Chapter 5

Bayesian Estimation of the Number of Species Using Poisson Lindley Stochastic Abundance Model

5.1 Introduction

Previous chapters are based on the lifetime problem. While this chapter deals with ecological problem to estimating the number of species are present in an organism. The problem of estimating the number of species has been discussed extensively in the biological and ecological literature (Wilson and Collins (1992), Colwell and Coddington (1994), Bunge et al. (1995)). Various approaches have been proposed like parametric and non-parametric respectively. Both of these approaches have some optimal properties. In a parametric distribution, we can fit the observed frequency counts and use the estimated parameter values to estimate the number of species see Greenwood and Yule (1920). A non-parametric approach of ML version has been given by Norris and Pollock (1998). In non-parametric, the estimators are based on the coverage of the sample and the fraction of the population. These concepts were first proposed by Chao and Lee (1992).

But authors are interested to estimate the total number of species. When the total number of species have not been caught during the experiment. The estimators have been developed in this chapter through a parametric approach when observed samples were induced a parametric model. The estimation methods needed in this chapter are based on a Poisson mixed sampling model. Because each species independently contributed as representatives of the sample according to a Poisson process. When the rate or abundance parameters for these processes are taken to be i.i.d. RV from some fixed well-known distribution, see Chao and Bunge (2002).

One parameter Lindley (1958) distribution has been used for this process. Ghitany et al. (2008) studied some properties of the one-parameter Lindley distribution. In the application part, they showed that it is more flexible and works better in modeling for different types of data than well-known exponential distribution. Now we mixed this distribution with Poisson, and get discrete Poisson Lindley distribution. For applicability of Poisson mixed distributions, authors are referring to see Sankaran (1970). Furthermore the distributions based on Poisson mixture model for species abundance problems have been study by many authors such as (Bulmer (1974), Ord and Whitmore (1986), Sichel (1986)) etc. But in estimation problem for the number of species, according to Fisher et al. (1943) and Sichel (1986), it has required a suitable Poisson mixed model for a given problem. Thus we need to Poisson mixed as well as in-truncated distributions as per the demand of the problem see, Leite et al. (2000). Bunge and Fitzpatrick (1993) shows an interesting review of the problem of estimating the number of species. Therefore, we motivated by the above study is that no attempt has been made to use Poisson Lindley distribution as a model in species problems. Therefore, in this chapter we propose to develop such an estimator and estimation procedure for the parameters. The details of the mathematical formulations are discussed in a further Section.

In the past few decades, Bayesian estimation for the number of species population parameter based on Poisson mixed models have been studied by several authors such as Lewins and Joanes (1984), Leite et al. (2000) and Barger et al. (2010), etc. Fully hierarchical and empirical Bayesian estimation of the number of species based on Poisson-Gamma mixed model has been discussed by Rodrigues et al. (2001), and for other Poisson-mixed models by Wang et al. (2007), Barger et al. (2010), etc. But, it seems as if no attempt has been made to develop Bayes estimators of the number of species based on Poisson mixed Lindley distribution. Although estimation of the number of species based on Poisson mixed models under classical set up has been attempted by Gotelli and Colwell (2011), Chao and Lee (1992), Sichel (1986), etc. Therefore, we propose to develop a Bayesian estimation procedure to obtain the estimate of the number of species (using a Lindley model as a stochastic abundance model in which the sample according to independent Poisson process i.e., Poisson Lindley). Jeffery's and Bernardo's reference priors have been obtaining and proposed the Bayes estimators of the number of species for this model. An important feature of this chapter is to develop the required mathematics for the number of species parameters and priors along with its application to biological data.

5.2 Model and Likelihood Function

In biological sampling there are S species present, it has for some time been observed. When the successive, independent and unequal samples with sizes $x_1, x_2, x_3, \cdots, x_S$ be taken from heterogeneous abundance of species. The number of individuals observed in different samples will vary in a different manner in study period $[0,t]$. The distribution of the number of observed species depends only on one parameter Poisson distribution $(t\lambda_i)$ may be easily expressed in terms of the number expected (λ_i), which is given $\frac{e^{-t\lambda}(t\lambda)^{x_i}}{x_i!}$, $i = 1, 2, 3, \cdots, S$. Where X_i is the variate representing the number, which has been observed in any sample. And λ_i is the parameter, which is average value of X_i, and need not be whole number. This is an extension of the Poisson process, and is provided by supposition that the values of λ are distributed as well-known Lindley distribution with density function f_η, where η is a low dimensional parameter vector. In the Lindley distribution case, $\eta = \theta$ and $f_\eta(\lambda) = f_\theta(\lambda)$ and empirical CDF is $F_\theta(\lambda)$. Thus λ must be positive, and it has followed a well known form the distribution of Lindley (θ), such that the element of frequency or probability with which it falls in any infinitesimal range.

$$F_\theta(\lambda) = F(\lambda|\theta) = 1 - \frac{(1+\theta+\lambda\theta)e^{-\lambda\theta}}{1+\theta}; \qquad \lambda > 0; \theta > 0,$$

$$f_\theta(\lambda) = f(\lambda|\theta) = \frac{\theta}{\theta+1}(1+\lambda)e^{-\lambda\theta}; \qquad \lambda > 0; \theta > 0.$$

We can only observe the number of individuals contributed to the sample by each species. When contribution is greater than 0 i.e. $X_i > 0$. The species that contribute zero individuals to the sample are unobserved. The observed data are therefore $S_j = \sum_{i=1}^{S} I(X_i = j)$ for $j \geq 1$. Thus, S_j represent the number of species that contribute j individuals to the sample. The observed number of species is $w = \sum_{j\geq 1} s_j$, and the observed number of individual is $s = \sum_{j\geq 1} js_j$, where s_j are realized values of S_j. The goal is to estimate S (or equivalently to predict s_0) based on the observed frequency counts $\{s_j : j \geq 1\}$. Without loss of generality, we can and do take $t = 1$ because the time scale does not affect any of our estimates of S.

Therefore, the marginal distribution of X_i is $p_\theta(j) = \int \frac{e^{-\lambda}\lambda^j}{j!} f(\lambda|\theta) d\lambda$ representing the zero truncated P-mixed Poisson distribution, where $f(\lambda|\theta) = \frac{\partial}{\partial\lambda} F(\lambda|\theta)$. Sankaran (1970) derived the zero truncated P-mixed Poisson Lindley distribution given below,

$$p_\theta(j) = \left(\frac{\theta}{1+\theta}\right)^2 \frac{j+\theta+2}{(\theta+1)^{j+1}}; \qquad \theta > 0; j = 0,1,2,3,\cdots, \tag{5.1}$$

when $j = 0$ then Equation (5.1) become

$$p_\theta(0) = \left(\frac{\theta}{1+\theta}\right)^2 \left(\frac{\theta+2}{\theta+1}\right); \qquad \theta > 0. \tag{5.2}$$

5.2.1 Likelihood and Information of Parameters

The likelihood can be written as

$$L(S,\theta|x) = \sum_{j\in\Delta} \prod_{j=1}^{S} p_\theta(j).$$

Where Δ is the set of $x_1, x_2, x_3, \cdots, x_S$ which correspond to the observed frequencies $(s_1, s_2, \cdots, s_S)$. Sanathanan (1972) has demonstrated that the likelihood can be written as

$$\begin{aligned}
L(S,\theta|x) &= \binom{S}{w}(1-p_\theta(0))^w (p_\theta(0))^{S-w} \frac{w!}{\prod_{j\geq 1} s_j!} \prod_{j\geq 1}\left(\frac{p_\theta(j)}{1-p_\theta(0)}\right)^{s_j} \\
&= \binom{S}{w}\left(1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^w \left(\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^{S-w} \\
&\quad \frac{w!}{\prod_{j\geq 1} s_j!} \prod_{j\geq 1}\left(\frac{\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{j+\theta+2}{(\theta+1)^{j+1}}\right)}{1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)}\right)^{s_j} \\
&= A(S,\theta)B(\theta), \qquad (5.3)
\end{aligned}$$

where $S \geq w$, i.e. $S - w = s_0$ is the number of unobserved species. Now, the likelihood are function of parameters S and θ in the Equation (5.3), where $\theta = (\theta_1, \theta_2, ..., \theta_m)$. Since, we consider θ is a nuisance parameter, and our interest is in estimating S. Which shows the likelihood can be factored into a binomial likelihood for w that corresponds $A(S,\theta)$, and a multinomial likelihood for the observed frequencies corresponds the $B(\theta)$. This factorization of the integrated likelihood has an important role. It was first formulated by Sanathanan (1972) who derived the asymptotic theory for the ML estimation for S and θ. Fisher Information matrix can only be found for likelihoods which are differentiable with respect to the parameters. In the species likelihood, S is discrete parameter, $S = 1, 2, ...$ This likelihood is not differentiable with respect to S. Lindsay and Roeder (1987) define information for discrete parameters using the LDS defined as

$$LDS(S) = \frac{L(S) - L(S-1)}{L(S)},$$

were $L(S)$ is the likelihood for an integer parameter S.

