Some Inferences for Lifetime and Ecological Models

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Dedicated to my Parents

DECLARATION

(*As required under clause 12 of Ordinance IIA of the Central University of Haryana)*

This is to certify that the material embodied in the present work, entitled **"Some Inferences for Lifetime and Ecological Models"**, is based on my original research work. The research work was carried out under the supervision of **Dr. Manoj Kumar**, Assistant Professor, Department of Statistics, Central University of Haryana, Mahendergarh. This work has not been submitted, in part or full, for any other diploma or degree of my University/Institution Deemed to be University and College/Institute of National Importance. References from other works have been duly cited at the relevant places.

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LIST OF RESEARCH PAPERS

Published

1. **Pathak A.**, Kumar M., Singh S. K. and Singh U. (2020). Assessing the effect of E-Bayesian Inference for Poisson inverse exponential distribution Parameters under different Loss functions and its application, *Communication in Statistics-Theory and Methods, Taylor & Francis*, : $1-45$, DOI: https://doi.org/10.1080/03610926.2020.184729.

2. **Pathak A.**, Kumar M., Singh S. K. and Singh U. (2020). Bayesian inference: Weibull Poisson model for censored data using the expectation–maximization algorithm and its application to bladder cancer data, *Journal of Applied Statistics, Taylor & Francis*, $47(16):1-23$, DOI: https://doi.org/10.1080/02664763.2020.1845626.

3. Kumar M., Singh S. K., Singh U. and **Pathak A.** (2019). Empirical Bayes estimator of parameter, reliability and hazard rate for Kumaraswamy distribution. *Life Cycle Reliability and Safety Engineering, Springer*, $8(3):243-256$, DOI: https://link.springer.com/article/10.1007%2Fs41872-019-00085-0.

Communicated

1. **Pathak A.**, Kumar M., Singh S. K., Singh U. and Kumar S. (2022). Bayesian Estimation of the Number of Species from Poisson Lindley Stochastic Abundance Model Using Non-Informative Priors. *Environmental and Ecological Statistics, Springer (SCIE, Scopus, UGC Care).*

Presented in Conferences

1. Presented a paper entitled **"Empirical and Hierarchical Bayesian estimation for the Number of species for Poisson Lindley mixture"** in the 24^{th} International Conference of International Academy of Physical Sciences on "Innovations in Physical Sciences" during 09-08-2019 to 11-08-2019, held at Chaudhary Charan Singh University, Meerut, India.

2. Presented a paper entitled **"Bayesian Inference for the Number of species using different priors"** in the "6^{th} INDIA BIODIVERSITY MEET - 2019 (International Conference)" during 14-02-2019 to 16-02-2019, held at Indian Statistical Institute, Kolkata, India.

3. Presented poster paper entitled **"E-Bayesian Estimation for the Parameter of PIED with Different Loss Function under Progressive Type II Censoring with Binomial Removals"** in the "International Conference on Emerging Methodologies in Theoretical and Applied Statistics (EMTAS)" on 18-09-2018 to 20-09-2018, organized by Department of Statistics, Institute of Science, Banaras Hindu University, Varanasi, India.

ABSTRACT

The present piece of work deals with two segments of the study, the first segment related to the lifetime study and the second segment related to the ecological study based on statistical models. The lifetime study in the thesis belongs to **Chapter 2, 3,** and **4** and the ecological modeling-based study belongs to **Chapters 5**. In this context, experimenters/doctors are dealing with lifetime data for patients' survival time. They are facing the difficulty of losing to follow-up of the patient, it is the problem of censoring. When patients are lost to follow-up in the duration of treatment due to unforeseen reasons which are beyond the control of experimenter/doctors such an appropriate censoring is known as progressive Type-II censoring with binomial removals. Such problems are shared in consecutive **Chapter**. **Chapter 2** deals with Bayesian estimation of the parameter of Weibull Poisson distribution under different loss functions using the Expectation-Maximization algorithm. A bladder cancer patient data has been used to show their applicability on Weibull Poisson distribution. **Chapter 3** presents Bayesian and E-Bayesian estimations for Poisson Inverted Exponential distribution under different loss functions. This approach allows and facilitate multiple myeloma patients' data. **Chapter 4**, covers Empirical Bayesian estimation under Linear Exponential loss function for Kumaraswamy distribution parameter, reliability, and hazard function. Also, ulcer patients' data are included in that **Chapter**. The problems their relations are presented at the end of the **Chapter**. Thus we cover all the aforesaid Chapters based on statistical inference of the lifetime models parameter used progressive censoring with Binomial removals. Last **Chapter 5**, concerned with the estimation of the number of species using Poisson Lindley as a stochastic abundance model. We have considered the classical estimation based on profile likelihood, conditional likelihood. For Bayesian estimation, Jeffery's priors and Bernardo's reference prior based on this **Chapter**. The proposed methods are illustrated through a microbial organisms species data.

The statistical **R** software is used for computation purposes in the thesis. The thesis contains a list of references at the end. We recognize that a comprehensive list of sources linked to

the issues mentioned in the thesis would be too long to present here. As a result, we've only included references that are cited in the thesis and are directly linked to our research.

LIST OF ABBREVIATION

Akaike Information Criterion	AIC
Bayesian Information Criterion	BIC
Coverage Probability	CP
Coverage Length	CL
Confidence Interval	CI
Confidence Intervals	CIs
Cumulative Distribution Function	CDF
Deviance Estimate	DE
Deviance Information Criteria	DIC
Expectation Maximization	EM
Exponential Poisson	EP
Exponentiated Exponential Distribution	EED
General Entropy Loss Function	GELF
Highest Posterior Density	HPD
Hazard Function	HF
Identically Independently Distributed	i.i.d.
Kumaraswamy Distribution	KD
Kolmogorov-Smirnov Statistic	K-S
Least Square	LS
Linear Difference Score	LDS
Likelihood Ratio	LR
Linear Exponential	LINEX
log-Likelihood	log-L

Maximum Likelihood	ML
Metropolis-Hastings	M-H
Markov Chain Monte Carlo	MCMC
Monte Carlo	MC
Newton-Raphson	NR
Over Estimation	o.e.
Poisson Inverted Exponential Distribution	PIED
Poisson Lindley Distribution	PLD
Poisson Lindley Distribution with Jeffery's prior	PLJ
Poisson Lindley Distribution with reference prior	PLR
Probability Density Function	PDF
Probability Mass Function	PMF
Probability-Probability	P-P
Progressive Type-II Censoring with Binomial removals	PT-II CBR
Quantiles-Quantiles	Q-Q
Random Variable	RV
Ratio of Expected Experiment Time	REET
Square Error Loss Function	SELF
Survival Function	SF
Total Time on Test	TTT
Under Estimation	u.e.
Weibull Poisson Distribution	WPD
Weibull Poisson	WP
Weibull Distribution	WD

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Chapter 1

Introduction: Definition and Terminologies

1.1 Introduction

We are frequently reminded of the fact that we are living in the information age. The information age is contemporary to the modern age of technological development and ecosystem. In the modern age of technological development, we have the requirement of ecological resources and technical items/devices to access the knowledge, benefiting in our daily life uses. The lifetime and ecological literature have adequacy to continue further study. Therefore, these fields are very attractive to draw the attention of researchers. In this context, we are focusing on lifetime and ecological study.

We make different choices in our lives to make our lives more convenient, to perform our duties more efficiently, and for a variety of purposes. However, statistics, with its many useful methods, aids us in making decisions under uncertainty, and as a result, it is gradually being embraced as the science of decision-making by many branches of science, applied disciplines, social sciences, and even literature. In statistics, uncertainty is defined as randomness, which is calculated in terms of probability. This theory serves as the foundation for a study in which decisions are taken based on a small number of data points (referred to as a sample) drawn from the population of interest. Variables such as human survival times and lifetime of other man-made systems/items, customer waiting times in line to receive service at a center, and other economic and demographic variables are often encountered in real-life situations. The term "lifetime" refers to the length of time it takes for an object to enter its failed state. It is worth noticing that all of these variables have the same level of support, namely the positive real-line number. Since statistical studies of such lifetimes play such a special and significant role in statistical procedures, they are classified as a distinct branch of statistics known as Survival/Reliability analysis. As a result, survival analysis is a technique for analyzing failure (death) time results (time to event data). Medicine, genetics, public health, epidemiology, engineering, economics, and demography are a few of the areas where it can be used.

Also, our daily life activity goes with the environment and ecology. The term ecology was coined by Ernst Haeckel in 1869. Ecology is the branch of science in which we study the presence of species in an organism and its environment, including individual habitat, population, community, ecosystem, and biospheres as a whole. Ecology is the study of households with emphasis on the totality or pattern of relationship between living organisms to one another and to their surroundings, their natural environment, and ecosystems. Ecology is defined as the study of an ecosystem. These household consists of non-living matter such as soil, water, light, wind, humidity, minerals, gases, etc., and living organisms such as micro-organisms, plants, animals, bacteria, and humans. An organism depends upon each other for its survival, existence, and continuance. Besides, living (biotic) organisms and their non-living (abiotic) environment are inseparably interrelated and interact with each other, see Dash (2001). It is a fascinating discipline because everyone is usually interested in knowing about his surroundings. Ecology is concerned with the biology of organisms, population, communities, etc., and their functional processes occurring in natural habitats like ponds, lakes, oceans, and land. A community or biotic community includes all the population of a given area, called the habitat. The community and the abiotic environment interact and function together as a system called the "ecological system" or "ecosystem", a term coined by the British ecologist Arthur Tansley in 1935. Tansley devised the concept to draw attention to the importance of transfers of materials between organisms and their environment. He regarded ecosystems not simply as natural units, but as mental isolates and later defined the spatial extent of ecosystem using the term ecotope. The functional form of the ecosystems is of great concern to ecologists. Ecosystems show large variations in their size, structure, composition, and so on. However, all the ecosystems are characterized by certain basic structural and functional features which are common. There can be different types of ecosystems such as forest ecosystem, desert ecosystem, and marine ecosystem.

Species richness estimation can be extended to both animals and organisms in biology. The term "species richness" is used to describe the number of species that live in a given biosphere or population. The number of species in an ecosystem will help assess its complexity. Finding a high level of species richness can aid in the identification of populations that have been under-sampled. The number of endangered or extinct organisms can be calculated using measurements taken over time. Species abundance is the number of individuals per species and relative abundance refers to the evenness of distribution of individuals among species in a community. Two communities may be equally rich in species but differ in relative abundance. Animal populations of interest can range from very large animals, such as whales Zeh et al. (1986), Raftery and Zeh (1998), to bacteria that can only be observed under a microscope Hong et al. (2006). Other interesting animal populations for which diversity is studied are fish Smith and Jones (2005), fossils Cobabe and Allmon (1994), and birds Borgella Jr and Gavin (2005); Walther and Martin (2001).

Initially, two papers based on estimating the number of classes was written by Good (1953) and Fisher et al. (1943), with the interest of estimating the frequencies of species in an animal population. Good (1953) estimates the probability of an unseen species as $n_1 = n$ where n_1 is the number of species represented by only one individual in the sample and n is the total number of observed individuals. Fisher et al. (1943) model the species abundances with a parametric gamma-mixed Poisson or negative binomial distribution. The negative binomial model is based on assuming that the numbers of individuals from each species are independent Poisson samples and that the means of these Poisson random variables follow a gamma distribution. Many other approaches, including Bayesian methods, have been developed for the species problem since these early works. For a review on this problem including other related models and additional applications see Bunge and Fitzpatrick (1993), Buckland et al. (2000), Pollock (2000), Schwarz and Seber (1999), and Seber and Schwarz (2002).

1.2 Classical Inference

In the classical inference, a population have some characteristic of the elements that can be represented by a RV X whose density is $f(X,\theta)$, where the form of the density is assumed to be known except that it contains an unknown parameter θ and make inferences about θ based on information contained in the observed sample only.

The classical school believes in *Fisher's Likelihood Principle*, which claims that all the information about the unknown parameter(s) is contained in the sample, as summarized by the likelihood function. This principle leads to *Fisher's* ML estimator. In spite of certain limitations, the ML estimators have a number of desirable properties and are extensively used in preference of the other classical estimators.

In ML estimation method, it seems that a good estimate of the unknown parameter θ would be the value of θ that maximizes the likelihood function. Parameter θ may be discrete and continuous. Suppose we have a random sample $(x_1,x_2,x_3,\ldots,x_n)$ for which the PDF of x_i is $f(x_i,\theta)$. Then the joint PDF or PMF of $(x_1,x_2,x_3,\ldots,x_n)$ is denoted by $L(x,\theta)$ as,

$$L(x,\theta) = \prod_{i=1}^{n} f(x_i,\theta). \tag{1.1}$$

Likelihood function in Equation (1.1) can be maximized through LS method for continuous parameter and LDS method for discrete parameter, see (Jain et al. (2003) and Lindsay and Roeder (1987)). There are many different methods of point estimation, mentioned as method of moment, method of least square (LS), and methods based on quantile/percentile, etc., are discussed in the literature. But the ML estimator is very popular and widely used since it has many optimum properties as consistency, invariance, and other asymptotic properties, see Rohatgi and Saleh (2015).

Point estimation, one may be interested in finding a set of values, say $A(\theta)$ such that $A(\theta)$ contains the true value of the parameter (θ) with a certain high probability $(1-\phi)$, $\phi \in (0,1)$. Let $T_1 = t_1(X_1, X_2, X_3, \cdots, X_n)$ and $T_2 = t_2(X_1, X_2, X_3, \cdots, X_n)$, $T_1 \leq T_2$ be two statistics such that

$$P_\theta[T_1 \leq \theta \leq T_2] = 1-\phi, \forall \theta \in \Theta,$$

where $(1-\phi)$ does not depend on θ. Then the random interval (T_1, T_2) is called the $100(1-\phi)\%$ confidence interval (CI) for θ. Exact CIs are generally not available for some distribution especially in the case of the multi-parameters model. In such a case, that the asymptotic distribution property of ML estimator is very useful in constructing asymptotic CIs principle. The asymptotic CIs are defined as

$$\{\hat{\theta} \mp Z_{\phi/2}\sqrt{var(\hat{\theta})}\}, \tag{1.2}$$

where, $\phi/2^{th}$ upper percentile of a standard normal variables, and $var(\hat{\theta})$ is asymptotic variance of θ, and is obtained as diagonal elements of inverse fisher's information matrix. Suppose there are k parameters $\theta = \theta_1, \theta_2, \cdots, \theta_k$, then variance-covariance matrix, say V, is defined by

$$V(\theta) = \begin{bmatrix} \delta_{11} & \delta_{12} & \cdots \delta_{1k} \\ \delta_{21} & \delta_{22} & \cdots \delta_{2k} \\ . & . & . \\ . & . & . \\ . & . & . \\ \delta_{k1} & \delta_{k2} & \cdots \delta_{kk} \end{bmatrix}^{-1}_{\theta=\hat{\theta}}$$

where, $\delta_{ij} = -\frac{\partial^2}{\partial\theta_i,\partial\theta_j} logL; (i,j) = 1,2,3,\cdots,k$, second derivatives of the likelihood function with respect to θ.

1.3 Bayesian Inference

In Bayesian inference, it is believed that we always have the availability to make use of subjective probabilities that measures the degrees of belief about the value or values of unknown parameter θ. These subjective probabilities are used to define prior distribution for the parameter θ, prior to sampling. In other words, the parameter θ may be treated as a RV with known prior distribution, say $g(\theta)$, see Berger (2013). The ML method, as well as other classical approaches, are based only on the empirical information provided by the available data. However, when there are some technical knowledge on the parameters of the distribution available, a Bayesian inference seems to be an attractive inferential method.

The goal of Bayesian inference is to represent prior uncertainty about model parameters with a probability distribution and to update this prior uncertainty with current data to produce a posterior probability distribution for the parameter that contains less uncertainty. The posterior distribution, denoted by $\pi(\theta|x)$ of the parameter, say θ given x is defined to be the conditional distribution of θ given the sample observations x and is given by

$$\pi(\theta|x) = \frac{g(\theta)L(x|\theta)}{\int_{\Theta} g(\theta)L(x|\theta)d\theta}, \tag{1.3}$$

where $g(\theta)$ is the prior distribution function that reflects beliefs about θ (prior to experimentation) and Equation (1.3) shows the updated belief about θ after observing the sample.

Clearly, the prior distribution plays a very crucial role in Bayesian of parameter. But the specification of the prior distribution is not an easy task. A vast Bayesian literature is fully devoted to finding the prior distribution of the parameter of interest, see for more details on prior specifications Jeffreys (1946), Zellner (1986a), Box and Tiao (2011), Berger (2013), Gelman et al. (2013). In the next section, we will discuss prior distributions.

1.4 Prior Distribution

The prior distributions are broadly classified into two categories, (i) Informative and (ii) Non-informative. Berger (2013) has given a useful discussion on various methods to construct a prior distribution is certainly a subjective manner. The main idea of subjective probability is to let the probability of an event reflect the personal belief in the chance of occurrence of the event, however, it is typically determined by introspection. The simplest way of determining subjective probabilities is to compare events, determining relative likelihoods. Several useful techniques are included (i) The Histogram Approach, (ii) The Relative Likelihood Approach, (iii) CDF Determination Approach, and (iv) Matching a Given Functional Form. The easiest and very simplified approach is to use a given functional form when a piece of prior information is available, and the problem is then reduced to a subjective determination of a few hyper-parameters (see Good (1950)).

For example, a gamma distribution is widely discussed in existing literature as a prior distribution for the parameters of numerous lifetime models. The gamma distribution as a prior have shape parameter α and scale parameter β for the parameter θ is defined as

$$g(\theta) = \frac{\alpha^{\beta}}{\gamma(\beta)} \theta^{\beta-1} e^{-\alpha\theta}; \qquad \theta > 0, \alpha > 0, \beta > 0. \tag{1.4}$$

Here, in the Bayesian paradigm, the parameters (α, β) of the prior distribution are referred to as hyper-parameters and are assumed to be known. The simplest way of eliciting these hyper-parameters is to set the prior moments equal to the values taken by the experimenter to guess the prior parameters. In the above case, the moments equations are

$$\frac{\beta}{\alpha} = M(say), \tag{1.5}$$

$$\frac{\beta}{\alpha^2} = V(say), \tag{1.6}$$

where the quantities M and V reflect the experimenter's beliefs about the mean and variance of the unknown parameter of the model. Note that the smaller, moderate and larger values of V respectively correspond to the low, moderate, and high magnitude of beliefs in the expected value of the parameter. The large values of V lead to a flat prior density whereas smaller values of V indicate a high picked prior density for θ. This facile methodology has been widely adopted by many authors for prior elicitation under Bayesian paradigm, see Kundu and Howlader (2010) and Singh et al. (2013a).

A large part of Bayesian literature is devoted to finding appropriate prior distributions for which $\pi(\theta|x)$ can be easily calculated. These are so-called conjugate priors and were developed extensively by Schlaifer and Raiffa (1961). Also defined is a family of prior distributions known as conjugate prior which eases the associated computational difficulties with Bayesian analysis. A prior is said to be conjugate if it's resulting posterior also belongs to the same family of distributions as that of the prior. A number of informative priors such as g-prior by Zellner (1986b), an independent t-gamma prior by Leamer and Leamer (1978) and maximum entropy prior by Berger (2013) have also been suggested in Bayesian literature.

A general class of prior that is often considered in Bayesian analysis is popularly known as non-informative priors. The non-informative priors are those priors that utilize very minimal or no prior information in their choice. Uniform and Jeffery's priors (Jeffreys (1946)) are widely used non-informative priors. If a non-informative prior density is desired, it seems reasonable to give equal weight to all possible values of θ, arriving at the uniform non-informative prior $g(\theta) \equiv c$. Although this was routinely done by Laplace (1812), it came under severe (though unjustified) criticism because of a lack of invariance property under transformation, see Jaynes (1983). The lack of invariance of the constant prior has led to a search of non-informative priors which are appropriately invariant under transformation. Efforts to derive non-informative priors through consideration of transformation of a problem had their beginnings with Jeffreys (1961). It has extensively used in Hartigan et al. (1964), Jaynes (1968), Jaynes (1983), Villegas (1977), Villegas (1981) and elsewhere. A very useful prior is Jeffrey's prior (Jeffreys (1961)). It satisfies the local uniformity property: a prior that does not change much over the region in which the likelihood is significant and does not assume large values outside that range. It is based on the Fisher information matrix $I(\theta)$, which is

$$g(\theta) = |I(\theta)|^{1/2}, \tag{1.7}$$

where $|.|$ denotes the determinant and Fisher information matrix under commonly satisfying assumptions (see Lehmann (1983)) this is given by

$$I(\theta) = -E_{\theta}\left[\frac{\partial^2 \log L(x|\theta)}{\partial \theta^2}\right]. \tag{1.8}$$

Jeffrey's prior is locally uniform and hence non-informative. It provides an automated scheme for finding a non-informative prior for any parametric model $f(x|\theta)$.

Another non-subjective prior is reference prior, introduced by Bernardo (1979) and further developed by Berger and Bernardo (1989), Berger and Bernardo (1992a), Berger and Bernardo (1992b) is, to the best of our knowledge, the only available method to derive non-subjective posterior distributions which satisfy all these property, (i) Invariance (ii) Consistent marginalization (iii) Consistent sampling properties (iv) Generality (v) Admissibility. Reference prior is non-informative prior and it also has improper nature. Reference prior is an objective prior which is based on maximizing the expected entropy provided by the prior. The amount of information to be expected from an experiment about some quantity of interest naturally depends on the available prior knowledge: the more prior information available, the less information may be expected to be learned from the data. An infinitely large experiment would eventually provide all missing information; thus, it is possible to obtain a measure of the amount of missing information as a limiting form of a functional of the prior distribution.

1.5 Empirical Bayesian Inference

A major drawback of the conventional approach is that it cannot use past data, while the Bayesian approach is that it requires an assumption for the prior distribution. It appears to be desirable to have an approach, which can make use of the past data and does not require an assumption for the prior distribution. It means we did not know the value of prior parameters. Many statisticians have considered working in areas where it is possible to specify prior distributions reasonably well. Among the others, empirical Bayesian have typically assumed that the parameters of interest come from some unknown prior distributions instead of specifying the prior distributions in advance. They have developed the technique, called the Empirical Bayesian approach, for eliciting the unknown prior distributions using the past available sample information. Borrowing the strengths of each, Herbert Robbins proposed an approach, known as empirical Bayesian approach, as a valid alternative to make use of past data to estimate the statistical form of prior information. Empirical Bayesian methods comparatively require fewer sample data to achieve the same quality of inference than the methods based on sampling theory. This is one important consideration when sample data is either expensive or difficult to obtain. Thus, the Empirical Bayesian approach sounds like a compromise between the conventional approach and the Bayesian approach. It is described extensively in the literature, e.g., Robbins (1955), Robbins (1964) and Maritz (1967), Sinha et al. (1976), Grabski and Sarhan (1996), Ahmad et al. (1997), Pensky and Singh (1999), Jaheen (2004). Recently, some more developments are Shojaee et al. (2012) that discuss the Empirical Bayes estimators for compound Rayleigh distribution parameter and reliability under record data.

Suppose that θ is a vector, consisting of components $(\theta_1,....,\theta_p)$ that are i.i.d. from the density π_0. Suppose also that the data X consists of independent components $(X_1,X_2,...,X_p)$ where each X_i has density $f(x_i|\theta_i)$. Then the common marginal distribution of each X_i is

$$m_0(x_i) \quad = \quad \int f(x_i|\theta_i)dF^{\pi_0}\theta_i, \tag{1.9}$$

and $(X_1, X_2, ..., X_p)$ can be considered to be a simple random sample from m_0. Note that this also follows from the direct calculation (assuming continuous densities for convenience)

$$\begin{aligned} m(x) &= \int f(x|\theta)\pi_0(\theta)d\theta \\ &= \int \left[\prod_{i=1}^{p} f(x_i|\theta_i)\right]\left[\prod_{i=1}^{p} \pi_0(\theta_i)\right] d\theta_i \\ &= \prod_{i=1}^{p} \int f(x_i|\theta_i)\pi_0(\theta_i)d\theta_i \\ &= \prod_{i=1}^{p} m_0(x_i). \end{aligned}$$

The data x can thus used to estimate m_0 (and hence m).

1.6 Hierarchical Bayesian Inference

An another important type of prior distribution is a hierarchical prior, also called a multistage prior, see Lindley and Smith (1972). The idea is that one may have structural and subjective prior information at the same time, it is often convenient to model at this stage. The hierarchical approach is most commonly used when the first stage, Γ, consists of prior of a certain functional form see Antoniak (1974), Berry et al. (1979), and Kuo (1986) being exceptions. Thus, if

$$\Gamma = \{g_1(\theta|\lambda) : g_1 \textit{ is of a given functional form and } \lambda \in \Lambda\}, \tag{1.10}$$

then the second stage would consist of putting a prior distribution, $g_2(\lambda)$, on the hyper parameter λ, which could be chosen for the hyper parameters according to subjective beliefs. Such a second stage prior is sometimes called a hyper prior for this reason. The difficulty of specifying the second stage prior has made common use of non-informative priors at the second stage. Note that there is no theoretical reason for limiting hierarchical priors to just two stages, but more than two are rarely useful in practice. see also Goel and Degroot (1981) and Goel (1983). As a final comment, note that a hierarchical structure is merely a convenient representation for a prior, rather than an entirely new entity: any hierarchical prior can be written as a standard prior. For instance, in this situation, the actual prior distribution is

$$g(\theta) = \int_{\Lambda} g_1(\theta|\lambda) dF^{g_2}(\lambda), \tag{1.11}$$

and any Bayesian analysis will actually be performed with respect to g. Attention will be restricted to two-stage priors. The first stage prior, $g_1(\theta|\lambda)$ where λ is a hyper parameter in Λ, can be thought of as the unknown prior in the empirical Bayes scenario. Instead of estimating λ, as in empirical Bayes analysis, however, λ will be given a second stage prior distribution $g_2(\lambda)$. This could be a proper prior but is often chosen to be a suitable non informative prior. It is frequently used to calculate and to write $\lambda = (\lambda^1, \lambda^2)$, represented g_2 as

$$\begin{aligned} g_2(\lambda) &= g_{2.1}(\lambda^1|\lambda^2) g_{2.2}(\lambda^2), \\ g(\theta|x) &= \int_{\Lambda} g_1(\theta|x,\lambda) g_{2.1}(\lambda^1|x,\lambda^2) g_{2.2}(\lambda^2|x) d\lambda. \end{aligned}$$

This process is called Hierarchical Bayesian process for estimating the hyper parameter, see Berger (2013).

1.7 E-Bayesian Inference

In the Bayesian inference, Posterior distribution is the basis of prior distribution and likelihood of the experiment. It depends on the selection of prior distribution and specification of loss functions. But, prior distribution may depend on the prior parameters i.e. hyper parameters. Hierarchical prior distribution has been used as prior for the unknown hyper parameters. Very first time Lindley and Smith (1972) introduced the idea of the hierarchical prior distribution. The hierarchical Bayesian inference have the requirement of prior at least two stages to finish the setting of the prior distribution. Hence, it is more robust as well as more efficient than Bayesian inference. The method for construction of hierarchical prior distribution has been developed by Han (2007). This method is known as E-Bayesian (Expected-Bayesian) estimation. The hierarchical Bayesian methods have confronted complicated integration, though some computational methods are available in the literature. However, it was observed that obtaining the E-Bayesian estimates of the unknown parameters is simpler than that of hierarchical Bayesian estimates, see Han (2007), Han (2011a), Han (2011b). Further, E-Bayesian estimation for the parameters of different lifetime distributions has been discussed by several authors, see Gupta (2017), Yousefzadeh (2017), Han (2017a), Han (2017b), Han (2019b) etc., and some authors also discussed the E-Bayesian estimation for parameters of lifetime distribution with type-II censoring, see Jaheen and Okasha (2011), Okasha (2014), Reyad and Ahmed (2016) etc. Recently, El-Sagheer (2017) has considered the Rayleigh distribution for E-Bayesian estimation under progressive type-II censoring and Kızılaslan (2017) discussed the hierarchical and E-Bayesian Bayesian estimations for the proportional reversed hazard rate model based on record values. Some recent literatures of E-Bayesian estimation to develop the E-posterior and E-MSE method, see Han (2018), Han (2019a), Han (2019b), Han (2019c) and Han (2020).

1.8 Loss Function

Statistical inference can always be viewed as a decision problem under prevailing uncertainties modeled in the form of parent population distribution. The statistical decisions are based on the sample information only in classical inferences; whereas in Bayesian decisions, in addition, to sample information, it also includes prior information. The overall purpose of statistical inference is to provide an optimal decision based on some evaluation criterion for the goodness of the decisions. The criterion assesses the average consequences of each decision. It is worthwhile to mention here that in statistical inference problems, we try to reveal the true state of affairs therefore two situations may arise. The first one is that the inference will be able to reveal the truth. In this case we are achieving what is intended. Thus there is no loss. In other cases, our inference may deviate from what the truth is and this leads to a loss. In other words, in developing the inferential procedures one should keep this point in mind by specifying an appropriate loss function. It may be noted that the considered loss functions should be non-negative functions as there is no chance of a gain. A statistical decision problem is formalized by specifying the set of elements (Ω, A, L), where, Ω is the set of all possible values of the parameter called as parameter space, A is set of all action (decision) that taken by statistician and L is the loss function which is a real-valued function of the decision and true state of nature. As mentioned earlier, the statistical decisions are based on the sample information, thus it may be defined as the function from sample space to action space. Thus, our aim is to select from the set of all possible decisions (called decision space), a decision (called optimal decision) for which the average loss (In non Bayesian or classical set up, the expected error is termed as risk defined as expected loss) is minimum. For example, in one-dimensional point estimation problems the true state of nature θ, called parameter, is unknown but one can always specify that it belongs to a set of real numbers $\Theta \in R$ takes $A = \Theta$ meaning that the decision rule will output estimates (guesses) of the true θ. Let us consider that θ is a parameter of some distribution $f(x|\theta)$ and suppose that the parameter θ is estimated by the decision rule $\delta(x)$. Then, the quantity $L(\theta, \delta)$ expresses how much wrong estimates are to be penalized. Then, the Bayes estimators δ^* of the parameter θ is defined as the estimator that minimizes the posterior expected loss i. e.

$$E_{\theta|x}[L(\theta, \delta^*)] = \int_{\Theta} L(\theta, \delta^*) \pi(\theta|x) d\theta. \tag{1.12}$$

In the statistical literature, several loss functions have been discussed namely, square error loss function (SELF), absolute loss function, $0-1$ loss function, quadratic loss function, Linear Exponential (LINEX) loss function, and general entropy loss function (GELF). For more detail about these loss functions, see Winkler (1972), Zellner (1996), Basu and Ebrahimi (1991),Calabria and Pulcini (1994), Box and Tiao (2011), Berger (2013) etc. Among the various loss functions, the most popular one is SELF which is initially proposed by Legendre (1806) and Gauss (1855) to develop the LS theory. Later, it was used in estimation problems when unbiased estimators of parameter θ were evaluated in terms of the risk function $R(\theta, \delta)$ which become nothing but the variance of the estimator. It was observed that SELF is a convex loss function and another beauty of this loss function is that it equally penalizes the overestimation as well as underestimation of equal magnitude. The mathematical form of the weighted quadratic loss function is given as follows

$$L(\theta,\delta) = \psi(\theta-\delta)^2. \tag{1.13}$$

If ψ is the function of θ then, the corresponding loss function is called a weighted quadratic loss function, and if $\psi = 1$ then we have SELF. i.e.

$$L(\theta,\delta) = (\theta-\delta)^2$$

Therefore, the Bayes estimate under SELF is

$$E_{\theta|x}[L(\theta,\delta)] = \int_{\Theta} L(\theta,\delta)\pi(\theta|x)d\theta,$$

after simplification, we get

$$\delta = \int_{\Theta} \theta\pi(\theta|x)d\theta, \tag{1.14}$$

here δ is Bayes estimate under SELF i.e posterior mean. SELF have the symmetric, it is justified for o.e. & u.e. with equal seriousness. Similarly, Bayes estimate under absolute loss function is posterior median and under $0-1$ loss function is posterior mode. However, assumption for real situation of symmetric loss function may not be appropriate. In these situations, when o.e. is more serious than u.e. or vice-versa. Then we have lot of asymmetric loss functions that are available in statistical literature.

The LINEX loss function suggested by Varian (1975) has been widely used by several authors, see Zellner (1986a), Schabe (1991), Pandey and Rai (1992), Ahmadi et al. (2005) and Doostparast (2009). This loss function rises approximately exponentially on one side of zero and approximately linearly on the other side. The mathematical form of this loss function is

$$L(\theta,\delta) = \nu_1\{e^{\nu_1(\theta-\delta)-\nu_1(\theta\delta)-1}\}; \qquad \nu_1 \neq 0, \tag{1.15}$$

where, ν_1 is the loss parameter. If $\nu_1 > 0$, then o.e. is more more serious than u.e. & vice versa. Under this loss function, the Bayes estimate is given by following equation

$$B_L(\delta) = -\frac{1}{\nu_1} log[e^{-\nu_1 \theta}]. \tag{1.16}$$

Despite the exibility and popularity of the LINEX loss function for the location parameter estimation, it appears to be unsuitable for the scale parameter, a similar comment can be found in Basu and Ebrahimi (1991) and Parsian and Sanjari Farsipour (1993). To provide a better asymmetric loss function for scale parameter, Basu and Ebrahimi (1991) defined a modified LINEX loss function. Calabria and Pulcini (1994) introduced an alternate of the modified LINEX loss function having the following form,

$$L_G(\theta, \delta) \propto \left\{ \left(\frac{\delta}{\theta} \right)^{\nu} - \nu \log \left(\frac{\delta}{\theta} \right) - 1 \right\}; \qquad \nu \neq 0. \tag{1.17}$$

When $\delta = \theta$, it has minimum. This loss function is generalization of the entropy loss function used by several authors Lindley (1980), Zellner (1986a), Dey et al. (1986), Basu and Ebrahimi (1991), Schabe (1991) and Singh et al. (2016), when shape parameter $\nu = 1$. Here, ν involved in above equation as shape parameter and it reflects the departure from symmetry. When $\nu > 0$ o.e. is considered to be more serious than u.e. of equal magnitude & vice versa. The Bayes estimator of θ under general entropy loss will be

$$B_G(\delta) = \left[E_{\theta}(\delta)^{-\nu} \right]^{-\frac{1}{\nu}}. \tag{1.18}$$

Provided that, $E_{\theta}(\delta)^{-\nu}$ exists and is finite. It may be noted here that, when $\nu = -1$ the Bayes estimate under GELF coincides with the Bayes estimate under the SELF.

1.9 Credible Intervals

Another approach to inference is to present a CI for model parameter. The Bayesian analog of a classical CI is called credible interval which is defined as

$$\int_{\theta_L}^{\theta_U} \pi(\theta|x)d\theta = 1-\varphi,$$

where, (θ_L, θ_U) is a $100(1-\varphi)\%$ credible interval for θ. Since the posterior distribution is an actual probability distribution on θ, this interval can be stated with a probabilistic statement. This is in contrast to classical CI which can only be interpreted in terms of CP. The equal tail credible interval for parameter θ, can be obtain by solving the following equation

$$\begin{aligned} \int_0^{\theta_L} \pi(\theta|x)d\theta &= \frac{\varphi}{2}, \\ \int_0^{\theta_U} \pi(\theta|x)d\theta &= 1-\frac{\varphi}{2}. \end{aligned}$$

In constructing a credible interval for the parameter θ, it is usually desirable to have those values in the interval which are more probable than those not included in the interval, and such an interval is called the highest posterior density (HPD) interval. The $100(1-\varphi)\%$ HPD credible interval (θ_L^h, θ_U^h) for θ must satisfy the following conditions provided the posterior distribution is unimodal and bell-shaped

$$\int_{\theta_L^h}^{\theta_U^h} \pi(\theta_U^h|x)d\theta = 1-\varphi; \qquad \theta, \in \Theta \tag{1.19}$$

where, $\pi(\theta_L^h|x) = \pi(\theta_U^h|x), \theta \in \Theta$.

