided to demonstrate the non-stability of the singular case.

Author Contributions: Conceptualization, K.T. and J.J. (Jyotsana Jakhar); formal analysis, A.H.A., R.C. and J.J. (Jagjeet Jakhar); investigation, K.T., J.J. (Jyotsana Jakhar), R.C. and J.M.R.; methodology, A.H.A., J.J. (Jyotsana Jakhar), and J.M.R.; writing—original draft, K.T., J.J. (Jagjeet Jakhar), and R.C.; writing—review and editing, K.T., A.H.A., J.J. (Jyotsana Jakhar), R.C., J.J. (Jagjeet Jakhar), and J.M.R.; All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to the Ministry of Education in KSA for funding this research work through project number KKU-IFP2-DC-3.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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