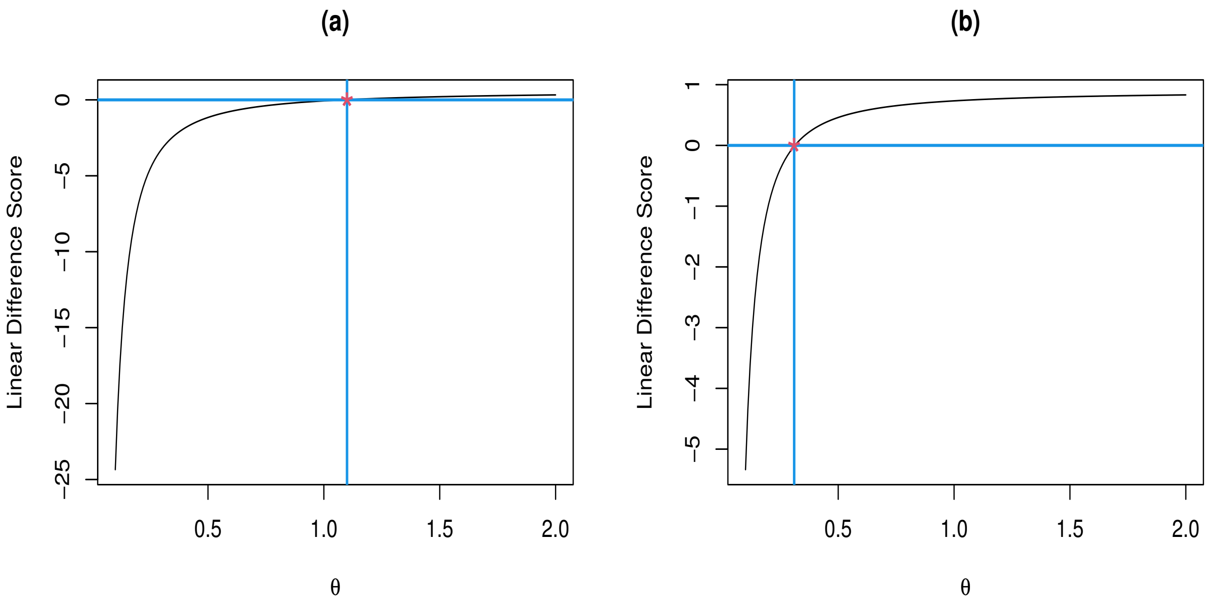

FIGURE 5.1: Plot of LDS for S with respect different θ for fixed in (a) $w = 30$, $N = 50$, and (b) $w = 45$, $N = 50$.

If $LDS(S)$ satisfies the form $LDS(S) = (Y - \mu_S)/c_S$, where μ_S and c_S are function of S and Y is random data, then $1/Var(LDS(S))$ is the information in S. In Figure (5.1), shows that $LDS(S) = 0$ then it gives the maxima of S with respective to different choice of θ. Using the method described by Lindsay and Roeder (1987) to calculate the information for S and θ, we obtain,

$$F(S,\theta) = \begin{pmatrix} \frac{1}{S}\frac{1-p_\theta(0)}{p_\theta(0)} & \left(-\frac{\partial}{\partial\theta} log p_\theta(0)\right)^T \\ -\frac{\partial}{\partial\theta} log p_\theta(0) & S(-\rho(\theta)) \end{pmatrix}.$$

Where, $\frac{\partial}{\partial\theta}p_\theta(0)$ is the column vector of partial derivatives. The $\rho(\theta) = \left(E_x \frac{\partial^2}{\partial\theta^2} log p_\theta(j)\right)$, has taken expectation with respect to p_θ. We may also observed that the diagonal elements of partitioned matrix contain elements which factor into a function of S times a function of θ. Thus we have,

$$F(S,\theta) = \begin{pmatrix} \frac{1}{S}\frac{(1+\theta)^3-\theta^2(\theta+2)}{\theta^2(\theta+1)} & -\left(\frac{2\theta^2+\theta-4}{\theta(\theta+1)(\theta+2)}\right)^T \\ -\left(\frac{2\theta^2+\theta-4}{\theta(\theta+1)(\theta+2)}\right) & S\left(\frac{2}{\theta} - \frac{2+8\theta+13\theta^2+10\theta^3+3\theta^4}{\theta(\theta+1)^5} + \psi(j,\theta)\right) \end{pmatrix}, \tag{5.4}$$

where, $\psi(j,\theta) = \sum_{j=0}^{\infty} \frac{e^{-\theta j}(j+1)\theta^2}{(\theta+1)(j+\theta+2)^2}$.

5.3 Bayes Estimators of Parameters

In Bayesian paradigm, the parameter of interest θ and S are consider to be RV, and having their prior distribution. The selection of prior distribution is often based on the type of prior information available to us. When we have minimal or no information about the parameter then a non-informative prior should be used.

5.3.1 Bayes Estimators of Parameters Using Jeffery's Priors

The Jeffrey's prior (see Jeffreys (1946)) is one of the general rule. Using the fisher information matrix as shown above $F(S,\theta)$ in Equation (5.4). The Jeffery's prior for (S,θ) is $g_J(S,\theta)$. It based on invariance property under one to one re-parameterization. The Jeffery's prior is defined to be proportional to the square root of the Fisher information matrix. For multidimensional model, the determinant of the Fisher information is used, which preserve the invariance property. By calculating the determinant of the partitioned matrix in Equation (5.4) is,

$$\begin{aligned} g_J(S,\theta) &\propto det[F(S,\theta)]^{1/2} \\ &\propto S^{\frac{m-1}{2}} g(\theta), \end{aligned} \tag{5.5}$$

where $g(\theta)$ is some function of θ, which will depend on the dimension of the information matrix. When the dimension increases this will become complex,

$$g_J(S,\theta) \propto \left(\left(\frac{(\theta+1)^3 - \theta^2(\theta+2)}{\theta^2(\theta+2)} \right) \left(\frac{2}{\theta^2} - \frac{2+8\theta+13\theta^2+10\theta^3+3\theta^4}{\theta(\theta+1)^5} + \psi(j,\theta) \right) - \left(\frac{2\theta^2+\theta-4}{\theta(\theta+1)(\theta+2)} \right)^2 \right)^{\frac{1}{2}}. \tag{5.6}$$

Several authors have presented a general rule, and for using Jeffery's prior for an exponential family by showing that a proper posterior is produced (Jeffreys (1961), Barger and Bunge (2008) and Barger et al. (2010)). The product of two independent prior has been discussed by Jeffery. They suggest to use an idea about reasonable Jeffery's prior for integer parameter S and continuous parameter θ. Integer parameter is the interest of our study parameter. Using Bayes theorem for computing likelihood in Equation (5.3) and Jeffery's prior in Equation (5.7). We get the joint posterior distribution of $\pi_J(S,\theta)$ is,

$$\pi_J(S,\theta|x) \propto L(S,\theta|x)g_J(S,\theta),$$

$$\pi_J(S,\theta|x) \propto \binom{S}{w}\left(1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^w\left(\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^{S-w}$$
$$\frac{w!}{\prod_{j\geq 1} s_j!}\prod_{j\geq 1}\left(\frac{\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{j+\theta+2}{(\theta+1)^{j+1}}\right)}{1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)}\right)^{s_j}$$
$$\left(\left(\frac{(\theta+1)^3-\theta^2(\theta+2)}{\theta^2(\theta+2)}\right)\left(\frac{2}{\theta^2}-\frac{2+8\theta+13\theta^2+10\theta^3+3\theta^4}{\theta(\theta+1)^5}+\psi(j,\theta)\right)\right.$$
$$\left.-\left(\frac{2\theta^2+\theta-4}{\theta(\theta+1)(\theta+2)}\right)^2\right)^{\frac{1}{2}}. \tag{5.7}$$