1.10 Advanced Statistical Computation Technique

The main obstructions to carrying out statistical inferences are the optimization of the likelihood function to obtain the ML estimator and the integration required to compute the Bayes estimator. The optimization of the likelihood function ultimately results in a solution of k likelihood equations, where k is the number of model parameters to be estimated from the data. The usual iterative methods for solving the likelihood equation

$$L'(\hat{\theta}) = \frac{d}{d\theta} L(\theta) \,\Big|_{\,\theta=\hat{\theta}} = 0, \tag{1.20}$$

are based on replacing the left side by the linear terms of its Taylor expansion about an approximate solution $\tilde{\theta}$. If $\hat{\theta}$ denotes a root of above equation, this leads to the approximation

$$0 = L'(\theta) = L'(\tilde{\theta}) + (\hat{\theta} - \tilde{\theta}) L''(\tilde{\theta}),$$

and hence to

$$\hat{\theta} = \tilde{\theta} - \frac{L'(\tilde{\theta})}{L''(\tilde{\theta})}. \tag{1.21}$$

The procedure is then iterated by replacing $\hat{\theta}$ with the value $\tilde{\theta}$ of the right side of the above equation, and so on. This method is referred as the Newton-Raphson iterative process, see Kale (1961) Kale (1962), Barnett (1966). Another method called the fixed-point iterative method alternative to Newton-Raphson (NR) is also used by many practitioners for solving the likelihood equations. In this method, likelihood Equation (1.20) is re-rewritten as

$$\theta = h(\theta),$$

in such a way that any solution of the above equation, which is a fixed point of h, is a solution of Equation (1.20). Reaching to the solution, the iterative process proceeds as

- Set initial solution as $\tilde{\theta}$.

- Update the initial $\tilde{\theta}$ as $\theta_{new} = h(\tilde{\theta})$ and set $\tilde{\theta} = \theta_{new}$.

- Repeat the above step till θ_{new} converges to a value $\hat{\theta}$. Now $\hat{\theta}$ is the solution of Equation (1.21).

For more detail on the fixed-point iterative method and NR method, readers may be referred to Jain et al. (2003). Moreover, these iterative methods may be used to solve complicated functions, sometimes, they do not converge. Then the corresponding point at which the convergence is obtained may not be the desired root. To overcome this difficulty, EM algorithm may be a choice, which is simple to apply and have sure convergence with any initial guess.

1.10.1 Expectation-Maximization Algorithm

Dempster et al. (1977) introduced the term EM algorithm to overcome the above difficulties. They synthesized the earlier formulation of this algorithm in many particular cases and presented a general formulation of this method in finding the ML estimates in a variety of problems. The main references for the EM algorithm are Schafer (1997), Little and Rubin (2019), Tanner (2012), McLachlan and Krishnan (2007) etc. In statistical inference, and EM algorithm is a method for finding ML or maximum posterior estimates of parameters in statistical models, where the model depends on unobserved latent variables. EM algorithm is an iterative method that alternates between performing an expectation (E) step, which computes the expectation of the log-L evaluated using the current estimate for the latent variables, and maximization (M) step, which computes parameters maximizing the expected log-L found on the E step. These parameter estimates are then used to determine the distribution of the variables for the next iteration in the next E step. The EM algorithm is an efficient iterative procedure to compute the ML estimate in the presence of missing or hidden data. In comparison to other optimization techniques, it is very simple and converges reliably. In the case of the unimodal and concave likelihood function, the EM algorithm converges to the global maxima from any starting value, see Wu (1983). Here, we have given a short description of the EM Algorithm. The EM algorithm has two main applications. The first case occurs when the data has missing values due to limitations or problems with the observation process. The second case occurs when the likelihood function can be simplified by assuming that there are additional but missing parameters. With missing values or parameters in the data which is generated by some distribution under the assumption, we call the data, X, the incomplete data. We assume that the complete data, $Z = (X,Y)$ exists with Y being missing data and that a joint density function also exists as follows,

$$p(Z|\theta) = p(X,Y|\theta) = p(Y|X,\theta)p(X|\theta), \tag{1.22}$$

where, θ is a set of unknown parameters from a distribution including a missing parameter. With the density function, we now define the complete-data likelihood as follows

$$L(\theta|Z) = L(\theta|X,Y) = p(X,y|\theta).$$

The likelihood $L(\theta|X)$ is known as incomplete data likelihood function. Since, we have missing data Y, which have unknown distribution by assumption, we can think of $L(\theta|X,Y)$ as a function of a random variable, Y , with constant values, X and θ.

$$L(\theta|X,Y) = f_{X|\theta}(Y).$$

Using the complete-data log-L function with respect to the missing data Y given the observed data X, the EM algorithm finds its expected value as well as the current parameter estimates at the E step and maximizes the expectation at the M step. Now, repeating E-step and M-step then the algorithm is guaranteed to converge to a local maximum of the likelihood function with each iteration increasing the log-L.

Expectation (E) Step: Firstly, we do the expectation of the complete-data log-L function as

$$Q(\theta|\theta^{(i-1)}) = E[\log\{p(X,Y,\theta|X,\theta^{(i-1)})\}],$$

where $\theta^{(i-1)}$ is a set of current parameters estimates that we use to evaluate the expectation and to increase Q with the new θ for optimization. Here, X and $\theta^{(i-1)}$ are known constants and θ is a variable to be adjusted. Since Y is a missing RV under an assumed distribution, $f(Y;X|\theta^{(i-1)})$, the expectation in the above equation can be written as

$$E[\log\{p(X,Y,\theta|X,\theta^{(i-1)})\}] = \int_{y\in\Theta} \log\{p(X,y|\theta)\}f(y|X,\theta^{(i-1)})dy,$$

where, Θ is the space of values where y can take values on and $f(y|X;\theta^{(i-1)})$ is the marginal distribution of the missing data Y depending on observed data and current parameters.

Maximization (M) Step: At the M step, we maximize the expectation then the E step, that is to find

$$\theta^{(i)} = arg \max_{\theta}\{\theta^{(i)},\theta^{(i-1)}\}.$$

1.10.2 Bayesian Computation

Bayes estimator required some numerical computation of integration since the integrals encountered in Bayesian analysis or inference of the parameters are often intractable and don't possess the analytical solution. For this reason, Bayesian analysis was often doubting before the invention of versatile computing. The developed computer-intensive sampling methods of estimation have revolutionized the application of Bayesian methods, and such methods now offer a comprehensive approach to complex model parameter estimation. The most common and basic approaches to computing the integrals are approximation methods and sampling methods. One of the simplest approximation methods for evaluating the integrals is called as Gauss-quadrature rule, which is stated as a weighted sum of function values at specified points within the domain of integration, see Golub and Welsch (1969), Hildebrand (1974). Although the numerical integration techniques mainly approximate the integrals by polynomials and were efficiently used in a variety of problems, see Smith et al. (1987), Shaw (1988), Tierney (1994), Smith (1991), but these integral techniques are quite complicated and not easy to use in case of higher-dimension scenarios, see Shaw (1988), Smith (1991), etc.

All of these approaches were methods of approximation, and hence formed a foundation for criticizing Bayesian analysis. Of course, it is true that the Bayesian central limit theorem shows that asymptotically most posterior distributions are normal, see Gelman et al. (2013) and the high dimensional integrals were solved by analytical approximation techniques based on normal approximation, see Heyde and Johnstone (1979), etc. The other approximations developed by Lindley (1980) and Tierney and Kadane (1986) received maximum attention in the Bayesian literature, see Singh et al. (2008a), Singh et al. (2008b), Singh et al. (2009) and references cited therein. These approximation methods were widely used, but these had limitations. If the dimension of the parameter to be estimated is high, these methods become unmanageable as clearly mentioned by Smith (1991). Sampling methods constitute an alternative to approximation methods. The logic of sampling is that we can generate/simulate a sample from the distribution of interest and use discrete formulas, applied to these samples to approximate the integrals of interest. The use of simulation methods for approximating integrals of the form

$$E_f[h(x)] = \int h(x)f(x)dx,$$

can be justified as: the above integration can be approximated by

$$\bar{h_n} = \frac{1}{n}\sum_{i=1}^{n} h(x_i),$$

where, $x's$ are i.i.d. sample of size n from the density $f(x)$, since $\bar{h_n}$ converges almost surely (i.e. for almost every generated sequence) to $E_f[h(x)]$ by the strong law of large numbers. This procedure is referred as Monte Carlo (MC) integration method, here MC refers to the random simulation/process.

Bayesian inference has now become closely linked to sampling-based estimation methods. There are various methods that have been suggested for sample generation. The idea of these procedures started with a concept of rejection sampling that provides a general method for simulation from an arbitrary posterior distribution, but it can be difficult to set up since it requires the construction of a suitable proposal density, see Robert and Casella (2013). In the case of high dimension model, it is suggested to use more advanced MCMC techniques. Robert and Casella (2013) stated that "a MCMC method for the simulation of a distribution f is any method producing an ergodic Markov chain $(X(t))$ whose stationary distribution is f". The MCMC techniques are relatively straightforward for a range of applications, involving sampling from one or more chains after convergence to a stationary distribution that approximates the posterior, see Gilks et al. (1996).

In fact, the development of MCMC sampling methods, coupled with exponential growth in computing capabilities, has made the use of Bayesian statistics more feasible because of their relative simplicity compared with traditional numerical methods. With the advent of MCMC sampling methods, more complicated and realistic applications can be undertaken, and there is no inherent reliance on asymptotic arguments and assumptions. In the early 1990s, however, the MCMC methods became standard for Bayesian analysis, but novices and other applied scientists who are really attracted to Bayesian methods were very curious about the implementation of MCMC methods and how these really work.

1.10.3 Gibbs Algorithm

Geman and Geman (1984), Geman and Geman (1993) proposed an MCMC algorithm known as Gibbs sampling or Gibbs sampler that is being found to be a very useful MCMC technique for Bayesian analysis under the assumption of high-dimensional models. This procedure permits the simulation from a model possessing high-dimensional parameters to be reduced to the simulation for its much simpler and lower dimensional parameter. Thus, one simulates p random variables sequentially from the p univariate conditionals rather than generating a single p-dimensional vector in a single pass using the full joint distribution. Suppose we wish to simulate a sample from a bivariate posterior, $\pi(\alpha,\beta|x)$ distribution. The two-stage $(p=2)$ Gibbs sampler algorithm can be stated as

- Simulate $\alpha_j \sim \pi_1(\alpha|\beta_{j-1},x)$ with $j=1,2,\cdots,N$ and β_0 is the initial value of β.

- Simulate $\beta_j \sim \pi_2(\beta|\alpha_j, x)$, where $j = 1, 2, \cdots, N$.

where $\pi_1(.)$ and $\pi_2(.)$ are the full conditionals, and N is the sample size to be required for posterior analysis. For large N, α_j and β_j converges to their stationary distributions. Gelfand and Smith (1990) reviewed Gibbs algorithm with another sampling approaches by revealing its potential in a wide variety of conventional statistical problems. Casella and George (1992) studied the properties of Gibbs algorithm with its convergence for many practical problems. Since then, this algorithm has been increasingly employed to perform Bayesian analysis of many real life problems, see Smith and Roberts (1993), Tierney (1994), Brooks (1998), Jackman (2000), Upadhyay and Smith (2001) and Upadhyay and Gupta (2010).

1.10.4 Metropolis-Hastings Algorithm

The Metropolis-Hastings (M-H) algorithm is the most popular MCMC method of sampling. This algorithm was initiated by Metropolis et al. (1953) and generalized by Hastings (1970). Like accept-reject sampling, the M-H algorithm needs to choose a proposal density, say $q(y|x)$ for which sample generation is easy to perform, to simulate a Markov chain from the target density, say f. The choice of proposal density $q(.|x)$ can be almost arbitrarily made in that the only theoretical requirements are that the ratio

$$\frac{f(y)}{q(y|x)},$$

is known up to a constant independent of x and that $q(y|x)$ has enough dispersion to lead to an exploration of the entire support of f. A key advantage of this algorithm over other methods of sampling, like inversion and accept-reject methods, is that it will effectively work with multivariate distributions and do not need an enveloping function as in rejection sampling. The M-H algorithm associated with the target density f and the conditional density q produces a Markov chain $(X^{(t)})$ through the following steps.

- Start with a value $x^{(t)}$ such that $f(x^{(t)}) > 0$.

- Generate $Y_t\ q(y|x^{(t)})$.
- Calculate the ratio

$$\rho(Y_t,x^{(t)}) = min\left(\frac{f(Y_t)q(x^{(t)}|Y_t)}{f(x^{(t)})q(Y_t|x^{(t)})},1\right).$$

- Accept $x^{(t+1)}$ as

$$x^{(t+1)} = \begin{cases} Y_t\ with\ probability\ \rho(Y_t,x^{(t)}) \\ x^{(t)}\ with\ probability\ 1-\rho(Y_t,x^{(t)}) \end{cases}$$

The probability $\rho(.|.)$ is the M-H acceptance probability. This algorithm always accepts values Y_t such that the ratio $\frac{f(Y_t)}{q(Y_t|x^{(t)})}$ is increased, compared to the previous value $\frac{f(x^{(t)})}{q(x^{(t)}|Y_t)}$. When the proposal density is symmetric i.e. $q(Y_t|x^{(t)}) = q(x^{(t)}|Y_t)$, the above M-H algorithm is identical with Metropolis algorithm having Metropolis acceptance function is

$$\rho(Y_t,x^{(t)}) = min\left(\frac{F(Y_t)}{f(x^{(t)})},1\right). \tag{1.23}$$

This algorithm is also referred to as an independent M-H algorithm. For more technical details about the properties and implementations of the M-H algorithm, readers may be referred to Robert and Casella (2013). There are cases, associated with the multi-parameters model, in which conditional distributions can not be easily derived or determined from the joint density, and so the Gibbs sampler can not be applied. A most common case in which Gibbs sampling seems to be inappropriate is that the conditional densities are not of known forms for the multi-parameters model. In such situations, a hybrid algorithm may be used. This algorithm tailors the M-H algorithm within a framework of the Gibbs sampler. Such a hybrid algorithm has the following steps to be executed (In the continuation of the algorithm of Gibbs sampler given in the previous section).

1− Start with $j = 1$ and initial values $\{\alpha^{(0)},\beta^{(0)}\}$.

2− Using initial values $\{\alpha^{(0)},\beta^{(0)}\}$, generate candidate (proposal) points $\{\alpha_c^{(j)},\beta_c^{(j)}\}$ from proposal densities $\{q_1(\alpha^{(j)}|\alpha^{(j-1)}),q_2(\beta^{(j)}|\beta^{(j-1)})\}$, respectively.

3− Calculate the ratios at the point $\alpha_c^{(j)}$ and previous point $\alpha^{(j)}$

$$\rho_1=\left\{\frac{\pi_1(\alpha_c^{(j)}|\beta^{(j-1)},x)q_1(\alpha^{(j-1)}|\alpha_c^{(j)})}{\pi_1(\alpha^{(j-1)}|\beta^{(j-1)},x)q_1(\alpha_c^{(j)}|\alpha^{(j-1)})}\right\}$$

4− Accept $\alpha^{(j)}$ as

$$\alpha^{(j)}=\begin{cases}\alpha_c^{(j)}\ with\ probability\ min\ (\rho_1,1)\\ \alpha_c^{(j-1)}\ with\ probability\ 1-min\ (\rho_1,1)\end{cases}$$

5− Then calculate the ratio for β

$$\rho_2=\left\{\frac{\pi_2(\beta_c^{(j)}|\alpha^{(j)},x)q_2(\beta^{(j-1)}|\beta_c^{(j)})}{\pi_2(\beta^{(j-1)}|\alpha^{(j)},x)q_2(\beta_c^{(j)}|\beta^{(j-1)})}\right\}$$

6− Accept $\beta^{(j)}$ as

$$\beta^{(j)}=\begin{cases}\beta_c^{(j)}\ with\ probability\ min\ (\rho_2,1)\\ \beta_c^{(j-1)}\ with\ probability\ 1-min\ (\rho_2,1)\end{cases}$$

7− Repeat steps 2 to 6 for all $j=1,2,\cdots,N$ and obtained $\{(\alpha^{(1)},\beta^{(1)}),(\alpha^{(2)},\beta^{(2)}),\cdots,(\alpha^{(N)},\beta^{(N)})\}$.

1.11 Censoring

The common characteristic of survival data is that one cannot always see the lifetimes of all items under study; rather, for some individuals, the genuine lifetime T is only known to be more or less than some value. Left censored data occurs whenever the censored data points are below a certain value but the degree by which they are below that value is unknown. Right-censored data, on the other hand, is defined as the set of censored data points that are above a certain value but not known to how much. For illustration, suppose a patient in a clinical trial is moved to another clinic/hospital for treatment and is no longer be eligible for treatment under the same study. The last day on which the patient reported to the clinic for a regular check-up and was known to be alive is the only information about the patient's survival. A patient's actual survival time can also be regarded as censored when death occurs due to a cause that is known to be unrelated to the treatment. Suppose a clinical trial is conducted in order to determine the survival times of some AIDS patients. Other causes of mortality, such as cancer, heart attack, and high blood pressure, may happen in a few patients. The survival times due to some causes other than AIDS are considered censored.

In other circumstances, the experimental items/units may be intentionally removed from the study to reduce the time required for data collecting to manageable levels, as individuals' lifespan can last an indefinite period of time. To reduce the time required to complete the experiment, various types of censoring are included in the design. The concept of censoring is accountable for some of the most significant advancements in survival analysis. For details see the books Lawless (2011), Nelson (2003), Lee and Wang (2003), Klein and Moeschberger (2006) which adduced the techniques for survival data analysis. There are various categories of right censoring, such as time censoring (Type-I censoring), failure censoring (Type-II censoring), and progressive censoring which are described below.

1.11.1 Type-I and Type-II Censoring

Suppose n identical units, here units may be regarded as human beings (patients) or electronic items/systems, are put on a life-test, performed under controlled environment, results in an i.i.d. failure lifetimes. In Type-I censoring, the experiment is terminated at a pre-determined time T_1, and the lifetime of k items failed by this time is observed, and the remaining $n_c = n - k$ items remained alive. The Type-I censored data consists of the lifetimes of k failed items and the censoring time T_1 for the remaining n_c items. Under this censoring, the time at which the experiment gets terminated is fixed but observable units are random. Type-II censoring allows us to terminate the experiment as soon as a prefixed number (say, $r < n$) of units have failed. Therefore, Type-II censoring ensures the availability of observation on the r failed units for the study but the duration of the life-test is random. For both Type-I and Type-II censoring, the likelihood function is defined by Cohen (1963), Cohen (1965) is given by

$$L(\Theta|x) = r!\binom{n}{r}\prod_{i=1}^{r} f_X(x_i|\Theta)[1 - F_X(T_0||\Theta)]^{n-r}, \tag{1.24}$$

where, $f_X(.)$ and $F_X(.)$ are the PDF and CDF respectively, Θ represents the model parameter and may be vector valued, and $\underline{x} = [x_1, x_2, \cdots, x_r]$ denotes the observed data. For Type-II censoring, $T_0 = x_r$, where, x_r denotes the lifetime of the r^{th} item, while for Type-I censoring, T_0 be a pre-determined time for the experiment and r be the number of units have failed by this time. Giving various real life examples with their associated inferences, Lawless (2011) and Nelson (2003) addressed the problem of estimation under survival analysis with Type-I and Type-II censoring. For some more citation one may refer to Sinha et al. (1976), Sinha (1986), Balakrishnan and Aggarwala (2000), Ashour and Afify (2008), Shah and Patel (2011).

1.11.2 Progressive Type-II Censoring with Binomial Removal

A disadvantage of Type-I and Type-II censoring method is that they do not allow the removal of active units during the experiment. Therefore, a general censoring scheme is introduced named as progressive Type-II censoring scheme. This scheme was firstly introduced by Cohen (1963). He suggests that this scheme may be used when test components are very expensive. Progressive censoring allows for both failure (Type-II censoring) and time censoring (Type-I censoring). In the progressive Type-II censoring , m effective sample units are removed at intermediate stage out of n sample units. In this censoring scheme removals are pre-fixed. But, if removals of censoring units are random then this censoring scheme are not suitable method. Then we consider another censoring method that having random removals with progressive

Type-II censoring is known as Progressive Type-II Censoring with Binomial removals (PT-II CBR). In this thesis, PT-II CBR is considered and it can be described as follows.

Let us assume that the experimenter conducts a life test experiment with n items/ units and decides to terminate the experiment as soon as m failure times are recorded. At first failure observed at X_1, R_1 out of the $n-1$ surviving items/ units are randomly removed from the experiment and the experiment continues. Similarly, at second failure observed X_2, R_2 of the remaining $n-R_1-2$ surviving items/ units are again randomly remove from the experiment and in a similar way the experiment continues till the m^{th} failure is recorded and at this stage all the remaining $(n-m-\sum_{i=1}^{m-1} R_i = (R_m))$ surviving items units are removed resulting to termination of the experiment. Since, R_i at i^{th} stage is the total removal out of surviving units, each experiencing the risk of removal with probability p; it is a random variable following the binomial distribution $B(n-m-\sum_{i=1}^{m-1} R_i, p)$. For details see. Viveros and Balakrishnan (1994) and Ng et al. (2004). Following Cohen (1963) for fixed removals, say $R_1 = r_1, R_2 = r_2, R_3 = r_3, \cdots, R_m = r_m$, the conditional likelihood function can be written as,

$$L(\Theta; x|R=r) = c\prod_{i=1}^{m} f(x_i|\Theta)[1-F(x_i|\Theta)]^{r_i}, \cdots -\infty < x_1 < \ldots < x_m < \infty, \tag{1.25}$$

here, Θ is the parameter space, n, m ε N, $1 \leq i \leq m$ and $c = \prod_{i=1}^{m} \gamma_i$ where $\gamma_i = \sum_{j=1}^{m} (r_j + 1)$. Substituting PDF and CDF into (1.25).

It may be noted that type-II censoring is a special case of PT-II CBR with $r = (0, 0, \cdots, 0, n-m)$. For more details on progressive censoring and its further development, readers may be referred to Balakrishnan and Aggarwala (2000), Balakrishnan (2007). There is the massive literature available that reports for estimation of parameters of several lifetime distributions based on progressive censored samples, see Cohen and Norgaard (1977), Davis and Feldstein (1979), Viveros and Balakrishnan (1994), Rastogi and Tripathi (2013), Krishna and Malik (2012), Krishna and Kumar (2013) and Singh et al. (2013b).

For PT-II CBR having recent developments discussed by Yuen and Tse (1996), Tse and Yuen (2000), Yuen and Tse (1996) and Tse et al. (2000).

The assumptions regarding the uniform distribution with equal chance for the number of removals, or binomial distribution with the fixed probability of a removal at each stage, do not seems to be realistic in the practical situations. Consider that a doctor starts an experiment with n patients. The patients may drop from the experiment due to various physical and psychological reasons. For example one of the reasons may be the duration of the cure. Thus at the early stages the chance of drop out will be small as compared to the later stages. The degree of belief, even if not cured completely, may be another factor. If the doctor's cure is not providing immediate relief the chances of drop out at the early stages are expected to be high as compared to the later stages. Hence, keeping these points in mind, it seems more reasonable to think that the number of removals follow a binomial distribution with random probability (p).

1.12 Summary of the Thesis

This thesis has a total of five chapters. Chapter 1 is the introductory definition and terminologies part of the thesis. It contains a brief explanation of the various terms and concepts which have been used in the rest of the thesis. Mainly this thesis work is based on estimation of the parameters for a few lifetime models and ecological model.

Chapter 2, deals with parameter estimation of experimental items/units from the WPD under PT-II CBRs. The EM algorithm has been used for ML estimators. The ML estimators and Bayes estimators have been obtained under symmetric and asymmetric loss functions. The performance of competitive estimators have been studied through their simulated risks. One sample Bayes prediction and expected experiment time have also been studied. Furthermore, through the real bladder cancer data set, the suitability of the considered model and proposed methodology has been illustrated.

The estimators $\hat{\alpha}_G, \hat{\beta}_G$ and $\hat{\lambda}_G$ perform better than all other considered competitive estimators, for ($\delta > 0$, δ is loss parameter) i.e., when o.e. is more serious than u.e. and for ($\delta < 0$) i.e., when u.e. is more serious than o.e., under both considered loss functions. Thus, the use of the proposed estimator $\hat{\alpha}_G, \hat{\beta}_G$ and $\hat{\lambda}_G$ are recommended under SELF and GELF. Moreover, a brief study has done on the expected experiment time by taking the various combinations of effective parameters n, p and m and it observed that on increases the value of p and m, the expected time to test increases. While, for fixed m, on increases the value of n, the expected time to test decreases. The LR test has performed the goodness of fit. The one sample Bayes prediction has also presented. Furthermore, a real data set is fitted to show the practical applicability of the model.

In Chapter 3, we present the E-Bayesian and Bayesian estimators of parameters of PIED under SELF, GELF, and LINEX for PT-II CBRs. The E-Bayesian and Bayesian estimators are compared through risk based on simulated samples. The effectiveness of proposed methodology is applied on the survival time of multiple myeloma patients' data.

The risk of the E-Bayesian and Bayesian estimators of λ and θ are compared under SELF, GELF and LINEX. Generally, we found that the estimated risk of the E-Bayesian estimate of λ and θ have minimum. Therefore, the simulated results shown in this chapter that the E-Bayesian estimation is more efficient and better to perform than Bayesian estimation.

Chapter 4, deals with empirical Bayes estimators of parameter, reliability, and hazard function for Kumaraswamy distribution under the LINEX loss function for PT-II CBRs and Type-II censored samples. The proposed estimators have been compared with the respective Bayes estimators for their simulated risks. The applicability of the proposed estimators has been illustrated through ulcer patient data.

We may conclude that the proposed empirical Bayes estimators $\hat{\lambda}_E, \hat{\lambda}_{E_2}$ and $\hat{h}_E(t), \hat{h}_{E_2}(t)$ are better than Bayes estimators $\hat{\lambda}_B, \hat{\lambda}_{B_2}$ and $\hat{h}_B(t), \hat{h}_{B_2}(t)$ for smaller or larger prior variance ($\sigma = 1,3$) of β with $a = \pm 1.5$. Also, we have seen that Table 4.1-4.2 under LINEX loss function for the estimators $\hat{R}_E(t)$ & $\hat{R}_{E_2}(t)$ are not always less than those of $\hat{R}_B(t)$ and $\hat{R}_{B_2}(t)$. Since the risks associated with $\hat{R}_B(t)$ and $\hat{R}_{B_2}(t)$ are smaller than the risk associated with reliability of the empirical estimators. Thus, the use of propose estimator $(\hat{\lambda}_E, \hat{R}_B(t), \hat{h}_E(t))$ and $(\hat{\lambda}_{E_2}, \hat{R}_{B_2}(t), \hat{h}_{E_2}(t))$ under PT-II CBRs and Type-II censoring are used under LINEX loss function respectively.

Chapter 5, deals with a Poisson Lindley distribution as a stochastic abundance model in which the sample is according to the independent Poisson process. We have obtained the maximum likelihood estimators through profile likelihood and the conditional likelihood of the number of species. In the Bayesian estimation of the number of species, we have considered two priors i.e., Jefferey's and reference priors. We obtain the Bayes estimators of the number of species through Jeffery's prior and reference prior. The proposed Bayes estimators have been compared with the corresponding profile and conditional ML estimators for their simulated samples. The Jeffery's and reference priors have considered and compared with the Bayesian approach based on biological data.

The biological data, shows similar results are obtained in estimating the number of species S between the posterior and the maximum likelihood estimators. The asymmetry in both the profile likelihood confidence intervals and the credible intervals accounts for skewed profile likelihoods and skewed posterior distribution. There is a large effect of the model on the estimates, showing a need for more models and a careful model selection technique. The methods of Jeffrey's prior and reference prior give us a way to construct priors that are defined to be non-informative or minimally informative. In the simulated result, we obtain model DIC lesser of Poisson Lindley model with reference prior (PLR) than Poisson Lindley model with Jeffery's prior (PLJ). Therefore, we can propose that Bayes estimate i.e. posterior mean from PLR gives the optimum number of species present there.

The **R** software is used for mathematical computations. This thesis contains a list of references at the end. We realize that an exhaustive list of references related to the problem discussed in the thesis is too big to be reproduced here. Therefore we have included only those references that are cited in the thesis and are directly related to our work.

Chapter 2

Bayesian Inference for Weibull Poisson Distribution Under Censored Data Using Expectation Maximization Algorithm *

2.1 Introduction

Statistical literature have numerous distributions for modeling life-time data. Due to the enormous use of the Poisson family distribution, we consider a very flexible Weibull Poisson Distribution (WPD). It is one of the recent compounding of two most greeted probability distributions i.e., Weibull and zero truncated Poisson distribution. This distribution was pioneered by Lu and Shi (2012). The CDF of WPD with (α, β, λ) is

$$F(x) = \frac{e^{\lambda e^{-\beta x^{\alpha}}} - e^{\lambda}}{1 - e^{\lambda}}; \quad \alpha > 0, \lambda > 0, \beta > 0, x > 0. \tag{2.1}$$

*Part of this chapter has been published in reputed peer-reviewed journals with indexing SCI, SCIE, SCOPUS, see Pathak et al. (2020b).

The PDF is given by

$$f(x) = \frac{\alpha\beta\lambda e^{-\lambda}}{1-e^{-\lambda}} e^{-\beta x^{\alpha}} x^{\alpha-1} e^{\lambda e^{-\beta x^{\alpha}}}; \quad \alpha > 0, \lambda > 0, \beta > 0, x > 0, \tag{2.2}$$

where, shape parameter α and scale parameters β of WPD, while λ is the rate parameter of zero truncated Poisson distribution. This distribution has an edge over other Poisson-based distributions like Poisson-gamma, Poisson-log normal etc in the sense that it covers all types of failure rates encountered in life testing experiments, see Gonzales-Barron and Butler (2011). We may note here a typical feature of life testing experiments is censoring because, situations do arise when items/ units are lost or removed from the experiment while they are alive; i.e., quite often, it is very much difficult to get failure times of all the items/units put on test experiments owing to various restriction related to time, cost and other resources. Type-I censoring takes place when experimental time is fixed and hence number of failures become random. While type-II censoring occurs when the number of failures is fixed, but experimental time remain random. Even under these conditions, some items/ units may drop out of the experiment randomly due to some unknown causes, which are beyond the control of the experimenter. For example, consider that a medical experiment starts with n patients but after the death of first patient, some patients who are alive leave the experiment and go for treatment elsewhere. Similarly, after death of second patient a few more are leave and the process continues till predetermined number of failure (say $m < n$) are recorded. It may be assumed here that at each stage participating patient may independently decide to leave the experiment with probability p. Thus the number of patients who leave the experiment at a specified stage will follow binomial distribution with probability p. It may be argued at this stage that probability p may vary at each stage. But sake of simplicity, we shall assume that p is same at each stages. Collecting information in this way results to a censored sample and the sampling technique used is called as PT-II CBRs. The mathematical formulation of PT-II CBRs is presented in next Section. For details, one can see Balakrishnan and Sandhu (1995), Balakrishnan and Aggarwala (2000).

In last few decades, parameter estimation for Weibull lifetime models based on progressive Type-II, PT-II CBRs and optimal progressive censoring schemes are studied by several authors

((Balasooriya et al., 2000), Tse et al. (2000), (Tang et al., 2003), Ng et al. (2004) etc.). Estimation of inverse Weibull parameters have been discussed by Sultan et al. (2014). Also, in last few years and for other lifetime models by Soliman et al. (2015), Singh et al. (2014), Kumar et al. (2015), Kumar et al. (2018), Kumar et al. (2019a), Kumar et al. (2019b) etc. But, it seems as if no attempt has been made to develop estimators for the parameters of WPD under PT-II CBRs; although estimation of parameters under classical set up has also been attempted by Lu and Shi (2012).

Therefore, in this chapter we propose to develop an estimation procedure to obtain the ML Estimators (using EM algorithm) and Bayes estimators for parameters of WPD under symmetric and asymmetric loss function when sample is obtained by the use of PT-II CBRs. An important feature of this chapter is to develop the required mathematics for PT-II CBRs, EM algorithm along with its application to the bladder cancer patients data (remission time in months).

