Chapter 4

Empirical Bayesian Estimation for Kumaraswamy Distribution Using Informative Prior *

4.1 Introduction

In the previous chapter, we have discussed the procedure for obtaining the classical, Bayesian and E-Bayesian estimation under PT-II CBRs. In this chapter, we are introducing the Empirical Bayes estimator of the Kumaraswamy distribution (KD). It is one of the simplest distribution in the sense of being parsimonious in parameter. It is applicable to many natural phenomena whose outcomes have lower and upper bound, such as the height of individuals, age of person, scores obtained on a test, atmospheric temperatures, hydro logical data such as daily rain fall, daily stream flow etc see Kumaraswamy (1980). The PDF and CDF of KD (α, λ) are given by

$$f(x;\alpha,\lambda) = \alpha\lambda x^{\alpha-1}(1-x^{\alpha})^{\lambda-1}; \qquad x>0, \quad \alpha>0, \quad \lambda>0, \tag{4.1}$$

*Part of this chapter has been published in reputed peer-reviewed journals with indexing EBSCO Discovery Service, see Kumar et al. (2019b).

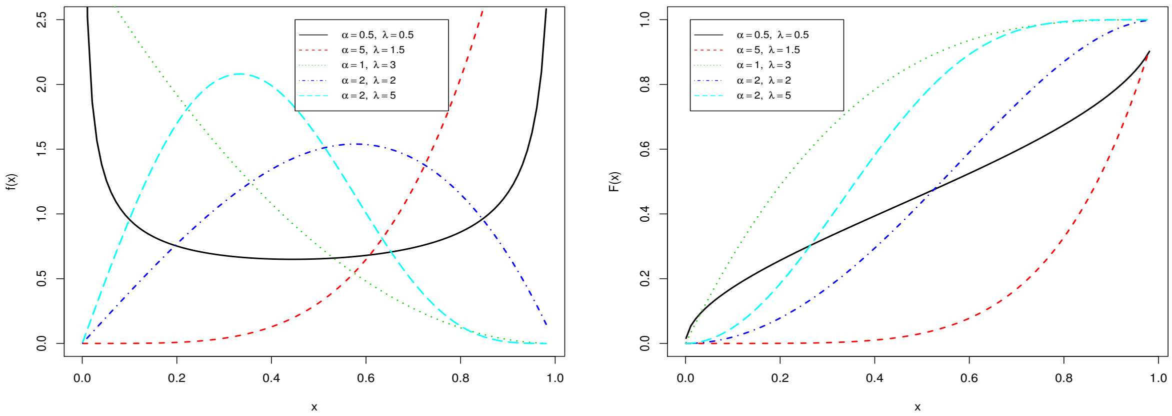

FIGURE 4.1: PDFs: left panel, CDFs: right panel of KD for different values of α and λ.

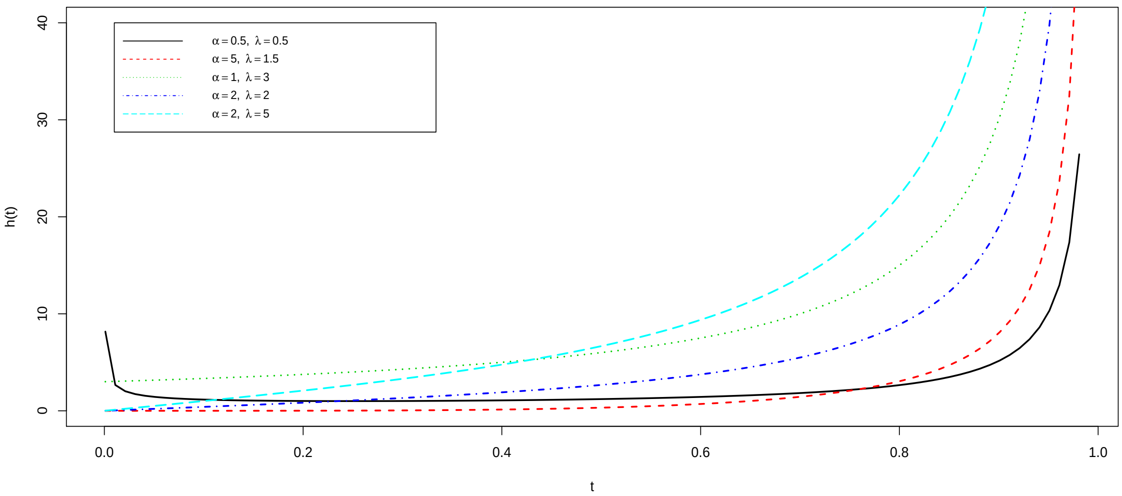

FIGURE 4.2: HF of KD for different values of α and λ.

and

$$F(x;\alpha,\lambda) = 1-(1-x^{\alpha})^{\lambda}; \qquad x>0, \quad \alpha>0, \quad \lambda>0, \tag{4.2}$$

respectively; where α and λ are shape parameters. Figure (4.1) shows PDF and CDF for $\alpha = 0.5, 5, 1, 2, 2$ and $\lambda = 0.5, 1, 3, 2, 5$ respectively.

The reliability function (i.e. the probability of failure after time t) and the HF for distribution Equation (4.1) are given by

$$R(t) = (1-t^{\alpha})^{\lambda},$$

and

$$h(t) = \frac{\alpha\lambda t^{\alpha-1}}{(1-t^{\alpha})}.$$

respectively. The HF $h(t)$ is shown in Figure (4.2) for $\alpha = 0.5, 5, 1, 2, 2$ and $\lambda = 0.5, 1.5, 3, 2, 5$ respectively. It may be noted here that the HF has a non-monotonic shape which decreases initially remains constant in the mid and lastly increases. For the statistical and probabilistic properties and other distributions obtained under the influence of KD for the use in life testing and reliability analysis, see Jones (2009), Lemonte (2011), Xiaohu et al. (2011), Santana et al. (2012) etc. The problem of the estimation of the parameters of KD have been discussed by Lemonte (2011) and Gholizadeh et al. (2011) etc. But it seems that the empirical Bayesian inferences have not been attempted to the extent of classical and Bayesian inferences, although it is well known that empirical Bayes is a good compromise between these two. According to Morris (1983), an empirical Bayesian inference, which is, as expected, a hybrid of frequentist and Bayesian inference.

The use of KD in life testing and reliability problems have been suggested by various authors, see Jones (2009), Lemonte (2011), Kohansal (2017), Amin (2017) etc. A general problem associated with life testing is that in most of the situation one can not wait for the failure of all the items put on test. In such situation, censoring becomes unavoidable. A number of censoring schemes are available in statistical literature. One of the popular censoring scheme under use is progressive Type-II censoring scheme. On one hand it provides flexibility because it allows the intermediate removals of the items from test and on other hand it guarantees for a minimum efficiency of the estimators by fixing the number of complete observations.

In the point estimation an important element is the loss function specification. A very popular loss function is SELF, used in estimation of parameter. Which can be appropriate on the space of minimum variance-unbiased estimation. However, main drawback of this loss function is that it have equal magnitude for o.e. and u.e., can say it is symmetric loss function. In the literature, many asymmetric loss functions are available, and one of the most frequently used asymmetric loss function is the LINEX loss function, originally it was proposed by Varian (1975) and popularized by Zellner (1986a), discussed in Chapter 1, Section (1.8).

In this chapter presented a piece of work, aims to develop the empirical Bayes estimators for an unknown shape parameter of KD based on PT-II CBR under LINEX loss function. In KD, one shape parameter known $\alpha > 1$ i.e. $\alpha = 2$ with $\lambda > 1$ (unknown) have been taken due to the distribution with one mode of the KD. For $\alpha > 1$, $\lambda > 1$, $\lim_{x\to 1} f(x;\alpha,\lambda) = 0$ and $\lim_{x\to 0} f(x;\alpha,\lambda) = 0$. Therefore, it is mathematically deal with, the characteristics of KD for different parameter values see Mitnik (2013).