Now full conditional posterior for S is

$$\begin{aligned}\pi_J(S|\theta,x) &\propto \binom{S}{w}\left(1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^{w+1}\left(\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^{S-w},\\ &\propto NB\left(w+1,\left(1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)\right),\end{aligned} \tag{5.8}$$

and full conditional posterior for θ is

$$\pi_J(\theta|S,x) \propto \frac{w!}{\prod_{j\geq 1} n_j!} \prod_{j\geq 1} \left(\left(\frac{\theta}{1+\theta} \right)^2 \left(\frac{j+\theta+2}{(\theta+1)^{j+1}} \right) \right)^{s_j} \left(\frac{1}{1-\left(\frac{\theta}{1+\theta}\right)^2 \left(\frac{\theta+2}{\theta+1}\right)} \right)^{w+1}$$
$$\left(\left(\frac{(\theta+1)^3 - \theta^2(\theta+2)}{\theta^2(\theta+2)} \right) \left(\frac{2}{\theta^2} - \frac{2+8\theta+13\theta^2+10\theta^3+3\theta^4}{\theta(\theta+1)^5} + \psi(j,\theta) \right) - \left(\frac{2\theta^2+\theta-4}{\theta(\theta+1)(\theta+2)} \right)^2 \right)^{\frac{1}{2}}. \quad (5.9)$$

Full conditional posterior for S, we can use direct sampling from negative binomial distribution with size $(w+1)$ and probability $\left(1-\left(\frac{\theta}{1+\theta}\right)^2 \left(\frac{\theta+2}{\theta+1}\right)\right)$, and full conditional posterior for θ does not come in closed form then using the M-H steps uses a normal proposal distribution to get posterior samples.

5.3.2 Bayes Estimators of Parameters Using Bernardo's Reference Priors

Now, we have proposed for considering the Bernardo's reference prior. This prior is a quite general and powerful tool for obtaining automatic prior to be used in Bayesian analysis. Because of that reference prior are firstly useful with large sample but may also be helpful where the data analysis is unsure whether a sample is large. Typically the Bernardo's reference prior is the same as the Jeffery's prior in the one dimensional case, but where the parameter space Θ is bivariate or more. Non-informative prior is Bernardo's reference prior (Bernardo (1979)) based on maximizing and expected entropy (measurement of loss of information). The Bernardo's reference prior algorithm take into account (see Bernardo and Ramon (1998). It may also noted that in the standard Bayesian approach, the Bernardo's reference prior is used to obtain the joint posterior for $(S,\theta) \equiv \Theta$. In this approach, we only discuss the two groups case, where the parametric space Θ or vector is split in the parameter of interest S, and the nuisance parameter, θ under certain regularity conditions (see Bernardo (1979), Bernardo and Ramon (1998), Bernardo and Smith (2009)) for the existence of a consistent and asymptotically normal estimator of the parameters. Thus the reference prior for S, when θ is known, is used Fisher information matrix in Equation (5.4). The construction of the reference prior takes into account the order of interest of the parameters S and θ is a nuisance parameter.

Now we obtain the Bernardo's reference prior for a nuisance parameter, $m = 1$. Thus, the Fisher information matrix in Equation (5.4) will be 2×2. Let us assume $H = F^{-1}$ be the variance-covariance matrix. $(h_{11})^{-1/2} = a_0(S)b_0(\theta)$ and $(f_{22})^{1/2} = a_1(S)b_1(\theta)$ are the elements of the covariance and information matrices, respectively. The nuisance parameter θ and number of species S are independent to each other. The joint Bernardo's reference prior $g_R(S,\theta)$ will become,

$$\begin{aligned} g_R(S,\theta) &\propto (a_0(S))^{-1/2}(b_1(\theta))^{1/2} \\ &\propto S^{-1/2}(\rho(\theta))^{1/2}. \end{aligned} \tag{5.10}$$

$$g_R(S,\theta) \propto S^{-1/2}\left(-\frac{2}{\theta^2} + \frac{2+8\theta+13\theta^2+10\theta^3+3\theta^4}{\theta(\theta+1)^5} - \psi(j,\theta)\right)^{1/2}. \tag{5.11}$$

From above Equation (5.11), we may also seen the joint $g_R(S,\theta)$ factorized into a marginal distribution function of S and θ. Now, the joint posterior distribution of $\pi(S,\theta)$ is based on likelihood and Bernardo's reference prior is,

$$\pi_R(S,\theta|x) \propto L(S,\theta|x)g_R(S,\theta),$$

$$\begin{aligned} \pi_R(S,\theta|x) \propto & \binom{S}{w}\left(1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^w \left(\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^{S-w} \\ & \frac{w!}{\prod_{j\geqslant 1} s_j!}\prod_{j\geqslant 1}\left(\frac{\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{j+\theta+2}{(\theta+1)^{j+1}}\right)}{1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)}\right)^{s_j} \\ & S^{-1/2}\left(-\frac{2}{\theta^2} + \frac{2+8\theta+13\theta^2+10\theta^3+3\theta^4}{\theta(\theta+1)^5} - \psi(j,\theta)\right)^{1/2}. \end{aligned} \tag{5.12}$$

Now full conditional posterior for S is

$$\pi_R(S|\theta,x) \propto S^{-1/2}\binom{S}{w}\left(1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^w\left(\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^{S-w}, \quad (5.13)$$

and full conditional posterior for θ is

$$\pi_R(\theta|S,x) \propto \frac{w!}{\prod_{j\geqslant 1} s_j!}\prod_{j\geqslant 1}\left(\frac{\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{j+\theta+2}{(\theta+1)^{j+1}}\right)}{1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)}\right)^{s_j}\Bigg(-\frac{2}{\theta^2}+ \frac{2+8\theta+13\theta^2+10\theta^3+3\theta^4}{\theta(\theta+1)^5}-\psi(j,\theta)\Bigg)^{1/2}. \quad (5.14)$$

Here, full conditional posterior distribution of S and θ are not obtainable in closed form then using the M-H steps. For posterior samples of S and θ, we used a negative binomial distribution and normal distribution as a proposal distribution for S and θ, respectively.

5.4 An application to Microbial Organisms Species Data

Let us consider a sample of microbial organisms species data set, taken from Barger and Bunge (2008). The data set may be assumed to be a sample from a P-mixed Poisson model having a non- monotonic HR as that of Poisson Lindley model. The data set was originally reported by Behnke et al. (2006), and it represents the classification of the organisms into species based on 18S rRNA similarity. The samples of microbes were taken from 18 meter below the water surface of the Framvaren Fjord in Norway. Diversity of these organisms is largely unknown and estimating the total number of species of microbes. Correspond the observed frequency (nonzero) and the number of species are listed as (j, s_j) : $(1,15),(2,6),(3,7),(4,2),(5,1),(6,1),(7,1),(8,1),(9,1),(12,1),(15,1),(20,1),(164,1)$. The observed number of species and observed number of individual organisms are found to be $w = 39$ and $s = 302$ respectively.

First of all, we checked the graphical method to the data set, have come from Poisson Lindley model. Figure (5.2) shows the (observed) relative frequency histogram and the postulated (or expected) relative histogram on the same graph. Which shows that Poisson Lindley model provides a satisfactory close to the agreement between two histogram appears. But there is little difference between the two histogram due to some chance fluctuation. Since, we study the chi-square test of goodness of fit. Hence, $\chi^2_{cal} = 3.70$ and $\chi^2_{tab,95\%,3} = 7.82$ then χ^2_{tab} is greater than χ^2_{cal}, so we can say that observed frequency has no significance difference between expected (hypothesized) frequency. Thus this data set has been proposed for Poisson Lindley model and compared with some well-established models, namely, Poisson and exponential-mixed Poisson model (discuss it in details Barger and Bunge (2008)). Here we used values of frequencies up to 10 selected by the criteria described therein (goodness-of-fit).