2.2 Classical and Bayesian Estimation Under PT-II CBRs

In this section, we follow the PT-II CBRs discussed in Chapter-1, Subsection 1.11.2. For details see. Viveros and Balakrishnan (1994) and Ng et al. (2004). Following Cohen (1963) for fixed removals, say $R_1 = r_1, R_2 = r_2, R_3 = r_3, \cdots, R_m = r_m$, the conditional likelihood function can be written as,

$$L(\alpha,\beta,\lambda;x|R=r) = c\prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{r_i}; \qquad -\infty < x_1 < \ldots < x_m < \infty, \tag{2.3}$$

$n,\ m\ \varepsilon$ N, $1 \le i \le m$ and $c = \prod_{i=1}^{m} \gamma_i$ where $\gamma_i = \sum_{j=1}^{m} (r_j+1)$. Substituting $f(x_i)$ and $F(x_i)$ from (2.1) and (2.2) into (2.3), we have

$$L(\alpha,\beta,\lambda;x|R=r) = c\prod_{i=1}^{m} \frac{\alpha\beta\lambda x_i^{\alpha-1}}{1-e^{-\lambda}} e^{-\lambda-\beta x_i^{\alpha}+\lambda e^{-\beta x_i^{\alpha}}} \left\{\frac{1-e^{\lambda e^{-\beta x_i^{\alpha}}}}{1-e^{\lambda}}\right\}^{r_i}. \tag{2.4}$$

As mentioned earlier, in the experiment removal of the number of items/units is random and independent of each other, therefore

$$p(R_1 = r_1; p) = \binom{n-m}{r_1} p^{r_1}(1-p)^{n-m-r_1} \tag{2.5}$$

and for $i = 2, 3, ..., m-1$

$$p(R_i; p) = p(R_i = r_i | R_{i-1} = r_{i-1}, ...R_1 = r_1)$$

$$= \binom{n-m-\sum_{l=0}^{i-1} r_l}{r_i} p^{r_i}(1-p)^{n-m-\sum_{l=0}^{i-1} r_l}. \tag{2.6}$$

Hence, likelihood function can be written as

$$L(\alpha, \beta, \lambda, p; x) = L(\alpha, \beta, \lambda; x | R = r) p(R = r; p) \tag{2.7}$$

where,

$$\begin{aligned} p(R = r; p) = p(R_1 = r_1) p(R_2 = r_2 | R_1 = r_1) p(R_3 = r_3 | R_2 = r_2, R_1 = r_1)... \\ p(R_{m-1} = r_{m-1} | R_{m-2} = r_{m-2}, ...R_1 = r_1). \end{aligned} \tag{2.8}$$

Substituting from Equation (2.5) and (2.6) into (2.8), we have

$$p(R = r; p) = \frac{(n-m)! p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i) r_i}}{(n-m-\sum_{l=1}^{i-1} r_l)! \prod_{i=1}^{m-1} r_i!}, \tag{2.9}$$

now using Equation (2.4), (2.7) and (2.9), the complete likelihood can be expressed in the following form,

$$L(\alpha, \beta, \lambda, p; x) = \Phi L_1(\alpha, \beta, \lambda) L_2(p)$$

where,

$$\Phi = \frac{c(n-m)!}{(n-m-\sum_{l=1}^{i-1} r_l)! \prod_{i=1}^{m-1} r_i!},$$

$$L_1(\alpha,\beta,\lambda;x|R=r) = c\prod_{i=1}^{m} \frac{\alpha\beta\lambda x_i^{\alpha-1}}{1-e^{-\lambda}} e^{-\lambda-\beta x_i^{\alpha}+\lambda e^{-\beta x_i^{\alpha}}} \left\{ \frac{1-e^{\lambda e^{-\beta x_i^{\alpha}}}}{1-e^{\lambda}} \right\}^{r_i}, \tag{2.10}$$

$$L_2(p) = p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}.$$

Now, ML Estimates of α, β and λ are computed by maximizing L_1 and MLE of p by maximizing L_2. Taking log of both sides to Equation (2.10), we get

$$\begin{aligned} l_1(\alpha,\beta,\lambda) = \ln(L_1(\alpha,\beta,\lambda)) &= m\ln\alpha + m\ln\beta + m\ln\lambda + (\alpha-1)\sum_{i=1}^{m}\ln x_i - m\lambda - \beta\sum_{i=1}^{m} x_i^{\alpha} \\ &\quad - m\ln(1-e^{-\lambda}) + \lambda\sum_{i=1}^{m} e^{-\beta x_i^{\alpha}} + \sum_{i=1}^{m} r_i \left(\ln(e^{\lambda e^{-\beta x_i^{\alpha}}} - 1) - \ln(e^{\lambda}-1) \right). \end{aligned} \tag{2.11}$$

Differentiating the Equation (2.11) with respect to parameter α, β and λ and equating to zero, we obtain following three normal equations. A simultaneous solution of these provide ML Estimates of the parameters.

$$\begin{aligned} \frac{\partial l_1(\alpha,\beta,\lambda)}{\partial\alpha} &= \frac{m}{\alpha} + \sum_{i=1}^{m}\ln x_i - \beta\sum_{i=1}^{m} x_i^{\alpha}\ln x_i - \lambda\beta\sum_{i=1}^{m} e^{-\beta x_i^{\alpha}}(x_i^{\alpha}\ln x_i) \\ &\quad + \sum_{i=1}^{m} r_i \left[\frac{\lambda e^{-\beta x_i^{\alpha}} e^{\lambda e^{-\beta x_i^{\alpha}}}}{1-e^{\lambda e^{-\beta x_i^{\alpha}}}} \beta x_i^{\alpha}\ln x_i \right] = 0, \end{aligned} \tag{2.12}$$

$$\frac{\partial l_1(\alpha,\beta,\lambda)}{\partial\beta} = \frac{m}{\beta} - \sum_{i=1}^{m} x_i^{\alpha} - \lambda x_i^{\alpha}\sum_{i=1}^{m} e^{-\beta x_i^{\alpha}} + \sum_{i=1}^{m} r_i \left[\frac{\lambda e^{-\beta x_i^{\alpha}} e^{\lambda e^{-\beta x_i^{\alpha}}}}{1-e^{\lambda e^{-\beta x_i^{\alpha}}}} x_i^{\alpha} \right] = 0, \tag{2.13}$$

$$\frac{\partial l_1(\alpha,\beta,\lambda)}{\partial\lambda} = \frac{m}{\lambda} - m - \sum_{i=1}^{m} e^{-\beta x_i^{\alpha}} - \frac{me^{-\lambda}}{1-e^{-\lambda}} - \sum_{i=1}^{m} r_i \left[\frac{e^{\lambda e^{-\beta x_i^{\alpha}}-\beta x_i^{\alpha}}}{1-e^{\lambda e^{-\beta x_i^{\alpha}}}} - \frac{me^{\lambda}}{1-e^{\lambda}} \right] = 0. \tag{2.14}$$

Unfortunately, Equation (2.12), (2.13) and (2.14) can not be analytically solved simultaneously. Hence we propose the use of numerical iterative procedure, namely i.e. NR method for solving these. The numerical procedure used here for obtaining the iteration function and the choice of initial guesses is based on maximum absolute row sum norms, which has been discussed by Jain et al. (2003). The EM algorithm has been proposed in this chapter to get the ML estimates of parameter α, β and λ, also discussed in Chapter 1, Subsection 1.10.1. Let Z_{ik} be the unobserved observation for the k^{th} items/ units moved out of the experiment at the time of observing i^{th} removal at time $X_i; i = 1,2,...,m$ and $k = 1,2,....,r_i$. Thus, the observed X_i's and Z_{ik}'s form the complete data. Hence the complete likelihood is

$$L(\alpha,\beta,\lambda) = \prod_{i=1}^{m}\left[\frac{\alpha\beta\lambda x_i^{\alpha-1}}{1-e^{-\lambda}} e^{-\lambda-\beta x_i^{\alpha}+\lambda e^{-\beta x_i^{\alpha}}} \prod_{k=1}^{r_i} \frac{\alpha\beta\lambda z_{ik}^{\alpha-1}}{1-e^{-\lambda}} e^{-\lambda-\beta z_{ik}^{\alpha}+\lambda e^{-\beta z_{ik}^{\alpha}}}\right].$$

The log-L function is

$$\begin{aligned}\ln L(\alpha,\beta,\lambda) = n\ln(\alpha)+n\ln(\beta)+n\ln(\lambda)-n\lambda-n\ln\left(1-e^{-\lambda}\right)\\ +(\alpha-1)\sum_{i=1}^{m}\ln x_i-\beta\sum_{i=1}^{m}x_i^{\alpha}+\lambda\sum_{i=1}^{m}e^{-\beta x_i^{\alpha}}\\ +(\alpha-1)\sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln z_{ik}-\beta\sum_{i=1}^{m}\sum_{k=1}^{r_i}z_{ik}^{\alpha}+\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}e^{-\beta z_{ik}^{\alpha}}.\end{aligned} \tag{2.15}$$

Hence, ML estimate of the parameters are, obtained the simultaneous solution of the following three nonlinear equations

$$\begin{aligned}\frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial\alpha} = \frac{n}{\alpha}-\alpha\beta\sum_{i=1}^{m}x_i^{\alpha-1}-\alpha\beta\lambda\sum_{i=1}^{m}x_i^{\alpha-1}e^{-\beta x_i^{\alpha}}+\sum_{i=1}^{m}\ln x_i\\ -\alpha\beta\sum_{i=1}^{m}\sum_{k=1}^{r_i}z_{ik}^{\alpha-1}-\alpha\beta\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}z_{ik}^{\alpha-1}e^{-\beta z_{ik}^{\alpha}}+\sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln z_{ik}=0,\end{aligned} \tag{2.16}$$

$$\begin{aligned}\frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial\beta} = \frac{n}{\beta}-\sum_{i=1}^{m}x_i^{\alpha}-\lambda\sum_{i=1}^{m}x_i^{\alpha}e^{-\beta x_i^{\alpha}}-\sum_{i=1}^{m}\sum_{k=1}^{r_i}z_{ik}^{\alpha}\\ -\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}z_{ik}^{\alpha}e^{-\beta z_{ik}^{\alpha}}=0,\end{aligned} \tag{2.17}$$

and

$$\frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial\lambda} = \frac{n}{\lambda}-n+\frac{ne^{-\lambda}}{\left(1-e^{-\lambda}\right)}+\sum_{i=1}^{m}e^{-\beta x_i^{\alpha}}+\sum_{i=1}^{m}\sum_{k=1}^{r_i}e^{-\beta z_{ik}^{\alpha}}=0. \tag{2.18}$$

Now, to perform the EM algorithm, joint distribution of x and z can be written as

$$f(x,z;\alpha,\beta,\lambda) = P(z;\lambda)f(x|z;\alpha,\beta),$$

where,

$$P(z;\lambda) = \frac{e^{-\lambda}\lambda^z}{z!\left[1-e^{-\lambda}\right]}; \quad \lambda > 0, \quad z = 1,2,3,\cdots.$$

Since, the conditional PDF is

$$P(z|x;\alpha,\beta,\lambda) = \frac{f(x,z;\alpha,\beta,\lambda)}{f(x;\lambda)} = \alpha\beta z x^{\alpha-1}e^{-\beta z x^{\alpha}}\lambda^z\Gamma^{-1}(z+1)\left(e^{\lambda}-1\right)^{-1}; \quad z = 1,2,3,\cdots, \tag{2.19}$$

where, $\alpha > 0, \beta > 0$ and $\lambda > 0$. The E-step of EM algorithm needs the computation of the conditional expectation $(Z|X,\alpha^t,\beta^t,\lambda^t)$, where, $(\alpha^t,\beta^t,\lambda^t)$ is the current estimates of (α,β,λ). Hence from Equation (2.19), we get

$$E(z|x;\alpha^t,\beta^t,\lambda^t) = \left(1+\lambda^t e^{-\beta^t x^{\alpha^t}}\right).$$

The EM algorithm is completed with M-step, with complete data, where missing Z's are replaced by their conditional expectations $(Z|X,\alpha^t,\beta^t,\lambda^t)$. Thus, an EM iteration, takes $(\alpha^t,\beta^t,\lambda^t)$ into $\left(\alpha^{t+1},\beta^{t+1},\lambda^{t+1}\right)$ obtained from the following

$$\frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial\alpha} = \frac{n}{\alpha} - \alpha\beta\sum_{i=1}^{m} x_i^{\alpha-1} - \alpha\beta\lambda\sum_{i=1}^{m} x_i^{\alpha-1}e^{-\beta x_i^{\alpha}} + \sum_{i=1}^{m}\ln x_i$$
$$-\alpha\beta\sum_{i=1}^{m}\sum_{k=1}^{r_i}\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha-1} - \alpha\beta\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha-1} e^{-\beta\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha}}$$
$$+\sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right) = 0,$$

$$\frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{m} x_i^{\alpha} - \lambda\sum_{i=1}^{m} x_i^{\alpha}e^{-\beta x_i^{\alpha}} - \sum_{i=1}^{m}\sum_{k=1}^{r_i}\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha}$$
$$-\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha} e^{-\beta\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha}} = 0,$$

and

$$\frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial \lambda} = \frac{n}{\lambda} - n + \frac{ne^{-\lambda}}{\left(1-e^{-\lambda}\right)} + \sum_{i=1}^{m} e^{-\beta x_i^{\alpha}} + \sum_{i=1}^{m}\sum_{k=1}^{r_i} e^{-\beta\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha}} = 0.$$

The iterative procedure obtained for EM algorithm is given below

$$\alpha^{t+1} = \frac{n}{\left\{\begin{array}{l} \alpha\beta\sum_{i=1}^{m} x_i^{\alpha-1} + \alpha\beta\lambda\sum_{i=1}^{m} x_i^{\alpha-1}e^{-\beta x_i^{\alpha}} - \sum_{i=1}^{m}\ln x_i + \alpha\beta\sum_{i=1}^{m}\sum_{k=1}^{r_i}\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha-1} \\ +\alpha\beta\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha-1} e^{-\beta\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha}} - \sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right) \end{array}\right\}}$$

$$\beta^{t+1} = \frac{n}{\left\{\begin{array}{l} \sum_{i=1}^{m} x_i^{\alpha} + \lambda\sum_{i=1}^{m} x_i^{\alpha}e^{-\beta x_i^{\alpha}} + \sum_{i=1}^{m}\sum_{k=1}^{r_i}\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha} \\ +\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha} e^{-\beta\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha}} \end{array}\right\}}$$

and

$$\lambda^{t+1} = \frac{n}{\left\{ n - \frac{ne^{-\lambda}}{\left(1-e^{-\lambda}\right)} - \sum_{i=1}^{m} e^{-\beta x_i^{\alpha}} - \sum_{i=1}^{m}\sum_{k=1}^{r_i} e^{-\beta\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha}} \right\}}.$$

Then $\left(\alpha^{t+1},\beta^{t+1},\lambda^{t+1}\right)$ is used as the current estimates of (α,β,λ) in the next iteration. The ML estimates of (α,β,λ) can be obtained by repeating the E-step and M-step until convergence is achieved.

2.2.1 Large Sample Test Procedure

Now, we shall discuss LR method for comparing the suitability of competitive models. Note that if we take $r_i = 0$ and $n = m$ in Equation (2.16), (2.17), (2.18), these reduce to complete sample normal Equations. The observed Fisher's Information matrix is

$$J_n(\alpha,\beta,\lambda)=\begin{pmatrix} -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\alpha^2} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\alpha\partial\beta} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\alpha\partial\lambda} \\ -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\beta\partial\alpha} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\beta^2} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\beta\partial\lambda} \\ -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\lambda\partial\alpha} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\lambda\partial\beta} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\lambda^2} \end{pmatrix}_{(\hat\alpha,\hat\beta,\hat\lambda)}$$

where,

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\alpha^2}=\frac{n}{\alpha^2}+\sum_{i=1}^{n}\beta x_i^{\alpha}(\log(x_i))^2(1+\lambda e^{-\beta x_i^{\alpha}}-\beta\lambda x_i^{\alpha}e^{-\beta x_i^{\alpha}}),$$

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\alpha\partial\beta}=\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\beta\partial\alpha}=\sum_{i=1}^{n}\beta x_i^{\alpha}\log(x_i)(1+\lambda e^{-\beta x_i^{\alpha}}-\beta\lambda x_i^{\alpha}e^{-\beta x_i^{\alpha}}),$$

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\alpha\partial\lambda}=\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\lambda\partial\alpha}=\sum_{i=1}^{n}\beta x_i^{\alpha}\log(x_i)e^{-\beta x_i^{\alpha}},$$

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\beta^2}=\frac{n}{\beta^2}-\lambda\sum_{i=1}^{n}(x_i^{\alpha})^2e^{-\beta x_i^{\alpha}},$$

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\beta\partial\lambda}=\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\lambda\partial\beta}=\sum_{i=1}^{n}x_i^{\alpha}e^{-\beta x_i^{\alpha}},$$

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\lambda^2}=\frac{n}{\lambda^2}-n\frac{e^{\lambda}}{(1-e^{\lambda})^2}.$$

Let $T_n(\alpha,\beta,\lambda)$ be the expectation of Fisher Information matrix, i.e.,

$$T_n(\alpha,\beta,\lambda)=E(J_n(\alpha,\beta,\lambda))=n\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

where,

$$T_{11}=\frac{1}{\alpha^2}+\beta E\left[x_z^{\alpha}(\log(x_z))^2(1+\lambda e^{-\beta x_z^{\alpha}}-\beta\lambda x_z^{\alpha}e^{-\beta x_z^{\alpha}})\right],$$

$$T_{12}=T_{21}=E\left[x_z^{\alpha}\log(x_z)(1+\lambda e^{-\beta x_z^{\alpha}}-\beta\lambda x_z^{\alpha}e^{-\beta x_z^{\alpha}})\right],$$

$$T_{13}=T_{31}=\beta E\left[x_z^{\alpha}\log(x_z)e^{-\beta x_z^{\alpha}}\right],$$

$$T_{22} = \frac{1}{\beta^2} - \lambda E\left[(x_z^{\alpha})^2 e^{-\beta x_z^{\alpha}}\right],$$

$$T_{23} = T_{32} = E\left[x_z^{\alpha} e^{-\beta x_z^{\alpha}}\right],$$

$$T_{33} = \frac{1}{\lambda^2} - \frac{e^{\lambda}}{(1-e^{\lambda})^2}.$$

For large n, under the usual regularity condition, we obtain that $(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ have multivariate normal distribution with attain mean (α, β, λ) and covariance matrix $T_n^{-1}(\alpha, \beta, \lambda)$. The asymptotic property of normality is useful for performing a goodness of fit test. Here, we can test the significance of the model parameters by comparing this full model with specified nested models based on the LR test. By considering null hypothesis $H_{01} : \alpha = 1$ against $H_{11} : \alpha \neq 1$ and $H_{02} : \lambda = 0$ against $H_{12} : \lambda \neq 0$, one can compare the suitability of Exponential Poisson and Weibull versus Weibull Poisson distribution respectively. The test statistic under $H_{0i}, \;\; i = 1, 2,$ are

$$R_1 = -2\ln\left(\frac{L(\alpha_0, \hat{\beta}, \hat{\lambda})}{L(\hat{\alpha}, \hat{\beta}, \hat{\lambda})}\right) \quad and \quad R_2 = -2\ln\left(\frac{L(\hat{\alpha}, \hat{\beta}, \lambda_0)}{L(\hat{\alpha}, \hat{\beta}, \hat{\lambda})}\right),$$

respectively, which are asymptotically distributed as χ^2 with degrees of freedom equal to the respective dimension of the parameter space under the null hypothesis.

2.2.2 Bayesian Estimation Under PT-II CBRs

To obtain the Bayes estimator of α, β and λ, we assume that these are independently distributed prior pdfs for α and λ are chosen by using Jeffery's method i.e., log of the parameters are uniformly distributed; resulting to the following distributions:

$$g_1(\alpha) \propto \frac{1}{\alpha}; \qquad \alpha > 0. \tag{2.20}$$

$$g_2(\lambda) \propto \frac{1}{\lambda}; \qquad \lambda > 0. \tag{2.21}$$

Keeping in mind the wide coverage of variety of prior beliefs, we have chosen gamma distribution given below as prior distribution; see for details, Nassar and Eissa (2005), Box and Tiao (2011).

$$g_3(\beta) \propto e^{-a\beta}\beta^{b-1}; \qquad a>0, b>0, \tag{2.22}$$

where, gamma distribution have scale parameter a and shape parameter b. Thus the posterior distribution of α, β and λ can easily be obtained as

$$\pi(\alpha,\beta,\lambda|x,r) \propto \frac{\alpha^{m-1}\lambda^{m-1}\beta^{m+b-1}e^{-m\lambda-\beta\sum_{i=1}^{m}x_i^{\alpha}-a\beta+\lambda\sum_{i=1}^{m}e^{-\beta x_i^{\alpha}}}}{(1-e^{\lambda})^m} \prod_{i=1}^{m}x_i^{\alpha-1}\left[\frac{1-e^{\lambda e^{-\beta x_i^{\alpha}}}}{1-e^{-\lambda}}\right]^{r_i},$$

and the respective marginal posterior pdfs of α, β and λ can be computed from the following

$$\pi_1(\alpha|x,r) = \int_0^{\infty}\int_0^{\infty}\pi(\alpha,\beta,\lambda|x,r)d\beta\, d\lambda,$$

$$\pi_2(\beta|x,r) = \int_0^{\infty}\int_0^{\infty}\pi(\alpha,\beta,\lambda|x,r)d\alpha\, d\lambda,$$

and

$$\pi_3(\lambda|x,r) = \int_0^{\infty}\int_0^{\infty}\pi(\alpha,\beta,\lambda|x,r)d\alpha\, d\beta.$$

Now, let us consider that the very much popular symmetric loss function i.e., SELF has equal weight to the o.e. and u.e. of the same magnitude. Also, consider the asymmetric loss function i.e. GELF has unequal weight to the o.e. is more serious than u.e. and vice versa. The SELF and GELF are discussed in Chapter 1, Subsection 1.8. The expressions for the Bayes estimators of the parameters α, β and λ, denoted by $\hat{\alpha}_G, \hat{\beta}_G$ and $\hat{\lambda}_G$ respectively, are given below

$$\hat{\alpha}_G = \left[\int_0^{\infty}\alpha^{-\delta}\pi_1(\alpha|x,r)d\alpha\right]^{-\frac{1}{\delta}}, \tag{2.23}$$

$$\hat{\beta}_G = \left[\int_0^{\infty}\beta^{-\delta}\pi_2(\beta|x,r)d\beta\right]^{-\frac{1}{\delta}}, \tag{2.24}$$

and

$$\hat{\lambda}_G = \left[\int_0^\infty \lambda^{-\delta} \pi_3(\lambda|x,r) d\lambda \right]^{-\frac{1}{\delta}}. \tag{2.25}$$

It may be noted that the integrals in Equation (2.23), (2.24) and (2.25) can not be reduced to closed forms. Hence, numerical computational techniques are suggested for their calculations following Tierney (1994). Who has suggested the use of well-known technique namely MCMC technique in which the samples are generated from posterior distribution by Gibbs sampler via M-H algorithms. The samples thus obtained are then used to evaluate the Bayes estimates under SELF and GELF. It may be noted that Gibbs sampler uses to generate samples from full conditionals to generate samples posterior distribution and for details Gelman et al. (2013). Full conditional posterior distributions of the parameters α, β, and λ can be written in the following form:

$$\pi_1^*(\alpha|\beta,\lambda,x,r) \propto \alpha^{m-1} e^{-\beta \sum_{i=1}^{m} x_i^\alpha + \lambda \sum_{i=1}^{m} e^{-\beta x_i^\alpha}} \prod_{i=1}^{m} x_i^{\alpha-1} \{1 - e^{\lambda e^{-\beta x_i^\alpha}}\}^{r_i}, \tag{2.26}$$

$$\pi_2^*(\beta|\alpha,\lambda,x,r) \propto \beta^{m+b-1} e^{-\beta \sum_{i=1}^{m} x_i^\alpha - a\beta + \lambda \sum_{i=1}^{m} e^{-\beta x_i^\alpha}} \prod_{i=1}^{m} \{1 - e^{\lambda e^{-\beta x_i^\alpha}}\}^{r_i}, \tag{2.27}$$

and

$$\pi_3^*(\lambda|\alpha,\beta,x,r) \propto \frac{\lambda^{m-1} e^{-m\lambda + \lambda \sum_{i=1}^{m} e^{-\beta x_i^\alpha}}}{(1-e^{-\lambda})^m} \prod_{i=1}^{m} \left\{ \frac{1 - e^{\lambda e^{-\beta x_i^\alpha}}}{1 - e^{\lambda}} \right\}^{r_i}. \tag{2.28}$$

The Bayes estimators of parameter α, β and λ are evaluated from the required sample of Equation (2.26), (2.27) and (2.28), generated by using MCMC procedure. The algorithm used for obtaining Bayes estimates and HPD credible intervals is given below:

I. Set α_0, β_0 and λ_0 be the initial guess of α, β and λ.

II. Set $i = 1$.

III. Generate α_i from $\pi_1^*(\alpha|\beta_{i-1},\lambda_{i-1},x,r)$, β_i from $\pi_2^*(\beta|\lambda_{i-1},\alpha_{i-1},x,r)$ and λ_i from $\pi_3^*(\lambda|\alpha_{i-1},\beta_{i-1},x,r)$ respectively.

IV. Repeat steps 2-3, N times.

V. Obtain the Bayes estimates of α, β and λ under GELF as $\left[E(\alpha^{-\delta}|x,r)\right]^{-\frac{1}{\delta}} = \left[\frac{1}{N-N_0}\sum_{i=1}^{N-N_0}\alpha_i^{-\delta}\right]^{-\frac{1}{\delta}}$, $\left[E(\beta^{-\delta}|x,r)\right]^{-\frac{1}{\delta}} = \left[\frac{1}{N-N_0}\sum_{i=1}^{N-N_0}\beta_i^{-\delta}\right]^{-\frac{1}{\delta}}$ and $\left[E(\lambda^{-\delta}|x,r)\right]^{-\frac{1}{\delta}} = \left[\frac{1}{N-N_0}\sum_{i=1}^{N-N_0}\lambda_i^{-\delta}\right]^{-\frac{1}{\delta}}$, where N_0 is the burn in period. Substituting $\delta = -1$ in step V, we get Bayes estimates of α, β and λ under SELF.

VI. For computing the highest posterior density (HPD) credible interval of α, β and λ. We order the MCMC sample values α, β and λ (say $\alpha_1,\alpha_2,\alpha_3,\cdots,\alpha_N$ as $\alpha_{(1)},\alpha_{(2)},\alpha_{(3)},\cdots,\alpha_{(N)}$, $\beta_1,\beta_2,\beta_3,\cdots,\beta_N$ as $\beta_{(1)},\beta_{(2)},\beta_{(3)},\cdots,\beta_{(N)}$ and $\lambda_1,\lambda_2,\lambda_3,\cdots,\lambda_N$ as $\lambda_{(1)},\lambda_{(2)},\lambda_{(3)},\cdots,\lambda_{(N)}$). Then construct all the 100(1-Ψ)% credible intervals of α, β and λ, say $\{(\alpha_{(1)},\alpha_{N[(1-\Psi)]+1}),\cdots,(\alpha_{[N\Psi]},\alpha_N)\},\{(\beta_{(1)},\beta_{N[(1-\Psi)]+1}),\cdots,(\beta_{[N\Psi]},\beta_N)\}$ & $\{(\lambda_{(1)},\lambda_{[N(1-\Psi)]+1}),\cdots,(\lambda_{[N\Psi]},\lambda_N)\}$ respectively. Where $[\eta]$ mentioned the largest integer less than or equal to η. Therefore, the HPD credible interval of α, β and λ is that interval which has the shortest length.

2.3 Bayes Prediction

In this Section, we have derived an expression for one sample Bayes prediction, if the experimenter is interested to know the lifetimes of the $(n-m)$ removed surviving units on the basis of observed sample. Let $Y_s = X_{m+s}, m < s \leq n$, represents the failure lifetime of the remaining units, then conditional distribution of $Y_{(s)}^{th}$ order statistics given PT-II CBRs sample **x** is given by, see Singh et al. (2013b)

$$f\left(y_{(s)}|x_{(m)},\alpha,\beta,\lambda\right) = \frac{(n-m)!\left[1-F\left(y_{(s)}\right)\right]^{n-m-s}}{(s-1)!(n-m-s)!\left[1-F\left(x_{(m)}\right)\right]^{n-m}}\left[F\left(y_{(s)}\right)-F\left(x_{(m)}\right)\right]^{s-1} f\left(y_{(s)}\right). \tag{2.29}$$

Substituting Equation (2.1) and Equation (2.2) in (2.29), we have

$$f\left(y_{(s)}|x_{(m)},\alpha,\beta,\lambda\right) = \alpha\beta y_{(s)}^{\alpha-1}\zeta(y_{(s)})log\left(\zeta(y_{(s)})\right)\frac{(n-m)!}{(s-1)!(n-m-s)!}$$
$$\left[\frac{1-\zeta(y_{(s)})}{1-\zeta(x_{(m)})}\right]^{n-m}\left[1-\zeta(y_{(s)})\right]^{-s}\left[\zeta(x_{(m)})-\zeta(y_{(s)})\right]^{s-1},$$

where, $\zeta(z) = e^{\lambda e^{-\beta z^{\alpha}}}$. One sample Bayes predictive density of $y_{(s)}^{th}$ ordered future sample can be obtained as follows

$$f\left(y_{(s)}|\mathbf{x}\right) = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} f\left(y_{(s)}|\mathbf{x},\alpha,\beta,\lambda\right)\pi\left(\alpha,\beta,\lambda|\mathbf{x}\right)d\alpha d\beta d\lambda$$

The above equation for $f\left(y_{(s)}|\mathbf{x}\right)$ cannot be expressed in closed form and hence it cannot be evaluated analytically. Therefore, MCMC techniques is proposed to be used for obtaining the approximate solution of the above predictive density.

$\{(\alpha_i,\beta_i,\lambda_i)\,; i = 1,2,\cdots,N-N_0\}$ obtained from $\pi\left(\alpha,\beta,\lambda|\mathbf{x}\right)$ using Gibbs sampling can be utilized to obtain the consistent estimate of $f\left(y_{(s)}|\mathbf{x}\right)$. It can be obtained by

$$f\left(y_{(s)}|\mathbf{x}\right) = \frac{1}{N-N_0}\sum_{i=1}^{N-N_0} f\left(y_{(s)}|\alpha_i,\beta_i,\lambda_i\right). \tag{2.30}$$

Thus, we can obtain the two-sided $100(1-\psi)\%$ prediction interval (l,u) for future sample by solving the following two equations:

$$P\left(Y_{(s)} > u|\mathbf{x}\right) = \frac{\psi}{2} \text{ and } P\left(Y_{(s)} > l|\mathbf{x}\right) = 1-\frac{\psi}{2}.$$

We are facing difficulties to obtain the explicit solution. Therefore, we need to apply as per required numerical technique for the purpose of solution of non-linear equations. Also we opted that an alternative method is MCMC discussed by Chen and Shao (1998), in the following way: Let $\left(y_{(i:s)}\right); i = 1,2,\cdots,N-N_0$ be the corresponding ordered MCMC sample of $\left(y_{i:s}\right); i = 1,2,\cdots,N-N_0$ from Equation (2.30). Then, the $100(1-\psi)\%$ HPD intervals for $y_{(s)}$ is $y_{(j*:s)}$, $y_{j*+[(1-\psi)M]:s}$, where $j*$ is chosen so that

$$y_{j*+[(1-\psi)N-N_0]:s} - y_{(j*:s)} = \min_{1\leq j\leq N-N_0-[(1-\psi)N-N_0]}\left[y_{j*+[(1-\psi)N-N_0]:s} - y_{(j*:s)}\right].$$

For considered real data set, we calculated the mean and 95% credible intervals (predictive bounds) for future samples using one sample prediction technique. The results are summarized in Table (2.4).

2.4 Expected Experiment Time

Cost is an very effective element in an experiment that is directly related to the time of experiment. Therefore, for a proper planning of the experimentation one is always interested in knowing the expected experiment time; which can be defined PT-II CBRs

$$E[X_m] = E_R[E[X_m|R=r]] \tag{2.31}$$

$$= \sum_{r_1=0}^{g(r_1)} \sum_{r_2=0}^{g(r_2)} \dots \sum_{r_{m-1}=0}^{g(r_{m-1})} p(R,p)E[X_{m:m:n}|R=r].$$

Where $g(r_i) = n-m-r_1-\dots-r_{i-1}$ and $p(R=r;p)$ is given in Equation (2.9). Conditioning on R the expected experiment time is

$$E[X_m|R] = \int_0^\infty x f_{X_m}(x)dx,$$

where, $f_{X_{(m)}} = C_{m-1} f(x) \sum_{j=1}^{m} a_{j,m}(1-F(x))^{\gamma_j}, 1 \le m \le n$ and $c_{m-1} = \prod_{i=1}^{m} \gamma_i, 1 \le m \le n$ and $a_{j,m} = \prod_{i=1}^{m} \frac{1}{\gamma_i - \gamma_j}; i \ne j, 1 \le j \le m \le n$. For more details about the procedure of evaluation of conditional expectation of X_m for given R, see Balakrishnan and Aggarwala (2000), Singh et al. (2013b), Tse et al. (2000). Using the suggested procedure, expected experiment times under PT-II CBRs are computed for different combinations of m and n listed in Table (2.1). The values of p, considered here are $0.1, 0.3, 0.5, 0.7$ and 0.9 while model parameters α, β and λ are arbitrarily taken as $1, 2$ and 2 respectively. The results obtained are summarized below

TABLE 2.1: Expected Experiment time $E[X_m]$ under PT-II CBRs.

n	m	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$
	10	0.15660	0.57392	0.86262	0.93033	0.96077
	15	0.37201	0.99414	1.09666	1.13149	1.13086
30	20	0.76695	1.23138	1.27059	1.30419	1.25337
	25	0.93638	1.35836	1.37069	1.34487	1.35232
	30	1.47650	1.45380	1.49828	1.45718	1.47108
	10	0.28018	0.73055	0.91508	0.95055	0.95818
20	15	0.71918	1.10303	1.13051	1.15929	1.14585
	20	1.27924	1.28742	1.27601	1.28292	1.28714
	3	0.08832	0.12157	0.19727	0.31295	0.43709
	4	0.13669	0.22246	0.36903	0.50305	0.58098
10	6	0.29297	0.51585	0.67842	0.75186	0.76813
	10	0.99404	0.98307	0.97925	0.98829	0.99048

Now we can obtain ratio of the expected experiment time (REET) between PT-II CBRs and the complete sampling as

$$REET = \frac{E[X_m] \;\; under \;\; PT-II \; CBRs}{E[X_n] \;\; under \;\; complete \;\; sampling}. \tag{2.32}$$

It may be noted that REET indicates the reduction in experiment time. Figure (2.1) shows REET for various values of n for $m = 10$ and different removal probability $p = 0.1, 0.3, 0.5, 0.7$ and 0.9. It can be seen from the Figure that for each values of p, the REET decreases as n increases. It may be, noted that for larger value of (> 0.5) and larger $n(> 25)$; the values of REET do not change for change in the value of p. For $p \leq 0.5$ and moderate sample size (25) larger valuers of REET is noted for smaller valuers of p.

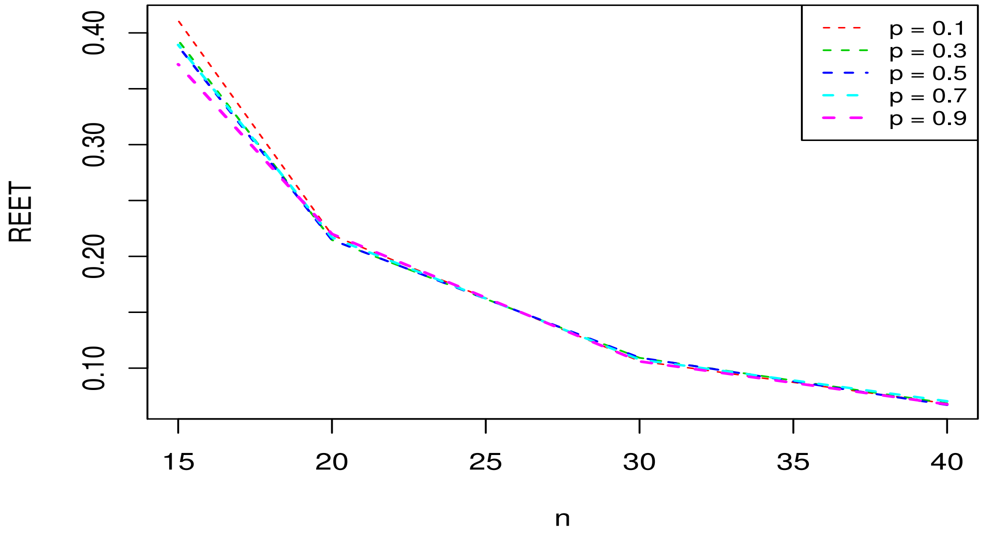

FIGURE 2.1: REET under PT-II CBRs to under complete sample.

2.5 Monte Carlo Simulation Study and Comparison of Estimators

We have seen above that proposed estimators are not obtained in the closed form; therefore, an analytical study of behavior of the estimators is not possible and we propose to study it numerically. For this purpose, we suggest the use of MCMC technique as suggested by Tierney (1994) also, for the calculation of risk (average loss over sample space) of estimators of the parameters α, β and λ. Hence, samples are generated from specified WPD and PT-II CBRs samples are obtained from these. ML estimator along with Bayes estimators under SELF and GELF are calculated. The ML estimators are denoted as; $\hat{\alpha}_M, \hat{\beta}_M$, $\hat{\lambda}_M$ where as $\hat{\alpha}_S, \hat{\beta}_S, \hat{\lambda}_S$ and $\hat{\alpha}_G, \hat{\beta}_G$, $\hat{\lambda}_G$ denote SELF and GELF estimates of the parameters α, β and λ, respectively. Similarly, $(\alpha_L^c, \alpha_U^c), (\beta_L^c, \beta_U^c), (\lambda_L^c, \lambda_U^c)$ and $(\alpha_L^h, \alpha_U^h), (\beta_L^h, \beta_U^h), (\lambda_L^h, \lambda_U^h)$ indicate $100(1-\Psi)\%$ CI and HPD credible intervals. Risk are estimated on the basis of 8000 samples. Since risk of the estimators under PT-II CBRs will be function of $n, m, p, \alpha, \beta, \lambda$, δ, a and b. The choice of hyper parameter are made by assuming that the prior information about the parameter is available in the form of its expected value μ and its variance σ^2 reflecting the confidence in expected value. Thus a and b are calculated from equations, which can be taken in such a way that if we consider any two independent pieces of information as prior mean and variance of β are $\mu = \frac{b}{a}$ and $\sigma^2 = \frac{b}{a^2}$, where μ is taken as true values of the parameter β and smaller, moderate and large values of variances namely $0.5, 1$ and 5 which gave $(a = 4,\ b = 8)$, $(a = 2,\ b = 4)$ and $(a = 0.4,\ b = 0.8)$ respectively. We vary the effective samples size $m = 10[5]30$. The value of α, β and λ are arbitrarily taken as $1,\ 2$ and 2 respectively. The value of loss parameter δ is taken as 1.5 for o.e. to be more serious than u.e. and see Singh et al. (2011). After an extensive study of results thus obtained, conclusions are drawn regarding the behavior of the estimators. It may be mention here that the space restriction, results of various variation in the parameters are not shown. Only selected Figures are included.

2.6 Discussion of Results

We shall discuss the impact of variation of effective sample size m under PT-II CBRs, and compare the risks of all estimators of α, β and λ, obtained under GELF with the corresponding Bayes estimators under SELF and ML estimator. We observed, the risks of all the estimators of α, β and λ decrease as effective sample observations m increases. The risks of $(\hat{\alpha}_G, \hat{\beta}_G)$ and $(\hat{\alpha}_S, \hat{\beta}_S)$ are found to be close respectively to each other for all the considered situations. A similar trend is observed for $\hat{\lambda}_G$ and $\hat{\lambda}_S$ also. It is further observed that, in general, the risks of the estimators under SELF and GELF decreases, as for $\delta = +1.5$ and $\delta = -1.5$ with each prior belief of the parameter β (see Figure $(2.2-2.4)$). For large number of effective sample sizes, the difference between the risks of the estimators are less. The decrease in the risks is more for $\hat{\alpha}_M$ as compared to the other estimators. For almost all values of prior belief of the parameter β and δ, the risk of $\hat{\alpha}_G$ under GELF is found to be least among the considered estimators. It is also interesting to remark here that $\hat{\alpha}_G$ has the least risk under SELF. For positive values of δ, the behavior of risks of estimators under GELF is more or less similar to the one obtained for negative δ (see Figure (2.2 – 2.4)).

Similarly, we have studied the risks of Bayes estimators β and λ respectively under SELF and GELF based on PT-II CBRs. The trend remains more or less the same as stated above under both loss functions see results in graphs, which has shown in supplementary material. Further we observed that the risk of $\hat{\beta}_G$ and $\hat{\lambda}_G$ under GELF and SELF are found to be least among the considered estimators respectively.