4.2 Likelihood Function under PT-II CBRs

Suppose that in a life testing experiment having items put on test, the lifetime of which follow the KD. Also, we considered that the lifetime experiment perform under PT-II CBR, discussed in Chapter 1, Subsection (1.11.2). The conditional likelihood function can be written as

$$L(\alpha,\lambda;x|R=r) = c\prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{r_i}; \qquad -\infty < x_1 < \ldots < x_m < \infty, \tag{4.3}$$

where $n = m + \sum_{i=1}^{m} r_i$, $n,m \in \mathrm{N}, 1 \le i \le m$ and $c = \prod_{i=1}^{m} \gamma_i$ where $\gamma_i = \sum_{j=1}^{m} (r_j + 1)$, $r_i \sim B(n-m-\sum_{l=0}^{i-1} r_l, p)$ for $i = 1,2,3,\ldots m-1$ and $r_0 = 0$ substituting $f(.)$ and $F(.)$ from Equation (4.1) and (4.2) respectively, into Equation (2.3), we have

$$L(\alpha,\lambda;x|R=r) = c\prod_{i=1}^{m} \alpha\lambda x_i^{\alpha-1}(1-x_i^{\alpha})^{\lambda-1}\left\{(1-x_i^{\alpha})^{\lambda}\right\}^{r_i}. \tag{4.4}$$

Since at the every stage the removals are independent of each other with probability p for each unit, the removals are following a binomial distribution i.e.,

$$r_i \sim B(n-m-\sum_{l=0}^{i-1} r_l, p),$$

where $i = 1, 2, 3, ...m-1$. Therefore;

$$p(R_1 = r_1; p) = \binom{n-m}{r_1} p^{r_1}(1-p)^{n-m-r_1}, \tag{4.5}$$

and for $i = 2, 3, ..., m-1$

$$p(R_i; p) = p(R_i = r_i | R_{i-1} = r_{i=1}, ...R_1 = r_1)$$

$$= \binom{n-m-\sum_{l=0}^{i-1} r_l}{r_i} p^{r_i}(1-p)^{n-m-\sum_{l=0}^{i-1} r_l}. \tag{4.6}$$

It is further assumed that $R_i s$ are independent of $X_{i:m:n}$ for all i. Thus full likelihood function can be written as:

$$L(\alpha, \lambda, p; x) = L(\alpha, \beta, \lambda; x | R = r) p(R = r; p), \tag{4.7}$$

where;

$$\begin{aligned} p(R = r; p) = p(R_1 = r_1) p(R_2 = r_2 | R_1 = r_1) p(R_3 = r_3 | R_2 = r_2, R_1 = r_1)... \\ p(R_{m-1} = r_{m-1} | R_{m-2} = r_{m-2}, ...R_1 = r_1). \end{aligned} \tag{4.8}$$

Making the substitution from the Equation (2.5) and (2.6) into Equation (2.8), we get

$$p(R = r; p) = \frac{(n-m)! p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i) r_i}}{(n-m-\sum_{l=1}^{i-1} r_l)! \prod_{i=1}^{m-1} r_i!}, \tag{4.9}$$

now using Equations (2.4), (2.7) and (2.9), the full likelihood can be represented in the following form:

$$L(\alpha, \lambda, p; x) = H L_1(\alpha, \lambda) L_2(p). \tag{4.10}$$

where

$$H = \frac{c(n-m)!}{(n-m-\sum_{l=1}^{i-1} r_l)! \prod_{i=1}^{m-1} r_i!},$$

$$L_1(\alpha,\lambda;x|R=r) = \prod_{i=1}^{m} \alpha\lambda x_i^{\alpha-1}\left(1-x_i^{\alpha}\right)^{-(\lambda(-1-r_i)+1)}, \tag{4.11}$$

$$L_2(p) = p^{\sum_{i=1}^{m-1} r_i}(1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}. \tag{4.12}$$

It may be noted here that the likelihood function is product of three terms H, L_1 and L_2 ; where H is a constant term, L_1 is function of the parameters but does not involve p and L_2 is function of p but does not involve other parameters.

4.3 Estimation of Parameters

4.3.1 Maximum Likelihood Estimator

As mentioned above, only L_1 involves the parameters, hence ML estimates of the parameters are those values which maximizes L_1, we have

$$\begin{aligned} \ln L_1(\alpha,\lambda) &= m\ln(\alpha) + m\ln(\lambda) + (\alpha-1)\sum_{i=1}^{m}\ln(x_i) \\ &\quad -\sum_{i=1}^{m}(\lambda(-1-r_i)+1)\ln(1-x_i^{\alpha}) \end{aligned} \tag{4.13}$$

Thus, the likelihood equations can be obtained by differentiating the log-L function given above with respect to parameter α and λ and equating to zero; i.e., ML estimates are $\hat{\alpha}$ and $\hat{\lambda}$ of α and λ respectively, can be obtained by simultaneously solving the likelihood equations:

$$\frac{m}{\alpha} + \sum_{i=1}^{m}\ln(x_i) + \sum_{i=1}^{m}(\lambda(-1-r_i)+1)\left(x_i^{-\alpha}-1\right)^{-1}\ln(x_i) = 0, \tag{4.14}$$

and

$$\frac{m}{\lambda} - \sum_{i=1}^{m} (-1 - r_i) \ln (1 - x_i^{\alpha}) = 0. \tag{4.15}$$

The above mentioned normal equation solved simultaneously but do not provided closed form solution for the estimators. Then we opted NR method to compute the ML estimators, then we are using the invariance property to the ML estimators of the reliability function $R(t)$ and the failure rate $h(t)$ at time t can be evaluated from the following:

$$\hat{R}(t) = (1 - t^{\alpha})^{\lambda}; \qquad t > 0, \tag{4.16}$$

and

$$\hat{h}(t) = \frac{\alpha \lambda t^{\alpha - 1}}{(1 - t^{\alpha})}; \qquad t > 0. \tag{4.17}$$

4.3.2 Bayes Estimator

In this sequence, we obtain the Bayes estimator of the parameter λ, when we assume that λ has a conjugate prior density,

$$\pi(\lambda, \beta) = \beta \exp(-\beta \lambda); \qquad \lambda > 0, \quad \beta > 0. \tag{4.18}$$

That is to say, we regard random variable λ with prior density an exponential distribution $\exp(\beta)$, which is used in detail Bayesian theory, see Berger (2013). It may be noted that, the exponential family prior $\pi(\lambda, \beta)$ has been used by Nassar and Eissa (2005), Kim et al. (2011) possibly because of the fact that it is flexible enough to cover a wide range of prior believes of the experimenter. Hence, mathematical formula to evaluate the posterior distribution of λ is given below,

$$\pi(\lambda | x) = \frac{\pi(\lambda, \beta) L_1(\alpha, \lambda; x | R = r)}{\int_0^{+\infty} \pi(\lambda, \beta) L_1(\alpha, \lambda; x | R = r) d\lambda}. \tag{4.19}$$

Substituting $L_1(\alpha,\lambda;x|R=r)$ and $\pi(\lambda;\beta)$ from Equation (2.10) and (4.18), respectively, in Equation (4.19). We obtain the posterior distribution after simplification as,

$$\begin{aligned}\pi(\lambda|T) &= \frac{\beta\exp(-\beta\lambda)\,c\prod_{i=1}^{m}\alpha\lambda x^{\alpha-1}(1-x^{\alpha})^{-(\lambda(-1-r_i)+1)}}{\int_0^{+\infty}\beta\exp(-\beta\lambda)\,c\prod_{i=1}^{m}\alpha\lambda x^{\alpha-1}(1-x^{\alpha})^{-(\lambda(-1-r_i)+1)}\,d\lambda}\\ &= \frac{(\beta+T)^{m+1}\lambda^{m}\exp(-\lambda(\beta+T))}{\Gamma(m+1)}. \end{aligned} \tag{4.20}$$

where $T=-\sum_{i=1}^{m}(r_i+1)\ln(1-x_i^{\alpha})$. Randomly generated posterior distribution for complete sample size 20 having $\mathbf{x} = (4.602501e-07, 9.335994E-07, 1.306320E-02, 1.351230E-02, 4.355106E-02, 9.641328E-02, 1.842315E-01, 2.266576E-01, 2.577245E-01, 4.145024E-01, 7.714380E-01, 7.891063E-01, 8.778412E-01, 8.926065E-01, 9.723090E-01, 9.963840E-01, 9.986856E-01, 9.990788E-01, 9.999941E-01, 1.000000E+00)$ are presented in Figure (4.3).