The full data includes observed frequencies greater than ten, but we only model the observed frequencies less than equal to ten. This can be interpreted as assuming the most abundant species are from a known sub population. For final estimate of the number of species were added later when observed frequencies greater than 10.

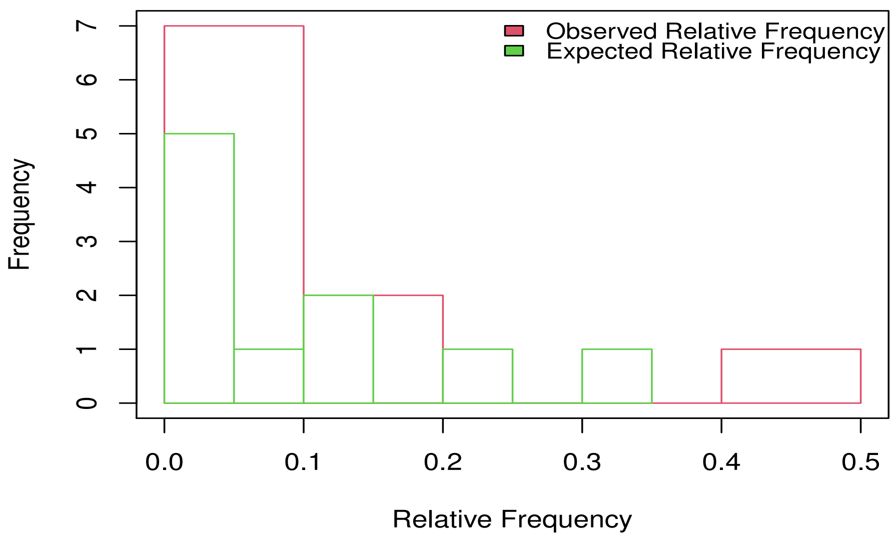

FIGURE 5.2: Observed and Expected relative frequency histogram plot of Poisson Lindley Model.

For Bayesian estimation we use MCMC sampler with M-H steps to simulate from the posterior distributions. Expression for the full conditionals posterior distribution of proposed model with Jeffery's prior is given by $\pi_J(S|\theta,x)$ and $\pi_J(\theta|S,x)$ in Equation (5.8) and (5.9), respectively and with Bernardo's reference prior is given by $\pi_R(S|\theta,x)$ and $\pi_R(\theta|S,x)$ in Equation (5.13) and (5.14), respectively. The posterior samples are taken to have an approximate effective sample size of 5000. Acceptance rates for parameters are kept below 40% and 30% for S and θ respectively.

In M-H step we use a normal proposal distribution for sampling of (nuisance) parameter θ. To obtain the sample from full conditional distribution for S in Equation (5.8) and (5.13). Figure (5.5) shows posterior simulations from each of the two models for species posterior distribution derived in Section (5.3). The proposed Poisson Lindley model parameter for Jeffrey's and Bernardo's reference priors, the posteriors are described in Subsection (5.3.1) and (5.3.2). It is well known that MCMC analysis provides reliable results only when the chains have run sufficiently large number of times and reached to the stationary distribution. In the existing literature of MCMC, a number of tools to assess the convergence of chain like mixing of chain and auto correlation are mentioned in Figure (5.3) and Figure (5.4). These Figures is enough to show that the chains in the present analysis have converged. Now, we may be mentioned here that Bayes estimators and credible intervals (with 95% confidence) have been obtained above using the MCMC procedures. The frequentist estimates for S are summarized in Table (5.1). While under Bayesian paradigm the estimate of S are summarized in Table (5.2). It has shown the posterior modes, means, median and central credible intervals. Also, we are drawn an comparison between Bayesian estimates and ML estimates; symmetric CI based on asymptotic normality, and asymptotic profile likelihood interval (described in Cormack (1992)) are included in Table (5.2). We can also see that the Bayesian estimates are always more than the ML estimates for PLJ and PLR. Further we observed that the profile likelihood intervals are comparable with credible interval estimates and the posterior mean estimates for S are also more than ML estimates. It may also notice that asymptotic 95% symmetric CI for the ML estimate in the consider Lindley model falls above the observed number of species, $w = 39$.

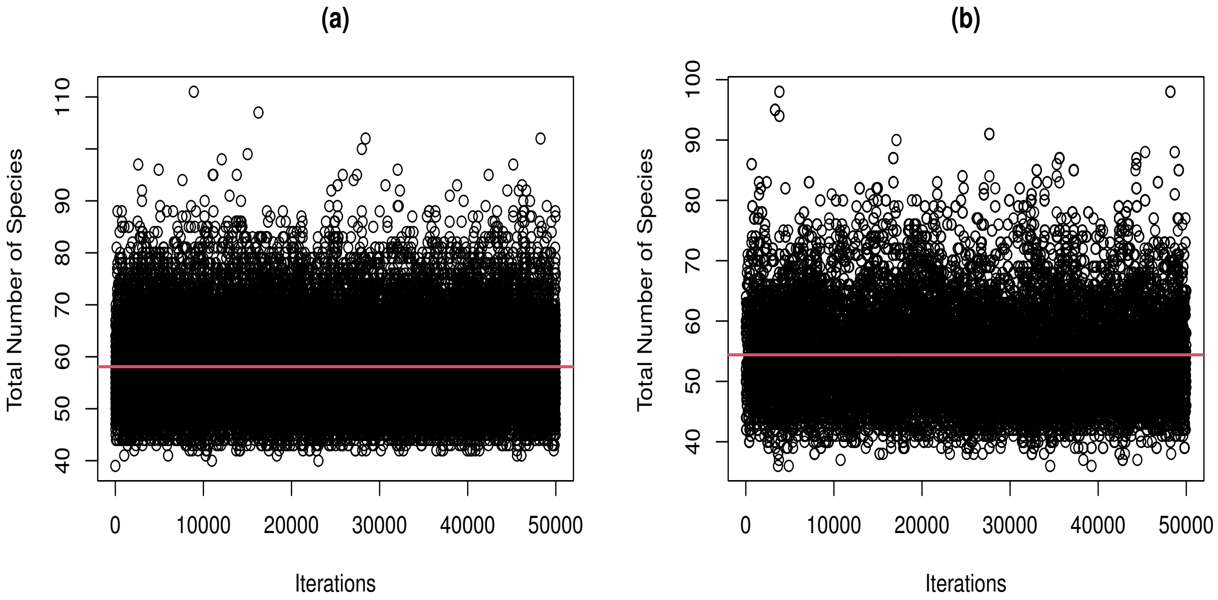

FIGURE 5.3: Trace plot of parameter S with (a) Jeffery's prior and (b) Bernardo's reference prior.

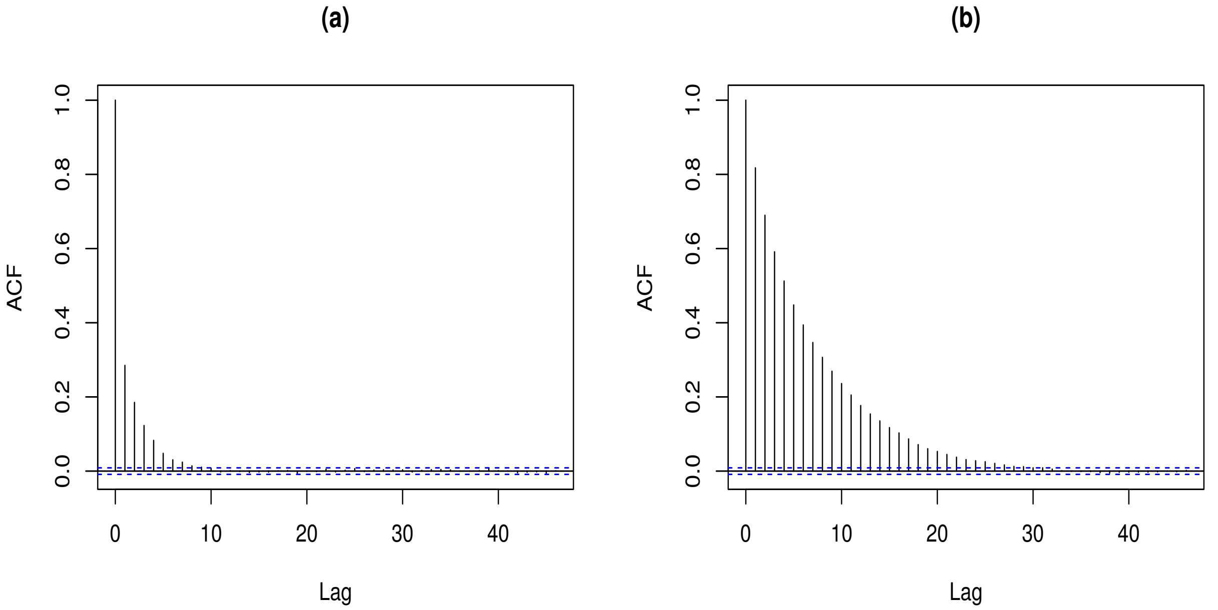

FIGURE 5.4: Auto Correlation Plot (ACF) of parameter S with (a) Jeffery's prior and (b) Bernardo's reference prior.