The Figure (2.5) shows the CI/HPD credible intervals for α. It may also noted, average CL of CI/HPD credible intervals consistently narrow down as m increases. The HPD credible intervals are better than CIs in respect of average CL. While studying the effect of large effective sample sizes m, the difference of average CL between the CIs and HPD credible intervals are negligibly small. For β and λ also, the trend of CI/HPD credible intervals, is similar to that of α. Due to space restriction, results for variations in m of CI/HPD credible intervals of β and λ are not shown here. The CI/HPD credible intervals of β and λ are given in supplementary material. Thus, we can not deny from the fact that estimates under Bayesian are more precise and accurate than ML estimates.

We also discussed the expected time to test and shown in Table (2.1), it is meaningful to comment that as the value p and m increase the expected time to test also increases. It is also observed that for fixed m, if increases the value of the sample size i.e., n, the expected time to test decreases.

2.7 An application to Bladder Cancer Data

For the application purpose, we have taken a real data set given by Lee and Wang (2003). It contains a set of remission times (in months) related to 137 cancer patients, and some patients are not present in the follow-up. The remission time in months are a subset of the data from a bladder cancer study. We have considered here a random set of 128 observations from it which are given follow: 4.50, 32.15, 3.88, 13.80, 19.13, 4.87, 5.85, 14.24, 5.71, 7.09, 7.87, 7.59, 20.28, 5.32, 5.49, 3.02, 46.12, 2.02, 4.51, 5.17, 2.83, 9.22, 1.05, 0.20, 8.37, 3.82, 9.47, 36.66, 14.77, 26.31, 79.05, 10.06, 8.53, 2.02, 4.98, 11.98, 2.62, 4.26, 5.06, 1.76, 0.90, 11.25, 16.62, 4.40, 21.73, 10.34, 12.07, 34.26, 10.66, 6.97, 2.07, 0.51, 12.03, 0.08, 17.12, 3.36, 2.64, 1.40, 12.63, 43.01, 14.76, 2.75, 7.66, 0.81, 1.19, 7.32, 4.18, 3.36, 8.66, 1.26, 13.29, 1.46, 14.83, 6.76, 23.63, 5.62, 3.25, 18.10, 7.62, 7.63, 17.14, 25.74, 3.52, 2.87, 15.96, 17.36, 9.74, 3.31, 7.28, 1.35, 0.40, 2.26, 4.33, 9.02, 5.41, 2.69, 22.69, 6.94, 2.54, 11.79, 2.46, 7.26, 2.69, 5.34, 3.48, 8.26, 6.93, 4.23, 3.70, 0.50, 10.75, 6.54, 3.64, 5.32, 13.11, 8.65, 3.57, 5.09, 7.39, 5.41, 11.64, 2.09, 2.23, 6.25, 7.93, 4.34, 25.82, 12.02.

First of all, we checked the suitability of WPD to the above said data and compared, some specified lifetime models; Exponential Poisson (EP) and Weibull distribution. For testing the goodness of fit we used the method based on ML function, the K-S distance, the AIC, proposed by Akaike (1978), BIC proposed by Schwarz et al. (1978). The best distribution is that which has the lowest -log-L, AIC, BIC and K-S statistic and corresponding highest p values. Further, we have used a goodness of fit of distributions. We draw a Q-Q plots for the said three lifetime distribution and are shown in the Figure (2.14). A Q-Q plot shows the points $\left\{F^{-1}\left(\frac{i-0.5}{n};\hat{\Theta}_M,x_{(i)}\right)\right\}, i=1,2,3,\cdots,n$, where $\hat{\Theta}_M$ is the ML estimates of the parameters of lifetime model. The values of ML estimates of the parameters of the considered lifetime models, -log-L, AIC, BIC, K-S statistic and their associated p values are reported in Table (2.2).

TABLE 2.2: The -log-L, K-S, p-value and the AIC and BIC values for the W), EP and Weibull fitted distributions.

	Estimates	-log-L	K-S	p-value	AIC	BIC
WP(α,β,λ)	(1.26853,0.01629,4.26518)	-410.189	0.046875	0.99896	826.3782	834.9343
EP(β,λ)	(0.106371,0.0000047)	-414.343	0.078125	0.82955	834.6856	843.2417
Weibull(α,β)	(1.04784,0.09389)	-414.087	0.0703125	0.90972	834.1738	842.7298

This Table shows that WPM provide better fit than EP and Weibull distribution. Further, we tested the hypothesis: $H_{01}:\alpha=1$ (Data follow Exponential Poisson) vs $H_{11}:\alpha\neq 1$ (Data follow Weibull Poisson) and $H_{02}:\lambda=0$ (Data follow Weibull) vs $H_{12}:\lambda\neq 0$ (Data follow Weibull Poisson), using the large sample test described in Subsection (2.2.1). The value of the test statistic R_1 and R_2 are obtained as 8.30737 and 7.79551 respectively. Which reject H_{01} and H_{02}.

Now for the purpose of illustrating the method discussed in this chapter, PT-II CBR samples are generated from this data set under different schemes. The number of removals are shown in Table (2.3) under different schemes. The ML estimates of parameter α, β and λ are used to compute by EM algorithm. The initial value of parameters are chosen through contour plots of parameters, and their corresponding log-L are plotted; using R software (Figure (2.16)).

As we have no prior information about the parameter β, and we use non informative prior for which the hyper parameter of β is taken to be $(a = 0 : 000001; b = 0 : 000001)$. When implementing MCMC algorithm, the values of ML estimates are used as initial guess and CUMSUM plots are plotted, and to verified the convergence of Markov chain. Then, we evaluate Bayes estimates and HPD intervals using the formulae given in previous Section (2.3) under different censoring schemes based on Table (2.3), the Bayes estimate of parameter α, β and λ under SELF and GELF for $\delta = \pm 1.5$ are presented in Table (2.5). It may be observed from Table (2.5) that various parameter estimates, obtained using PT-II CBRs, are quite close to those obtained under complete samples.

TABLE 2.3: PT-II CBR samples under different censoring scheme $(S_{n:m})$ for fixed $n = 128, p = 0.5$.

	$S_{128:64}$		$S_{128:77}$				$S_{128:102}$			
i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i
1	0.08	23	0.08	18	20.28	0	0.08	7	10.34	0
2	2.69	17	2.26	9	21.73	0	1.19	4	10.66	0
3	4.23	7	3.02	8	22.69	0	1.76	3	10.75	0
4	4.98	2	3.7	6	23.63	0	2.09	4	11.25	0
5	5.17	3	4.34	2	25.74	0	2.62	0	11.64	0
6	5.41	1	4.51	3	25.82	0	2.64	2	11.79	0
7	5.49	5	5.09	2	26.31	0	2.75	2	11.98	0
8	6.76	4	5.32	1	32.15	0	3.02	0	12.02	0
9	7.26	0	5.41	1	34.26	0	3.25	2	12.03	0
10	7.28	1	5.49	0	36.66	0	3.36	0	12.07	0

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Table 2.3 – *Continued from previous page*

	$S_{128:64}$		$S_{128:77}$				$S_{128:102}$			
i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i
11	7.39	1	5.62	0	43.01	0	3.48	1	12.63	0
12	7.62	0	5.71	0	46.12	0	3.57	0	13.11	0
13	7.63	0	5.85	0	79.05	0	3.64	0	13.29	0
14	7.66	0	6.25	1			3.7	1	13.8	0
15	7.87	0	6.76	0			3.88	0	14.24	0
16	7.93	0	6.93	0			4.18	0	14.76	0
17	8.26	0	6.94	0			4.23	0	14.77	0
18	8.37	0	6.97	0			4.26	0	14.83	0
19	8.53	0	7.09	0			4.33	0	15.96	0
20	8.65	0	7.26	0			4.34	0	16.62	0
21	8.66	0	7.28	0			4.4	0	17.12	0
22	9.02	0	7.32	0			4.5	0	17.14	0
23	9.22	0	7.39	0			4.51	0	17.36	0
24	9.47	0	7.59	0			4.87	0	18.1	0
25	9.74	0	7.62	0			4.98	0	19.13	0
26	10.06	0	7.63	0			5.06	0	20.28	0
27	10.34	0	7.66	0			5.09	0	21.73	0
28	10.66	0	7.87	0			5.17	0	22.69	0
29	10.75	0	7.93	0			5.32	0	23.63	0
30	11.25	0	8.26	0			5.32	0	25.74	0
31	11.64	0	8.37	0			5.34	0	25.82	0
32	11.79	0	8.53	0			5.41	0	26.31	0
33	11.98	0	8.65	0			5.41	0	32.15	0
34	12.02	0	8.66	0			5.49	0	34.26	0
35	12.03	0	9.02	0			5.62	0	36.66	0
36	12.07	0	9.22	0			5.71	0	43.01	0
37	12.63	0	9.47	0			5.85	0	46.12	0
38	13.11	0	9.74	0			6.25	0	79.05	0
39	13.29	0	10.06	0			6.54	0		
40	13.8	0	10.34	0			6.76	0		
41	14.24	0	10.66	0			6.93	0		
42	14.76	0	10.75	0			6.94	0		
43	14.77	0	11.25	0			6.97	0		
44	14.83	0	11.64	0			7.09	0		

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Table 2.3 – *Continued from previous page*

	$S_{128:64}$		$S_{128:77}$				$S_{128:102}$			
i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i
45	15.96	0	11.79	0			7.26	0		
46	16.62	0	11.98	0			7.28	0		
47	17.12	0	12.02	0			7.32	0		
48	17.14	0	12.03	0			7.39	0		
49	17.36	0	12.07	0			7.59	0		
50	18.1	0	12.63	0			7.62	0		
51	19.13	0	13.11	0			7.63	0		
52	20.28	0	13.29	0			7.66	0		
53	21.73	0	13.8	0			7.87	0		
54	22.69	0	14.24	0			7.93	0		
55	23.63	0	14.76	0			8.26	0		
56	25.74	0	14.77	0			8.37	0		
57	25.82	0	14.83	0			8.53	0		
58	26.31	0	15.96	0			8.65	0		
59	32.15	0	16.62	0			8.66	0		
60	34.26	0	17.12	0			9.02	0		
61	36.66	0	17.14	0			9.22	0		
62	43.01	0	17.36	0			9.47	0		
63	46.12	0	18.1	0			9.74	0		
64	79.05	0	19.13	0			10.06	0		

TABLE 2.4: Mean and 95 % predictive bounds for future ordered observations from the bladder cancer data set.

One sample prediction			
s	Mean	Bounds	
		l	u
1	79.04829	77.18001	80.46525
2	79.42236	78.31463	80.52509
3	79.59276	78.47721	80.69601
4	79.89351	78.78346	81.01349

TABLE 2.5: Bayes and ML estimates, CI/HPD interval for WPD parameters α, β and λ with pre-defined censoring schemes for the bladder cancer data set.

Scheme	Parameter	MLE	SELF	GELF		CI		HPD	
$S_{n:m}$				$\delta = -1.5$	$\delta = 1.5$	θ_L^C	θ_U^C	θ_L^h	θ_U^h
$S_{128:64}$	α	1.752554	1.753276	1.753308	1.753111	1.722654	1.782455	1.723427	1.78316
	β	0.001915	0.015308	0.016975	0.00012	3.52E-07	0.031815	4.53E-07	0.034268
	λ	4.664404	4.664876	4.665002	4.664246	4.569793	4.759014	4.571824	4.760873
$S_{128:77}$	α	1.690516	1.708283	1.709049	1.704437	1.550406	1.830627	1.569133	1.853984
	β	0.002777	0.080686	0.087632	0.000115	1.26E-07	0.469512	1.37E-07	0.168236
	λ	4.652349	4.671151	4.674187	4.655874	4.185615	5.119084	4.209607	5.1385
$S_{128:102}$	α	1.523263	1.511028	1.513516	1.498465	1.279468	1.767057	1.267033	1.742682
	β	0.006193	0.006015	0.006602	0.000115	4.43E-07	0.013385	8.43E-07	0.012561
	λ	4.499809	4.739747	4.982471	0.314773	1.033094	7.966523	3.946512	6.012466

2.8 Conclusion

On the basis of the discussion of results given in the previous Section, we may conclude that the proposed estimators $\hat{\alpha}_G, \hat{\beta}_G$ and $\hat{\lambda}_G$ perform better than all other considered competitive estimators, for $(\delta > 0)$ i.e., when o.e. is more serious than u.e. and for $(\delta < 0)$, when u.e. is more serious than o.e., under both the loss functions. Thus, the use of the proposed estimator $\hat{\alpha}_G, \hat{\beta}_G$ and $\hat{\lambda}_G$ are recommended under SELF and GELF. Moreover, a brief study has done on the expected experiment time by taking the various combinations of effective parameters n, p and m and it observed that on increases the value of p and m, the expected time to test increases. While, for fixed m, on increases the value of n, the expected time to test decreases. The LR test has performed the goodness of fit. The one sample Bayes prediction has also presented. Furthermore, a real data set is fitted to show the practical applicability of WPD.

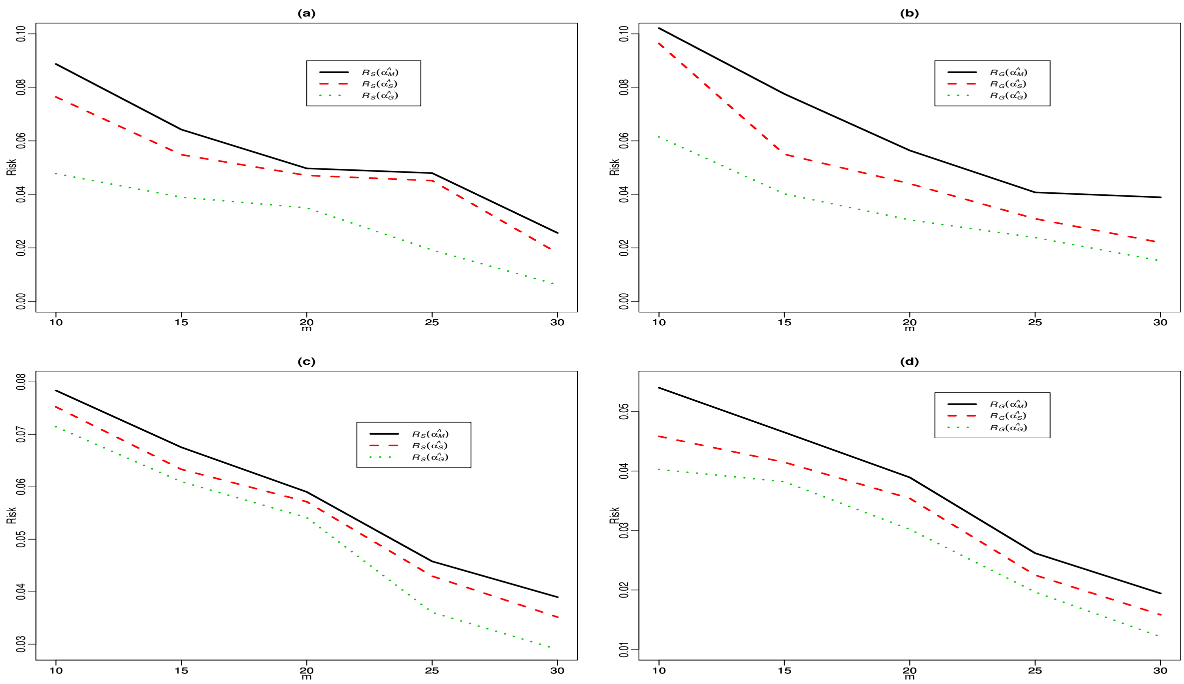

FIGURE 2.2: Risks for the estimators of parameter α for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with small prior variance, $\beta = 0.5$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

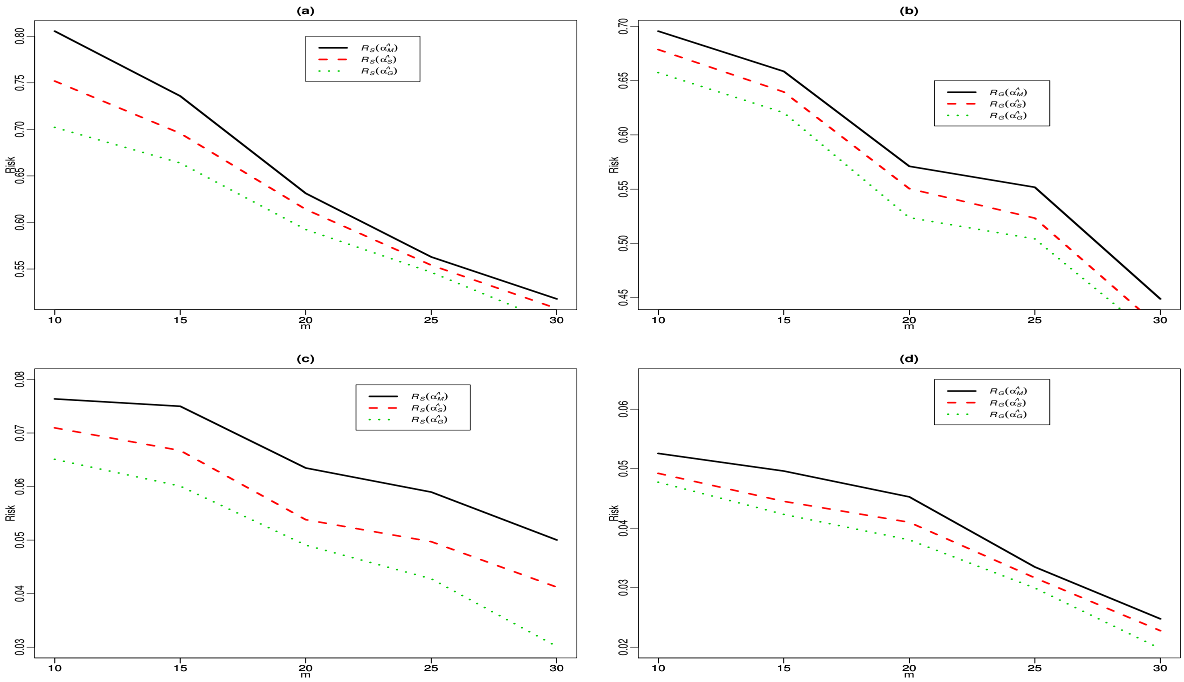

FIGURE 2.3: Risks for the estimators of parameter α for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with moderate prior variance, $\beta = 1$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

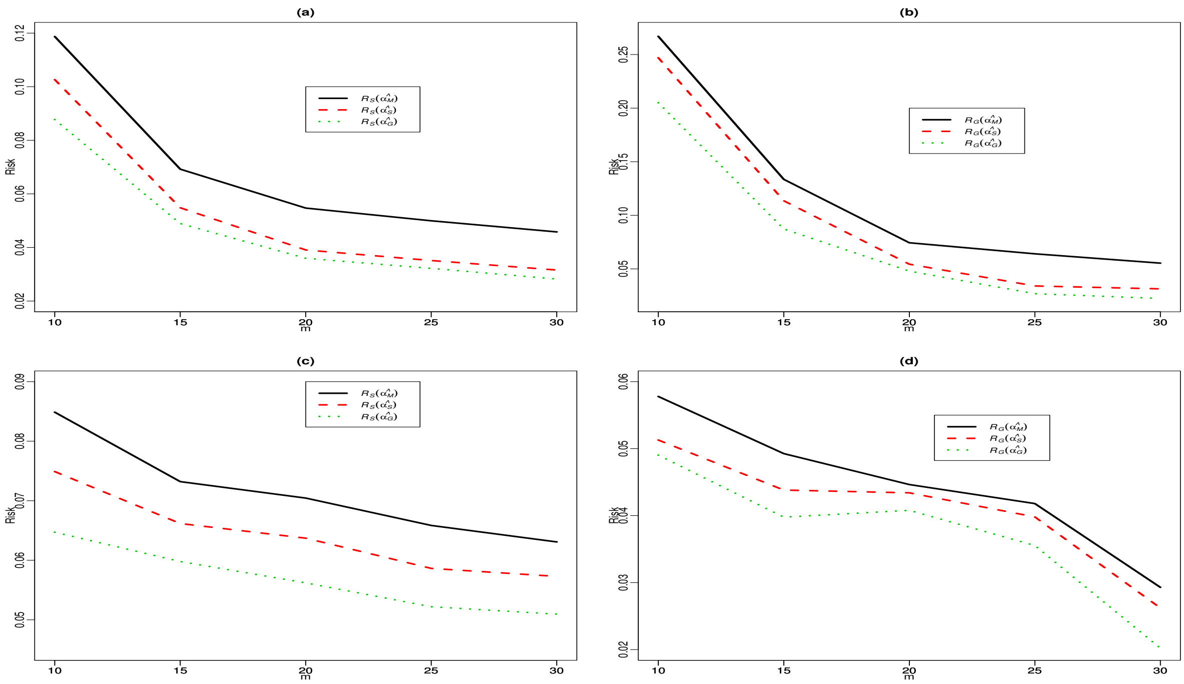

FIGURE 2.4: Risks for the estimators of parameter α for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with high prior variance, $\beta = 5$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

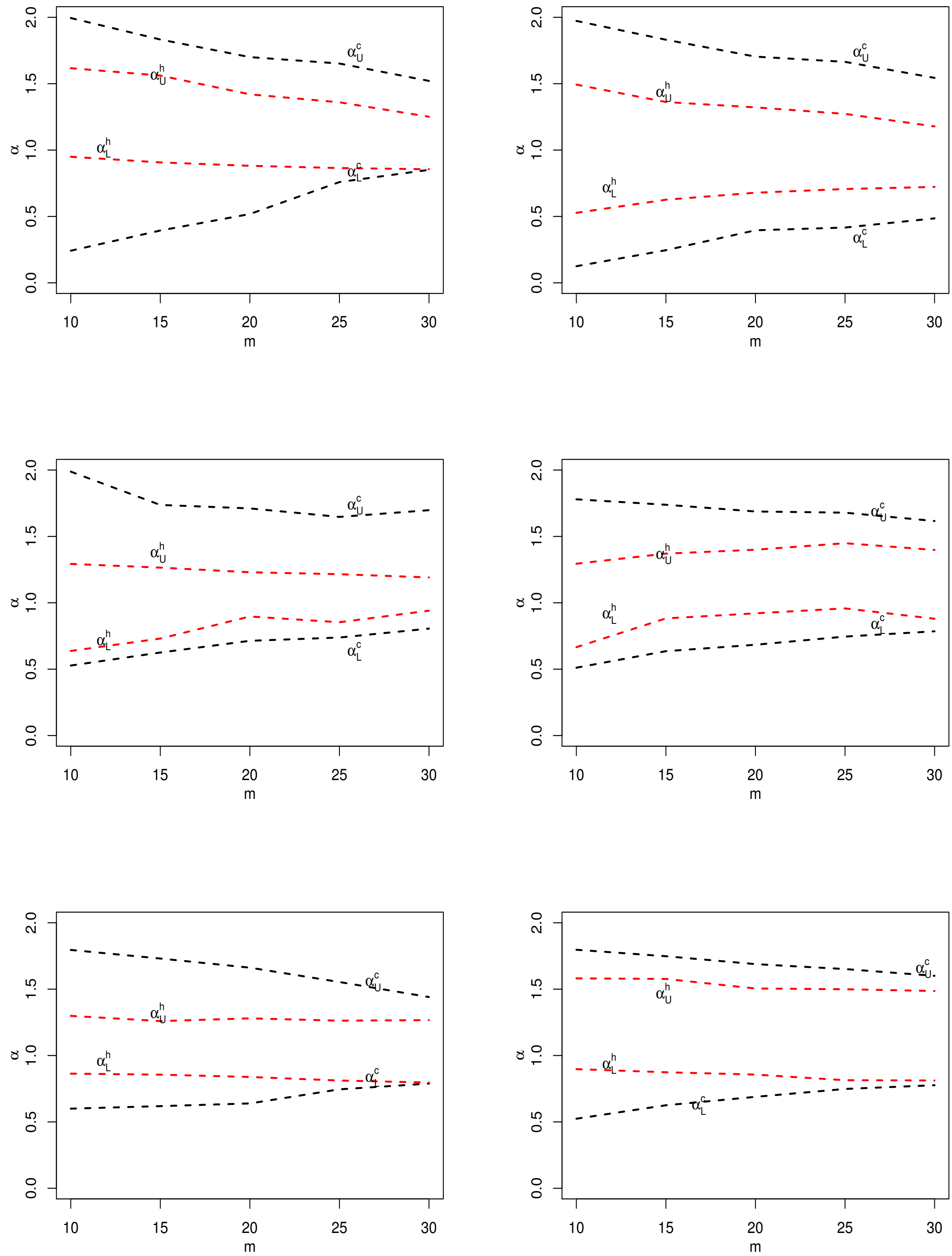

FIGURE 2.5: The CI and HPD intervals for α when prior variance is $0.5, 1$ and 5 with left panel: $\delta = 1.5$; right panel: $\delta = -1.5$, respectively.

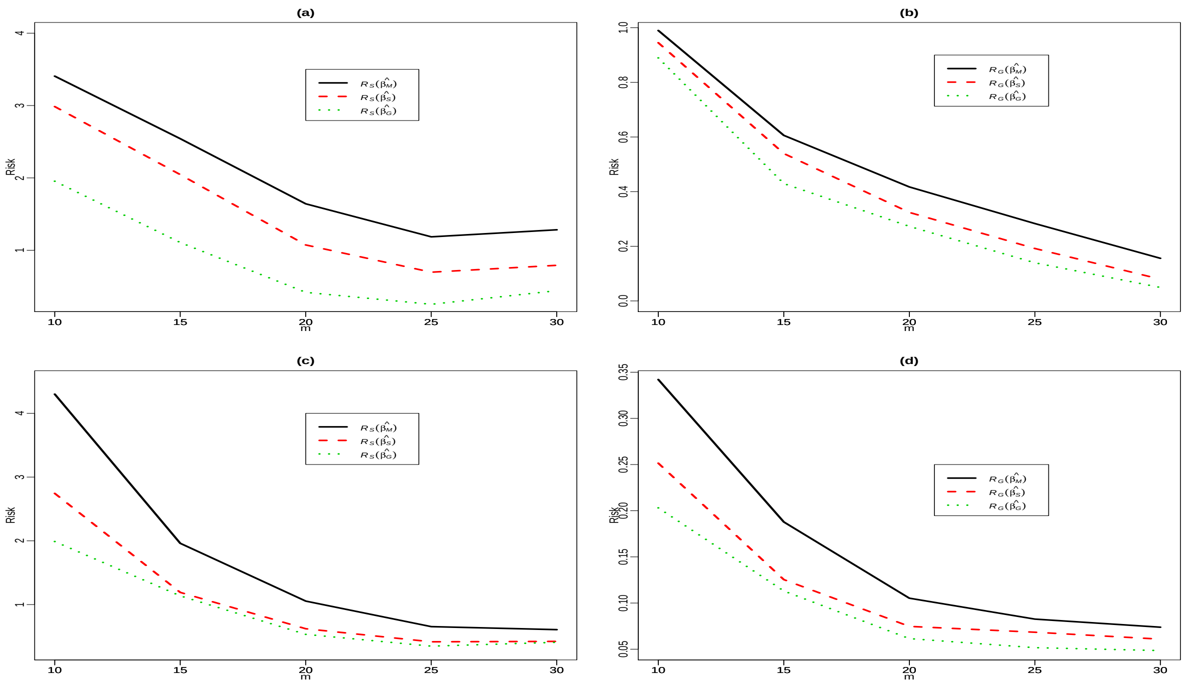

FIGURE 2.6: Risks for the estimators of parameter β for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with small prior variance, $\beta = 0.5$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

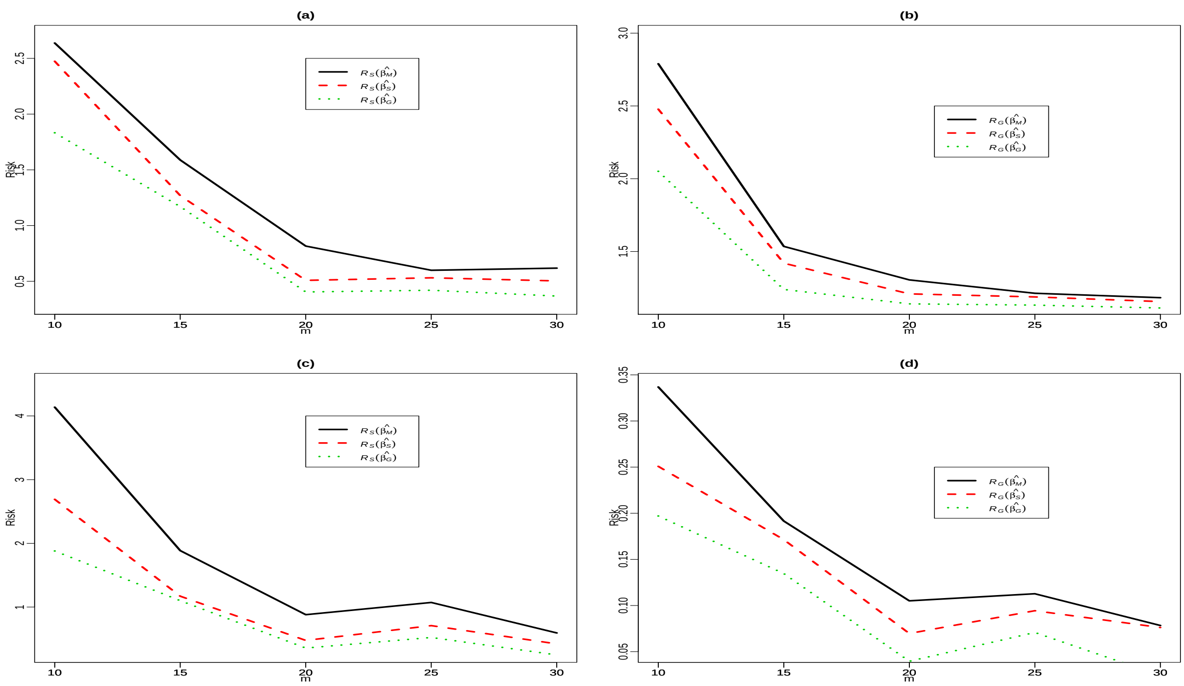

FIGURE 2.7: Risks for the estimators of parameter β for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with moderate prior variance, $\beta = 1$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

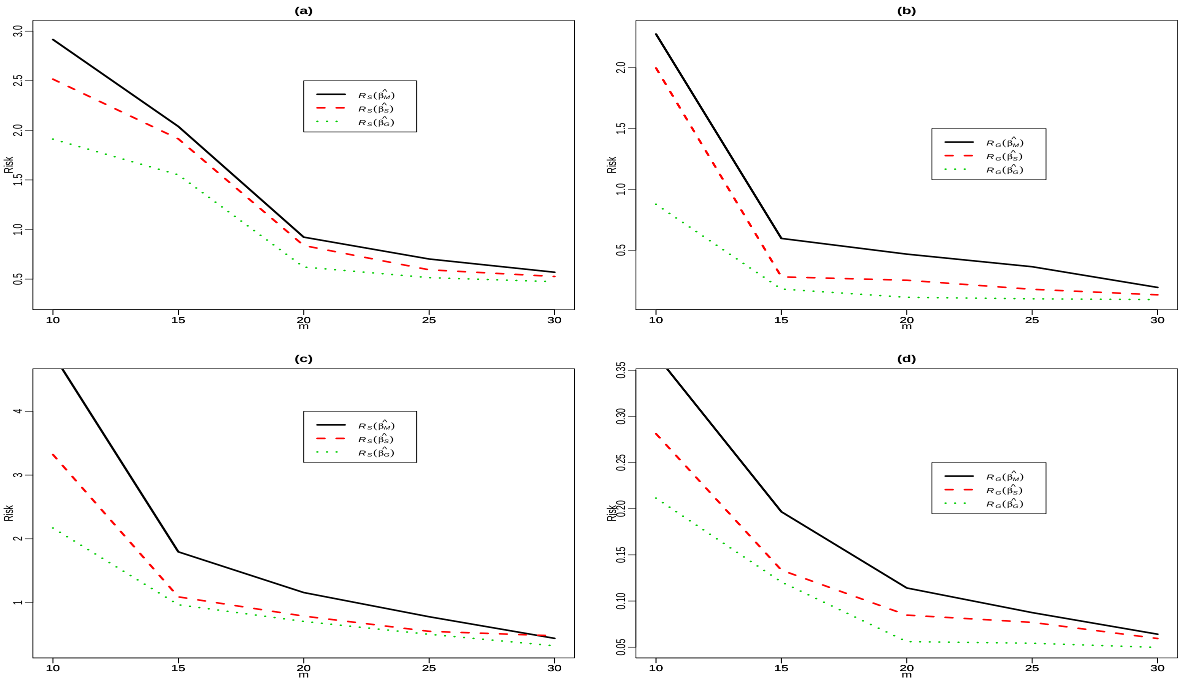

FIGURE 2.8: Risks for the estimators of parameter β for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with high prior variance, $\beta = 5$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

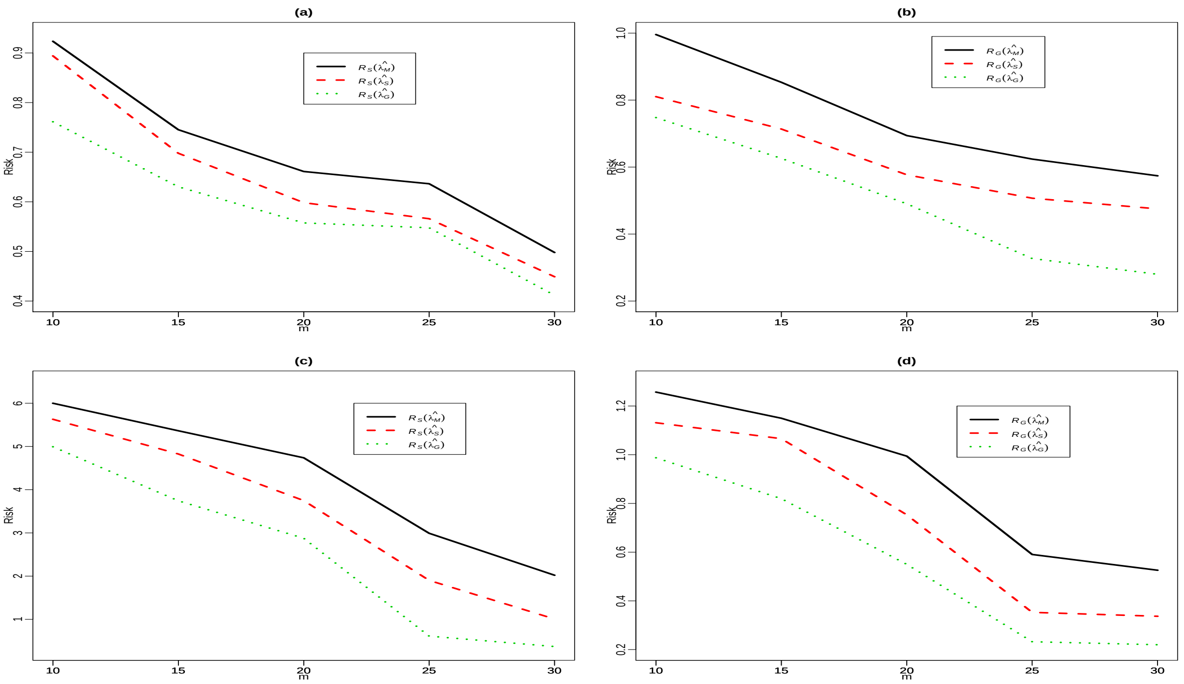

FIGURE 2.9: Risks for the estimators of parameter λ for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with small prior variance, $\beta = 0.5$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