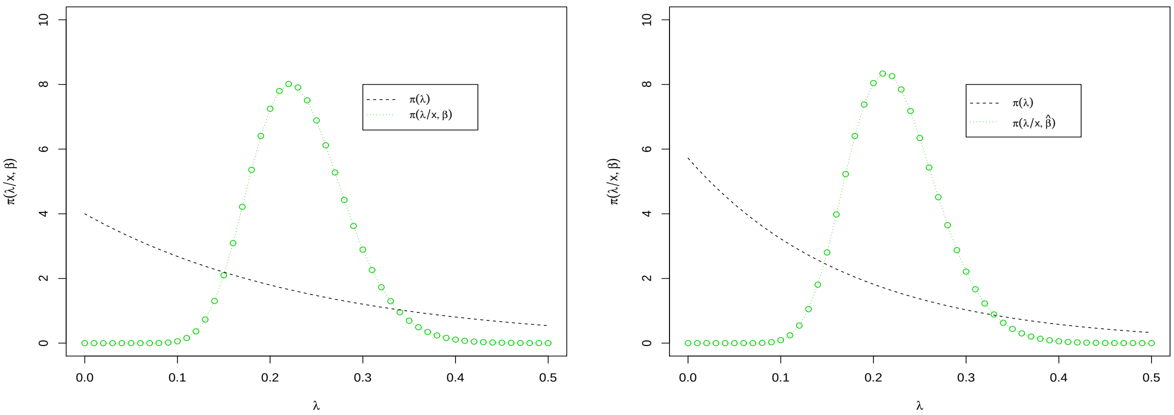

FIGURE 4.3: Informative prior $\pi(\lambda)$ and the posterior $\pi(\lambda|x,\beta)$: left panel, Informative prior $\pi(\lambda)$ and the posterior $\pi(\lambda|x,\hat{\beta})$: right panel of λ.

Note that the posterior distribution of λ is gamma distribution with parameters $(m+1)$ and $(\beta+T)$. The Bayes estimator of λ under LINEX loss function for posterior Equation (4.20) is obtained, after simplification, as

$$\hat{\lambda}_B = -\frac{1}{a}\ln\int_0^{\infty}e^{-a\lambda}\pi(\lambda|T)d\lambda = \frac{m+1}{a}\ln\left(1+\frac{a}{\beta+T}\right). \tag{4.21}$$

Similarly, the Bayes estimators of $R(t)$ and $h(t)$ at time t are obtained under LINEX loss function

$$\hat{R}_B(t) = -\frac{1}{a}\ln\int_0^\infty e^{-a(1-t^\alpha)^\lambda}\pi^*(\lambda|T)d\lambda = -\frac{1}{a}\ln\left(\sum_{s=0}^{\infty}\frac{(-a)^s}{s!}\left(1-\frac{s\ln(1-t^\alpha)}{(\beta+T)}\right)^{-(m+1)}\right), \tag{4.22}$$

and

$$\hat{h}_B(t) = -\frac{1}{a}\ln\int_0^\infty e^{\frac{-a\alpha\lambda t^{\alpha-1}}{(1-t^\alpha)}}\pi^*(\lambda|T)d\lambda = \frac{m+1}{a}\ln\left(1+\frac{a\alpha t^{\alpha-1}}{(\beta+T)(1-t^\alpha)}\right), \tag{4.23}$$

respectively.

4.3.3 Empirical Bayes Estimator

In view of this fact, Shi et al. (2005) and Yan and Gendai (2003) used the ML estimator to estimate hyper parameter of prior distribution for analyzing the Bayesian reliability quantitative indexes of cold stand by system. In Equation (4.21), the hyper parameter β is an unknown constant, so λ can not be used directly. Therefore, we make use of the ML estimator to estimate β.

$$\begin{aligned} f(x) &= \int_0^\infty f(x;\alpha,\lambda)\pi(\lambda;\beta)d\lambda \\ &= \int_0^\infty \alpha\lambda x^{\alpha-1}(1-x^\alpha)^{\lambda-1}\beta\exp(-\beta\lambda)\,d\lambda \\ &= \frac{\alpha\beta x^{\alpha-1}}{(1-x^\alpha)(\beta-\ln(1-x^\alpha))^2}, \end{aligned}$$

and

$$\begin{aligned}1-F(x) &= \int_x^\infty f(x)dx \\ &= \int_x^\infty \frac{\alpha\beta x^{\alpha-1}}{(1-x^\alpha)(\beta-\ln(1-x^\alpha))^2}dx \\ &= \frac{\beta}{\beta-\ln(1-x^\alpha)}.\end{aligned}$$

Hence, Equation (2.3) can be expressed as

$$L(\alpha,\lambda;x|R=r) = c\prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{r_i} \tag{4.24}$$

substituting $f(x)$ and $F(x)$ in to (4.24)

$$L(\alpha,\lambda;x|R=r) = c\prod_{i=1}^{m}\frac{\alpha\beta x^{\alpha-1}}{(1-x^\alpha)(\beta-\ln(1-x^\alpha))^2}\left(\frac{\beta}{\beta-\ln(1-x^\alpha)}\right)^{r_i},$$

$$\begin{aligned}\ln L(\alpha,\lambda;x|R=r) = \ln c + m\ln\alpha + m\ln\beta + (\alpha-1)\sum_{i=1}^{m}\ln x - \sum_{i=1}^{m}\ln(1-x^\alpha) \\ + \sum_{i=1}^{m} r_i\ln\beta - \sum_{i=1}^{m}(r_i+2)\ln(\beta-(1-x^\alpha)),\end{aligned}$$

$$\frac{\partial\ln L(\alpha,\lambda;x|R=r)}{\partial\beta} = \frac{m}{\beta} + \sum_{i=1}^{m} r_i\left(\frac{1}{\beta} - \frac{1}{(\beta-\ln(1-x^\alpha))}\right) - 2\sum_{i=1}^{m}\frac{1}{(\beta-\ln(1-x^\alpha))}.$$

Now, we have considered,

$$k_1(\beta) = \frac{m}{\beta} + \sum_{i=1}^{m} r_i\left(\frac{1}{\beta} - \frac{1}{(\beta-\ln(1-x^\alpha))}\right),\quad k_2(\beta) = 2\sum_{i=1}^{m}\frac{1}{(\beta-\ln(1-x^\alpha))}.$$

Using iterative numerical computing method to obtain the ML estimate of β. We just draw a conclusion that $k_1(\beta) = k_2(\beta)$ has a root i.e, $\hat{\beta}$, numerically solved through R software. Since the empirical Bayes estimate of λ is

$$\hat{\lambda}_E = \frac{m+1}{a} \ln\left(1 + \frac{a}{\hat{\beta}+T}\right) \tag{4.25}$$

where β is replaced by $\hat{\beta}$ in Equation (4.21). Substituting $\hat{\beta}$ in Equation (4.22), the empirical Bayes estimation of $\hat{R}(t)$ is obtained

$$\hat{R}_E(t) = -\frac{1}{a} \ln\left(\sum_{s=0}^{\infty} \frac{(-a)^s}{s!} \left(1 - \frac{s\ln(1-t^{\alpha})}{(\hat{\beta}+T)}\right)^{-(m+1)}\right). \tag{4.26}$$