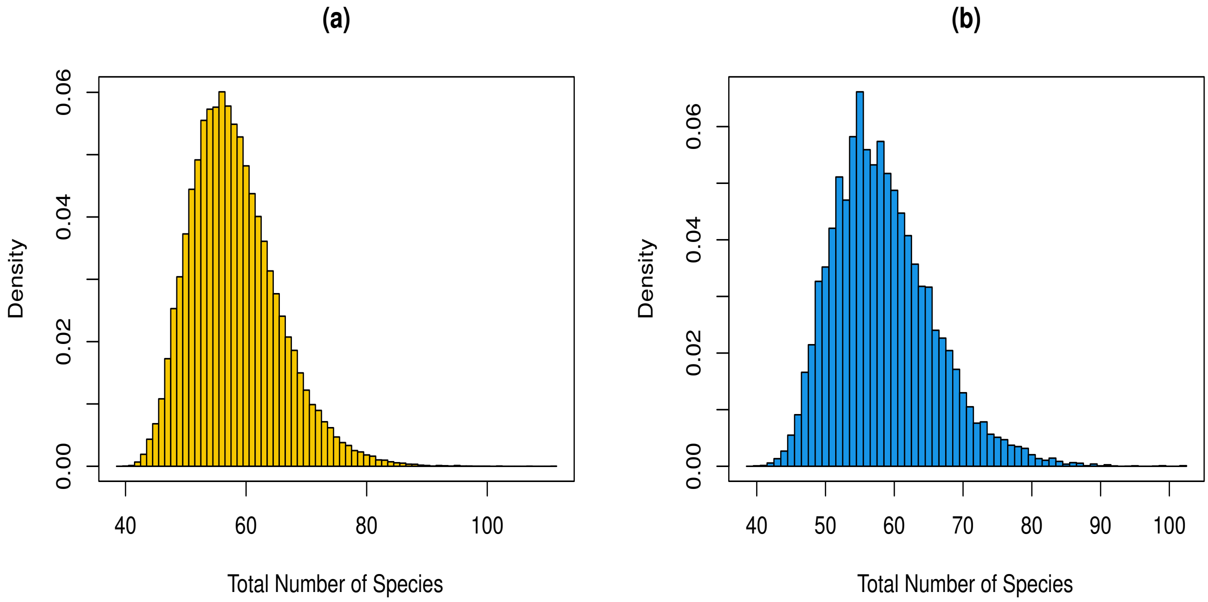

FIGURE 5.5: Posterior histogram plot of parameter S with (a) Jeffery's prior and (b) Bernardo's reference prior.

TABLE 5.1: ML estimate and 95% CI of parameter S obtained with profile likelihood θ_p and conditional likelihood θ_c.

Model	MLE	95%Confidence Interval
Poisson Lindley model	$S_p = 52.67544$	$(42.56637, 62.78451)$
	$S_c = 53.01631$	$(42.77726, 63.25537)$

TABLE 5.2: Summary statistics for posterior $\pi(S|x)$ with PLJ and PLR.

	PLJ	PLR
Mode	55	58
Mean	58.02068	58.37274
Median	57	58
95% Credible Interval	$(47, 74)$	$(47, 75)$

We next check the fit of the each model as well as the relative fit among the models. For the relative fit of models we have derived the deviance averaged over values from posterior sample for each considered model. The model deviance is defined as

$$DE(x,S,\theta) = -2\log L(S,\theta|x). \tag{5.15}$$

Now, using the posterior samples θ^i, $k = 1,2,...,N$ to obtain the average deviance, where I is the total number of posterior samples. Model deviance estimates (DE) formula is as given below,

$$\hat{DE}(x) = \frac{1}{N}\sum_{k=1}^{N} DE(x,\theta^j). \tag{5.16}$$

Table (5.3) is shown to DE of the models, lower DE shows a better fit. From Barger and Bunge (2008) considered the same data set for various models, that mentioned in the Table (5.3), such as PJ: Poisson model with Jeffrey's prior, PR: Poisson model with Bernardo's reference prior; EJ: exponential-mixed Poisson model with Jeffrey's prior and ER: exponential-mixed Poisson model with Bernardo's reference prior. Here we consider these models to compare deviance of PLJ and PLR. We obtained PLR have very minimum model DE as well as better fit for the data set.

TABLE 5.3: DIC for PJ: Poisson model with Jeffrey's prior; PR: Poisson model with Bernardo's reference prior; EJ: exponential-mixed Poisson model with Jeffrey's prior; ER:exponential-mixed Poisson model with Bernardo's reference prior; PLJ and PLR.

Model	DIC
PJ	58.05805
PR	58.06852
EJ	35.63834
ER	35.61999
PLJ	**32.5128**
PLR	**30.83205**

In this sequence, we plot the expected frequencies using posterior samples of the parameters S for PLJ and PLR. We see that for this small data set PLR fit is acceptable, see in Figure (5.6).

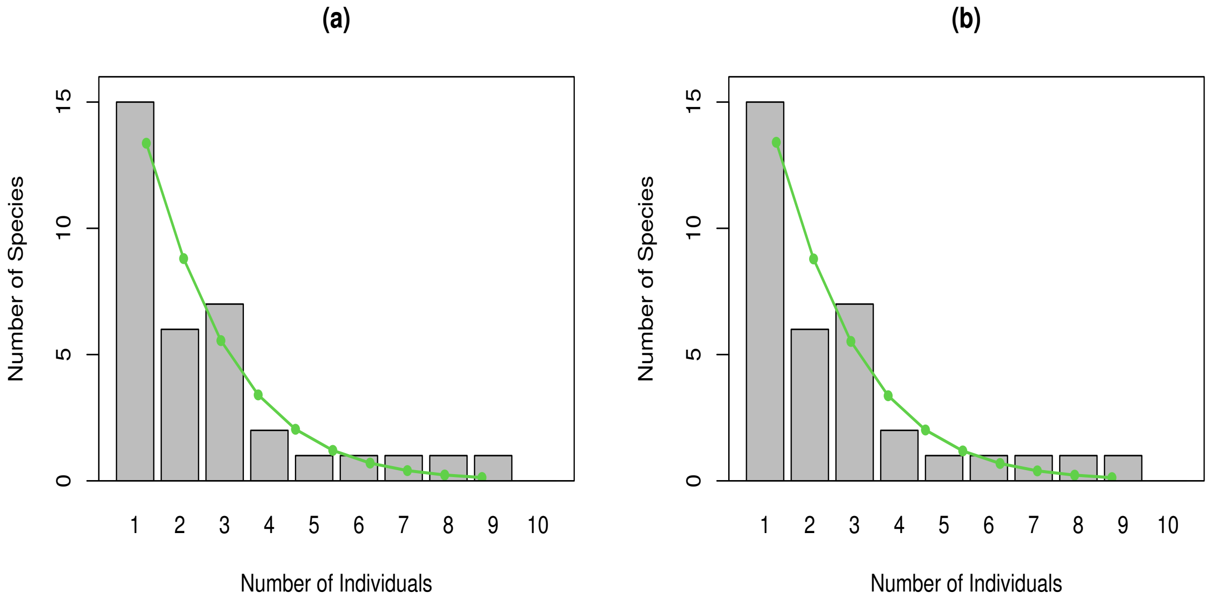

FIGURE 5.6: Expected frequency plot of S with (a) Jeffrey's prior and (b) Bernardo's reference prior.