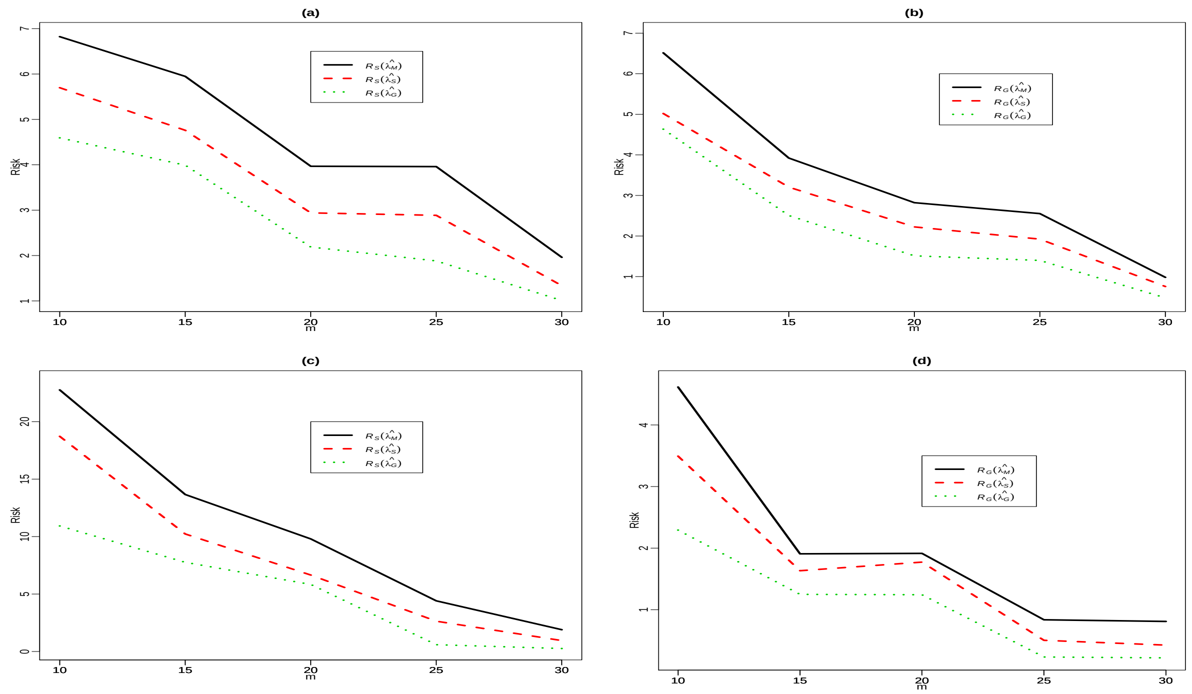

FIGURE 2.10: Risks for the estimators of parameter λ for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with moderate prior variance, $\beta = 1$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

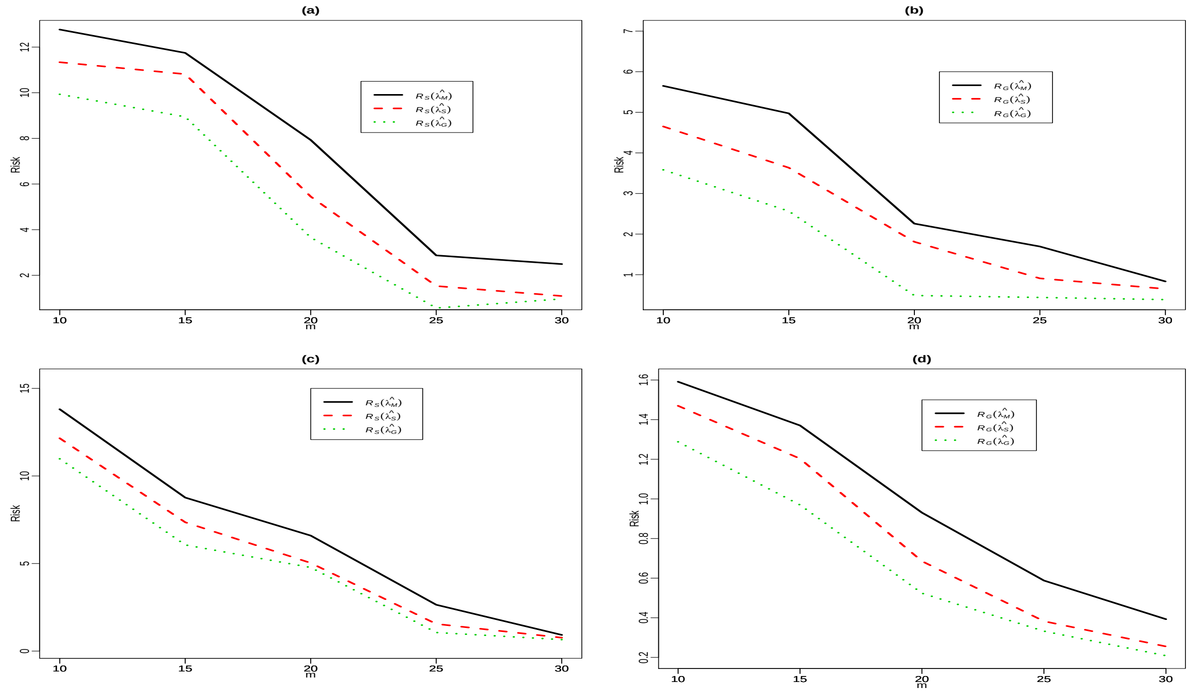

FIGURE 2.11: Risks for the estimators of parameter λ for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with high prior variance, $\beta = 5$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

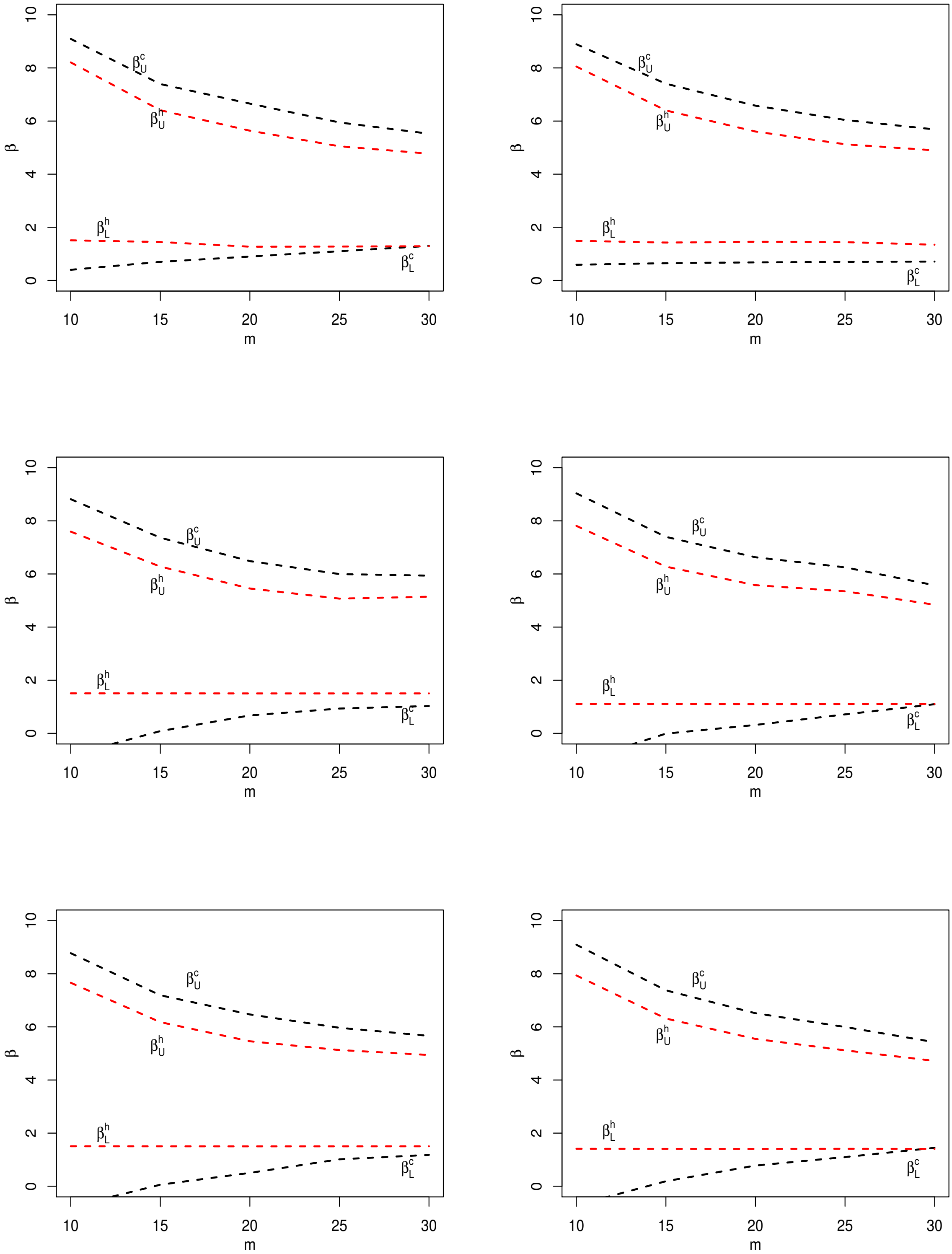

FIGURE 2.12: The CI and HPD intervals for β when prior variance is $0.5, 1$ and 5 with left panel: $\delta = 1.5$; right panel: $\delta = -1.5$, respectively.

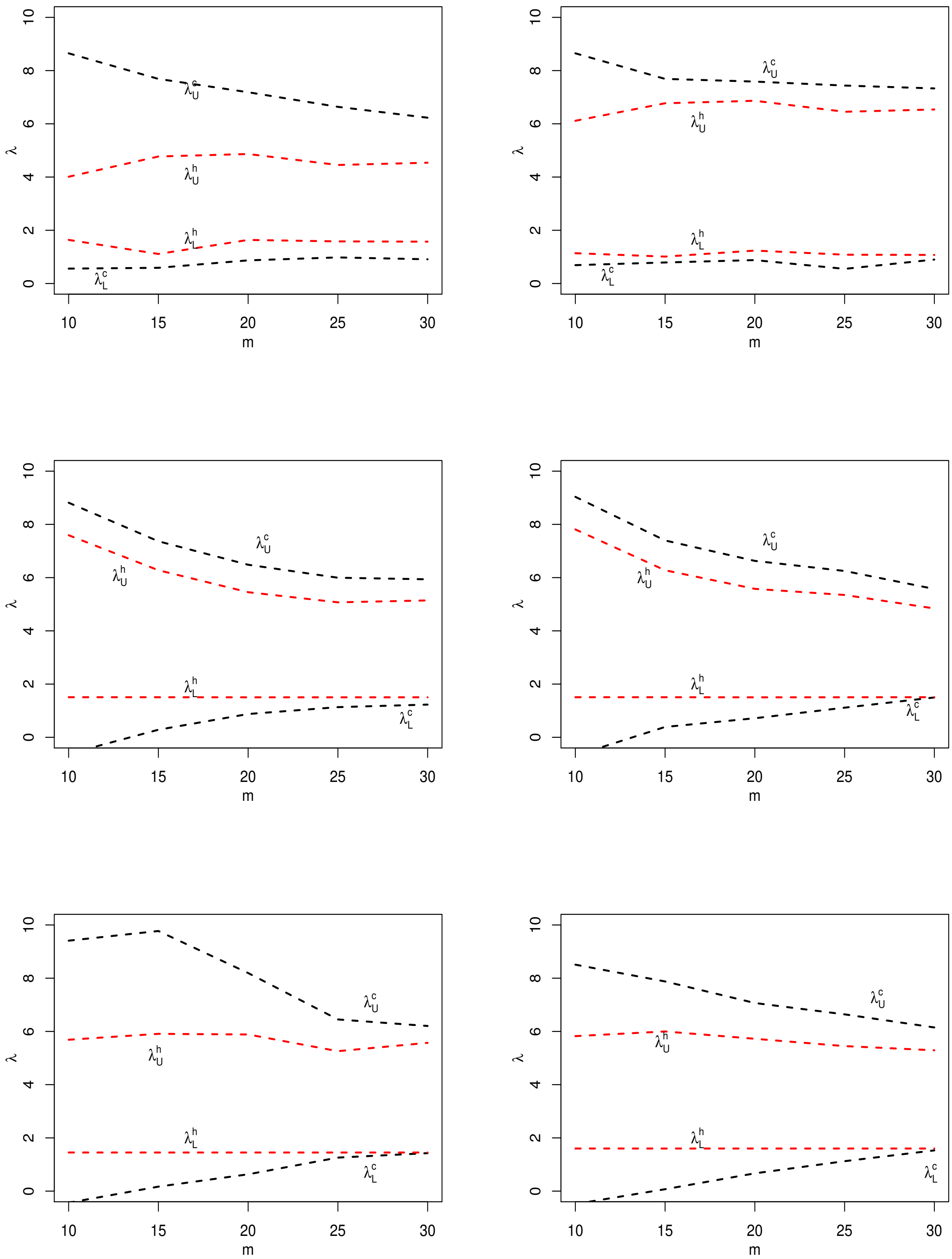

FIGURE 2.13: The CI and HPD interval for λ when prior variance is $0.5, 1$ and 5 with left panel: $\delta = 1.5$; right panel: $\delta = -1.5$, respectively.

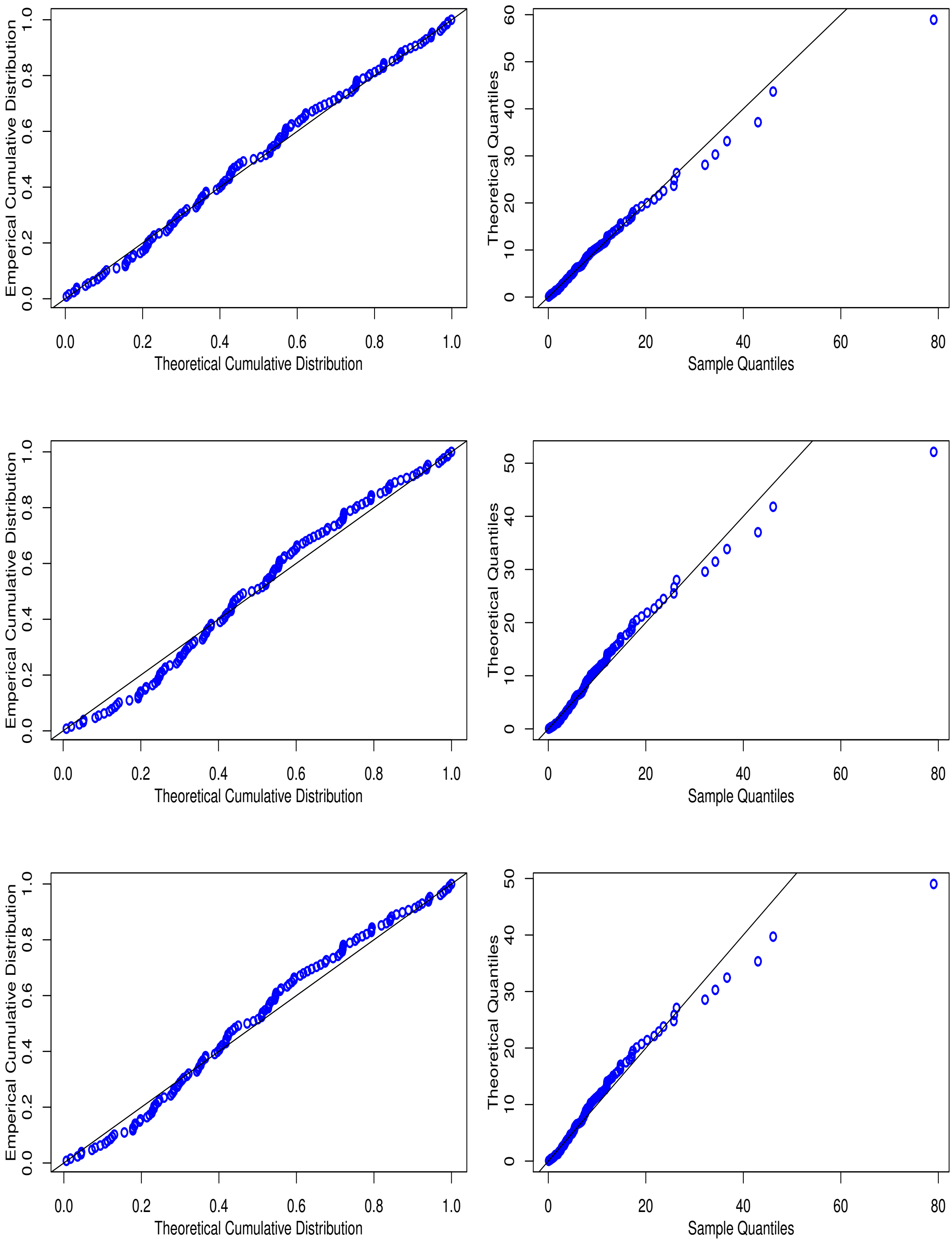

FIGURE 2.14: Top row: WP, Middle row: EP, Last row: WD shows the P-P and Q-Q plot for bladder cancer data set.

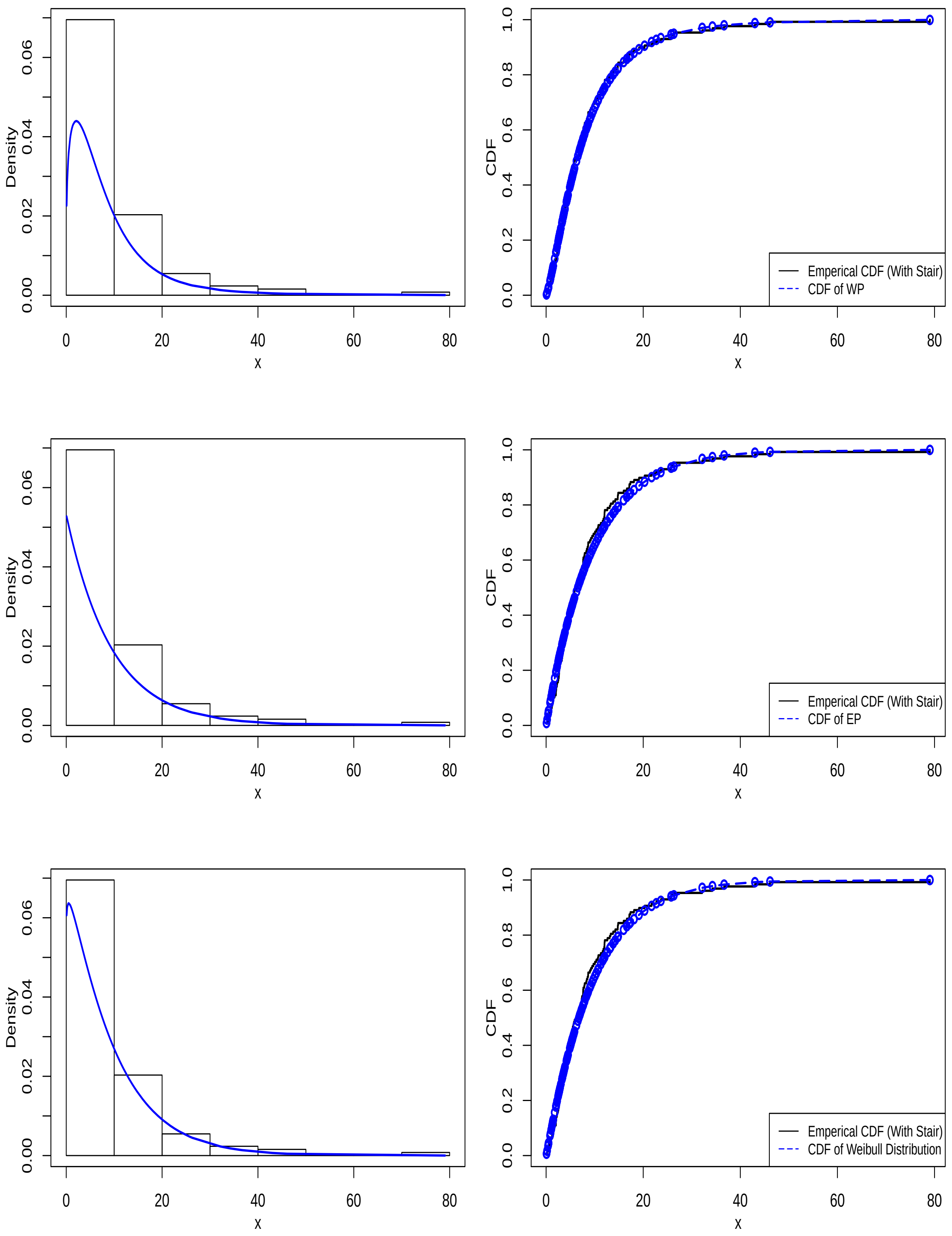

FIGURE 2.15: Top row: WP, Middle row: EP, Last row: WD shows the PDF and CDF Plot of bladder cancer data set.

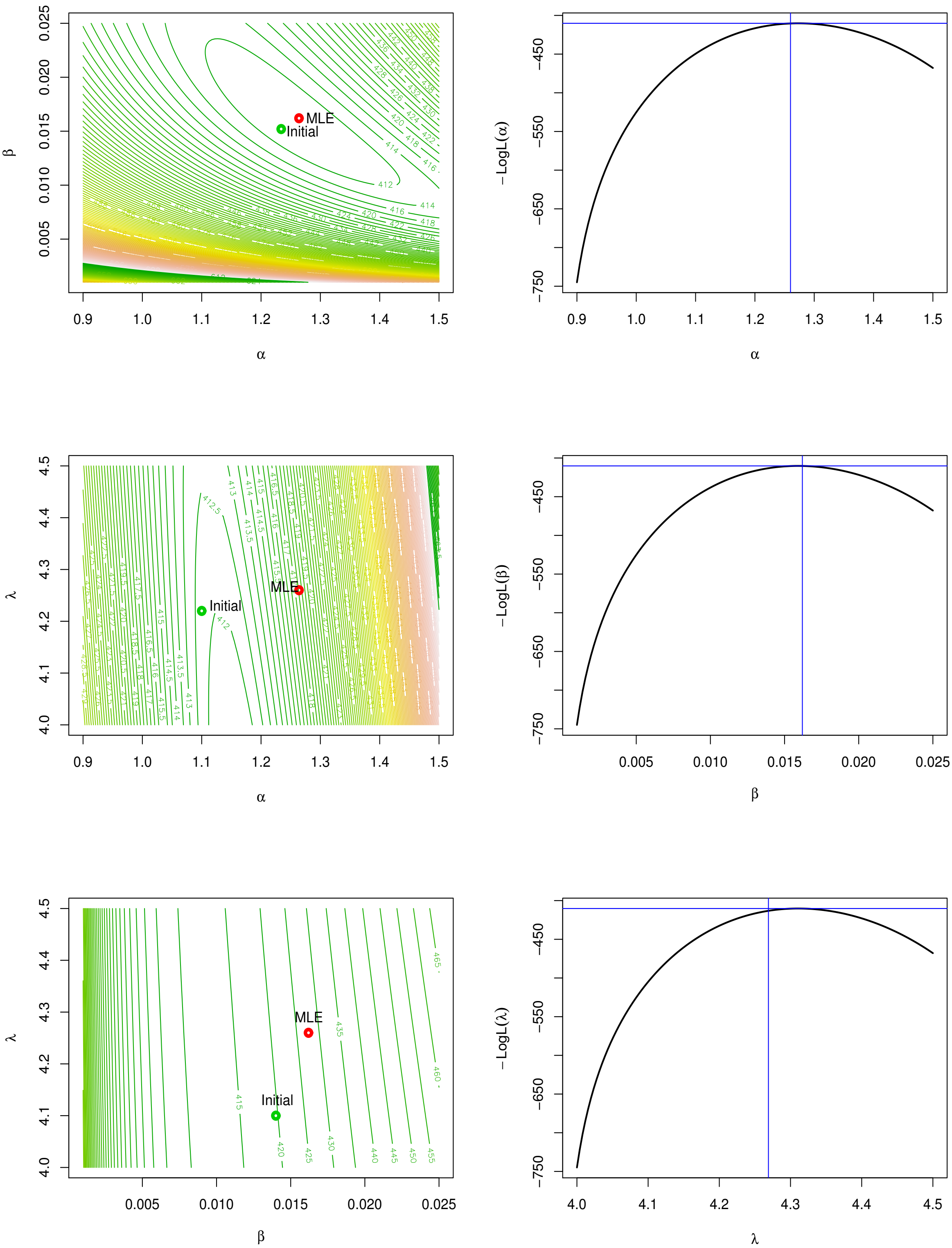

FIGURE 2.16: The Contour and -log-L plot of α, β and λ for bladder cancer data set.

Chapter 3

E-Bayesian Inference for Poisson Inverse Exponential Distribution Under Different Loss Functions *

3.1 Introduction

In the previous chapter, we have discussed the procedure for obtaining the classical and Bayesian estimation under PT-II CBRs. This chapter, we obtain the E- Bayes estimator of the considered Poisson family, we know that the distribution as a Poisson inverse exponential distribution (PIED). Here, experimental units generally deal with truncated data in studies pertaining to medical and survival analysis. However, data might be a set of some samples or censored data from an experiment. The literature has witnessed the different type of censoring techniques to effectively reduce the time and cost of the life testing experimentation. The Type-I and Type-II censoring are the very useful real life problems. Type-I (pertains to time constraint) and Type-II (pertains to cost constraint) censoring; time and number of experimental units are pre-specified

* Part of this chapter has been published in reputed peer-reviewed journals with indexing SCIE, SCOPUS, PubMed see Pathak et al. (2020a).

or fixed, which has been controlled by experimenter. However, it may not be always true for data related with medical studies. Because some of the surviving experimental units are removed randomly at each stage of failure from the experiment due to some unforeseen reasons, which are beyond the control of the experimenter. For example, let us suppose that n, number of multiple myeloma patients have a malignant disease characterized by the accumulation of abnormal plasma cell, a type of white blood cell, in the bone marrow, and are put under medication in hospital. It is decided to observe the survival lifetime of m patients out of n. During medication it may happens that after death (first death, second death or so on) some patients may leave the hospital due to various reasons like loss in faith with hospital, treatment etc. The process of taking observations continues till survival times of m patients are recorded. It may also be noted that the number of patients dropping out from the test at each stage is random and can not be predetermined. The above said censoring and their mathematical formulation, expression have been discussed in Chapter 1, Subsection 1.11.2.

In last few years, the estimation of parameters of different lifetime distribution based on progressive censored samples have been discussed by several authors such as (Childs and Balakrishnan (2000), Kundu (2008), Kim and Han (2009), Gholizadeh et al. (2011), Kim et al. (2011), Huang and Wu (2012)).

Some early works based on the estimation of parameters of different lifetime distribution under PT-II CBRs has been done by (Tse and Yuen (2000), Wu and Chang (2002), Singh et al. (2013b), Singh et al. (2014), Kumar et al. (2015), Kumar et al. (2018), Kumar et al. (2019a)).

Firstly, han has discussed the E-Bayesian inference, which is an alternative to Bayesian inference. This method has been used as prior for the unknown hyper parameters. The hierarchical prior distribution may be used as prior for the unknown hyper parameters, and has required to set the at least two stages of prior setting (see Lindley and Smith (1972)). But in practice, under censoring mechanism through this prior, the Bayesian estimates of the unknown parameters that have been obtained is quite complicated for the purpose of data analysis as well as computations. For this contrary, we are approaching to follow E-Bayesian estimation method for PIED parameters under PT-II CBRs.

The detailed discussions about the E-Bayesian method see, (Han (2007), Han (2011a), Han (2011b)). The E-Bayesian estimation for the parameters of different lifetime distributions (see Gupta (2017), Yousefzadeh (2017), Han (2017a), Han (2017b), Han (2019b) etc.). Furthermore, a few number of authors dealt with the E-Bayesian estimation of parameters for lifetime distribution with type-II censoring (see Jaheen and Okasha (2011), Okasha (2014), Reyad and Ahmed (2016) etc.).

Also, El-Sagheer (2017) considered the Rayleigh distribution for E-Bayesian estimation under progressive type-II censoring. The hierarchical and E-Bayesian estimations for the proportional reversed hazard rate model based on record values have been discussed by Kızılaslan (2017). Moreover, some more relevant literature related to the study of E-Bayesian estimation for E-posterior and E-MSE has been done (see Han (2018), Han (2019a), Han (2019b), Han (2019c), Han (2020)).

Recently, PIED as a parametric compounding based upon Poisson lifetime distribution which has been introduced by Kumar et al. (2018). The various parametric compounding based on Poisson lifetime distributions ((Barreto-Souza and Cribari-Neto (2009), Louzada-Neto et al. (2011), Lu and Shi (2012), Kumar et al. (2018)) have been used for parameter estimation, but no one attempted to work on E-Bayesian inference for the parameters under PT-II CBRs. This is the beauty of this chapter.

Finally, in this chapter we obtain the E-Bayesian and Bayesian estimators of parameters of PIED under SELF, GELF and LINEX for PT-II CBRs. The E-Bayesian estimators are compared with Bayesian estimators and obtained under different loss function in terms of their risks.

3.2 Mathematical Formulation

We begin by summarizing PIED model and likelihood under PT-II CBRs along with Bayesian and E-Bayesian approach that will be used through out the chapter.

3.2.1 The Model

The two-parameter PIED is one of the latest compounding of two most useful probability distributions termed as Poisson inverse exponential distribution i.e., zero truncated Poisson and inverse exponential, and their PDF is

$$f(x;\theta,\lambda) = \frac{\theta\lambda e^{-\theta-\frac{\lambda}{x}+\theta e^{-\frac{\lambda}{x}}}}{\left(1-e^{-\theta}\right)x^2}; \qquad x>0, \lambda>0, \theta>0, \tag{3.1}$$

where, the parameter θ and λ are represented as shape and scale, respectively. The corresponding CDF of PIED (λ,θ) is given by

$$F(x;\theta,\lambda) = \frac{e^{-\theta}\left(e^{\theta e^{-\frac{\lambda}{x}}}-1\right)}{1-e^{-\theta}}; \qquad x>0, \lambda>0, \theta>0. \tag{3.2}$$

It is a lifetime distribution with initially increasing then decreasing failure distribution. The HF can be defined as

$$h(x;\theta,\lambda) = \frac{f(x;\theta,\lambda)}{1-F(x;\theta,\lambda)} = \frac{\theta\lambda e^{-\theta-\frac{\lambda}{x}+\theta e^{-\frac{\lambda}{x}}}}{x^2\left(1-e^{-\theta+\theta e^{-\frac{\lambda}{x}}}\right)}; \qquad x>0, \lambda>0, \theta>0. \tag{3.3}$$

It is very much plausible statistical distribution and alternative to common mixture of lifetime distributions when it is heavy-tailed with monotone failure data. As per applied mathematical formulation of PT-II CBR mechanism are discussed in Chapter 1, Section 1.11.2. The conditional likelihood function can be written as (see Cohen (1963) and Kamps and Cramer (2001))

$$\begin{aligned} L(\lambda,\theta;x|R=r) &= f_{(X_1,\cdots,X_m)}(x_1,\cdots,x_m) \\ &= c\prod_{i=1}^{m} f(x_i)\left[1-F(x_i)\right]^{r_i}; \quad -\infty < x_1 < \cdots < x_m < \infty, \end{aligned} \tag{3.4}$$

where $n = m+\sum_{i=1}^{m} r_i, n,m \in N, r_i \in N_0, 1 \leq i \leq m, r_i \sim B(n-m-\sum_{l=0}^{i-1} r_l, p)$ for $i = 1,2,3,\cdots,m-1$ and $r_0 = 0$ and $c = \prod_{i=1}^{m} \gamma_i$ with $\gamma_i = \sum_{j=i}^{m}(r_j+1)$ and for $\gamma_1 = n$. Substituting $f(x_i)$ and $F(x_i)$ from Equations (3.1) and (3.2) into Equation (3.4), it reduces to

$$L(\lambda,\theta;x|R=r) = c\prod_{i=1}^{m} \frac{\theta\lambda e^{-\theta-\frac{\lambda}{x_i}+\theta e^{-\frac{\lambda}{x_i}}}}{(1-e^{-\theta})x_i^{2}} \left\{1-\frac{e^{-\theta}\left(e^{\theta e^{-\frac{\lambda}{x_i}}}-1\right)}{(1-e^{-\theta})}\right\}^{r_i}. \tag{3.5}$$

The number of the experimental unit R_i removed at i^{th} failure X_i; $i = 1,2,\cdots(m-1)$, follows a Binomial distribution with parameters $\left(n-m-\sum_{l=1}^{i-1} r_i, p\right)$. Therefore,

$$P(R_1 = r_1;p) = \binom{n-m}{r_1} p^{r_1}(1-p)^{n-m-r_1}, \tag{3.6}$$

and for $i = 2,3,\cdots,m-1$,

$$\begin{aligned} P(R_i;p) &= P(R_i = r_i|R_{i-1} = r_{i-1},\cdots R_1 = r_1) \\ &= \binom{n-m-\sum_{l=0}^{i-1} r_l}{r_i} p^{r_i}(1-p)^{n-m-\sum_{l=0}^{i-1} r_l}. \end{aligned} \tag{3.7}$$

We also assume that $R_i s$ are independent of X_i for all i. Thus, the joint likelihood function $X_i, i = 1,2,3,\cdots,m$ and $R_i,\ i = 1,2,3,\cdots,m$ can take the following form

$$L(\theta,\lambda,p;x) = L(\theta,\lambda;x|R=r)\,P(R=r;p), \tag{3.8}$$

where

$$P(R=r;p)=P(R_1=r_1)P(R_2=r_2|R_1=r_1)P(R_3=r_3|R_2=r_2,R_1=r_1)$$
$$\cdots P(R_{m-1}=r_{m-1}|R_{m-2}=r_{m-2},\cdots R_1=r_1). \tag{3.9}$$

Substituting Equations (3.6) and (3.7) into Equation (3.9), we get

$$P(R=r;p)=\frac{(n-m)!p^{\sum_{i=1}^{m-1}r_i}(1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}}{\left(n-m-\sum_{l=1}^{i-1}r_i\right)!\prod_{i=1}^{m-1}r_i!}. \tag{3.10}$$

Now using Equations (3.5), (3.8) and (3.10) the complete likelihood function can be written as

$$L(\lambda,\theta,p;x)=\eta L_1(\lambda,\theta)L_2(p), \tag{3.11}$$

where $\eta=\frac{c(n-m)!}{(n-m-\sum_{l=1}^{i-1}r_i)!\prod_{i=1}^{m-1}r_i!}$,

$$L_1(\lambda,\theta)=\prod_{i=1}^{m}\frac{\theta\lambda e^{-\theta-\frac{\lambda}{x_i}+\theta e^{-\frac{\lambda}{x_i}}}}{(1-e^{-\theta})x_i{}^2}\left\{1-\frac{e^{-\theta}\left(e^{\theta e^{-\frac{\lambda}{x_i}}}-1\right)}{(1-e^{-\theta})}\right\}^{r_i}, \tag{3.12}$$

and

$$L_2(p)=p^{\sum_{i=1}^{m-1}r_i}(1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}. \tag{3.13}$$

3.3 Maximum Likelihood Estimation under PT-II CBRs

3.3.1 Point Estimation

As we observed that $L_1(\lambda,\theta)$ has independent of $L_2(p)$. Therefore, the ML estimates of λ and θ can derived by maximizing Equation (3.12) directly. The log-L function of the above

Equation (3.12) becomes

$$\ln L_1(\lambda,\theta) = -m\theta + m\ln(\theta) + m\ln(\lambda) - \lambda\sum_{i=1}^{m}\frac{1}{x_i} + \theta\sum_{i=1}^{m} e^{-\frac{\lambda}{x_i}}$$
$$-\left(m+\sum_{i=1}^{m} r_i\right)\ln\left(1-e^{-\theta}\right) - \sum_{i=1}^{m} 2\ln(x_i) + \sum_{i=1}^{m} r_i \ln\left(1-e^{-\theta\left(1-e^{-\frac{\lambda}{x_i}}\right)}\right). \tag{3.14}$$

The ML estimates of (λ,θ) can be directly obtained by maximizing the log-L function Equation (3.14), or alternatively, by finding the solution for the following two nonlinear Equations,

$$\frac{m}{\lambda} - \sum_{i=1}^{m}\frac{1}{x_i} - \theta\sum_{i=1}^{m}\frac{1}{x_i}e^{-\frac{\lambda}{x_i}} - \sum_{i=1}^{m}\frac{r_i e^{-\frac{\lambda}{x_i}}}{x_i\left(e^{\theta\left(1-e^{-\frac{\lambda}{x_i}}\right)}-1\right)} = 0, \tag{3.15}$$

and

$$\frac{m}{\theta} - m + \sum_{i=1}^{m} e^{-\frac{\lambda}{x_i}} - \left(m+\sum_{i=1}^{m} r_i\right)\frac{e^{-\theta}}{1-e^{-\theta}} - \sum_{i=1}^{m}\frac{r_i(1-e^{-\frac{\lambda}{x_i}})}{\left(e^{\theta\left(1-e^{-\frac{\lambda}{x_i}}\right)}-1\right)} = 0. \tag{3.16}$$

The above Equations (3.15) and (3.16) can not be solved simultaneously to provide any explicit solution for $\Psi = (\lambda,\theta)$. Therefore, these normal equations are to be solved numerically using some adequate iteration such as the NR method or an algorithm such as *nlm* of software R (Ihaka and Gentleman (1996)).