Similarly, the empirical Bayes estimation of $\hat{h}(t)$ is given as

$$\hat{h}_E(t) = \frac{m+1}{a} \ln\left(1 + \frac{a\alpha t^{\alpha-1}}{(\hat{\beta}+T)(1-t^{\alpha})}\right) \tag{4.27}$$

As it has been mentioned earlier, using $(R_i = r_i = 0; i = 1, \cdots, m-1)$ in Equation (2.3) and Equation (4.24) and proceed to above subsequent equations, we can get the Bayes and empirical estimators $\hat{\lambda}_{B_2}, \hat{\lambda}_{E_2}$, $\hat{R}_{B_2}(t), \hat{R}_{E_2}(t)$ and $\hat{h}_{B_2}(t), \hat{h}_{E_2}(t)$ of $\lambda, R(t), h(t)$ for Type-II censoring at time t, respectively. For the assessment of the above equations, we numerically calculate through R software.

4.4 Monte Carlo Simulation Study and Comparison of Estimators

An analytical study of the behavior of the estimators are not possible. Therefore, we make a study based on simulated results and hence, we need to simulate PT-II CBR samples from KD. The algorithm proposed by Balakrishnan and Sandhu (1995) have been used for simulation of samples, Since, we simulate PT-II CBR from specified KD and propose the use of following algorithm

i. Specify the value of n.

ii. Specify the value of m.

iii. Specify the value of parameters α, λ and p.

iv. Generate random number r_i from $B\left(n-m-\sum_{l=0}^{i-1} r_l, p\right)$, for $i = 1,2,3,\cdots,m-1$.

v. Set r_m according to the following relation.

vi. $r_m = \begin{cases} n-m-\sum_{l=1}^{m-1} r_l & \text{if } n-m-\sum_{l=1}^{m-1} r_l > 0 \\ 0 & \text{otherwise} \end{cases}$

vii. Generate m independent $U(0,1)$ random variables $W_1, W_2, \cdots, W_m$.

viii. For given values of the progressive type-II censoring scheme $r_i (i = 1,2,\cdots,m)$ set $V_i = W_i^{1/(i+r_m+\cdot+r_{m-i+1})} (i = 1,2,\cdots,m)$.

ix. Set $U_i = 1 - V_m V_{m-1} \cdots V_{m-i+1} (i = 1,2,\cdots,m)$, then $U_1, U_2, \cdots, U_m$ are PT-II CBR samples of size m from $U(0,1)$.

x. Finally, for given values of parameters α and λ, set $x_i = F^{-1}(U) (i = 1,2,\cdots,m)$. Then $(x_1, x_2, \cdots, x_m)$ is the required PT-II CBR sample of size m from the KD.

Comparison of Estimators

Here, we compare the different estimators obtained through PT-II CBR and Type-II censored samples. The comparison of the risks (average loss over sample space) under LINEX loss function. The estimators $\hat{\lambda}_B, \hat{\lambda}_{B_2}, \hat{\lambda}_E, \hat{\lambda}_{E_2}; \hat{R}_B(t), \hat{R}_{B_2}(t), \hat{R}_E(t), \hat{R}_{E_2}(t)$ and $\hat{h}_B(t), \hat{h}_{B_2}(t), \hat{h}_E(t), \hat{h}_{E_2}(t)$ of $\lambda, R(t)$ and $h(t)$ are respective Bayes and empirical Bayes estimators for PT-II CBR and Type-II censoring samples under LINEX loss function, respectively. Through MC simulation obtained the risks of the estimators of 1000 samples. Here, we note that the risks of the estimators are function of $n, m, a, \alpha, \lambda, \beta$ and t. The choice of hyper parameters of the prior distribution of λ can be taken in such a way that if we consider any two independent information as prior mean and variance of λ, then, $(\mu = 1/\beta, \sigma^2 = 1/\beta^2)$ whereas μ is considered as true values of the parameter λ for different confidence in terms of smaller and larger variances. On the basis of this information, the hyper parameter of λ can be easily evaluated from this relation, $(\beta = \mu/\sigma^2)$.

In order to consider the variation of these values, we obtained the simulated risks for $n = 20[10]90, m = 10[10]80, t = 0.2, \alpha = 2$(known), $\lambda = 2 = \mu$ (say prior mean of λ), $\sigma^2 = (1,3)$ (say prior variance of λ), since $\beta = (2/1, 2/3)$, $a = \pm 1.5$. We use the symbol R_L to denote the risk under LINEX loss function, and the simulated risks under LINEX loss functions are given in Tables (4.1 − 4.2). Table (4.1) present the risks of estimators for PT-II CBR. The next Table (4.2) show the risks of estimators for Type-II censored samples. From Table (4.1), we can observe that for PT-II CBR, the risk of the estimators of $\hat{\lambda}_E$ and $\hat{h}_E(t)$ under LINEX loss function is the least (for both small and large prior variances i.e. $\sigma^2 = 1,3$) for both $a = +1.5$ (when o.e. is more serious than u.e.) and $a = -1.5$ (when u.e. is more serious than o.e.). But the risk of the estimators of $\hat{R}_B(t)$ under PT-II CBR is minimum for LINEX loss function with $a = \pm 1.5$ for small and large prior variances. Due to the change in the value of n and m (effective sample size), the risks of the estimators change, but follow a particular trend. Further, the risk of the estimator $\hat{\lambda}_E$ and $\hat{h}_E(t)$ under LINEX loss function was found to be least always. It is also observed that as the failure proportion (m/n) increases, the magnitude of the risk of the estimator $\hat{\lambda}_E$ and $\hat{h}_E(t)$ decreases. However, the magnitude of the risk of the estimator $\hat{R}_B(t)$ increases as failure proportion increases.

From Table (4.2), we can observe that for Type-II censoring, the risk of the estimators of $\hat{\lambda}_{E_2}, \hat{h}_{E_2}(t)$ and $\hat{R}_{B_2}(t)$ have also the least (for both small and large prior variances) for $a = \pm 1.5$ under LINEX loss function. When the change in the value of (n,m) with respective for small and large prior variances, the risks of the estimators change, they have follow a similar trend as discuss above in Table (4.1). But, the risk of the estimators at $\hat{\lambda}_{E_2}, \hat{h}_{E_2}(t)$ and $\hat{R}_{B_2}(t)$ were also found to be the least always. From Tables (4.1 − 4.2), it can be seen that the behavior of the risks of the estimators under PT-II CBR is more similar to that of the estimators under

Type-II censoring. The risks were found to be least for the empirical Bayes estimators $\hat{\lambda}_E, \hat{\lambda}_{E_2}$ and $\hat{h}_E(t), \hat{h}_{E_2}(t)$ of λ and $h(t)$ with an informative prior $\Gamma(1,\beta)$ respectively. Therefore, we propose that empirical Bayesian estimator of parameter and reliability characteristics can use planning of the experiment. Hence, the reliability practitioners can save much time and cost of the experiment.