The Jeffrey's and Bernardo's reference prior for the Poisson Lindley model give very similar results. These priors have been very minimal effect on resulting estimates. Hence, the model selection is highly influence by the final estimates. It is a very important problem for the choice of models.

5.5 Monte Carlo Simulation Study and Comparison of Estimators

We shall compute and set side by side the estimators obtained under ML and Bayesian estimators. The estimators $\hat{S}_p$, $\hat{S}_c$, $\hat{S}_J$ and $\hat{S}_R$ denotes the profile ML estimator, conditional ML estimator, Bayes estimator with Jeffery's prior and Bernardo's reference prior, respectively. Here, S stands for the total number of species as a discrete parameter and θ were abundance parameter generated from Lindley distribution as a nuisance parameter. For the stochastic abundance of the model we have stopping time t and w stands for the observed number of species in the sample experiment. The comparisons are based on the square root of average risk (expected loss over sample space) of the estimators of the parameters S, denoted by $R(S)$.

In the simulation study number of species was to be fixed to be $S = 50, 60$ and 70. The abundance parameters were generated from Lindley distribution with parameter $\theta = 0.5$ and 1.2. We considered the two stopping time i.e. $t = 0.4$ and 1.2. So we observed that the capture fraction are (C.F.=$\frac{w}{S} * 100$) lies between 70% and 98%. For each simulated data set, four estimates were reported $\hat{S}_p$, $\hat{S}_c$, $\hat{S}_J$ and $\hat{S}_R$. The estimates average risk based on the asymptotic formula for each estimator was also obtained.

We excluded those data sets for which the iterative steps for any ML estimates did not get solution or the overlap fraction was negative. (This occurred only when the capture fraction was 50%). The procedure continued until 5000 data sets had been generated. The estimates are obtained through ML method using NR iterative method for nuisance parameter. For the proposed estimator also observed the acceptance rate through M-H algorithm nearly 28% and 40%. For the 5000 generated datasets, the average estimates and their square root of average risk of parameter estimates were given in Table (5.4) and (5.5). All estimates were computed using frequencies f_j. The problem of cut off point selected did not arise because only few species were observed more than 10 times in most generated data sets. In the mentioned Table (5.4) and (5.5), we observed on various fixed frequencies j, when it was increases then coverage fraction also increases and we obtain in a trend that risk of the estimators decreases gradually. We also list the observed CI/HPD interval for the nominal 95%. Also, in this Table (5.4) and (5.5), we observed that coverage fraction increase as increases the frequencies than the observed number of species gets more closer to estimated number of species (as most of the time we got over estimate of parameter).

5.6 Conclusion

We observed that the ML estimation plays an important role in estimating the number of observed species or unobserved species. Intuitively, when there are low coverage fractions i.e. few overlaps of observed species and estimated species, we know that the true number of species is much higher than the observed. On the other hand, if the coverage fraction is high then we are likely to have seen most of the species. Based on this idea, we have proposed a consistent estimator for the number of species, under a P-mixed Poisson Lindley model. Here the model has low dimensional parameter space then computation became easy. Also, in the parameter space known as hyper parameter or nuisance parameter. For these hyper parameters we have non-informative prior or objective prior i.e. Jeffery's and Bernardo's reference prior, it can be based on one's belief. Both Jeffrey's and Bernardo's reference prior have simple forms in the case when there is only one nuisance parameter, and become increasingly complex as the dimension of the parameter space grows. For the comparison of these considered models based on model deviance criteria in Table (5.3), PLR has the lesser deviance then we can say that PLR gives a more optimum estimate of the number of species. In simulated Table (5.4) and (5.5), shows the posterior mean (estimate of number of species) as Bayes estimate under squared error loss function (see Pathak et al. (2020a)) and square root of average risk. When j increases then CF increases but the estimate of S and $R(S)$ decreases and the estimate of the number of species $\hat{S}$ is the case of o.e., i.e. $\hat{S} > w$. We observed that $R(S)$ under Bayes estimate of Bernardo's reference prior have a minimum than $R(S)$ under Bayes estimate of Jeffery's prior.

TABLE 5.4: ML estimates and Bayes estimates of number of species S and square root of average risk $R(S)$ for Poisson Lindley Model with fixed $\theta = 0.5$.

				$t = 0.4$				$t = 1.2$		
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
50	5	77.8	$\hat{S}_p$	76.2915	28.9324	(59.37,93.25)	78.36	76.5461	28.7271	(59.6793.41)
			$\hat{S}_c$	76.9169	29.5737	(59.77,94.05)		77.1656	29.3547	(60.08,94.24)
			$\hat{S}_J$	79.8932	33.4933	(54.54,107.03)		80.3515	33.0884	(54.25,108.73)
			$\hat{S}_R$	79.0514	32.3035	(54.75,107.03)		78.6280	30.6371	(54.88,108.73)
	6	85.44	$\hat{S}_p$	69.5552	20.5514	(56.57,82.53)	89.18	68.9346	19.8140	(56.91,80.95)
			$\hat{S}_c$	69.9760	20.9738	(56.85,83.11)		69.3029	20.1864	(57.15,81.4)
			$\hat{S}_J$	71.3986	22.4712	(53.54,89.7)		70.5116	21.4886	(54.33,86.79)
			$\hat{S}_R$	71.6397	22.6900	(53.95,89.7)		70.7497	21.6782	(54.77,86.79)
	7	90.92	$\hat{S}_p$	67.0534	17.9341	(56.05,78.12)	93	65.5289	16.3219	(55.39,75.66)
			$\hat{S}_c$	67.3796	18.2695	(56.21,78.55)		65.8139	16.6204	(55.58,76.04)
			$\hat{S}_J$	68.3769	19.3467	(53.86,82.91)		66.5949	17.4944	(53.75,79.59)
			$\hat{S}_R$	68.6156	19.5206	(54.27,82.91)		66.9356	17.7806	(54.18,79.59)
	8	94.26	$\hat{S}_p$	64.2468	14.8935	(54.81,73.68)	95.72	62.4558	12.8146	(53.92,70.98)
			$\hat{S}_c$	64.5011	15.1549	(54.97,74.03)		62.6723	13.0348	(54.06,71.27)
			$\hat{S}_J$	65.1451	15.8593	(53.35,76.73)		63.1699	13.5595	(52.97,73.31)
			$\hat{S}_R$	65.4924	16.1759	(53.81,76.73)		63.4992	13.8800	(53.24,73.31)

Continued on next page

Table 5.4 – *Continued from previous page*

				$t = 0.4$				$t = 1.2$		
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
	9	96.2	$\hat{S}_p$	62.0680	12.4986	(53.78,70.35)	96.84	60.7107	11.0496	(53.05,68.36)
			$\hat{S}_c$	62.2752	12.7117	(53.91,70.63)		60.8942	11.2383	(53.16,68.62)
			$\hat{S}_J$	62.7266	13.1984	(52.86,72.48)		61.2626	11.6359	(52.33,70.07)
			$\hat{S}_R$	63.0553	13.5120	(53.21,72.48)		61.6034	11.9750	(52.63,70.07)
	10	97.44	$\hat{S}_p$	60.7469	11.1974	(53.21,68.28)	98.88	58.4898	8.7739	(52.11,64.86)
			$\hat{S}_c$	60.9253	11.3820	(53.31,68.53)		58.6251	8.9134	(52.19,65.05)
			$\hat{S}_J$	61.2840	11.7706	(52.51,69.9)		58.8438	9.1489	(51.74,65.71)
			$\hat{S}_R$	61.5844	12.0653	(52.8,69.9)		59.1375	9.4437	(51.94,65.71)
60	5	77	$\hat{S}_p$	90.4434	31.8015	(72.06,108.82)	78.68	88.4186	30.0404	(71.06,105.76)
			$\hat{S}_c$	91.0633	32.4214	(72.48,109.64)		88.9895	30.6258	(71.46,106.51)
			$\hat{S}_J$	93.7993	35.3535	(65.86,123.84)		91.0347	32.6833	(65.78,117.57)
			$\hat{S}_R$	92.7634	33.9882	(66.21,123.84)		91.3674	33.2955	(65.9,117.57)
	6	83.78	$\hat{S}_p$	85.0163	26.5623	(69.91,100.11)	87.87	83.7267	24.8610	(69.95,97.49)
			$\hat{S}_c$	85.4789	27.0317	(70.23,100.72)		84.1218	25.2577	(70.23,98.01)
			$\hat{S}_J$	87.0985	28.7321	(66.04,109.02)		85.4179	26.5925	(66.64,104.24)
			$\hat{S}_R$	87.0159	28.4700	(66.13,109.02)		85.6225	26.7642	(67.32,104.24)