3.3.2 Confidence Intervals

Now, we discussed CIs of the parameters λ and θ under PT-II CBRs. Therefore we have,

$$\begin{bmatrix} Var(\hat{\lambda}_M) & Cov(\hat{\lambda}_M,\hat{\theta_M}) \\ Cov(\hat{\lambda}_M,\hat{\theta_M}) & Var(\hat{\theta}_M) \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 \ln L_1(\lambda,\theta)}{\partial\lambda^2} & -\frac{\partial^2 \ln L_1(\lambda,\theta)}{\partial\lambda\partial\theta} \\ -\frac{\partial^2 \ln L_1(\lambda,\theta)}{\partial\theta\partial\lambda} & -\frac{\partial^2 \ln L_1(\lambda,\theta)}{\partial\theta^2} \end{bmatrix}^{-1}_{\lambda=\hat{\lambda}_M,\theta=\hat{\theta}_M}, \tag{3.17}$$

where,

$$\frac{\partial^2 \ln L_1(\lambda,\theta)}{\partial \lambda^2} = -\frac{m}{\lambda^2} + \theta \sum_{i=1}^{m} \left(\frac{1}{x_i}\right)^2 e^{-\frac{\lambda}{x_i}} - \sum_{i=1}^{m} T_i,$$

and,

$$T_i = \frac{r_i e^{-\frac{\lambda}{x_i}} \left(1 - e^{\theta\left(1-e^{-\frac{\lambda}{x_i}}\right)} - e^{\theta\left(1-e^{-\frac{\lambda}{x_i}}\right)} e^{-\frac{\lambda}{x_i}}\right)}{x_i^2 \left(e^{\theta\left(1-e^{-\frac{\lambda}{x_i}}\right)} - 1\right)^2},$$

$$\frac{\partial^2 \ln L_1(\lambda,\theta)}{\partial \lambda \partial \theta} = -\sum_{i=1}^{m} \frac{e^{-\frac{\lambda}{x_i}}}{x_i} + \sum_{i=1}^{m} \frac{r_i e^{-\frac{\lambda}{x_i}} e^{\theta\left(1-e^{-\frac{\lambda}{x_i}}\right)} \left(1 - e^{-\frac{\lambda}{x_i}}\right)}{x_i \left(e^{\theta\left(1-e^{-\frac{\lambda}{x_i}}\right)} - 1\right)^2},$$

$$\frac{\partial^2 \ln L_1(\lambda,\theta)}{\partial \theta \partial \lambda} = -\sum_{i=1}^{m} \frac{e^{-\frac{\lambda}{x_i}}}{x_i} + \sum_{i=1}^{m} \frac{r_i e^{-\frac{\lambda}{x_i}} \left(e^{\theta\left(1-e^{-\frac{\lambda}{x_i}}\right)} \left(\theta(1 - e^{-\frac{\lambda}{x_i}}) - 1\right) + 1\right)}{x_i \left(e^{\theta\left(1-e^{-\frac{\lambda}{x_i}}\right)} - 1\right)^2},$$

$$\frac{\partial^2 \ln L_1(\lambda,\theta)}{\partial \theta^2} = -\frac{m}{\theta^2} + \left(m + \sum_{i=1}^{m} r_i\right) \frac{e^{-\theta}}{(1-e^{-\theta})^2} + \sum_{i=1}^{m} \frac{r_i e^{\theta\left(1-e^{-\frac{\lambda}{x_i}}\right)} \left(1 - e^{-\frac{\lambda}{x_i}}\right)^2}{\left(e^{\theta\left(1-e^{-\frac{\lambda}{x_i}}\right)} - 1\right)^2}.$$

The $\hat{\lambda}_M$ and $\hat{\theta}_M$ are denoted as ML estimates of λ and θ. For the asymptomatic variance-covariance of λ and θ are computed by invert og the Fisher's information matrix,

$$I = E \begin{bmatrix} -\frac{\partial^2 \ln L_1(\lambda,\theta)}{\partial \lambda^2} & -\frac{\partial^2 \ln L_1(\lambda,\theta)}{\partial \lambda \partial \theta} \\ -\frac{\partial^2 \ln L_1(\lambda,\theta)}{\partial \theta \partial \lambda} & -\frac{\partial^2 \ln L_1(\lambda,\theta)}{\partial \theta^2} \end{bmatrix}.$$

Thus, an approximate $100(1-\alpha)\%$ CIs for the parameters λ and θ are given by

$$\left(\hat{\lambda}_M - z_{\alpha/2}\sqrt{Var(\hat{\lambda}_M)}, \qquad \hat{\lambda}_M + z_{\alpha/2}\sqrt{Var(\hat{\lambda}_M)}\right)$$

and

$$\left(\hat{\theta}_M - z_{\alpha/2}\sqrt{Var(\hat{\theta}_M)}, \qquad \hat{\theta}_M + z_{\alpha/2}\sqrt{Var(\hat{\theta}_M)}\right),$$

respectively, where $z_{\alpha/2}$ is the percentile $(\alpha/2)^{th}$ of the standard normal distribution, also $Var(\hat{\lambda}_M)$ and $Var(\hat{\theta}_M)$ represent asymptotic variances of ML estimates.

3.4 Bayesian and E-Bayesian Estimation

We discuss the process of obtaining the Bayesian and E-Bayesian estimator of PIED parameters λ and θ based under PT-II CBRs. Bayesian and E-Bayesian estimators of the respective parameters are shown below.

3.4.1 Bayesian Estimation

Based on PT-II CBRs, observations from the PIED, the likelihood function given by Equation (3.12), prior distributions for the parameters in the distribution is $g_1(\lambda|a,b)$ of λ is given by Equation (3.18) and $g_2(\theta|\alpha,\beta)$ of θ is given by Equation (3.19) respectively,

$$g_1(\lambda|a,b) = \frac{b^a}{\Gamma(a)} e^{-b\lambda}\lambda^{a-1}; \qquad \lambda > 0, a > 0, b > 0, \tag{3.18}$$

$$g_2(\theta|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\theta}\theta^{\alpha-1}; \qquad \theta > 0, \alpha > 0, \beta > 0. \tag{3.19}$$

It may also be noted that the gamma prior $g_1(\lambda|a,b)$ and $g_2(\theta|\alpha,\beta)$ are independent and take on wide variety of shapes (prior believes of experimenter) depending on the value of hyper parameters, the joint prior pdf of λ and θ is

$$g(\lambda,\theta) = g_1(\lambda|a,b) * g_2(\theta|\alpha,\beta). \tag{3.20}$$

Combining the priors given by Equation (3.20) with likelihood given by Equation (3.12), we can easily obtain joint posterior PDF of (λ, θ) as

$$\pi(\lambda, \theta|x) = \frac{\zeta_0}{\zeta}, \tag{3.21}$$

where

$$\zeta_0 = \prod_{i=1}^{m} \frac{\theta\lambda e^{-\theta-\frac{\lambda}{x_i}+\theta e^{-\frac{\lambda}{x_i}}} \left(1 - e^{-\theta\left(1-e^{-\frac{\lambda}{x_i}}\right)}\right)^{r_i} b^a \beta^\alpha e^{-b\lambda-\beta\theta} \lambda^{a-1} \theta^{\alpha-1}}{\Gamma(a)\Gamma(\alpha)(1-e^{-\theta})^{r_i+1} {x_i}^2}$$

and

$$\zeta = \int_0^\infty \int_0^\infty \zeta_0 d\lambda d\theta.$$

Hence, the respective marginal posterior PDF's of λ and θ are given by

$$\pi_1(\lambda|x, r) = \int_0^\infty \frac{\zeta_0}{\zeta} d\theta, \tag{3.22}$$

$$\pi_2(\theta|x, r) = \int_0^\infty \frac{\zeta_0}{\zeta} d\lambda. \tag{3.23}$$

Further, in this context of loss function is the very essential element of the parameter estimation problem. SELF is very frequently used loss function and the weakness of this loss function is symmetric and put on equal weight to o.e. and u.e. of the same magnitude. Also, considered asymmetric loss function is GELF and LINEX are discussed in Chapter 1, Section 1.8.

Expressions for the Bayesian estimators $\left(\hat{\lambda}_S, \hat{\theta}_S\right)$, $\left(\hat{\lambda}_L, \hat{\theta}_L\right)$ and $\left(\hat{\lambda}_G, \hat{\theta}_G\right)$ for (λ, θ) under SELF, LINEX and GELF respectively can be given as

$$\left.\begin{aligned} \hat{\lambda}_S &= \int_0^\infty \lambda \pi_1(\lambda|x,r)d\lambda, \\ \hat{\theta}_S &= \int_0^\infty \theta \pi_2(\theta|x,r)d\theta, \end{aligned}\right\} \tag{3.24}$$

$$\left.\begin{aligned} \hat{\lambda}_L &= -\frac{1}{\delta}\ln\left(\int_0^\infty e^{-\delta\lambda}\pi_1(\lambda|x,r)d\lambda\right), \\ \hat{\theta}_L &= -\frac{1}{\delta}\ln\left(\int_0^\infty e^{-\delta\theta}\pi_2(\theta|x,r)d\theta\right), \end{aligned}\right\} \tag{3.25}$$

$$\left.\begin{aligned} \hat{\lambda}_G &= \left(\int_0^\infty \lambda^{-\delta}\pi_1(\lambda|x,r)d\lambda\right)^{-\frac{1}{\delta}}, \\ \hat{\theta}_G &= \left(\int_0^\infty \theta^{-\delta}\pi_2(\theta|x,r)d\theta\right)^{-\frac{1}{\delta}}. \end{aligned}\right\} \tag{3.26}$$

Here, δ is the shape parameter of loss function. Substituting the posterior PDF from Equation (3.22) and (3.23) in Equations (3.24), (3.25) and (3.26) respectively and then simplifying, we get the Bayesian estimators $\left(\hat{\lambda}_S, \hat{\theta}_S\right)$, $\left(\hat{\lambda}_L, \hat{\theta}_L\right)$ and $\left(\hat{\lambda}_G, \hat{\theta}_G\right)$ of (λ, θ). It may also be noted that the above integrals involved in the expressions for the Bayesian estimators $\left(\hat{\lambda}_S, \hat{\theta}_S\right)$, $\left(\hat{\lambda}_L, \hat{\theta}_L\right)$ and $\left(\hat{\lambda}_G, \hat{\theta}_G\right)$ are not possible to reduce in closed form. Therefore, we propose the use of numerical method for obtaining the estimates. We have used MCMC method. For this, we proceed by generating observations from posteriors Equations (3.22) and (3.23) respectively. These posteriors does not follow any standard form density, since with help of Metropolis and Ulam (1949), we use M-H algorithm to generate sample observations from each of these posterior distributions as, see Gelman et al. (2013)

$$\pi_1^*(\lambda|\theta,x,r) \propto \lambda^{m+a-1} e^{-\lambda\sum_{i=1}^m\left(\frac{1}{x_i}+b\right)} e^{\theta\sum_{i=1}^m e^{-\frac{\lambda}{x_i}}} \prod_{i=1}^m \left(1-e^{-\theta\left(1-e^{-\frac{\lambda}{x_i}}\right)}\right)^{r_i}, \tag{3.27}$$

$$\pi_2^*(\theta|\lambda,x,r) \propto \frac{\theta^{m+\alpha-1}e^{-\theta(m+\beta)}e^{\theta\sum_{i=1}^m e^{-\frac{\lambda}{x_i}}}\prod_{i=1}^m\left(1-e^{-\theta\left(1-e^{-\frac{\lambda}{x_i}}\right)}\right)^{r_i}}{(1-e^{-\theta})^{\sum_{i=1}^m r_i+m}}, \tag{3.28}$$

respectively. For detailed see Aggarwala and Balakrishnan (1998), Louzada-Neto et al. (2011), Wu and Chang (2002)

3.4.2 E-Bayesian Estimation

The prior distribution of λ and θ are $g_1(\lambda|a,b)$ and $g_2(\theta|\alpha,\beta)$ respectively, which are decreasing function of $\lambda,\ \theta$ with respective hyper parameter (a,b) and (α,β), for more detailed see Han (2007). The derivative of $g_1(\lambda|a,b)$ and $g_2(\theta|\alpha,\beta)$ with respect to λ,θ is

$$\frac{d[g_1(\lambda|a,b)]}{d\lambda} = \frac{b^a\lambda^{a-2}e^{-b\lambda}}{\Gamma(a)}[(a-1)-b\lambda], \tag{3.29}$$

$$\frac{d[g_2(\theta|\alpha,\beta)]}{d\theta} = \frac{\beta^\alpha\theta^{\alpha-2}e^{-\beta\theta}}{\Gamma(\alpha)}[(\alpha-1)-\beta\theta], \tag{3.30}$$

where $(a,b)>0,(\alpha,\beta)>0$ and $(\lambda,\theta)>0$. From above Equation (3.29) and Equation (3.30), it is clear that for $0<(a,\alpha)<1,(b,\beta)>0$, and since $g_1(\lambda|a,b)$, $g_2(\theta|\alpha,\beta)$ is decreasing function of λ,θ. It also observed for given $0<(a,\alpha)<1$ and $(b,\beta)>0$ are, the very less probability in the tail of the gamma density function. According to Berger (2013), the thinner tailed prior distribution often reduces the robustness of Bayesian estimate. Since, the value of (b,β) should not be larger than a given upper bound (c,γ), where $(c,\gamma)>0$ is a given upper bound. Therefore, the hyper parameter (a,b) and (α,β) should be selected with the restriction of $0<(a,\alpha)<1$ and $0<b<c$, $0<\beta<\gamma$ (where constant (c,γ) would be considered later in simulation study). Then we are obtained the E-Bayesian estimate of λ and θ based on three different distribution of the hyper parameters (a,b) and (α,β). These distributions are used to investigate the influence of the different prior distributions on the E-Bayesian estimation of λ

and θ. The following distribution of hyper parameter a,b are used for λ

$$\left.\begin{aligned}
\pi_{11}(a,b) &= \frac{1}{c(B(u,v))}a^{(u-1)}(1-a)^{(v-1)}; && 0<a<1; 0<b<c,\\
\pi_{12}(a,b) &= \frac{2(c-b)}{c^2B(u,v)}a^{(u-1)}(1-a)^{(v-1)}; && 0<a<1; 0<b<c,\\
\pi_{13}(a,b) &= \frac{2b}{c^2B(u,v)}a^{(u-1)}(1-a)^{(v-1)}; && 0<a<1; 0<b<c.
\end{aligned}\right\} \tag{3.31}$$

We may also notice from above Equations (3.31), if $u>1$, $v>1$ then $\pi_{1i}(a,b)\to 0$ as $a\to 0$ or $a\to 1; i=1,2,3$. But $0<u<1$, then $\pi_{1i}(a,b)\to\infty$ as $a\to 0$, and if $0<v<1$, then $\pi_{1i}(a,b)\to\infty$ as $a\to 1; i=1,2,3$. Therefore, $u>1$, $v>1$ then $\pi_{1i}(a,b)$ has unique optimal solutions for each $i=1,2,3$. Also, distribution of hyper parameter α,β are used for θ

$$\left.\begin{aligned}
\pi_{21}(\alpha,\beta) &= \frac{1}{\gamma B(u_1,v_1)}\alpha^{(u_1-1)}(1-\alpha)^{(v_1-1)}; && 0<\alpha<1; 0<\beta<\gamma,\\
\pi_{22}(\alpha,\beta) &= \frac{2(\gamma-\beta)}{\gamma^2B(u_1,v_1)}\alpha^{(u_1-1)}(1-\alpha)^{(v_1-1)}; && 0<\alpha<1; 0<\beta<\gamma,\\
\pi_{23}(\alpha,\beta) &= \frac{2\beta}{\gamma^2B(u_1,v_1)}\alpha^{(u_1-1)}(1-\alpha)^{(v_1-1)}; && 0<\alpha<1; 0<\beta<\gamma.
\end{aligned}\right\} \tag{3.32}$$

Similarly, from above Equations (3.32), if $u_1>1$, $v_1>1$ then $\pi_{2i}(\alpha,\beta)\to 0$ as $\alpha\to 0$ or $\alpha\to 1; i=1,2,3$. But $0<u_1<1$, then $\pi_{2i}(\alpha,\beta)\to\infty$ as $\alpha\to 0$, and if $0<v_1<1$, then $\pi_{2i}(\alpha,\beta)\to\infty$ as $\alpha\to 1; i=1,2,3$. Therefore, $u_1>1$, $v_1>1$ then $\pi_{2i}(\alpha,\beta)$ has unique optimal solutions for each $i=1,2,3$. Since, the above Equations (3.31) and (3.32) are said to be the prior of the hyper parameter a and b for λ; α and β for θ respectively. The corresponding E-Bayesian estimation of λ and θ under SELF, GELF and LINEX are given as follows:

$$\left.\begin{aligned}
\hat{\lambda}_{EBSi} &= \int\int_D \hat{\lambda}_S\pi_{1i}(a,b)\,da\,db; && i=1,2,3; D\in\{(a,b):\ 0<a<1,\ 0<b<c\},\\
\hat{\theta}_{EBSi} &= \int\int_D \hat{\theta}_S\pi_{2i}(\alpha,\beta)\,d\alpha\,d\beta; && i=1,2,3; D\in\{(\alpha,\beta):\ 0<\alpha<1,\ 0<\beta<\gamma\},
\end{aligned}\right\} \tag{3.33}$$

$$\left.\begin{aligned}\hat{\lambda}_{EBGi} &= \int\int_D \hat{\lambda}_G \pi_{1i}(a,b)dadb; \qquad i=1,2,3; D\in\{(a,b):\ 0<a<1,\ 0<b<c\},\\ \hat{\theta}_{EBGi} &= \int\int_D \hat{\theta}_G \pi_{2i}(\alpha,\beta)d\alpha d\beta; \qquad i=1,2,3; D\in\{(\alpha,\beta):\ 0<\alpha<1,\ 0<\beta<\gamma\},\end{aligned}\right\} \tag{3.34}$$

and

$$\left.\begin{aligned}\hat{\lambda}_{EBLi} &= \int\int_D \hat{\lambda}_L \pi_{1i}(a,b)dadb; \qquad i=1,2,3; D\in\{(a,b):\ 0<a<1,\ 0<b<c\},\\ \hat{\theta}_{EBLi} &= \int\int_D \hat{\lambda}_L \pi_{2i}(a,b)dadb; \qquad i=1,2,3; D\in\{(\alpha,\beta):\ 0<\alpha<1,\ 0<\beta<\gamma\}.\end{aligned}\right\} \tag{3.35}$$

We observe the above Equations (3.33), (3.34) and (3.35) in integrals form of E-Bayesian estimate under SELF, GELF and LINEX loss functions are not possible to get the solution. Therefore, we perform the computational method Markov Chain Monte Carlo (MCMC) through R software and get the estimates of E-Bayesian under SELF, GELF and LINEX (see Zellner (1994) and Zellner (1986a)). In the next subsection, we deal with the MCMC algorithm.

3.4.3 The MCMC Algorithm

We now generate sample observation from $\pi_1^*(\lambda|\theta,x,r)$ and $\pi_2^*(\theta|\lambda,x,r)$, and taking normal distribution, $N\left(\hat{\lambda}_M, Var\left(\hat{\lambda}_M\right)\right)$ and $N\left(\hat{\theta}_M, Var\left(\hat{\theta}_M\right)\right)$ as proposal density respectively. The following steps of the algorithm is given as

(i) Set the initial guess of λ and θ say $\lambda_0 = \hat{\lambda}_M$ and $\theta_0 = \hat{\theta}_M$

(ii) Set $i = 1$

(iii) Generate a candidate point λ_i^* and θ_i^* from proposal distribution $\Phi_1 \sim N\left(\hat{\lambda}_M, Var\left(\hat{\lambda}_M\right)\right)$ and $\Phi_2 \sim N\left(\hat{\theta}_M, Var\left(\hat{\theta}_M\right)\right)$ respectively and take a point $\mathbf{x}$ from uniform distribution $U(0,1)$.
Let $\tau_1\left(\lambda_i^{(i-1)}, \lambda^*\right) = min\left(\frac{\pi_1^*(\lambda_1^*|\lambda_{(i-1)},x,r)\Phi_1(\lambda^{i-1})}{\pi_1^*(\lambda^*|\lambda_{i-1},x,r)\Phi_1(\lambda^*)}, 1\right)$,

$\tau_2\left(\theta_i^{(i-1)}, \theta^*\right) = min\left(\frac{\pi_2^*(\theta_1^*|\theta_{(i-1)},x,r)\Phi_2(\theta^{i-1})}{\pi_2^*(\theta^*|\theta_{i-1},x,r)\Phi_2(\theta^*)}, 1\right)$
Set $\lambda^{(i)} = \lambda^*$, $\theta^{(i)} = \theta^*$ if $u \leq \tau_1\left(\lambda_i^{(i-1)}, \lambda^*\right)$, $u \leq \tau_2\left(\theta_i^{(i-1)}, \theta^*\right)$ otherwise set $\lambda^{(i)} = \lambda^{(i-1)}$, $\theta^{(i)} = \theta^{(i-1)}$

(iv) Set $i = i+1$

(v) Repeat steps (ii)-(iv) for sufficiently large number of time i.e., N sufficiently large number. After the convergence of chain of observations, we have $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_N$ and $\theta_1, \theta_2, \theta_3, \cdots, \theta_N$.

(vi) Obtain the Bayesian estimates of λ and θ under GELF and LINEX as
$\hat{\lambda}_G = \left[\frac{1}{N-N_0}\sum_{i=1}^{N-N_0}\lambda_i^{-\delta}\right]^{-\frac{1}{\delta}}$, $\hat{\theta}_G = \left[\frac{1}{N-N_0}\sum_{i=1}^{N-N_0}\theta_i^{-\delta}\right]^{-\frac{1}{\delta}}$ and
$\hat{\lambda}_L = -\frac{1}{\delta}\ln\left[\frac{1}{N-N_0}\sum_{i=1}^{N-N_0}e^{-\delta\lambda_i}\right]$, $\hat{\theta}_L = -\frac{1}{\delta}\ln\left[\frac{1}{N-N_0}\sum_{i=1}^{N-N_0}e^{-\delta\theta_i}\right]$, where, N_0 is the burn-in-period of Markov Chain. When substituting δ equal to -1 in step (vi), of $\hat{\lambda}_G, \hat{\theta}_G$, we get Bayesian estimates of λ and θ under SELF.

(vii) The E-Bayesian estimates of λ and θ under different loss functions, firstly we generate a and b from Beta distribution and Uniform distribution for the values of prior parameters (u, v) and $(0, c)$ respectively as stated Equation (3.31). Similarly, generate α and β from Beta distribution and Uniform distribution for the values of prior parameters (u_1, v_1) and $(0, \gamma)$ respectively as stated Equation (3.32). Using step (vi) with above Equations (3.31), (3.32) to get E-Bayesian estimates of λ and θ under SELF, GELF and LINEX respectively.

3.4.4 Bayesian Intervals

In this subsection, we obtain Bayesian credible and HPD intervals of λ and θ, which is analog of a frequentest method of CIs. We use the sample λ and θ (say $\lambda_{[1]}, \lambda_{[2]}, \lambda_{[3]}, \cdots, \lambda_{[N]}$) and $(\theta_{[1]}, \theta_{[2]}, \theta_{[3]}, \cdots, \theta_{[N]})$ obtained from posterior distribution through the MCMC method in the previous subsection and apply the algorithm of Wu and Chang (2002) to get Bayesian credible and HPD intervals for λ and θ. In this algorithm, we first order the obtained sample observations as $(\lambda_{[1]} < \lambda_{[2]} < \lambda_{[3]} < \cdots < \lambda_{[N]})$ and $(\theta_{[1]} < \theta_{[2]} < \theta_{[3]} < \cdots < \theta_{[N]})$ and then

(i) The $100(1-\psi)\%$ Bayesian credible interval of λ and θ are given as $(\lambda_{[(\psi/2)N]}, \lambda_{[(1-\psi/2)N]})$ and $(\theta_{[(\psi/2)N]}, \theta_{[(1-\psi/2)N]})$; where $[(\psi/2)N]$ and $[(1-\psi/2)N]$ are integer parts of $[(\psi/2)N]$ and $[(1-\psi/2)N]$ respectively.

(ii) For HPD of λ and θ, we first obtain all $100(1-\psi)\%$ credible intervals given as

$$(\lambda_{[i]}, \lambda_{[i+(1-\psi/2)N]}); \qquad i = 1,2,3,\cdots,\psi N;$$

$$(\theta_{[i]}, \theta_{[i+(1-\psi/2)N]}); \qquad i = 1,2,3,\cdots,\psi N;$$

along with their corresponding lengths

$$L(\lambda)_i = \lambda_{[i+(1-\psi/2)N]} - \lambda_{[i]}; \qquad i = 1,2,3,\cdots,\psi N;$$

$$L(\theta)_i = \theta_{[i+(1-\psi/2)N]} - \theta_{[i]}; \qquad i = 1,2,3,\cdots,\psi N;$$

and thereafter pick up the interval of λ and θ of which have smallest length $L(\lambda)_i$ and $L(\theta)_i$ respectively.

3.5 Monte Carlo Simulation Study and Comparison of Estimators

We are obtaining some numerical illustrations based on MC simulation study. Because, analytically the Bayesian and E-Bayesian estimators are not obtained in closed form. Therefore, we need to simulate propose estimators under PT-II CBR samples from PIED. The Balakrishnan and Sandhu (1995)'s algorithm has been used for simulation under PT-II CBR samples. Then, we also compare the various estimators computed in Sections 3.3 and 3.4 under PT II-CBRs. The set of estimators $\left(\hat{\lambda}_M, \hat{\theta}_M\right)$, $\left(\hat{\lambda}_S, \hat{\theta}_S\right)$, $\left(\hat{\lambda}_G, \hat{\theta}_G\right)$ and $\left(\hat{\lambda}_L, \hat{\theta}_L\right)$ denote the ML estimators and Bayesian estimators of the parameters (λ, θ) under SELF, GELF and LINEX respectively, while $\left(\hat{\lambda}_{EBSi}, \hat{\theta}_{EBSi}\right)$, $\left(\hat{\lambda}_{EBGi}, \hat{\theta}_{EBGi}\right)$ and $\left(\hat{\lambda}_{EBLi}, \hat{\theta}_{EBLi}\right)$ with $i = 1,2,3$ are corresponding E-Bayesian estimators under SELF, respectively. Now, we are comparing the estimators obtained under symmetric and asymmetric loss functions with corresponding Bayesian and E-Bayesian estimators respectively. The comparisons are based on the simulated risks under SELF, GELF and LINEX. It may also be mentioned here that the exact expressions for the risks can not be computed because the estimators are not found in closed form. Therefore, the risks of the estimators are estimated on the basis of MC simulation study of 10000 samples. It may also be cleared that the risks of the estimators will depend on values of $n, m, p, \lambda, \theta, \gamma$, δ and c.

In this study the effect of variation in the value of the combination of total sample size n with different size of effective sample size m, we have obtained the simulated risks for $n = 20, m = 12\,[2]\,4$; $n = 30, m = 18\,[3]\,6$ and $n = 40, m = 24\,[4]\,8$ and with loss parameter $\delta = +0.1$, (o.e. is more serious than u.e.), $\delta = -0.1$ (u.e. is more serious than o.e.), $p = 0.05$ (probability of removals). The selection of the hyper parameter based on the prior distribution setting, here we are choosing the prior mean as a true parameter value and prior variance is $\sigma^2 = 0.9$. Then the hyper parameter $a = \frac{\lambda^2}{\sigma^2}$ and $b = \frac{\lambda}{\sigma^2}$, similarly $\alpha = \frac{\theta^2}{\sigma^2}$ and $\beta = \frac{\theta}{\sigma^2}$ and hyper prior parameter is $u = a,\; v = b,\; c = 4$ and $u_1 = \alpha,\; v_1 = \beta,\;\; \gamma = 3$. Using PT-II CBR samples and obtained simulated risks for the estimators of λ and θ under SELF, GELF and LINEX have been obtained for selected values of $n, m, p, \lambda, \theta, \gamma$, δ and c. The results are summarized in Table (3.2), Table (3.4), Table (3.3) and Table (3.5) respectively.

The computations in Table (3.2) and Table (3.3) show that ML estimates, Bayesian and E-Bayesian estimates of parameter λ and θ based on SELF, GELF and LINEX. The estimated risks of the different estimators are compared on SELF. Table (3.2) shows that the O.e. is more serious than u.e. cases i.e, for $\delta > 0$ while Table (3.3) represents that for u.e. is more serious than o.e. cases i.e, for $\delta < 0$ of $\left(\hat{\lambda}_G, \hat{\theta}_G\right)$ and $\left(\hat{\lambda}_L, \hat{\theta}_L\right)$; $\left(\hat{\lambda}_{EBGi}, \hat{\theta}_{EBGi}\right)$ and $\left(\hat{\lambda}_{EBLi}, \hat{\theta}_{EBLi}\right)$ with $i = 1,2,3$ are corresponding E-Bayesian estimators. We observed from Table (3.2), Table (3.3) that estimated risks of Bayesian and E-Bayesian estimators decrease as effective sample size m (and fixed n) increases. Generally, in most of the cases, risks of the proposed E-Bayesian estimator of λ and θ i.e, $\left(\hat{\lambda}_{EBL3}, \hat{\theta}_{EBL3}\right)$ has minimum risks as compared to other competitive estimators of λ and θ. Also, in Table (3.2) and Table (3.3) show average Bayesian and E-Bayesian (in parenthesis) estimate, average length of CI and HPD intervals. The Average length of CI and HPD interval of λ and θ are also decreases when m increases.

However, in Table (3.4) shows that the risks of estimators of λ and θ under GELF and LINEX for $\delta > 0$. Under both losses we show that the proposed E-Bayesian estimator $\left(\hat{\lambda}_{EBS3}, \hat{\theta}_{EBS3}\right)$ has minimum risk as compared to other competitive estimators and the trend of risks of the estimators of λ and θ have similar trend as previous tables. But Table (3.5) shows risks of estimators of λ and θ under GELF and LINEX for $\delta < 0$. In this table we also found that the proposed E-Bayesian estimator $\left(\hat{\lambda}_{EBL3}, \hat{\theta}_{EBL3}\right)$ perform better than other estimators. For fixed n, when the effective sample size m increases risks of Bayesian and E-Bayesian estimators decreases.

3.6 An application to Survival of Multiple Myeloma Patients Data

Here, we consider a Multiple myeloma patients data set from Collett (2014). The observed data of survival time (in months) of multiple myeloma patients are 13, 52, 6, 40, 10, 7, 66, 10, 10, 14, 16, 4, 65, 5, 11, 10, 15, 5,76, 56, 88, 24, 51, 4, 40, 8, 18, 5, 16, 50, 40, 1, 36, 5, 10, 91, 18, 1, 18, 6, 1, 23, 15, 18, 12, 12, 17, 3, which are related to 48 patients, all of whom are aged between 50 to 80 years. Suppose the survival time of multiple myeloma patients who have a malignant disease characterized by the accumulation of abnormal plasma cell, a type of white blood cell, in the bone marrow can be modeled by PIED as a lifetime model.

Goodness of Fit Tests

The χ^2 goodness of fit and the K-S test to check whether PIED has properly accommodate the data. The simultaneous optimal solution of λ and θ are 2.575034 and 3.872177 respectively, which is verified by Figure (3.3). Now we want to test the null hypothesis that the distribution function $F(x)$ from which the data came PIED with $\hat{\lambda}$ = 2.575034 and $\hat{\theta}$ = 3.872177 respectively. Thus, $F(x)$ is completely specified.

The χ^2 test

The random sample of size 48 is drawn from a population with unknown CDF $F_0(x)$. We wish to test the null hypothesis

$$H_0 : F_0(x) = F(x),$$

$$H_1 : F_0(x) \neq F(x).$$

Let us start with six intervals $(0,5], (5,8], (8,14], (14,23], (23,53]$ and $(53,100]$ with equal bins. The sample size of each interval is $Y_1 = 6,\ Y_2 = 7,\ Y_3 = 10,\ Y_4 = 10,\ Y_5 = 9$, and $Y_6 = 6$ respectively. Corresponding probabilities are $p(Y_1) = 0.19366,\ p(Y_2) = 0.13691,\ p(Y_3) = 0.18103,\ p(Y_4) = 0.14482,\ p(Y_5) = 0.17226$ and $p(Y_6) = 0.07559$. The calculated value of $\chi^2_{cal} = 4.34604$ has less than tabulated value of $\chi^2_{3,0.05} = 7.81473$. Sine, we cannot reject H_0 at $\alpha\%$ level of significance. Thus $F(x)$ is suitable for the data set. But the χ^2 test is essentially applicable for large samples. Although it is also observed that the latter treats individual observations directly, whereas the former discretized the data and sometimes loses information through grouping. Therefore, the K-S test is applicable even in the case of very small samples as well as large samples.

The K–S test

This test assumes continuous of the distribution function, to check difference between $F_n(x)$ and $F(x)$. Since, to test

$$H_0 : F_n(x) = F(x),$$

$$H_1 : F_n(x) \neq F(x),$$

where $F_n(x)$ is the sample (empirical) distribution function, $F(x)$ is specified for all x. The test statistic

$$D_n = sup|F_n(x) - F(x)|,$$

is less then tabulated value of K-S distance $D_{n,\alpha}$ then accept H_0. For complete data set K-S distance and p-value are 0.125 and 0.8475, respectively. Also for various censoring schemes under PT-II CBRs, we calculate the K-S distance and corresponding p-value, see in Table (3.1). Other hand, the D_n statistic is used to obtain the confidence bands on $F_n(x)$ for all x, where $F_n(x)$ is a consistent estimator for CDF $F(x)$. The number $D_{n,\alpha}$ is obtain from the K-S table (Standard), such that

$$P\left[sup|F_n(x) - F(x)| < D_{n,\alpha}\right] = 1 - \alpha,$$

where, $0 \leq F(x) \leq 1 \; \forall x$. Thus we define

$$L_n(x) = max\left[F_n(x) - D_{n,\alpha}, 0\right],$$

and

$$U_n(x) = min\left[F_n(x) + D_{n,\alpha}, 1\right],$$

where $L_n(x)$ and $U_n(x)$ are lower and upper confidence band for the cdf $F(x)$, with $(1-\alpha)\%$ confidence coefficient. Of course, the $F(x)$ lies completely within the limits if and only if the hypothesis cannot be rejected at $\alpha\%$ level of significance. The K-S bound for various scheme are shown in the Figure (3.7).

TABLE 3.1: Real data analysis of various schemes to obtain estimates of parameter, log-L, K-S distance, p-value.

$\hat{\lambda}$	$\hat{\theta}$	log-L	K-S	p-value	Sample
2.677374	4.60836	120.5736	0.17857	0.76364	1,3,4,5,6,8,10,10,11,12,12,15,16,16,17,18,18,18,23,24,40,40,51,52,56,65,66,76
2.749227	3.828369	101.981	0.16667	0.8928	1,1,4,5,7,10,10,10,12,12,13,14,15,16,17,18,24,40,40,40,50,51,56,65
2.848905	3.543231	83.73955	0.15789	0.9718	1,1,4,5,8,10,11,12,16,18,18,24,36,40,40,51,52,56,65
2.275538	3.984181	54.4534	0.21429	0.9048	1,4,5,8,10,10,12,15,16,17,18,24,36,40
2.112439	3.18371	35.77236	0.4	0.4005	1,3,6,10,10,12,14,16,18,18

Also, we considered, a graphical method based on TTT plot as a crude indicator see Aarset (1987). The empirical TTT is given as

$$\frac{S_r}{S_n} = \frac{\sum_{i=1}^{r} x_{(i)} + (n-r)x_{(r)}}{\sum_{i=1}^{n} x_{(i)}},$$

where $r = 1,2,\cdots,n$ and $x_{(r)}$ is the order statistics of the sample. Figure (3.1), represents the TTT plot for given data set and it indicate the increasing then decreasing failure rate functions along with PDF/CDF plot. Figure (3.2), represents the PP plot, sample QQ plot, and KMP plot i.e., Kaplan–Meier plot respectively, which can be suitable to PIED. Figure (3.3), represents the K-S plot, hazard plot and Likelihood plots, Contour plot, Contour3D plot respectively.