4.5 An application to Ulcer Patients Data

Now, we extract 43 primary disease (ulcer) patients data set from Collett (2014) to show practical applicability of proposed work. It have been taken for the analysis of PT-II CBRs discussed in the context of a study based on age $\left((10^{-2}) * age\right)$ data. In order to have an idea about the associated primary disease (ulcer) patient's age failure rate, we considered, a graphical method based on TTT plot as a crude indicator see Aarset (1987). The empirical TTT is given as $T(\frac{r}{n}) = \frac{\sum_{i=1}^{r} x_{(i)} + (n-r)x_{(r)}}{\sum_{i=1}^{n} x_{(i)}}$, where $r = 1,2,\cdots,n$ and $x_{(r)}$ is the order statistics of the sample. For this data set in Figure (4.4) shows concave TTT plots, indicating increasing failure rate functions along with Figure (4.5), (4.6) and (4.7) represent PDF/CDF plot, sample Q-Q plot and hazard plots respectively, which can be properly accommodated by KD. However, we fitted three competitive distributions, $F(x;\alpha,\lambda) = \left(1-e^{-\lambda x}\right)^{\alpha}$, $x > 0,\ \alpha > 0, \lambda > 0$ and $F(x;\alpha,\lambda) = 1-e^{-(\lambda x)^{\alpha}}$, $x > 0,\ \alpha > 0, \lambda > 0$ are CDFs of the EED (Exponentiated exponential distribution) and WD (Weibull distribution) respectively. Table (4.3) provides the -log-L values and the AIC, BIC and p-values for these distributions. They indicate evidence in favor of KD. The ML estimates (and their corresponding standard errors in parentheses) of the KD, EED and WD parameters are given by $\hat{\alpha} = 3.2490(0.0108917), \hat{\lambda} = 5.64104(0.03766)$; $\hat{\alpha} = 15.44731(0.12571), \hat{\lambda} = 6.531165(0.01933)$ and $\hat{\alpha} = 3.55875(0.01012), \hat{\lambda} = 1.76975(0.00186)$ respectively. But for the purpose of illustrating the method discussed in this chapter, PT-II CBR samples are generated from this data set under different schemes see Table (4.6). The box plot of different censoring schemes as well as descriptive statistics is also presented in Figure (4.8) and Table (4.7) respectively. The required numerical calculations for the considered schemes are carried out using the formula given in Section (4.3) through R software see Ihaka and Gentleman (1996). The Bayes estimates, empirical Bayes estimates of λ_B, $R_B(.)$, $h_B(.)$ and λ_E,$R_E(.)$, $h_E(.)$ under LINEX loss for $a = \pm 1.5$ are presented in Table (4.4). While, Table (4.5) shows the Bayes, empirical Bayes estimates of λ_{B_2}, $R_{B_2}(.)$, $h_{B_2}(.)$ and λ_{E_2},$R_{E_2}(.)$, $h_{E_2}(.)$ for Type-II censoring under LINEX loss for $a = \pm 1.5$. From Tables $(4.4 - 4.5)$, it may also be observed that the behavior of the estimators under PT-II CBRs are more similar to that of the estimators under Type-II censoring. The estimates were found to be decreases as effective sample size increases.

4.6 Conclusion

On the basis of the previous discussion given in the above Section (4.5), we may conclude that the proposed empirical Bayes estimators $\hat{\lambda}_E, \hat{\lambda}_{E_2}$ and $\hat{h}_E(t), \hat{h}_{E_2}(t)$ are better than Bayes estimators $\hat{\lambda}_B, \hat{\lambda}_{B_2}$ and $\hat{h}_B(t), \hat{h}_{B_2}(t)$ for smaller or larger prior variance $(\sigma = 1, 3)$ of β with $a = \pm 1.5$. Also, we have seen that Table $(4.1 - 4.2)$ under LINEX loss function for the estimators $\hat{R}_E(t)$ and $\hat{R}_{E_2}(t)$ is not always less than those of $\hat{R}_B(t)$ and $\hat{R}_{B_2}(t)$. Since the risks associated with $\hat{R}_B(t)$ and $\hat{R}_{B_2}(t)$ is smaller than the risk associated with reliability of the empirical estimators. Thus, the use of propose estimator $(\hat{\lambda}_E, \hat{R}_B(t), \hat{h}_E(t))$ and $(\hat{\lambda}_{E_2}, \hat{R}_{B_2}(t), \hat{h}_{E_2}(t))$ under PT-II CBRs and Type-II are recommended under LINEX loss function respectively.

TABLE 4.1: Risks of the estimators of λ, R and h under LINEX loss function for fixed $\alpha = 2, \lambda = 2$ and $t = 0.2$ under PT-II CBR.

σ	n	m	$R_L(\hat{\lambda}_B)$	$R_L(\hat{\lambda}_E)$	$R_L(\hat{R}_B(t))$	$R_L(\hat{R}_E(t))$	$R_L(\hat{h}_B(t))$	$R_L(\hat{h}_E(t))$	$R_L(\hat{\lambda}_B)$	$R_L(\hat{\lambda}_E)$	$R_L(\hat{R}_B(t))$	$R_L(\hat{R}_E(t))$	$R_L(\hat{h}_B(t))$	$R_L(\hat{h}_E(t))$
			$a = -1.5$						$a = +1.5$					
	20	10	2.0554	1.7067	2.2062	2.2484	7.7037	6.8461	0.7931	0.6942	0.7718	0.7807	1.5106	1.4218
	30	20	0.9801	0.7438	2.2609	2.2924	6.3574	5.7448	0.4950	0.4049	0.7837	0.7903	1.3732	1.3034
	40	30	0.5118	0.3769	2.3032	2.3312	5.4718	4.9748	0.3271	0.2584	0.7915	0.7972	1.2841	1.2241
1	50	40	0.3493	0.2581	2.3235	2.3473	5.0844	4.6820	0.2452	0.1924	0.7956	0.8004	1.2368	1.1862
	60	50	0.2605	0.1944	2.3374	2.3578	4.8334	4.4966	0.1859	0.1471	0.7993	0.8034	1.1959	1.1511
	70	60	0.2050	0.1541	2.3472	2.3651	4.6637	4.3741	0.1537	0.1233	0.8013	0.8050	1.1723	1.1329
	80	70	0.1555	0.1200	2.3601	2.3766	4.4506	4.1934	0.1224	0.0992	0.8036	0.8070	1.1463	1.1105
	90	80	0.1319	0.1014	2.3642	2.3789	4.3803	4.1522	0.1111	0.0901	0.8041	0.8071	1.1402	1.1086
	20	10	1.5727	1.5716	2.2474	2.2521	6.8021	6.7789	0.6867	0.6842	0.7798	0.7807	1.4286	1.4218
	30	20	0.7183	0.7126	2.2933	2.2982	5.7144	5.6381	0.4101	0.4049	0.7894	0.7903	1.3132	1.3034
	40	30	0.4267	0.4147	2.3207	2.3253	5.1583	5.0838	0.2659	0.2584	0.7962	0.7972	1.2343	1.2241
3	50	40	0.2565	0.2460	2.3418	2.3460	4.7584	4.6900	0.1992	0.1924	0.7995	0.8004	1.1953	1.1862
	60	50	0.1979	0.1903	2.3559	2.3599	4.5271	4.4659	0.1521	0.1471	0.8027	0.8034	1.1595	1.1511
	70	60	0.1518	0.1459	2.3650	2.3685	4.3731	4.3190	0.1274	0.1233	0.8043	0.8050	1.1405	1.1329
	80	70	0.1266	0.1220	2.3723	2.3756	4.2577	4.2091	0.1022	0.0992	0.8063	0.8070	1.1176	1.1105
	90	80	0.1040	0.1001	2.3775	2.3805	4.1727	4.1287	0.0931	0.0901	0.8065	0.8071	1.1148	1.1086

TABLE 4.2: Risks of the estimators of λ, R and h under LINEX loss function for fixed $\alpha = 2, \lambda = 2$ and $t = 0.2$ under Type-II censoring.