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Table 5.4 – *Continued from previous page*

		$t = 0.4$					$t = 1.2$			
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
	7	89.95	$\hat{S}_p$	80.5178	21.4039	(68.17,92.85)	92.5	79.8360	20.5628	(68.25,91.41)
			$\hat{S}_c$	80.8551	21.7486	(68.41,93.29)		80.1377	20.8704	(68.46,91.80)
			$\hat{S}_J$	81.8485	22.7658	(65.69,98.15)		80.9866	21.7665	(66.09,95.74)
			$\hat{S}_R$	82.1537	23.0656	(66.11,98.15)		81.3036	22.0575	(66.41,95.74)
	8	92.53	$\hat{S}_p$	77.5003	18.4388	(66.67,88.32)	95.67	76.0319	16.9688	(66.34,85.71)
			$\hat{S}_c$	77.7768	18.7250	(66.86,88.69)		76.2616	17.2124	(66.51,86.02)
			$\hat{S}_J$	78.5471	19.6118	(64.8,92.07)		76.8104	17.8450	(64.96,88.49)
			$\hat{S}_R$	78.8317	19.8316	(65.31,92.07)		77.1322	18.1022	(65.19,88.49)
	9	96.18	$\hat{S}_p$	74.4787	14.9179	(65.39,83.56)	96.82	73.7224	14.2342	(65.03,82.41)
			$\hat{S}_c$	74.6861	15.1294	(65.53,83.84)		73.9155	14.4340	(65.16,82.66)
			$\hat{S}_J$	75.1361	15.6065	(64.25,85.75)		74.3160	14.8792	(64.1,84.34)
			$\hat{S}_R$	75.4869	15.9535	(64.59,85.75)		74.6553	15.2032	(64.27,84.34)
	10	97.43	$\hat{S}_p$	72.7065	13.4414	(64.52,80.88)	98.3	70.8159	11.0848	(63.46,78.16)
			$\hat{S}_c$	72.8828	13.6272	(64.63,81.12)		70.9639	11.2367	(63.55,78.36)
			$\hat{S}_J$	73.2362	14.0305	(63.69,82.56)		71.2179	11.5070	(62.96,79.35)
			$\hat{S}_R$	73.5298	14.3036	(63.98,82.56)		71.5269	11.8238	(63.07,79.35)
70	5	73.69	$\hat{S}_p$	103.6304	36.6810	(83.29,123.96)	77.56	105.1706	37.0413	(85.57,124.76)

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Table 5.4 – *Continued from previous page*

				$t = 0.4$				$t = 1.2$		
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
			$\hat{S}_c$	104.2959	37.3353	(83.75,124.83)		105.7769	37.6502	(86.07,125.54)
			$\hat{S}_J$	107.8331	41.5143	(76.88,139.73)		108.1667	40.1847	(79.13,138.5)
			$\hat{S}_R$	106.9667	40.4973	(76.16,139.73)		108.2674	41.0500	(79.48,138.5)
	6	83.66	$\hat{S}_p$	98.9745	30.7352	(82.71,115.23)	87.49	100.2521	31.8337	(84.57,115.93)
			$\hat{S}_c$	99.4332	31.1990	(83.04,115.82)		100.6760	32.2682	(84.87,116.47)
			$\hat{S}_J$	101.1014	32.9567	(78.17,124.5)		102.1903	33.9195	(80.3,123.89)
			$\hat{S}_R$	100.9205	32.5382	(78.51,124.5)		102.2757	33.8317	(80.95,123.89)
	7	90.13	$\hat{S}_p$	95.0908	26.7498	(81.48,108.69)	90.87	94.6346	25.8961	(81.31,107.95)
			$\hat{S}_c$	95.4368	27.1127	(81.73,109.13)		94.9700	26.2487	(81.55,108.38)
			$\hat{S}_J$	96.5155	28.3390	(78.27,114.38)		96.0162	27.4287	(78.34,113.44)
			$\hat{S}_R$	96.6356	28.2402	(78.83,114.38)		96.1968	27.5364	(78.83,113.44)
	8	92.59	$\hat{S}_p$	90.9369	21.9594	(79.09,102.77)	94.51	89.0097	19.5819	(78.17,99.84)
			$\hat{S}_c$	91.2182	22.2539	(79.29,103.14)		89.2532	19.8317	(78.34,100.16)
			$\hat{S}_J$	91.9776	23.1059	(76.91,106.84)		89.8473	20.4659	(76.5,102.91)
			$\hat{S}_R$	92.3141	23.4021	(77.18,106.84)		90.1793	20.7751	(76.78,102.91)
	9	95.14	$\hat{S}_p$	87.8295	18.4818	(77.49,98.16)	96.91	85.9045	16.5534	(76.58,95.22)
			$\hat{S}_c$	88.0565	18.7156	(77.65,98.46)		86.0963	16.7532	(76.71,95.47)

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Table 5.4 – *Continued from previous page*

				$t = 0.4$				$t = 1.2$		
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
			$\hat{S}_J$	88.5855	19.2908	(75.99,100.91)		86.4736	17.1578	(75.45,97.23)
			$\hat{S}_R$	88.9589	19.6409	(76.32,100.91)		86.8590	17.5628	(75.75,97.23)
	10	98.07	$\hat{S}_p$	84.4702	14.9258	(75.86,93.07)	98.29	83.4878	14.0530	(75.27,91.71)
			$\hat{S}_c$	84.6377	15.0978	(75.97,93.29)		83.6439	14.2149	(75.37,91.91)
			$\hat{S}_J$	84.9569	15.4426	(75.03,94.6)		83.9271	14.5218	(74.6,93.08)
			$\hat{S}_R$	85.2850	15.7594	(75.32,94.6)		84.2193	14.8126	(74.68,93.08)

TABLE 5.5: ML estimates and Bayes estimates of number of species S and square root of average risk $R(S)$ for Poisson Lindley Model with fixed $\theta = 1.2$.

				$t=0.4$				$t=1.2$		
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
50	5	69.1	$\hat{S}_p$	74.1148	26.3888	(55.88,92.34)	78.58	73.2085	25.2695	(57.57,88.84)
			$\hat{S}_c$	74.8486	27.0927	(56.35,93.34)		73.7696	25.8190	(57.93,89.59)
			$\hat{S}_J$	77.9722	30.1572	(49.74,109.39)		76.1269	28.2345	(53.1,100.67)
			$\hat{S}_R$	76.5438	28.5623	(49.76,109.39)		75.7119	27.6897	(53.35,100.67)
	6	76.78	$\hat{S}_p$	71.8245	24.3243	(56.14,87.51)	86.12	68.9278	19.9289	(56.28,81.57)
			$\hat{S}_c$	72.4037	24.9170	(56.52,88.28)		69.3331	20.3418	(56.55,82.11)
			$\hat{S}_J$	74.8051	28.1481	(51.8,98.95)		70.6759	21.7849	(53.3,88.42)
			$\hat{S}_R$	74.5278	27.2211	(52.25,98.95)		70.8299	21.8530	(53.74,88.42)
	7	81.76	$\hat{S}_p$	69.1489	20.3952	(55.52,82.76)	91.26	65.7806	16.4045	(55.23,76.33)
			$\hat{S}_c$	69.6124	20.8574	(55.83,83.39)		66.0849	16.7135	(55.42,76.73)
			$\hat{S}_J$	71.6071	22.9607	(52.17,91.85)		66.9679	17.6509	(53.41,80.54)
			$\hat{S}_R$	71.2835	22.5005	(52.48,91.85)		67.2743	17.9216	(53.69,80.54)
	8	87.7	$\hat{S}_p$	67.9515	19.2072	(55.96,79.94)	94.78	64.0293	14.6732	(54.76,73.29)
			$\hat{S}_c$	68.3262	19.5954	(56.21,80.44)		64.2763	14.9230	(54.92,73.63)
			$\hat{S}_J$	69.6012	20.9793	(53.34,86.14)		64.8531	15.5358	(53.42,76.15)
			$\hat{S}_R$	69.7103	20.9837	(53.8,86.14)		65.2441	15.9165	(53.8,76.15)