Data Analysis

Figure (3.4), shows the PDF in first column and CDF in second column, Figure (3.5) shows the P-P plot in first column and Q-Q plot in second column, Figure (3.6) shows the TTT plot in first column and KM plot in second column, it all for different combinations $n = 48, m = 10, 19, 24, 28$. These figures are helpful for showing which features of the data sets are well captured by the PIED model. We may also see from Figure (3.7) at several combination $(n = 48, m = 10, 19, 24, 28)$ of EDF (empirical distribution function) plot with K-S bound and hazard plot. From all the graphs, we have not shown the major discrepancies between the sampled distribution (Observed values) and PIED. Further, Table (3.6) shows various combinations of censoring schemes (n,m) i.e., $(48,28),(48,24),(48,19),(48,14),(48,10)$ under PT-II CBR for survival of multiple myeloma patients data set with arbitrary choice of probability of removal $p = 0.05$. Based on all these combinations (n,m) of PT-II CBR, we obtained ML, Bayesian, E-Bayesian estimates of λ,θ under SELF, GELF and LINEX loss function, when $\delta > 0$ and $\delta < 0$ are presented in Tables ((3.8), (3.8)) and Tables ((3.9), (3.10)) with different set of values $(c,\gamma) = (3,2);(4,3)$. We also observed that for all combination of censoring schemes ML, Bayesian and E-Bayesian estimates of λ,θ under SELF, GELF and LINEX loss functions always lies in CI and HPD interval. Even though the each an every combination of censoring scheme for Table (3.6), the length of HPD always less than length of CI, see Tables (3.7), (3.8), (3.9) and Table (3.10) respectively. We now discuss some quantiles and estimate of λ, θ results obtained from PIED model with different samples, which are shown in Table (3.11) and Table (3.12) for $\delta = 0.1$ and $\delta = -0.1$ respectively. It is very interesting to note that through E-Bayesian approach covers the more survival time of myeloma patients in respect of two existing approach i.e. ML method and Bayesian method.

3.7 Conclusion

In this chapter, firstly we have studied on E-Bayesian method to compare with Bayesian estimators for both unknown parameters of PIED under PT-II CBRs. On the other hand, the risk of the E-Bayesian and Bayesian estimators of λ and θ are compared under SELF, GELF and LINEX. Generally, we found that the estimated risk of the E-Bayesian estimate of λ and θ have minimum. Therefore, the simulated results showed that the E-Bayesian estimation method is more efficient and better to perform than Bayesian estimation. Beside this, we have shown the interest with application to survival time of multiple myeloma patients data and applying E-Bayesian inferential procedures for the PIED as underline distribution with PT-II CBRs.

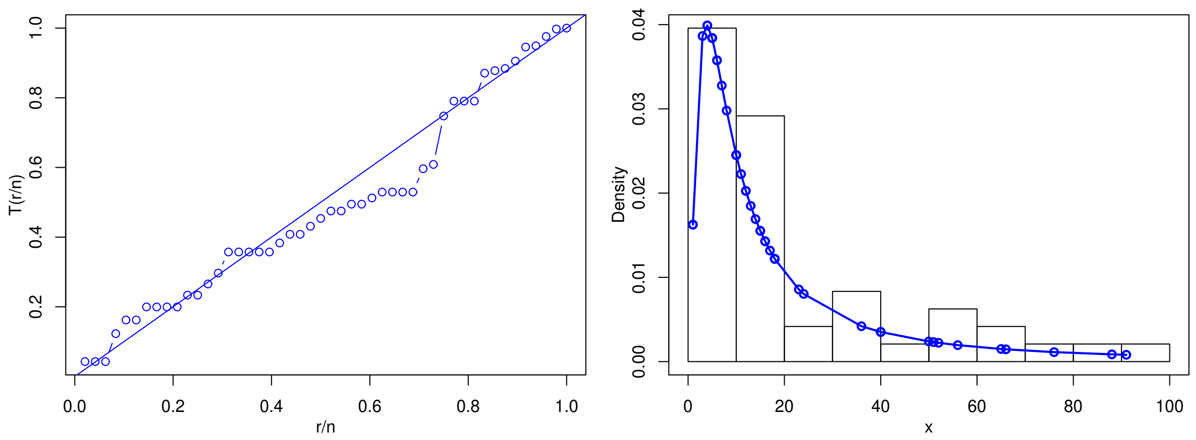

FIGURE 3.1: The Survival time of multiple myeloma patients data, Left panel: TTT plot; Right panel: PDF plot

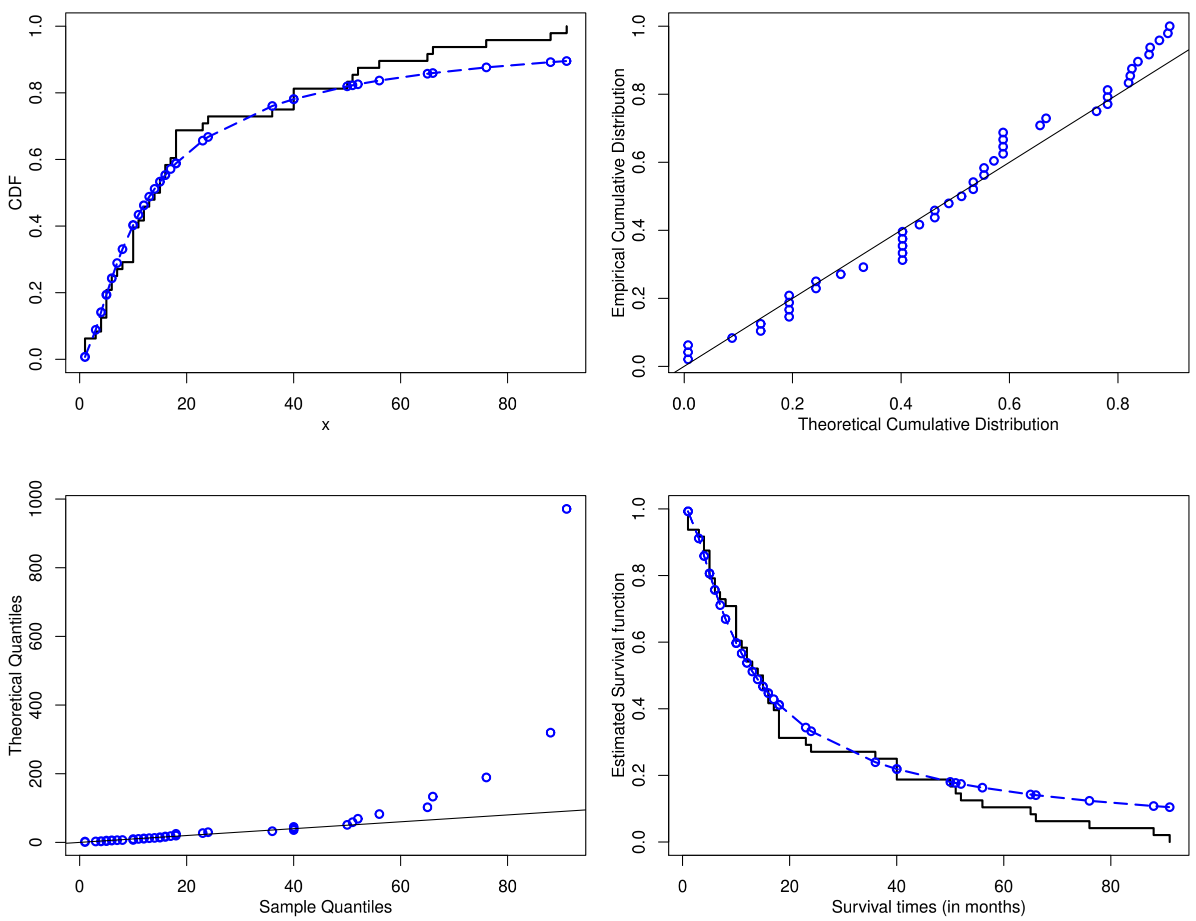

FIGURE 3.2: The survival time of multiple myeloma patients data, Left panel: CDF plot and Q-Q plot, Right panel: P-P plot and KMP plot

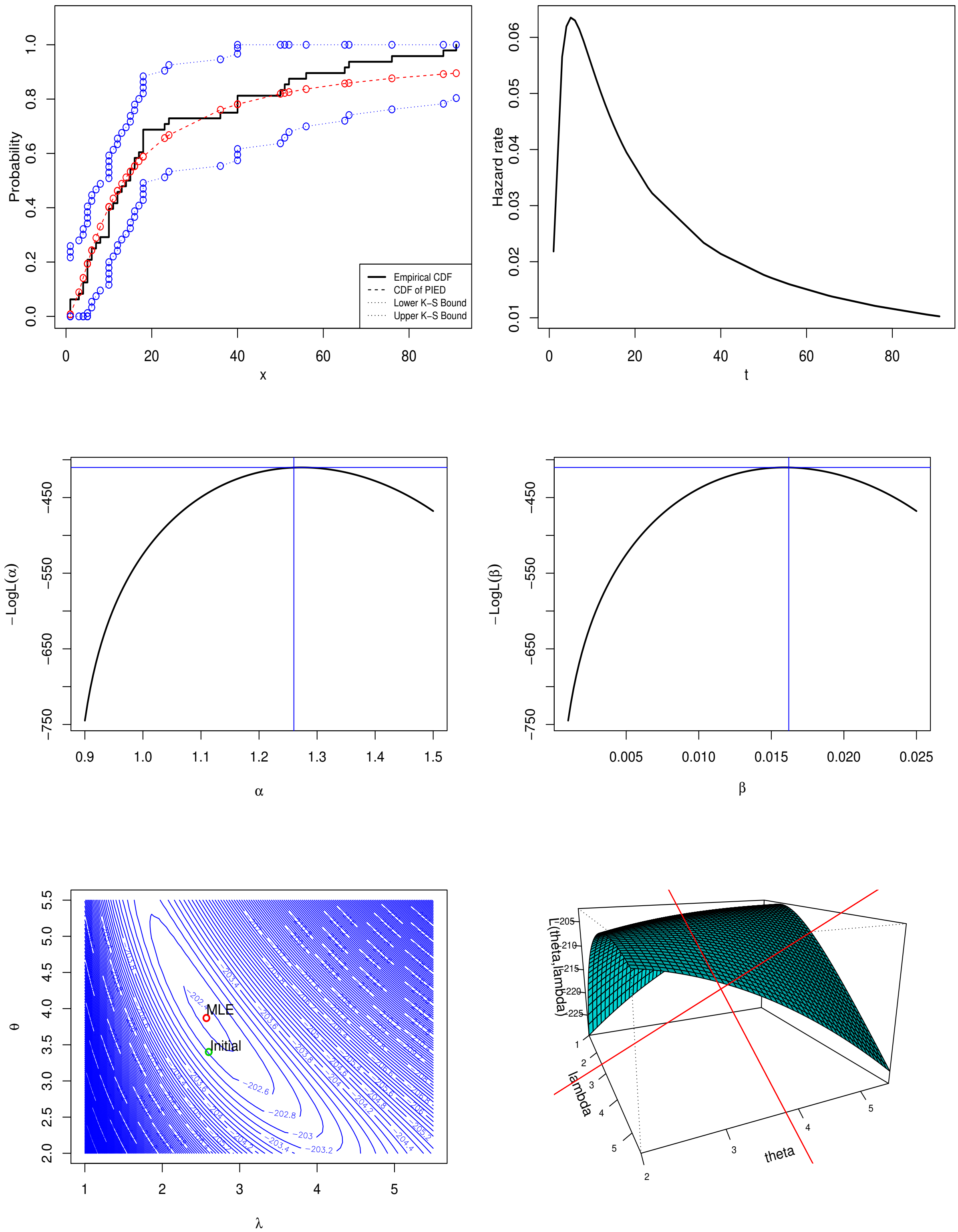

FIGURE 3.3: K-S plot, hazard plot and Likelihood plots, Contour plot, Contour3D plot shows the ML estimates of λ and θ based on the survival time of multiple myeloma patients data.

TABLE 3.2: Risks and different estimators, CI and HPD interval for parameters λ and θ under SELF for fixed $\lambda = 1.1$, $\theta = 0.8$, $\delta = 0.1, p = 0.05, c = 4, \gamma = 3$.

(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD
(20,12)	$\hat{R}_S(.)$	0.07569	0.07580	0.07569	0.07573	(0.41713, 1.54596)	0.30367	0.35083	0.30565	0.31641	(0.07498, 0.40008)
	$B(.)$	(0.98338)	(0.98241)	(0.98304)	(0.98155)		(0.24899)	(0.20788)	(0.24719)	(0.23753)	
	$\hat{R}_{EBS1}(.)$	0.07567	0.07578	0.07567		(0.91379, 1.05288)	0.30366	0.35081	0.30564		(0.0829, 0.41571)
	$EBS_1(.)$	(0.98341)	(0.98245)	(0.98308)			(0.24900)	(0.20790)	(0.24720)		
	$\hat{R}_{EBS2}(.)$	0.07536	0.07547	0.07535			0.30364	0.35080	0.30562		
	$EBS_2(.)$	(0.98370)	(0.98273)	(0.98337)			(0.24902)	(0.20791)	(0.24722)		
	$\hat{R}_{EBS3}(.)$	0.07605	0.07616	0.07605			0.30368	0.35083	0.30566		
	$EBS_3(.)$	(0.98313)	(0.98216)	(0.98279)			(0.24898)	(0.20789)	(0.24718)		
(20,10)	$\hat{R}_S(.)$	0.10622	0.10650	0.10626	0.10710	(0.33668, 1.45314)	0.36819	0.41628	0.36962	0.38088	(0.05158, 0.31412)
	$B(.)$	(0.89721)	(0.89617)	(0.89687)	(0.89492)		(0.19323)	(0.15505)	(0.19205)	(0.18285)	
	$\hat{R}_{EBS1}(.)$	0.10630	0.10657	0.10633		(0.82849, 0.96591)	0.36810	0.41621	0.36954		(0.05786, 0.32853)
	$EBS_1(.)$	(0.89712)	(0.89608)	(0.89679)			(0.19330)	(0.15511)	(0.19212)		
	$\hat{R}_{EBS2}(.)$	0.10598	0.10625	0.10601			0.36798	0.41609	0.36942		
	$EBS_2(.)$	(0.89711)	(0.89608)	(0.89678)			(0.19340)	(0.15520)	(0.19222)		
	$\hat{R}_{EBS3}(.)$	0.10668	0.10695	0.10671			0.36823	0.41632	0.36966		
	$EBS_3(.)$	(0.89713)	(0.89609)	(0.89680)			(0.19320)	(0.15502)	(0.19202)		

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Table 3.2 – *Continued from previous page*

(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD
(20,8)	$\hat{R}_S(.)$	0.13512	0.13563	0.13522	0.13587	(0.25083, 1.37284)	0.42862	0.47833	0.42965	0.43943	(0.03085, 0.24338)
	$B(.)$	(0.81319)	(0.81203)	(0.81286)	(0.81183)		(0.14531)	(0.10848)	(0.14453)	(0.13711)	
	$\hat{R}_{EBS1}(.)$	0.13507	0.13559	0.13517		(0.74408, 0.88234)	0.42863	0.47834	0.42966		(0.03354, 0.25507)
	$EBS_1(.)$	(0.81317)	(0.81201)	(0.81284)			(0.14530)	(0.10847)	(0.14452)		
	$\hat{R}_{EBS2}(.)$	0.13511	0.13562	0.13521			0.42869	0.47838	0.42971		
	$EBS_2(.)$	(0.81297)	(0.81181)	(0.81263)			(0.14526)	(0.10844)	(0.14448)		
	$\hat{R}_{EBS3}(.)$	0.13509	0.13560	0.13519			0.42858	0.47830	0.42961		
	$EBS_3(.)$	(0.81338)	(0.81222)	(0.81304)			(0.14535)	(0.10850)	(0.14456)		
(20,6)	$\hat{R}_S(.)$	0.19109	0.19194	0.19126	0.19269	(0.15097, 1.28286)	0.48536	0.53862	0.48606	0.49300	(0.012782, 0.182939)
	$B(.)$	(0.71886)	(0.71752)	(0.71851)	(0.71692)		(0.10333)	(0.06615)	(0.10282)	(0.09786)	
	$\hat{R}_{EBS1}(.)$	0.19115	0.19200	0.19132		(0.64896, 0.78815)	0.48538	0.53864	0.48608		(0.005229, 0.18151)
	$EBS_1(.)$	(0.71873)	(0.71739)	(0.71838)			(0.10333)	(0.06615)	(0.10283)		
	$\hat{R}_{EBS2}(.)$	0.19096	0.19181	0.19112			0.48535	0.53861	0.48605		
	$EBS_2(.)$	(0.71873)	(0.71739)	(0.71838)			(0.10333)	(0.06615)	(0.10283)		
	$\hat{R}_{EBS3}(.)$	0.19139	0.19224	0.19156			0.48541	0.53866	0.48611		
	$EBS_3(.)$	(0.71902)	(0.71769)	(0.71867)			(0.10329)	(0.06611)	(0.10279)		

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Table 3.2 – *Continued from previous page*

(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD
(20,4)	$\hat{R}_S(.)$	0.28077	0.28223	0.28104	0.28269	(0.02582, 1.17301)	0.54067	0.59060	0.54109	0.54392	(-0.00233, 0.12731)
	$B(.)$	(0.60146)	(0.59982)	(0.60110)	(0.59941)		(0.06470)	(0.03152)	(0.06441)	(0.06249)	
	$\hat{R}_{EBS1}(.)$	0.28090	0.28236	0.28117		(0.53087, 0.67187)	0.54068	0.59060	0.54110		(0.00004, 0.1212)
	$EBS_1(.)$	(0.60135)	(0.59972)	(0.60100)			(0.06469)	(0.03152)	(0.06441)		
	$\hat{R}_{EBS2}(.)$	0.28086	0.28232	0.28113			0.54064	0.59057	0.54106		
	$EBS_2(.)$	(0.60164)	(0.60001)	(0.60129)			(0.06472)	(0.03153)	(0.06444)		
	$\hat{R}_{EBS3}(.)$	0.28096	0.28242	0.28124			0.54073	0.59062	0.54114		
	$EBS_3(.)$	(0.60106)	(0.59943)	(0.60071)			(0.06466)	(0.03150)	(0.06438)		
(30,18)	$\hat{R}_S(.)$	0.05510	0.05521	0.05512	0.05569	(0.51424, 1.41804)	0.29228	0.31250	0.29360	0.31208	(0.10564, 0.37713)
	$B(.)$	(0.97010)	(0.96947)	(0.96989)	(0.96614)		(0.25942)	(0.24104)	(0.25820)	(0.24138)	
	$\hat{R}_{EBS1}(.)$	0.05513	0.05524	0.05515		(0.91447, 1.02564)	0.29224	0.31247	0.29357		(0.12318, 0.39602)
	$EBS_3(.)$	(0.96988)	(0.96925)	(0.96966)			(0.25945)	(0.24106)	(0.25823)		
	$\hat{R}_{EBS1}(.)$	0.05505	0.05516	0.05507			0.29233	0.31255	0.29365		
	$EBS_2(.)$	(0.96960)	(0.96897)	(0.96939)			(0.25937)	(0.24099)	(0.25815)		
	$\hat{R}_{EBS3}(.)$	0.05527	0.05539	0.05529			0.29216	0.31239	0.29349		

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Table 3.2 – *Continued from previous page*

(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD
	$EBS_3(.)$	(0.97015)	(0.96953)	(0.96994)			(0.25953)	(0.24113)	(0.25830)		
(30,15)	$\hat{R}_S(.)$	0.08814	0.08837	0.08819	0.08970	(0.43187, 1.32347)	0.36033	0.37980	0.36127	0.37894	(0.07605, 0.29282)
	$B(.)$	(0.88239)	(0.88172)	(0.88218)	(0.87767)		(0.19974)	(0.18374)	(0.19895)	(0.18443)	
	$\hat{R}_{EBS1}(.)$	0.08811	0.08834	0.08816		(0.82775, 0.93721)	0.36027	0.37975	0.36122		(0.09078, 0.30932)
	$EBS_1(.)$	(0.88231)	(0.88164)	(0.88211)			(0.19979)	(0.18378)	(0.19900)		
	$\hat{R}_{EBS2}(.)$	0.08826	0.08849	0.08831			0.36022	0.37970	0.36116		
	$EBS_2(.)$	(0.88237)	(0.88170)	(0.88216)			(0.19983)	(0.18382)	(0.19905)		
	$\hat{R}_{EBS3}(.)$	0.08801	0.08824	0.08806			0.36033	0.37980	0.36127		
	$EBS_3(.)$	(0.88226)	(0.88159)	(0.88206)			(0.19974)	(0.18374)	(0.19896)		
(30,12)	$\hat{R}_S(.)$	0.12723	0.12762	0.12732	0.12973	(0.34415, 1.23115)	0.42223	0.44340	0.42289	0.43857	(0.05051, 0.22506)
	$B(.)$	(0.79251)	(0.79177)	(0.79230)	(0.78765)		(0.15022)	(0.13416)	(0.14970)	(0.13776)	
	$\hat{R}_{EBS1}(.)$	0.12731	0.12771	0.12741		(0.73799, 0.84706)	0.42222	0.44340	0.42289		(0.06156, 0.23872)
	$EBS_1(.)$	(0.79248)	(0.79174)	(0.79227)			(0.15022)	(0.13416)	(0.14971)		
	$\hat{R}_{EBS2}(.)$	0.12714	0.12754	0.12723			0.42211	0.44330	0.42278		
	$EBS_2(.)$	(0.79241)	(0.79167)	(0.79220)			(0.15030)	(0.13423)	(0.14979)		
	$\hat{R}_{EBS3}(.)$	0.12753	0.12793	0.12762			0.42233	0.44349	0.42300		

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Table 3.2 – *Continued from previous page*

(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD
	$EBS_3(.)$	(0.79255)	(0.79181)	(0.79235)			(0.15014)	(0.13409)	(0.14962)		
(30,9)	$\hat{R}_S(.)$	0.19194	0.19256	0.19207	0.19479	(0.24596, 1.13782)	0.48073	0.50515	0.48118	0.49303	(0.02841, 0.16727)
	$B(.)$	(0.69648)	(0.69563)	(0.69627)	(0.69189)		(0.10666)	(0.08928)	(0.10633)	(0.09784)	
	$\hat{R}_{EBS1}(.)$	0.19198	0.19260	0.19211		(0.64199, 0.75135)	0.48068	0.50511	0.48114		(0.03512, 0.17785)
	$EBS_1(.)$	(0.69641)	(0.69556)	(0.69620)			(0.10669)	(0.08931)	(0.10636)		
	$\hat{R}_{EBS2}(.)$	0.19241	0.19303	0.19255			0.48056	0.50500	0.48102		
	$EBS_2(.)$	(0.69609)	(0.69556)	(0.69620)			(0.10669)	(0.08931)	(0.10636)		
	$\hat{R}_{EBS3}(.)$	0.19158	0.19220	0.19171			0.48080	0.50522	0.48126		
	$EBS_3(.)$	(0.69673)	(0.69556)	(0.69620)			(0.10669)	(0.08931)	(0.10636)		
(30,6)	$\hat{R}_S(.)$	0.28659	0.28760	0.28678	0.29050	(0.12579, 1.03098)	0.53727	0.56821	0.53755	0.54439	(0.00956, 0.11478)
	$B(.)$	(0.58245)	(0.58140)	(0.58224)	(0.57839)		(0.06701)	(0.04622)	(0.06682)	(0.06218)	
	$\hat{R}_{EBS1}(.)$	0.28653	0.28755	0.28673		(0.52707, 0.63826)	0.53727	0.56822	0.53756		(0.01005, 0.12021)
	$EBS_1(.)$	(0.58246)	(0.58141)	(0.58225)			(0.06701)	(0.04621)	(0.06682)		
	$\hat{R}_{EBS2}(.)$	0.28649	0.28750	0.28668			0.53729	0.56823	0.53757		
	$EBS_2(.)$	(0.58249)	(0.58144)	(0.58227)			(0.06700)	(0.04621)	(0.06681)		
	$\hat{R}_{EBS3}(.)$	0.28661	0.28763	0.28680			0.53726	0.56821	0.53754		
	$EBS_3(.)$	(0.58243)	(0.58144)	(0.58227)			(0.06700)	(0.04621)	(0.06681)		

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Table 3.2 – *Continued from previous page*

(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD
(40,24)	$\hat{R}_S(.)$	0.04457	0.04465	0.04459	0.04533	(0.58674, 1.38123)	0.28437	0.29737	0.28538	0.30786	(0.12499, 0.36536)
	$B(.)$	(0.98973)	(0.98925)	(0.98957)	(0.98398)		(0.26679)	(0.25472)	(0.26584)	(0.24518)	
	$\hat{R}_{EBS1}(.)$	0.04459	0.04467	0.04461		(0.94095, 1.03846)	0.28430	0.29730	0.28531		(0.14675, 0.38678)
	$EBS_1(.)$	(0.98969)	(0.98922)	(0.98953)			(0.26685)	(0.25479)	(0.26591)		
	$\hat{R}_{EBS2}(.)$	0.04491	0.04498	0.04492			0.28427	0.29727	0.28528		
	$EBS_2(.)$	(0.98944)	(0.98922)	(0.98953)			(0.26685)	(0.25479)	(0.26591)		
	$\hat{R}_{EBS3}(.)$	0.04434	0.04442	0.04436			0.28433	0.29733	0.28534		
	$EBS_3(.)$	(0.98994)	(0.98922)	(0.98953)			(0.26685)	(0.25479)	(0.26591)		
(40,20)	$\hat{R}_S(.)$	0.06843	0.06861	0.06848	0.07039	(0.49933, 1.27881)	0.35422	0.36643	0.35493	0.37614	(0.09132, 0.28209)
	$B(.)$	(0.89571)	(0.89521)	(0.89556)	(0.88907)		(0.20485)	(0.19468)	(0.20425)	(0.18671)	
	$\hat{R}_{EBS1}(.)$	0.06845	0.06863	0.06850		(0.84803, 0.94363)	0.35411	0.36632	0.35483		(0.10949, 0.30039)
	$EBS_1(.)$	(0.89573)	(0.89522)	(0.89557)			(0.20494)	(0.19477)	(0.20434)		
	$\hat{R}_{EBS2}(.)$	0.06832	0.06849	0.06836			0.35412	0.36633	0.35483		
	$EBS_2(.)$	(0.89617)	(0.89566)	(0.89601)			(0.20494)	(0.19476)	(0.20434)		
	$\hat{R}_{EBS3}(.)$	0.06865	0.06883	0.06869			0.35411	0.36632	0.35482		
	$EBS_3(.)$	(0.89528)	(0.89566)	(0.89601)			(0.20494)	(0.19476)	(0.20434)		

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Table 3.2 – *Continued from previous page*

(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD
(40,16)	$\hat{R}_S(.)$	0.11770	0.11800	0.11777	0.12114	(0.40579, 1.17501)	0.41857	0.43076	0.41907	0.43734	(0.06251, 0.21487)
	$B(.)$	(0.79740)	(0.79685)	(0.79724)	(0.79040)		(0.15304)	(0.14368)	(0.15265)	(0.13869)	
	$\hat{R}_{EBS1}(.)$	0.11771	0.11801	0.11778		(0.75031, 0.8444)	0.41857	0.43076	0.41907		(0.07666, 0.22974)
	$EBS_1(.)$	(0.79746)	(0.79691)	(0.79730)			(0.15304)	(0.14368)	(0.15265)		
	$\hat{R}_{EBS2}(.)$	0.11760	0.11790	0.11767			0.41866	0.43084	0.41916		
	$EBS_2(.)$	(0.79786)	(0.79691)	(0.79730)			(0.15304)	(0.14368)	(0.15265)		
	$\hat{R}_{EBS3}(.)$	0.11787	0.11816	0.11794			0.41848	0.43068	0.41898		
	$EBS_3(.)$	(0.79706)	(0.79651)	(0.79690)			(0.15311)	(0.14375)	(0.15272)		
(40,12)	$\hat{R}_S(.)$	0.19020	0.19068	0.19030	0.19543	(0.30012, 1.05721)	0.47822	0.49165	0.47856	0.49276	(0.03778, 0.15829)
	$B(.)$	(0.68513)	(0.68451)	(0.68498)	(0.67866)		(0.10847)	(0.09883)	(0.10822)	(0.09804)	
	$\hat{R}_{EBS1}(.)$	0.19027	0.19075	0.19038		(0.63886, 0.73133)	0.47825	0.49168	0.47859		(0.04714, 0.16906)
	$EBS_1(.)$	(0.68506)	(0.68445)	(0.68492)			(0.10844)	(0.09881)	(0.10820)		
	$\hat{R}_{EBS2}(.)$	0.19030	0.19078	0.19040			0.47828	0.49171	0.47862		
	$EBS_2(.)$	(0.68497)	(0.68435)	(0.68483)			(0.10842)	(0.09878)	(0.10818)		
	$\hat{R}_{EBS3}(.)$	0.19028	0.19076	0.19039			0.47822	0.49165	0.47856		
	$EBS_3(.)$	(0.68516)	(0.68454)	(0.68501)			(0.10847)	(0.09883)	(0.10822)		

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Table 3.2 – *Continued from previous page*

(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD
(40,8)	$\hat{R}_S(.)$	0.28359	0.28435	0.28374	0.28939	(0.18689, 0.9705)	0.53493	0.55352	0.53514	0.54410	(0.01665, 0.10809)
	$B(.)$	(0.58468)	(0.58391)	(0.58453)	(0.57868)		(0.06861)	(0.05603)	(0.06847)	(0.06237)	
	$\hat{R}_{EBS1}(.)$	0.28374	0.28450	0.28389		(0.53677, 0.6326)	0.53492	0.55351	0.53513		(0.02147, 0.11579)
	$EBS_1(.)$	(0.58451)	(0.58373)	(0.58435)			(0.06862)	(0.05603)	(0.06848)		
	$\hat{R}_{EBS2}(.)$	0.28389	0.28465	0.28404			0.53489	0.55349	0.53510		
	$EBS_2(.)$	(0.58456)	(0.58378)	(0.58440)			(0.06864)	(0.05605)	(0.06849)		
	$\hat{R}_{EBS3}(.)$	0.28362	0.28437	0.28376			0.53495	0.55353	0.53516		
	$EBS_3(.)$	(0.58447)	(0.58369)	(0.58431)			(0.06860)	(0.05602)	(0.06846)		

~ *CI - Confidence Interval

~ *HPD - Height Posterior Density

~ *MLE - Maximum Likelihood Estimator

~ *Bayesian estimate (in parenthesis)

TABLE 3.3: Risks and different estimators, CI and HPD interval for parameters λ and θ under SELF for fixed $\lambda = 1.1$ and $\theta = 0.8,\ \delta = -0.1,\ p = 0.05,\ c = 4,\ \gamma = 3$.

(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD
(20,12)	$\hat{R}_S(.)$	0.07569	0.07573	0.07570	0.07573	(0.41713, 1.54597)	0.30367	0.31311	0.30170	0.31641	(0.07498, 0.40008)
	$B(.)$	(0.98338)	(0.98306)	(0.98372)	(0.98155)		(0.24899)	(0.24048)	(0.25078)	(0.23753)	
	$\hat{R}_{EBS1}(.)$	0.07567	0.07571	0.07568		(0.91379, 1.05288)	0.30366	0.31309	0.30168		(0.0829, 0.41571)
	$EBS_1(.)$	(0.98341)	(0.98309)	(0.98375)			(0.24900)	(0.24050)	(0.25080)		
	$\hat{R}_{EBS2}(.)$	0.07536	0.07540	0.07537			0.30364	0.31307	0.30166		
	$EBS_2(.)$	(0.98370)	(0.98338)	(0.98404)			(0.24902)	(0.24052)	(0.25082)		
	$\hat{R}_{EBS3}(.)$	0.07605	0.07609	0.07606			0.30368	0.31311	0.30171		
	$EBS_3(.)$	(0.98313)	(0.98280)	(0.98346)			(0.24898)	(0.24048)	(0.25078)		
(20,10)	$\hat{R}_S(.)$	0.10622	0.10631	0.10619	0.10710	(0.33669, 1.45315)	0.36819	0.37714	0.36676	0.38088	(0.05159, 0.31412)
	$B(.)$	(0.89721)	(0.89686)	(0.89754)	(0.89492)		(0.19323)	(0.18590)	(0.19441)	(0.18285)	
	$\hat{R}_{EBS1}(.)$	0.10630	0.10639	0.10626		(0.82849, 0.96591)	0.36810	0.37705	0.36667		(0.05786, 0.32851)
	$EBS_1(.)$	(0.89712)	(0.89678)	(0.89746)			(0.19330)	(0.18597)	(0.19448)		
	$\hat{R}_{EBS3}(.)$	0.10598	0.10607	0.10595			0.36798	0.37693	0.36655		
	$EBS_2(.)$	(0.89711)	(0.89677)	(0.89745)			(0.19340)	(0.18607)	(0.19458)		
	$\hat{R}_{EBS3}(.)$	0.10668	0.10677	0.10665			0.36823	0.37718	0.36680		
	$EBS_3(.)$	(0.89713)	(0.89678)	(0.89746)			(0.19320)	(0.18587)	(0.19438)		

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Table 3.3 – *Continued from previous page*

(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD
(20,8)	$\hat{R}_S(.)$	0.13512	0.13529	0.13502	0.13587	(0.2508, 1.37284)	0.42862	0.43736	0.42760	0.43943	(0.03084, 0.24338)
	$B(.)$	(0.81319)	(0.81281)	(0.81353)	(0.81183)		(0.14531)	(0.13867)	(0.14610)	(0.13711)	
	$\hat{R}_{EBS1}(.)$	0.13507	0.13525	0.13497		(0.74408, 0.88233)	0.42863	0.43737	0.42761		(0.03354, 0.25507)
	$EBS_1(.)$	(0.81317)	(0.81279)	(0.81351)			(0.14530)	(0.13866)	(0.14609)		
	$\hat{R}_{EBS2}(.)$	0.13511	0.13528	0.13500			0.42869	0.43743	0.42766		
	$EBS_2(.)$	(0.81297)	(0.81258)	(0.81331)			(0.14526)	(0.13862)	(0.14605)		
	$\hat{R}_{EBS3}(.)$	0.13509	0.13526	0.13499			0.42858	0.43732	0.42756		
	$EBS_3(.)$	(0.81338)	(0.81299)	(0.81372)			(0.14535)	(0.13870)	(0.14613)		
(20,6)	$\hat{R}_S(.)$	0.19109	0.19137	0.19093	0.19269	(0.15097, 1.28286)	0.48536	0.49411	0.48466	0.49300	(0.01278, 0.18294)
	$B(.)$	(0.71886)	(0.71841)	(0.71920)	(0.71692)		(0.10333)	(0.09708)	(0.10383)	(0.09786)	
	$\hat{R}_{EBS1}(.)$	0.19115	0.19143	0.19099		(0.64896, 0.78815)	0.48538	0.49413	0.48468		(0.00523, 0.1815)
	$EBS_1(.)$	(0.71887)	(0.71843)	(0.71922)			(0.10331)	(0.09706)	(0.10382)		
	$\hat{R}_{EBS2}(.)$	0.19096	0.19124	0.19079			0.48535	0.49410	0.48465		
	$EBS_2(.)$	(0.71873)	(0.71828)	(0.71907)			(0.10333)	(0.09708)	(0.10384)		
	$\hat{R}_{EBS3}(.)$	0.19139	0.19167	0.19123			0.48541	0.49415	0.48471		
	$EBS_3(.)$	(0.71902)	(0.71858)	(0.71937)			(0.10329)	(0.09704)	(0.10379)		

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Table 3.3 – *Continued from previous page*