σ	n	m	$R_L(\hat{\lambda}_{B_2})$	$R_L(\hat{\lambda}_{E_2})$	$R_L(\hat{R}_{B_2}(t))$	$R_L(\hat{R}_{E_2}(t))$	$R_L(\hat{h}_{B_2}(t))$	$R_L(\hat{h}_{E_2}(t))$	$R_L(\hat{\lambda}_{B_2})$	$R_L(\hat{\lambda}_{E_2})$	$R_L(\hat{R}_{B_2}(t))$	$R_L(\hat{R}_{E_2}(t))$	$R_L(\hat{h}_{B_2}(t))$	$R_L(\hat{h}_{E_2}(t))$
			a=-1.5						a=+1.5					
	20	10	15.4053	15.4024	1.9869	1.9869	15.8211	15.8200	2.0154	2.0152	0.7243	0.7243	2.0365	2.0365
	30	20	14.8668	14.8624	1.9911	1.9911	15.6074	15.6056	1.9872	1.9870	0.7252	0.7252	2.0257	2.0256
	40	30	14.3394	14.3333	1.9954	1.9954	15.3939	15.3914	1.9593	1.9590	0.7262	0.7262	2.0150	2.0148
1	50	40	13.8365	13.8289	1.9996	1.9997	15.1863	15.1832	1.9301	1.9297	0.7271	0.7272	2.0037	2.0035
	60	50	13.3365	13.3276	2.0039	2.0040	14.9758	14.9720	1.9030	1.9024	0.7281	0.7281	1.9932	1.9930
	70	60	12.8660	12.8558	2.0082	2.0083	14.7738	14.7693	1.8747	1.8741	0.7290	0.7290	1.9823	1.9820
	80	70	12.4045	12.3931	2.0124	2.0125	14.5717	14.5667	1.8469	1.8462	0.7299	0.7300	1.9715	1.9712
	90	80	11.9724	11.9600	2.0166	2.0167	14.3788	14.3732	1.8170	1.8162	0.7310	0.7310	1.9599	1.9596
	20	10	15.3963	15.3956	1.9869	1.9869	15.8176	15.8173	2.0149	2.0148	0.7243	0.7243	2.0364	2.0363
	30	20	14.8698	14.8688	1.9911	1.9911	15.6086	15.6082	1.9873	1.9873	0.7252	0.7252	2.0258	2.0257
	40	30	14.3382	14.3368	1.9954	1.9954	15.3934	15.3929	1.9587	1.9586	0.7262	0.7262	2.0147	2.0147
3	50	40	13.8395	13.8379	1.9996	1.9996	15.1876	15.1870	1.9310	1.9309	0.7271	0.7271	2.0040	2.0040
	60	50	13.3360	13.3340	2.0039	2.0040	14.9756	14.9748	1.9022	1.9020	0.7281	0.7281	1.9929	1.9928
	70	60	12.8641	12.8618	2.0082	2.0082	14.7729	14.7719	1.8743	1.8742	0.7290	0.7290	1.9821	1.9821
	80	70	12.3757	12.3732	2.0127	2.0127	14.5589	14.5578	1.8446	1.8444	0.7300	0.7300	1.9706	1.9705
	90	80	11.9640	11.9613	2.0166	2.0167	14.3750	14.3738	1.8188	1.8186	0.7309	0.7309	1.9606	1.9605

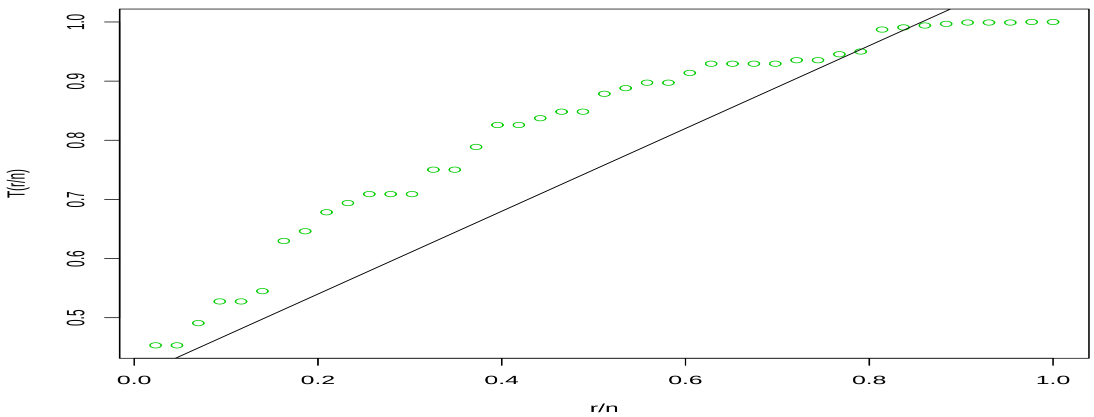

FIGURE 4.4: TTT plot for an ulcer patient with different ages $\left((10^{-2}) * age\right)$ for the primary disease.

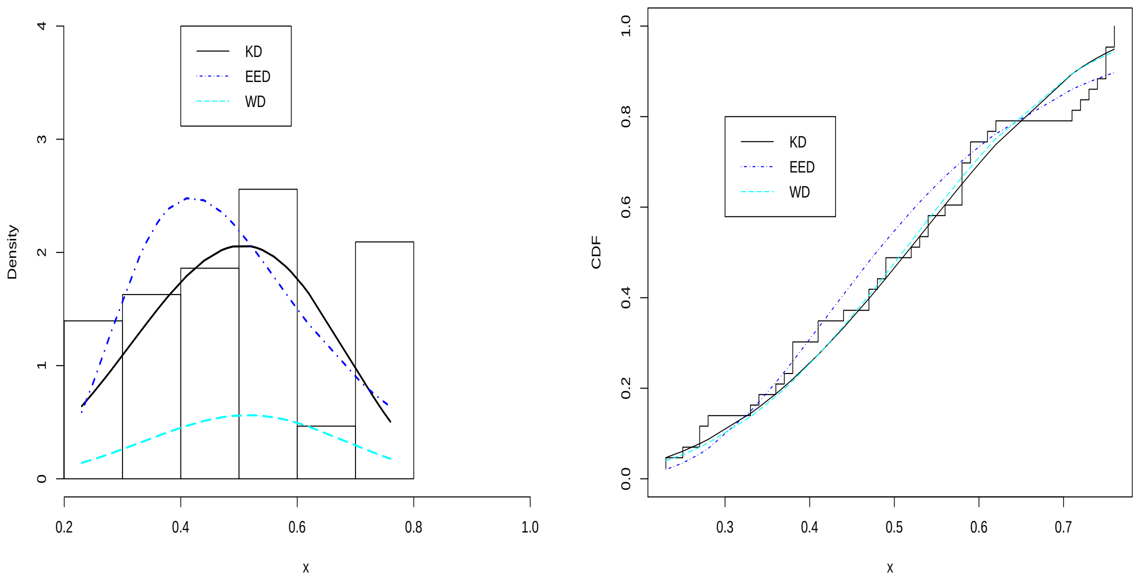

FIGURE 4.5: The PDF and CDF plots via the KD, EED and WD, for an ulcer patient with different ages $\left((10^{-2}) * age\right)$ for the primary disease. Left panel: PDF; right panel: CDF.