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Table 5.5 – *Continued from previous page*

		$t = 0.4$					$t = 1.2$			
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
	9	90.24	$\hat{S}_p$	65.1419	15.9535	(54.61,75.67)	97.48	61.6460	12.1177	(53.77,69.52)
			$\hat{S}_c$	65.4504	16.2714	(54.81,76.09)		61.8358	12.3131	(53.89,69.78)
			$\hat{S}_J$	66.4124	17.3559	(52.8,80.18)		62.2131	12.7169	(52.96,71.3)
			$\hat{S}_R$	66.6825	17.5443	(53.15,80.18)		62.5770	13.0802	(53.28,71.3)
	10	93.52	$\hat{S}_p$	63.6564	14.3420	(54.29,73.02)	98.6	59.8611	10.2902	(52.89,66.83)
			$\hat{S}_c$	63.9114	14.6084	(54.45,73.36)		60.0173	10.4519	(52.98,67.04)
			$\hat{S}_J$	64.5567	15.3228	(52.86,76.07)		60.3057	10.7678	(52.3,68.02)
			$\hat{S}_R$	64.9064	15.6527	(53.07,76.07)		60.6127	11.0681	(52.62,68.02)
60	5	68	$\hat{S}_p$	91.0171	34.2648	(69.94,112.08)	76.5	89.3137	31.3091	(71.15,107.50)
			$\hat{S}_c$	91.8144	35.0148	(70.47,113.15)		89.9317	31.9256	(71.54,108.31)
			$\hat{S}_J$	98.6509	45.6611	(63.47,135.94)		93.0898	35.5656	(65.33,122.91)
			$\hat{S}_R$	94.5460	38.3863	(62.14,135.94)		91.9702	34.1541	(65.48,122.91)
	6	76.55	$\hat{S}_p$	85.6931	27.2472	(68.60,102.78)	84.88	82.9419	24.0653	(68.74,97.14)
			$\hat{S}_c$	86.2705	27.8175	(68.99,103.54)		83.3651	24.4880	(69.03,97.69)
			$\hat{S}_J$	89.3155	31.7867	(63.46,116.12)		84.9407	26.1157	(65.08,104.91)
			$\hat{S}_R$	88.1867	29.9551	(63.61,116.12)		85.0612	26.2612	(65.74,104.91)

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Table 5.5 – *Continued from previous page*

				$t = 0.4$				$t = 1.2$		
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
	7	83.37	$\hat{S}_p$	82.0504	23.2424	(67.73,96.36)	90.78	80.6208	21.4043	(68.43,92.80)
			$\hat{S}_c$	82.4883	23.6751	(68.03,96.94)		80.9496	21.7418	(68.66,93.23)
			$\hat{S}_J$	84.2316	25.4629	(64.08,105.27)		81.9179	22.7546	(65.95,97.8)
			$\hat{S}_R$	84.0615	25.2632	(64.47,105.27)		82.2095	23.0186	(66.3,97.8)
	8	87.92	$\hat{S}_p$	80.7663	21.6105	(67.83,93.63)	93.67	77.5658	18.8377	(66.94,88.19)
			$\hat{S}_c$	81.1308	21.9815	(68.14,94.12)		77.8341	19.1331	(67.15,88.55)
			$\hat{S}_J$	82.3833	23.3154	(65.08,99.84)		78.7200	20.6560	(65.09,92.62)
			$\hat{S}_R$	82.5667	23.4472	(65.38,99.84)		78.8093	20.0348	(65.58,92.62)
	9	90.65	$\hat{S}_p$	78.1751	19.1565	(66.72,89.62)	96.53	75.4442	16.1671	(66.13,84.75)
			$\hat{S}_c$	78.4786	19.4666	(66.93,90.02)		75.6591	16.3918	(66.28,85.03)
			$\hat{S}_J$	79.3394	20.3871	(64.73,94.09)		76.1513	16.9503	(64.88,87.15)
			$\hat{S}_R$	79.6369	20.6902	(64.89,94.09)		76.4896	17.2611	(65.2,87.15)
	10	92.53	$\hat{S}_p$	77.6222	18.3225	(66.75,88.49)	97.37	72.1140	12.5488	(64.15,80.12)
			$\hat{S}_c$	77.9000	18.6103	(66.94,88.86)		72.2842	12.7234	(64.21,80.35)
			$\hat{S}_J$	78.6466	19.4320	(64.8,92.23)		72.6064	13.0685	(63.32,81.63)
			$\hat{S}_R$	78.9632	19.7020	(65.21,92.23)		72.9243	13.3803	(63.6,81.63)
70	5	69.89	$\hat{S}_p$	106.6320	39.1589	(84.42,128.85)	75.8	103.8727	35.6500	(84.18,123.58)

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Table 5.5 – *Continued from previous page*

				$t = 0.4$				$t = 1.2$		
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
			$\hat{S}_c$	107.3826	39.9099	(84.95,129.83)		104.4938	36.2573	(84.61,124.38)
			$\hat{S}_J$	110.5150	43.7610	(76.11,146.65)		106.7120	38.5803	(77.22,136.99)
			$\hat{S}_R$	110.2608	43.8537	(76.31,146.65)		106.2939	38.0505	(77.52,136.99)
	6	78.3	$\hat{S}_p$	103.5278	35.6120	(84.56,122.52)	83.2	98.9265	30.4256	(82.54,115.34)
			$\hat{S}_c$	104.1108	36.2054	(84.96,123.25)		99.3953	30.9075	(82.85,115.94)
			$\hat{S}_J$	106.2811	38.3011	(78.23,135.21)		101.2658	32.9727	(77.97,125.06)
			$\hat{S}_R$	106.2227	38.6766	(79.07,135.21)		101.0559	32.5457	(78.44,125.06)
	7	82.63	$\hat{S}_p$	99.8923	31.1279	(83.09,116.69)	89.07	94.8172	25.6796	(81.08,108.69)
			$\hat{S}_c$	100.3756	31.6211	(83.46,117.39)		95.1733	26.0420	(81.21,109.04)
			$\hat{S}_J$	102.3696	33.7969	(78.08,127.07)		96.2523	27.1824	(77.77,114.47)
			$\hat{S}_R$	102.1939	33.4930	(78.46,127.07)		96.5055	27.3575	(78.29,114.47)
	8	87.59	$\hat{S}_p$	94.9343	26.0750	(80.78,109.09)	93.73	89.7655	20.3863	(78.56,101.07)
			$\hat{S}_c$	95.3066	26.4556	(81.042,109.57)		90.0235	20.6525	(78.76,101.39)
			$\hat{S}_J$	96.4929	27.7044	(77.43,115.52)		90.6547	21.3275	(76.52,104.35)
			$\hat{S}_R$	96.7343	27.9168	(77.95,115.52)		91.0031	21.6521	(76.85,104.35)
	9	89.54	$\hat{S}_p$	92.4444	23.4610	(79.45,105.47)	96.39	86.5609	17.2582	(76.92,96.23)
			$\hat{S}_c$	92.7732	23.7999	(79.68,105.87)		86.7631	17.4668	(77.06,96.46)

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Table 5.5 – *Continued from previous page*

				$t = 0.4$				$t = 1.2$		
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
			$\hat{S}_J$	93.8092	24.9482	(76.7,110.74)		87.1982	17.9444	(75.63,98.4)
			$\hat{S}_R$	94.0383	25.1089	(76.98,110.74)		87.5530	18.2800	(76,98.4)
	10	91.89	$\hat{S}_p$	90.2471	21.6947	(78.48,102.03)	97.47	83.9483	14.5015	(75.38,92.53)
			$\hat{S}_c$	90.5305	21.9985	(78.64,102.48)		84.1160	14.6755	(75.49,92.76)
			$\hat{S}_J$	91.4327	23.2054	(76.2,106.49)		84.4377	15.0264	(74.53,94.07)
			$\hat{S}_R$	91.5546	23.0372	(76.46,106.49)		84.7767	15.3642	(74.82,94.07)