TABLE 4.3: The -log-L values and the AIC and BIC values for the KD, EED and WD fitted distributions.

distribution	-log-L	AIC	BIC	KS	p-value
KD	15.6765	27.35302	23.83062	0.082175	.9923
EED	18.6053	33.21062	29.68822	0.102652	.9333
WD	18.0699	32.13973	28.61733	0.086067	.9901

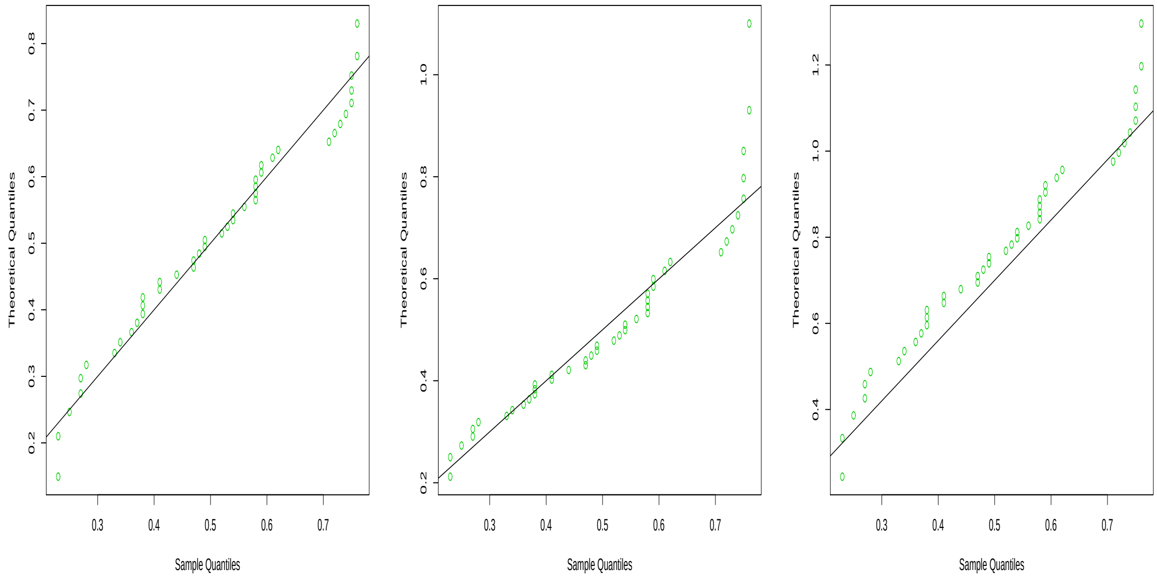

FIGURE 4.6: The sample Q-Q plots via the KD, EED and WD, for an ulcer patient with different ages $\left((10^{-2}) * age\right)$ for the primary disease. Left panel: KD; middle panel: EED; right panel:WD.

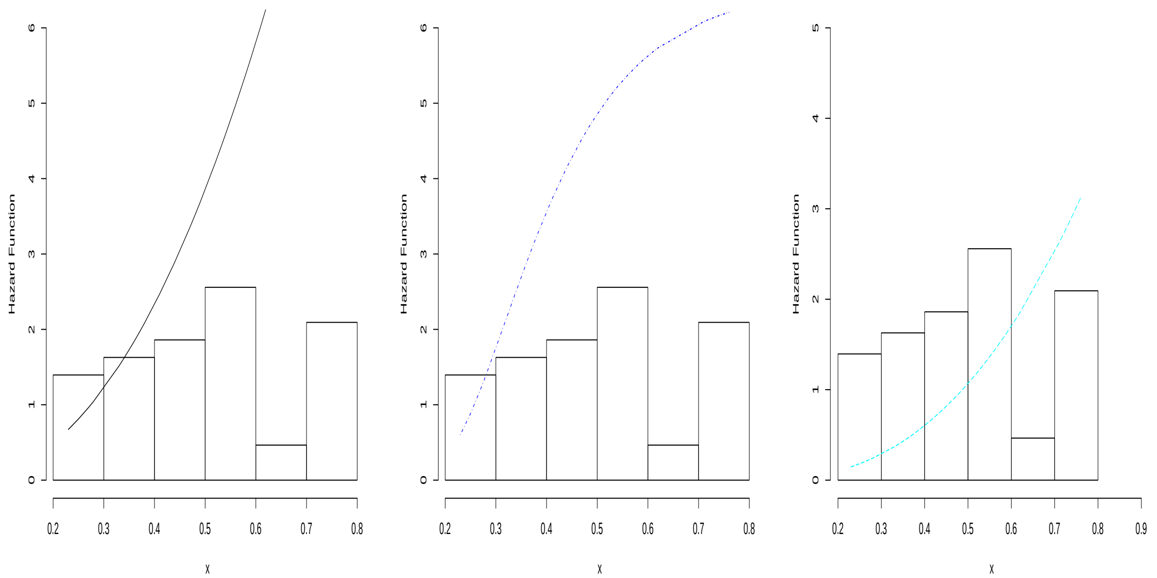

FIGURE 4.7: The hazard plots via the KD, EED and WD, for an ulcer patient with different ages $\left((10^{-2}) * age\right)$ for the primary disease. Left panel: KD; middle panel: EED; right panel:WD.

TABLE 4.4: Bayes and empirical Bayes estimates of λ, $R()$ and $h()$ under LINEX loss function for an ulcer patient with different ages $\left((10^{-2})*age\right)$ for the primary disease with fixed $n=43, p=0.5$, and $t=0.5074419$ under PT-II CBR.

Scheme	$\hat{\lambda}_B$		$\hat{\lambda}_E$		$\hat{R}_B(t)$		$\hat{R}_E(t)$		$\hat{h}_B(t)$		$\hat{h}_E(t)$	
$S_{n:m}$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$
$S_{43:20}$	22.4656	8.2216	20.7328	8.0140	0.8580	0.8568	0.8613	0.8601	2.0805	1.8104	2.0172	1.7724
$S_{43:23}$	19.8084	8.5843	18.7520	8.3980	0.8326	0.8312	0.8375	0.8361	2.4059	2.0935	2.3170	2.0321
$S_{43:25}$	15.3871	8.0207	14.7764	7.8589	0.7944	0.7924	0.7999	0.7981	2.8227	2.4187	2.7266	2.3519
$S_{43:30}$	14.1203	8.3109	13.7254	8.1758	0.7410	0.7387	0.7477	0.7453	3.4093	2.9418	3.3229	2.8555
$S_{43:33}$	10.6401	7.2122	10.3965	7.1005	0.6808	0.6777	0.6847	0.6818	3.9226	3.3225	3.8441	3.2937
$S_{43:35}$	9.8288	6.9530	9.6264	6.8518	0.6497	0.6465	0.6527	0.6494	4.1826	3.5656	4.1370	3.5207
$S_{43:38}$	7.9651	6.0887	7.8212	6.0045	0.5928	0.5893	0.5959	0.5921	4.5155	3.8749	4.5134	3.8106
$S_{43:40}$	7.7514	6.0319	7.6227	5.9539	0.5692	0.5654	0.5738	0.5702	4.7356	4.0268	4.6237	3.9718

TABLE 4.5: Bayes and empirical Bayes estimates of λ, $R()$ and $h()$ under LINEX loss function for an ulcer patient with different ages $\left((10^{-2}) * age\right)$ for the primary disease with fixed $n = 43$ and $t = 0.5074419$ under Type-II censoring.

Schame	$\hat{\lambda}_{B_2}$		$\hat{\lambda}_{E_2}$		$\hat{R}_{B_2}(t)$		$\hat{R}_{E_2}(t)$		$\hat{h}_{B_2}(t)$		$\hat{h}_{E_2}(t)$	
(n,m)	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$
(43,20)	18.5013	7.7001	17.2407	7.4957	0.8554	0.8541	0.8578	0.8566	2.0807	1.8017	2.0223	1.7769
(43,23)	18.5814	8.3666	17.6334	8.1843	0.8277	0.8261	0.8309	0.8295	2.4480	2.1160	2.3925	2.0837
(43,25)	16.7217	8.3504	16.0359	8.1849	0.8041	0.8024	0.8055	0.8038	2.7107	2.3522	2.6790	2.3306
(43,30)	13.7657	8.1897	13.3810	8.0551	0.7434	0.7410	0.7464	0.7442	3.3630	2.8924	3.3011	2.8610
(43,33)	11.5500	7.6144	11.2843	7.4993	0.6925	0.6896	0.6953	0.6926	3.8517	3.2836	3.8030	3.2554
(43,35)	9.7266	6.9021	9.5257	6.8010	0.6490	0.6459	0.6551	0.6522	4.1296	3.5610	4.0747	3.5005
(43,38)	8.5165	6.4045	8.3633	6.3178	0.6015	0.5977	0.6065	0.6029	4.5553	3.8370	4.4561	3.7968
(43,40)	7.7646	6.0399	7.6356	5.9617	0.5673	0.5635	0.5725	0.5686	4.7640	4.0620	4.7010	3.9839

TABLE 4.6: PT-II CBR under different censoring schemes $(S_{n:m})$ for fixed $n = 43$ and $p = 0.5$ for an ulcer patient with different ages $\left((10^{-2}) * age\right)$ for the primary disease.

$S_{n:m}$	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$S_{43:20}$	X_i	0.23	0.37	0.49	0.58	0.58	0.58	0.58	0.59	0.59	0.61	0.62	0.71	0.72	0.73	0.74
	R_i	8	10	5	0	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.75	0.75	0.75	0.76	0.76										
	R_i	0	0	0	0	0										
$S_{43:23}$	X_i	0.23	0.38	0.47	0.52	0.53	0.54	0.58	0.58	0.58	0.58	0.59	0.59	0.61	0.62	0.71
	R_i	9	5	4	0	1	1	0	0	0	0	0	0	0	0	0
	X_i	0.72	0.73	0.74	0.75	0.75	0.75	0.76	0.76							
	R_i	0	0	0	0	0	0	0	0							
$S_{43:25}$	X_i	0.23	0.36	0.41	0.49	0.49	0.53	0.54	0.54	0.58	0.58	0.58	0.58	0.59	0.59	0.61
	R_i	7	5	4	0	1	0	0	1	0	0	0	0	0	0	0
	X_i	0.62	0.71	0.72	0.73	0.74	0.75	0.75	0.75	0.76	0.76					
	R_i	0	0	0	0	0	0	0	0	0	0					
$S_{43:30}$	X_i	0.23	0.38	0.41	0.47	0.47	0.48	0.49	0.49	0.52	0.53	0.54	0.54	0.56	0.58	0.58
	R_i	10	1	2	0	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.58	0.58	0.59	0.59	0.61	0.62	0.71	0.72	0.73	0.74	0.75	0.75	0.75	0.76	0.76
	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$S_{43:33}$	X_i	0.23	0.28	0.37	0.38	0.41	0.44	0.47	0.47	0.48	0.49	0.49	0.52	0.53	0.54	0.54
	R_i	4	3	1	2	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.56	0.58	0.58	0.58	0.58	0.59	0.59	0.61	0.62	0.71	0.72	0.73	0.74	0.75	0.75
	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.75	0.76	0.76												
	R_i	0	0	0												
$S_{43:35}$	X_i	0.23	0.28	0.37	0.38	0.38	0.41	0.41	0.44	0.47	0.47	0.48	0.49	0.49	0.52	0.53
	R_i	4	3	0	0	1	0	0	0	0	0	0	0	0	0	0
	X_i	0.54	0.54	0.56	0.58	0.58	0.58	0.58	0.59	0.59	0.61	0.62	0.71	0.72	0.73	0.74
	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.75	0.75	0.75	0.76	0.76										
	R_i	0	0	0	0	0										
$S_{43:38}$	X_i	0.23	0.23	0.27	0.34	0.37	0.38	0.38	0.38	0.41	0.41	0.44	0.47	0.47	0.48	0.49
	R_i	0	2	2	1	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.49	0.52	0.53	0.54	0.54	0.56	0.58	0.58	0.58	0.58	0.59	0.59	0.61	0.62	0.71
	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.72	0.73	0.74	0.75	0.75	0.75	0.76	0.76							
	R_i	0	0	0	0	0	0	0	0							
$S_{43:40}$	X_i	0.23	0.27	0.28	0.33	0.34	0.36	0.37	0.38	0.38	0.38	0.41	0.41	0.44	0.47	0.47
	R_i	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.48	0.49	0.49	0.52	0.53	0.54	0.54	0.56	0.58	0.58	0.58	0.58	0.59	0.59	0.61
	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.62	0.71	0.72	0.73	0.74	0.75	0.75	0.75	0.76	0.76					
	R_i	0	0	0	0	0	0	0	0	0	0					

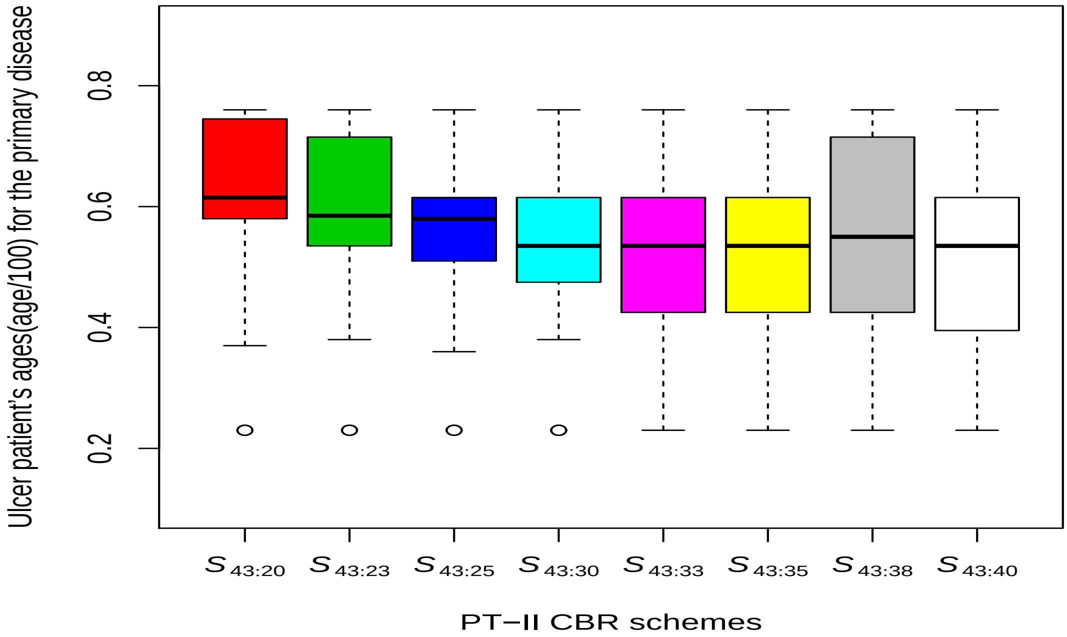

FIGURE 4.8: Box plot for PT-II CBR under different censoring schemes $S_{n:m}$ for an ulcer patient with different ages $\left((10^{-2})*age\right)$ for the primary disease.

TABLE 4.7: Summary of the different censoring schemes ($S_{n:m}$) for PT-II CBR.

$S_{n:m}$	Min	Q_1	Median	Mean	Q_3	Max	SD	Skewness	Kurtosis
$S_{43:20}$	0.23	0.580	0.6150	0.62450	0.74250	0.76	0.1402807	-1.1887960	0.9533382
$S_{43:23}$	0.23	0.560	0.5900	0.61170	0.73500	0.76	0.1344907	-0.9352590	0.6451634
$S_{43:25}$	0.23	0.540	0.5900	0.59960	0.73000	0.76	0.1370669	-0.7446627	0.1144300
$S_{43:30}$	0.23	0.498	0.5800	0.58370	0.71750	0.76	0.129973	-0.4230723	-0.1033113
$S_{43:33}$	0.23	0.480	0.5800	0.56360	0.71000	0.76	0.140886	-0.3267876	-0.5493007
$S_{43:35}$	0.23	0.470	0.5600	0.55400	0.66500	0.76	0.1423789	-0.1981708	-0.7473111
$S_{43:38}$	0.23	0.433	0.5500	0.54630	0.71250	0.76	0.156757	-0.2054257	-0.9473064
$S_{43:40}$	0.23	0.403	0.5350	0.52680	0.61250	0.76	0.1522546	-0.0128692	-1.0644150