(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD
(20,4)	$\hat{R}_S(.)$	0.28077	0.28126	0.28050	0.28269	(0.02582, 1.17301)	0.54067	0.54924	0.54026	0.54392	(-0.00233, 0.12731)
	$B(.)$	(0.60146)	(0.60091)	(0.60181)	(0.59941)		(0.06470)	(0.05890)	(0.06498)	(0.06249)	
	$\hat{R}_{EBS1}(.)$	0.28090	0.28138	0.28062		(0.53086, 0.67187)	0.54068	0.54924	0.54026		(0.00003, 0.1212)
	$EBS_1(.)$	(0.60135)	(0.60081)	(0.60171)			(0.06469)	(0.05889)	(0.06498)		
	$\hat{R}_{EBS2}(.)$	0.28086	0.28134	0.28059			0.54064	0.54920	0.54022		
	$EBS_2(.)$	(0.60164)	(0.60110)	(0.60200)			(0.06472)	(0.05892)	(0.06501)		
	$\hat{R}_{EBS3}(.)$	0.28096	0.28145	0.28069			0.54073	0.54928	0.54031		
	$EBS_3(.)$	(0.60106)	(0.60052)	(0.60142)			(0.06466)	(0.05887)	(0.06495)		
(30,18)	$\hat{R}_S(.)$	0.05510	0.05513	0.05508	0.05569	(0.51424, 1.41804)	0.29228	0.29785	0.29096	0.31208	(0.10564, 0.37712)
	$B(.)$	(0.97010)	(0.96989)	(0.97031)	(0.96614)		(0.25942)	(0.25429)	(0.26064)	(0.24138)	
	$\hat{R}_{EBS1}(.)$	0.05513	0.05517	0.05511		(0.91447, 1.02564)	0.29224	0.29782	0.29093		(0.12318, 0.39602)
	$EBS_1(.)$	(0.96988)	(0.96967)	(0.97009)			(0.25945)	(0.25432)	(0.26067)		
	$\hat{R}_{EBS2}(.)$	0.05505	0.05509	0.05503			0.29233	0.29791	0.29101		
	$EBS_2(.)$	(0.96960)	(0.96939)	(0.96981)			(0.25937)	(0.25424)	(0.26060)		
	$\hat{R}_{EBS3}(.)$	0.05527	0.05531	0.05525			0.29216	0.29774	0.29085		

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Table 3.3 – *Continued from previous page*

(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD
	$EBS_3(.)$	(0.97015)	(0.96994)	(0.97036)			(0.25953)	(0.25439)	(0.26075)		
(30,15)	$\hat{R}_S(.)$	0.08814	0.08822	0.08809	0.08970	(0.43187, 1.32346)	0.36033	0.36555	0.35939	0.37894	(0.07605, 0.29282)
	$B(.)$	(0.88239)	(0.88217)	(0.88260)	(0.87767)		(0.19974)	(0.19541)	(0.20052)	(0.18443)	
	$\hat{R}_{EBS1}(.)$	0.08811	0.08819	0.08806		(0.82775, 0.93721)	0.36027	0.36549	0.35933		(0.09078, 0.30932)
	$EBS_1(.)$	(0.88231)	(0.88209)	(0.88252)			(0.19979)	(0.19545)	(0.20057)		
	$\hat{R}_{EBS2}(.)$	0.08826	0.08834	0.08821			0.36022	0.36544	0.35928		
	$EBS_2(.)$	(0.88237)	(0.88214)	(0.88257)			(0.19983)	(0.19550)	(0.20062)		
	$\hat{R}_{EBS3}(.)$	0.08801	0.08809	0.08796			0.36033	0.36555	0.35939		
	$EBS_3(.)$	(0.88226)	(0.88204)	(0.88247)			(0.19974)	(0.19541)	(0.20052)		
(30,12)	$\hat{R}_S(.)$	0.12723	0.12736	0.12714	0.12973	(0.34415, 1.23115)	0.42223	0.42725	0.42156	0.43857	(0.05051, 0.22501)
	$B(.)$	(0.79251)	(0.79226)	(0.79271)	(0.78765)		(0.15022)	(0.14636)	(0.15073)	(0.13776)	
	$\hat{R}_{EBS1}(.)$	0.12731	0.12745	0.12722		(0.73799, 0.84707)	0.42222	0.42725	0.42156		(0.06156, 0.23872)
	$EBS_1(.)$	(0.79248)	(0.79223)	(0.79268)			(0.15022)	(0.14636)	(0.15073)		
	$\hat{R}_{EBS2}(.)$	0.12714	0.12727	0.12705			0.42211	0.42714	0.42145		
	$EBS_2(.)$	(0.79241)	(0.79216)	(0.79261)			(0.15030)	(0.14644)	(0.15082)		
	$\hat{R}_{EBS3}(.)$	0.12753	0.12766	0.12744			0.42233	0.42736	0.42167		

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Table 3.3 – *Continued from previous page*

(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD
	$EBS_3(.)$	(0.79255)	(0.79230)	(0.79275)			(0.15014)	(0.14628)	(0.15065)		
(30,9)	$\hat{R}_S(.)$	0.19194	0.19215	0.19181	0.19479	(0.24596, 1.13783)	0.48073	0.48580	0.48027	0.49303	(0.02841, 0.16727)
	$B(.)$	(0.69648)	(0.69620)	(0.69669)	(0.69189)		(0.10666)	(0.10301)	(0.10699)	(0.09784)	
	$\hat{R}_{EBS1}(.)$	0.19198	0.19218	0.19185		(0.64198, 0.75135)	0.48068	0.48575	0.48022		(0.03512, 0.17786)
	$EBS_1(.)$	(0.69641)	(0.69612)	(0.69662)			(0.10669)	(0.10304)	(0.10702)		
	$\hat{R}_{EBS2}(.)$	0.19241	0.19262	0.19228			0.48056	0.48564	0.48010		
	$EBS_2(.)$	(0.69609)	(0.69580)	(0.69629)			(0.10678)	(0.10313)	(0.10711)		
	$\hat{R}_{EBS3}(.)$	0.19158	0.19179	0.19145			0.48080	0.48587	0.48034		
	$EBS_3(.)$	(0.69673)	(0.69645)	(0.69694)			(0.10661)	(0.10296)	(0.10694)		
(30,6)	$\hat{R}_S(.)$	0.28659	0.28692	0.28639	0.29050	(0.12579, 1.03098)	0.53727	0.54262	0.53699	0.54439	(0.00956, 0.11479)
	$B(.)$	(0.58245)	(0.58210)	(0.58266)	(0.57839)		(0.06701)	(0.06337)	(0.06721)	(0.06218)	
	$\hat{R}_{EBS1}(.)$	0.28653	0.28687	0.28634		(0.52709, 0.63821)	0.53727	0.54262	0.53699		(0.01005, 0.12021)
	$EBS_1(.)$	(0.58246)	(0.58211)	(0.58267)			(0.06701)	(0.06337)	(0.06720)		
	$\hat{R}_{EBS2}(.)$	0.28649	0.28682	0.28629			0.53729	0.54264	0.53700		
	$EBS_2(.)$	(0.58249)	(0.58214)	(0.58270)			(0.06700)	(0.06336)	(0.06720)		
	$\hat{R}_{EBS3}(.)$	0.28661	0.28695	0.28641			0.53726	0.54261	0.53698		
	$EBS_3(.)$	(0.58243)	(0.58208)	(0.58265)			(0.06702)	(0.06338)	(0.06721)		

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Table 3.3 – *Continued from previous page*

(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD
(40,24)	$\hat{R}_S(.)$	0.04457	0.04460	0.04456	0.04533	(0.58674, 1.38123)	0.28437	0.28839	0.28336	0.30786	(0.12499, 0.36536)
	$B(.)$	(0.98973)	(0.98957)	(0.98989)	(0.98398)		(0.26679)	(0.26303)	(0.26773)	(0.24518)	
	$\hat{R}_{EBS1}(.)$	0.04459	0.04462	0.04458		(0.94095, 1.03845)	0.28430	0.28832	0.28329		(0.14676, 0.38678)
	$EBS_1(.)$	(0.98969)	(0.98954)	(0.98985)			(0.26685)	(0.26310)	(0.26780)		
	$\hat{R}_{EBS2}(.)$	0.04491	0.04493	0.04489			0.28427	0.28829	0.28326		
	$EBS_2(.)$	(0.98944)	(0.98929)	(0.98960)			(0.26689)	(0.26313)	(0.26784)		
	$\hat{R}_{EBS3}(.)$	0.04434	0.04437	0.04433			0.28433	0.28835	0.28332		
	$EBS_3(.)$	(0.98994)	(0.98979)	(0.99011)			(0.26682)	(0.26306)	(0.26777)		
(40,20)	$\hat{R}_S(.)$	0.11770	0.11780	0.11763	0.12114	(0.49933, 1.27881)	0.41857	0.42210	0.41807	0.43734	(0.09132, 0.28209)
	$B(.)$	(0.79740)	(0.79721)	(0.79755)	(0.79040)		(0.15304)	(0.15032)	(0.15342)	(0.13869)	
	$\hat{R}_{EBS1}(.)$	0.11771	0.11781	0.11764		(0.84803, 0.94363)	0.41857	0.42209	0.41807		(0.10949, 0.30039)
	$EBS_1(.)$	(0.79746)	(0.79727)	(0.79761)			(0.15304)	(0.15032)	(0.15342)		
	$\hat{R}_{EBS2}(.)$	0.11760	0.11770	0.11753			0.41866	0.42218	0.41816		
	$EBS_2(.)$	(0.79786)	(0.79767)	(0.79801)			(0.15297)	(0.15025)	(0.15336)		
	$\hat{R}_{EBS3}(.)$	0.11787	0.11797	0.11779			0.41848	0.42201	0.41798		
	$EBS_3(.)$	(0.79706)	(0.79687)	(0.79721)			(0.15311)	(0.15038)	(0.15349)		

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Table 3.3 – *Continued from previous page*

(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD
(40,16)	$\hat{R}_S(.)$	0.06843	0.06849	0.06839	0.07039	(0.4058, 1.17501)	0.35422	0.35793	0.35351	0.37614	(0.06251, 0.21487)
	$B(.)$	(0.89571)	(0.89554)	(0.89587)	(0.88907)		(0.20485)	(0.20174)	(0.20545)	(0.18671)	
	$\hat{R}_{EBS1}(.)$	0.06845	0.06851	0.06841		(0.75031, 0.8444)	0.35411	0.35783	0.35340		(0.07666, 0.22974)
	$EBS_1(.)$	(0.89573)	(0.89556)	(0.89588)			(0.20494)	(0.20183)	(0.20554)		
	$\hat{R}_{EBS2}(.)$	0.06832	0.06837	0.06827			0.35412	0.35783	0.35341		
	$EBS_2(.)$	(0.89617)	(0.89600)	(0.89632)			(0.20494)	(0.20182)	(0.20554)		
	$\hat{R}_{EBS3}(.)$	0.06865	0.06871	0.06860			0.35411	0.35782	0.35340		
	$EBS_3(.)$	(0.89528)	(0.89512)	(0.89544)			(0.20495)	(0.20183)	(0.20555)		
(40,12)	$\hat{R}_S(.)$	0.19020	0.19036	0.19009	0.19543	(0.30012, 1.05722)	0.47822	0.48169	0.47789	0.49276	(0.03778, 0.15829)
	$B(.)$	(0.68513)	(0.68492)	(0.68527)	(0.67866)		(0.10847)	(0.10596)	(0.10871)	(0.09804)	
	$\hat{R}_{EBS1}(.)$	0.19027	0.19043	0.19017		(0.63886, 0.73133)	0.47825	0.48172	0.47791		(0.04714, 0.16906)
	$EBS_1(.)$	(0.68506)	(0.68486)	(0.68521)			(0.10844)	(0.10594)	(0.10869)		
	$\hat{R}_{EBS2}(.)$	0.19030	0.19046	0.19019			0.47828	0.48175	0.47795		
	$EBS_2(.)$	(0.68497)	(0.68477)	(0.68512)			(0.10842)	(0.10592)	(0.10867)		
	$\hat{R}_{EBS3}(.)$	0.19028	0.19044	0.19018			0.47822	0.48169	0.47788		
	$EBS_3(.)$	(0.68516)	(0.68495)	(0.68530)			(0.10847)	(0.10596)	(0.10871)		

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Table 3.3 – *Continued from previous page*

(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD
(40,8)	$\hat{R}_S(.)$	0.28359	0.28385	0.28345	0.28939	(0.186898, 0.97046)	0.53493	0.53859	0.53472	0.54410	(0.01665, 0.10809)
	$B(.)$	(0.58468)	(0.58443)	(0.58484)	(0.57868)		(0.06861)	(0.06611)	(0.06876)	(0.06237)	
	$\hat{R}_{EBS1}(.)$	0.28374	0.28400	0.28360		(0.53677, 0.6326)	0.53492	0.53858	0.53471		(0.02147, 0.11579)
	$EBS_1(.)$	(0.58451)	(0.58425)	(0.58467)			(0.06862)	(0.06612)	(0.06876)		
	$\hat{R}_{EBS2}(.)$	0.28389	0.28414	0.28375			0.53489	0.53855	0.53468		
	$EBS_2(.)$	(0.58456)	(0.58430)	(0.58472)			(0.06864)	(0.06614)	(0.06878)		
	$\hat{R}_{EBS3}(.)$	0.28362	0.28387	0.28347			0.53495	0.53861	0.53474		
	$EBS_3(.)$	(0.58447)	(0.58421)	(0.58462)			(0.06860)	(0.06610)	(0.06875)		

~ *CI - Confidence Interval
~ *HPD - Height Posterior Density
~ *MLE - Maximum Likelihood Estimator
~ *Bayesian estimate (in parenthesis)

TABLE 3.4: Risks of estimators of λ and θ under GELF and LINEX with fixed value $\lambda = 1.1$, $\theta = 0.8$, $\delta = 0.1, p = 0.05, c = 4, \gamma = 3$.

		GELF						LINEX					
(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$
	$\hat{R}_{ML}$	0.00982			0.15217			0.00922			0.03609		
(20,12)	$\hat{R}_G(.)$	0.00978	0.00981	0.00979	0.14164	0.18492	0.14325	0.00923	0.00923	0.00922	0.03470	0.03982	0.03492
	$\hat{R}_{EBG1}(.)$	0.00977	0.00981	0.00978	0.14163	0.18491	0.14324	0.00922	0.00923	0.00922	0.03470	0.03982	0.03491
	$\hat{R}_{EBG2}(.)$	0.00973	0.00976	0.00973	0.14163	0.18490	0.14323	0.00919	0.00920	0.00918	0.03470	0.03982	0.03491
	$\hat{R}_{EBG3}(.)$	0.00983	0.00986	0.00984	0.14166	0.18493	0.14326	0.00927	0.00928	0.00927	0.03470	0.03982	0.03492
	$\hat{R}_{ML}$	0.01516			0.21613			0.01282			0.04307		
(20,10)	$\hat{R}_G(.)$	0.01497	0.01503	0.01498	0.20191	0.26791	0.20347	0.01273	0.01276	0.01273	0.04170	0.04686	0.04186
	$\hat{R}_{EBG1}(.)$	0.01498	0.01504	0.01499	0.20183	0.26781	0.20339	0.01274	0.01277	0.01274	0.04169	0.04685	0.04185
	$\hat{R}_{EBG2}(.)$	0.01496	0.01502	0.01497	0.20170	0.26768	0.20326	0.01269	0.01272	0.01270	0.04168	0.04684	0.04183
	$\hat{R}_{EBG3}(.)$	0.01501	0.01507	0.01502	0.20197	0.26796	0.20353	0.01279	0.01282	0.01279	0.04171	0.04686	0.04186
	$\hat{R}_{ML}$	0.02148			0.29597			0.01588			0.04933		
(20,8)	$\hat{R}_G(.)$	0.02133	0.02144	0.02135	0.27911	0.37076	0.28066	0.01579	0.01585	0.01581	0.04818	0.05345	0.04829
	$\hat{R}_{EBG1}(.)$	0.02132	0.02143	0.02134	0.27913	0.37079	0.28068	0.01579	0.01585	0.01580	0.04818	0.05345	0.04829
	$\hat{R}_{EBG2}(.)$	0.02132	0.02143	0.02135	0.27922	0.37089	0.28077	0.01579	0.01585	0.01580	0.04819	0.05346	0.04830
	$\hat{R}_{EBG3}(.)$	0.02132	0.02143	0.02135	0.27906	0.37071	0.28061	0.01579	0.01585	0.01580	0.04818	0.05345	0.04829
	$\hat{R}_{ML}$	0.03312			0.40037			0.02224			0.05501		
(20,6)	$\hat{R}_G(.)$	0.03278	0.03297	0.03282	0.38282	0.53828	0.38438	0.02207	0.02216	0.02208	0.05420	0.05979	0.05427

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Table 3.4 – *Continued from previous page*

		GELF						LINEX					
(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$
	$\hat{R}_{EBG1}(.)$	0.03279	0.03298	0.03283	0.38287	0.53832	0.38443	0.02207	0.02217	0.02209	0.05420	0.05980	0.05428
	$\hat{R}_{EBG2}(.)$	0.03277	0.03295	0.03280	0.38281	0.53826	0.38437	0.02205	0.02214	0.02207	0.05420	0.05979	0.05427
	$\hat{R}_{EBG3}(.)$	0.03283	0.03301	0.03287	0.38294	0.53840	0.38451	0.02210	0.02220	0.02212	0.05420	0.05980	0.05428
	$\hat{R}_{ML}$	0.05479			0.55437			0.03211			0.06035		
(20,4)	$\hat{R}_G(.)$	0.05427	0.05464	0.05433	0.54190	0.82765	0.54348	0.03190	0.03206	0.03193	0.06001	0.06521	0.06005
	$\hat{R}_{EBG1}(.)$	0.05430	0.05467	0.05436	0.54193	0.82768	0.54351	0.03191	0.03207	0.03194	0.06001	0.06521	0.06006
	$\hat{R}_{EBG2}(.)$	0.05431	0.05467	0.05437	0.54178	0.82751	0.54335	0.03190	0.03206	0.03193	0.06001	0.06521	0.06005
	$\hat{R}_{EBG3}(.)$	0.05430	0.05467	0.05437	0.54210	0.82787	0.54368	0.03192	0.03208	0.03195	0.06002	0.06521	0.06006
	$\hat{R}_{ML}$	0.00714			0.14851			0.00675			0.03562		
(30,18)	$\hat{R}_G(.)$	0.00703	0.00705	0.00704	0.13266	0.14891	0.13368	0.00668	0.00670	0.00669	0.03345	0.03566	0.03360
	$\hat{R}_{EBG1}(.)$	0.00704	0.00706	0.00704	0.13264	0.14889	0.13366	0.00669	0.00670	0.00669	0.03345	0.03566	0.03359
	$\hat{R}_{EBG2}(.)$	0.00703	0.00706	0.00704	0.13271	0.14896	0.13373	0.00668	0.00669	0.00668	0.03346	0.03567	0.03360
	$\hat{R}_{EBG3}(.)$	0.00705	0.00707	0.00706	0.13258	0.14883	0.13360	0.00671	0.00672	0.00671	0.03344	0.03565	0.03359
	$\hat{R}_{ML}$	0.01308			0.21387			0.01058			0.04286		
(30,15)	$\hat{R}_G(.)$	0.01280	0.01284	0.01281	0.19355	0.21491	0.19453	0.01041	0.01044	0.01041	0.04085	0.04295	0.04096
	$\hat{R}_{EBG1}(.)$	0.01280	0.01284	0.01281	0.19349	0.21485	0.19447	0.01041	0.01043	0.01041	0.04085	0.04295	0.04095
	$\hat{R}_{EBG2}(.)$	0.01281	0.01285	0.01282	0.19343	0.21479	0.19442	0.01043	0.01045	0.01043	0.04084	0.04294	0.04094
	$\hat{R}_{EBG3}(.)$	0.01279	0.01283	0.01280	0.19356	0.21492	0.19454	0.01039	0.01042	0.01040	0.04085	0.04295	0.04096

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Table 3.4 – *Continued from previous page*

		GELF						LINEX					
(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$
	$\hat{R}_{ML}$	0.02025			0.29458			0.01514			0.04924		
(30,12)	$\hat{R}_G(.)$	0.01980	0.01988	0.01982	0.26964	0.30356	0.27061	0.01485	0.01489	0.01486	0.04750	0.04975	0.04757
	$\hat{R}_{EBG1}(.)$	0.01982	0.01989	0.01984	0.26964	0.30356	0.27061	0.01486	0.01491	0.01487	0.04750	0.04975	0.04757
	$\hat{R}_{EBG2}(.)$	0.01979	0.01987	0.01981	0.26949	0.30340	0.27046	0.01484	0.01488	0.01485	0.04749	0.04974	0.04756
	$\hat{R}_{EBG3}(.)$	0.01985	0.01993	0.01987	0.26981	0.30373	0.27078	0.01489	0.01493	0.01490	0.04751	0.04976	0.04758
	$\hat{R}_{ML}$	0.03305			0.40040			0.02247			0.05501		
(30,9)	$\hat{R}_G(.)$	0.03246	0.03259	0.03249	0.37267	0.43203	0.37366	0.02215	0.02222	0.02216	0.05371	0.05628	0.05376
	$\hat{R}_{EBG1}(.)$	0.03247	0.03260	0.03250	0.37257	0.43192	0.37356	0.02215	0.02222	0.02217	0.05370	0.05628	0.05375
	$\hat{R}_{EBG2}(.)$	0.03256	0.03269	0.03259	0.37232	0.43166	0.37330	0.02220	0.02227	0.02222	0.05369	0.05627	0.05374
	$\hat{R}_{EBG3}(.)$	0.03240	0.03253	0.03242	0.37284	0.43221	0.37383	0.02211	0.02218	0.02212	0.05372	0.05629	0.05377
	$\hat{R}_{ML}$	0.05524			0.55618			0.03304			0.06040		
(30,6)	$\hat{R}_G(.)$	0.05427	0.05452	0.05431	0.52934	0.67039	0.53037	0.03261	0.03272	0.03263	0.05965	0.06288	0.05968
	$\hat{R}_{EBG1}(.)$	0.05425	0.05450	0.05430	0.52936	0.67040	0.53038	0.03260	0.03272	0.03262	0.05966	0.06289	0.05968
	$\hat{R}_{EBG2}(.)$	0.05424	0.05449	0.05428	0.52941	0.67046	0.53044	0.03260	0.03271	0.03262	0.05966	0.06289	0.05969
	$\hat{R}_{EBG3}(.)$	0.05428	0.05453	0.05433	0.52932	0.67036	0.53035	0.03261	0.03272	0.03263	0.05965	0.06288	0.05968
	$\hat{R}_{ML}$	0.00568			0.14500			0.00551			0.03516		
(40,24)	$\hat{R}_G(.)$	0.00555	0.00556	0.00555	0.12669	0.13661	0.12744	0.00542	0.00543	0.00542	0.03259	0.03401	0.03270

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Table 3.4 – *Continued from previous page*

		GELF						LINEX					
(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$
	$\hat{R}_{EBG1}(.)$	0.00555	0.00556	0.00555	0.12663	0.13656	0.12739	0.00543	0.00543	0.00543	0.03258	0.03400	0.03269
	$\hat{R}_{EBG2}(.)$	0.00559	0.00560	0.00559	0.12663	0.13655	0.12738	0.00547	0.00547	0.00547	0.03258	0.03400	0.03269
	$\hat{R}_{EBG3}(.)$	0.00552	0.00554	0.00553	0.12665	0.13658	0.12741	0.00539	0.00540	0.00540	0.03258	0.03401	0.03269
	$\hat{R}_{ML}$	0.00996			0.21070			0.00833			0.04256		
(40,20)	$\hat{R}_G(.)$	0.00963	0.00966	0.00964	0.18727	0.20001	0.18799	0.00810	0.00812	0.00811	0.04019	0.04151	0.04027
	$\hat{R}_{EBG1}(.)$	0.00963	0.00966	0.00964	0.18716	0.19990	0.18788	0.00810	0.00812	0.00811	0.04018	0.04150	0.04026
	$\hat{R}_{EBG2}(.)$	0.00962	0.00965	0.00963	0.18717	0.19991	0.18789	0.00809	0.00811	0.00809	0.04018	0.04150	0.04026
	$\hat{R}_{EBG3}(.)$	0.00966	0.00968	0.00966	0.18717	0.19990	0.18789	0.00813	0.00815	0.00813	0.04018	0.04150	0.04026
	$\hat{R}_{ML}$	0.01836			0.29260			0.01417			0.04911		
(40,16)	$\hat{R}_G(.)$	0.01778	0.01783	0.01779	0.26439	0.28239	0.26510	0.01378	0.01381	0.01379	0.04711	0.04841	0.04716
	$\hat{R}_{EBG1}(.)$	0.01778	0.01784	0.01780	0.26439	0.28239	0.26510	0.01378	0.01381	0.01379	0.04711	0.04841	0.04716
	$\hat{R}_{EBG2}(.)$	0.01776	0.01781	0.01777	0.26453	0.28253	0.26524	0.01377	0.01380	0.01378	0.04712	0.04842	0.04717
	$\hat{R}_{EBG3}(.)$	0.01782	0.01787	0.01783	0.26427	0.28227	0.26499	0.01380	0.01383	0.01380	0.04710	0.04840	0.04715
	$\hat{R}_{ML}$	0.03260			0.39974			0.02256			0.05498		
(40,12)	$\hat{R}_G(.)$	0.03153	0.03163	0.03155	0.36734	0.39717	0.36805	0.02197	0.02202	0.02198	0.05345	0.05486	0.05348
	$\hat{R}_{EBG1}(.)$	0.03155	0.03164	0.03157	0.36740	0.39723	0.36811	0.02198	0.02203	0.02199	0.05345	0.05487	0.05348
	$\hat{R}_{EBG2}(.)$	0.03156	0.03165	0.03158	0.36748	0.39731	0.36819	0.02198	0.02204	0.02199	0.05345	0.05487	0.05349
	$\hat{R}_{EBG3}(.)$	0.03154	0.03164	0.03156	0.36734	0.39717	0.36805	0.02198	0.02203	0.02199	0.05344	0.05486	0.05348

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Table 3.4 – *Continued from previous page*

		GELF						LINEX					
(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$
	$\hat{R}_{ML}$	0.05449			0.55504			0.03293			0.06037		
(40,8)	$\hat{R}_G(.)$	0.05308	0.05327	0.05311	0.52098	0.59591	0.52172	0.03229	0.03238	0.03231	0.05941	0.06135	0.05943
	$\hat{R}_{EBG1}(.)$	0.05311	0.05330	0.05315	0.52094	0.59587	0.52168	0.03231	0.03239	0.03233	0.05941	0.06135	0.05943
	$\hat{R}_{EBG2}(.)$	0.05318	0.05337	0.05322	0.52085	0.59577	0.52159	0.03233	0.03241	0.03234	0.05941	0.06135	0.05943
	$\hat{R}_{EBG3}(.)$	0.05305	0.05324	0.05309	0.52104	0.59598	0.52179	0.03230	0.03238	0.03232	0.05941	0.06135	0.05943

TABLE 3.5: Risks of estimators of λ and θ under GELF and LINEX with fixed value $\lambda = 1.1$, $\theta = 0.8$, $\delta = -0.1, p = 0.05, c = 4, \gamma = 3$.

		GELF						LINEX					
(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$
(20,12)	$\hat{R}_{ML}$	0.01103			0.22833			0.00978			0.04354		
	$\hat{R}_G(.)$	0.01098	0.01099	0.01097	0.20924	0.22329	0.20641	0.00977	0.00977	0.00977	0.04170	0.04306	0.04142
	$\hat{R}_{EBG1}(.)$	0.01096	0.01098	0.01096	0.20923	0.22327	0.20640	0.00976	0.00977	0.00976	0.04170	0.04306	0.04141
	$\hat{R}_{EBG2}(.)$	0.01091	0.01092	0.01090	0.20922	0.22326	0.20639	0.00972	0.00973	0.00972	0.04170	0.04306	0.04141
	$\hat{R}_{EBG3}(.)$	0.01103	0.01104	0.01102	0.20927	0.22331	0.20644	0.00981	0.00982	0.00981	0.04170	0.04306	0.04142
(20,10)	$\hat{R}_{ML}$	0.01763			0.35392			0.01406			0.05291		
	$\hat{R}_G(.)$	0.01739	0.01742	0.01737	0.32460	0.34499	0.32146	0.01394	0.01395	0.01393	0.05105	0.05236	0.05085
	$\hat{R}_{EBG1}(.)$	0.01740	0.01743	0.01738	0.32442	0.34481	0.32128	0.01395	0.01396	0.01394	0.05104	0.05235	0.05083
	$\hat{R}_{EBG2}(.)$	0.01739	0.01742	0.01737	0.32417	0.34455	0.32103	0.01391	0.01392	0.01390	0.05102	0.05233	0.05082
	$\hat{R}_{EBG3}(.)$	0.01742	0.01745	0.01741	0.32471	0.34511	0.32157	0.01399	0.01400	0.01398	0.05106	0.05237	0.05085
(20,8)	$\hat{R}_{ML}$	0.02600			0.53381			0.01825			0.06154		
	$\hat{R}_G(.)$	0.02581	0.02586	0.02578	0.49367	0.52581	0.49006	0.01815	0.01817	0.01813	0.05994	0.06123	0.05978
	$\hat{R}_{EBG1}(.)$	0.02579	0.02584	0.02576	0.49372	0.52586	0.49011	0.01814	0.01816	0.01813	0.05994	0.06123	0.05979
	$\hat{R}_{EBG2}(.)$	0.02580	0.02584	0.02577	0.49393	0.52608	0.49032	0.01814	0.01817	0.01813	0.05995	0.06124	0.05979
	$\hat{R}_{EBG3}(.)$	0.02580	0.02585	0.02577	0.49356	0.52570	0.48995	0.01814	0.01817	0.01813	0.05993	0.06122	0.05978
(20,6)	$\hat{R}_{ML}$	0.04164			0.80896			0.02621			0.06952		
	$\hat{R}_G(.)$	0.04118	0.04127	0.04113	0.75944	0.81642	0.75512	0.02599	0.02603	0.02596	0.06837	0.06968	0.06827

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Table 3.5 – *Continued from previous page*

		GELF						LINEX					
(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$
	$\hat{R}_{EBG1}(.)$	0.04120	0.04128	0.04115	0.75958	0.81656	0.75525	0.02600	0.02604	0.02597	0.06838	0.06969	0.06827
	$\hat{R}_{EBG2}(.)$	0.04117	0.04125	0.04112	0.75942	0.81640	0.75510	0.02597	0.02601	0.02595	0.06837	0.06968	0.06827
	$\hat{R}_{EBG3}(.)$	0.04124	0.04133	0.04119	0.75979	0.81679	0.75547	0.02603	0.02607	0.02600	0.06838	0.06969	0.06828
(20,4)	$\hat{R}_{ML}$	0.07282			1.30357			0.03911			0.07718		
	$\hat{R}_G(.)$	0.07202	0.07219	0.07192	1.25934	1.38146	1.25382	0.03883	0.03890	0.03879	0.07669	0.07798	0.07662
	$\hat{R}_{EBG1}(.)$	0.07206	0.07224	0.07197	1.25945	1.38157	1.25393	0.03885	0.03892	0.03881	0.07669	0.07798	0.07663
	$\hat{R}_{EBG2}(.)$	0.07208	0.07226	0.07199	1.25893	1.38101	1.25341	0.03884	0.03891	0.03881	0.07668	0.07798	0.07662
	$\hat{R}_{EBG3}(.)$	0.07205	0.07223	0.07196	1.26006	1.38222	1.25453	0.03886	0.03893	0.03882	0.07670	0.07799	0.07663
(30,18)	$\hat{R}_{ML}$	0.00793			0.22163			0.00721			0.04291		
	$\hat{R}_G(.)$	0.00779	0.00780	0.00778	0.19329	0.20094	0.19153	0.00713	0.00713	0.00712	0.04006	0.04086	0.03987
	$\hat{R}_{EBG1}(.)$	0.00780	0.00780	0.00779	0.19325	0.20090	0.19148	0.00713	0.00714	0.00713	0.04006	0.04086	0.03987
	$\hat{R}_{EBG2}(.)$	0.00779	0.00780	0.00779	0.19338	0.20103	0.19161	0.00712	0.00713	0.00712	0.04007	0.04087	0.03988
	$\hat{R}_{EBG3}(.)$	0.00781	0.00782	0.00780	0.19315	0.20080	0.19139	0.00715	0.00715	0.00715	0.04005	0.04085	0.03986
(30,15)	$\hat{R}_{ML}$	0.01524			0.34921			0.01192			0.05263		
	$\hat{R}_G(.)$	0.01489	0.01491	0.01488	0.30770	0.31880	0.30574	0.01170	0.01171	0.01169	0.04991	0.05067	0.04977
	$\hat{R}_{EBG1}(.)$	0.01488	0.01490	0.01487	0.30758	0.31868	0.30562	0.01170	0.01171	0.01169	0.04990	0.05066	0.04976
	$\hat{R}_{EBG2}(.)$	0.01490	0.01491	0.01488	0.30747	0.31856	0.30552	0.01172	0.01173	0.01171	0.04989	0.05065	0.04975
	$\hat{R}_{EBG3}(.)$	0.01489	0.01490	0.01487	0.30773	0.31883	0.30577	0.01169	0.01170	0.01168	0.04991	0.05067	0.04977

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Table 3.5 – *Continued from previous page*

		GELF						LINEX					
(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$
(30,12)	$\hat{R}_{ML}$	0.02451			0.53047			0.01744			0.06141		
	$\hat{R}_G(.)$	0.02370	0.02374	0.02368	0.47163	0.48883	0.46940	0.01709	0.01711	0.01708	0.05899	0.05973	0.05889
	$\hat{R}_{EBG1}(.)$	0.02372	0.02375	0.02370	0.47163	0.48883	0.46940	0.01710	0.01712	0.01709	0.05899	0.05973	0.05889
	$\hat{R}_{EBG2}(.)$	0.02369	0.02372	0.02367	0.47129	0.48848	0.46906	0.01708	0.01710	0.01707	0.05897	0.05972	0.05888
	$\hat{R}_{EBG3}(.)$	0.02376	0.02380	0.02374	0.47202	0.48923	0.46979	0.01713	0.01715	0.01712	0.05901	0.05975	0.05891
(30,9)	$\hat{R}_{ML}$	0.04120			0.80902			0.02650			0.06952		
	$\hat{R}_G(.)$	0.04040	0.04046	0.04037	0.73141	0.76209	0.72873	0.02610	0.02613	0.02608	0.06768	0.06844	0.06761
	$\hat{R}_{EBG1}(.)$	0.04042	0.04048	0.04039	0.73114	0.76182	0.72847	0.02610	0.02613	0.02609	0.06767	0.06843	0.06761
	$\hat{R}_{EBG2}(.)$	0.04054	0.04060	0.04051	0.73046	0.76111	0.72778	0.02616	0.02619	0.02615	0.06766	0.06842	0.06759
	$\hat{R}_{EBG3}(.)$	0.04032	0.04038	0.04028	0.73189	0.76259	0.72921	0.02605	0.02608	0.02603	0.06769	0.06845	0.06762
(30,6)	$\hat{R}_{ML}$	0.07261			1.31007			0.04012			0.07725		
	$\hat{R}_G(.)$	0.07119	0.07131	0.07113	1.21556	1.28549	1.21203	0.03956	0.03961	0.03953	0.07617	0.07698	0.07613
	$\hat{R}_{EBG1}(.)$	0.07117	0.07129	0.07110	1.21562	1.28555	1.21209	0.03955	0.03960	0.03952	0.07617	0.07698	0.07613
	$\hat{R}_{EBG2}(.)$	0.07113	0.07125	0.07107	1.21581	1.28575	1.21228	0.03955	0.03959	0.03952	0.07618	0.07698	0.07613
	$\hat{R}_{EBG3}(.)$	0.07122	0.07134	0.07115	1.21550	1.28543	1.21197	0.03956	0.03961	0.03954	0.07617	0.07698	0.07613
(40,24)	$\hat{R}_{ML}$	0.00622			0.21525			0.00585			0.04230		
	$\hat{R}_G(.)$	0.00606	0.00606	0.00605	0.18287	0.18810	0.18157	0.00575	0.00575	0.00574	0.03893	0.03951	0.03878

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Table 3.5 – *Continued from previous page*

		GELF						LINEX					
(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$
	$\hat{R}_{EBG1}(.)$	0.00606	0.00607	0.00606	0.18277	0.18800	0.18147	0.00575	0.00575	0.00575	0.03892	0.03949	0.03877
	$\hat{R}_{EBG2}(.)$	0.00610	0.00611	0.00610	0.18276	0.18800	0.18147	0.00579	0.00579	0.00579	0.03891	0.03949	0.03877
	$\hat{R}_{EBG3}(.)$	0.00603	0.00604	0.00603	0.18280	0.18803	0.18151	0.00572	0.00572	0.00572	0.03892	0.03950	0.03878
(40,20)	$\hat{R}_{ML}$	0.02168			0.52569			0.01623			0.06123		
	$\hat{R}_G(.)$	0.02096	0.02098	0.02095	0.45956	0.47119	0.45794	0.01576	0.01577	0.01575	0.05845	0.05897	0.05838
	$\hat{R}_{EBG1}(.)$	0.02096	0.02099	0.02095	0.45957	0.47120	0.45795	0.01576	0.01577	0.01575	0.05845	0.05897	0.05838
	$\hat{R}_{EBG2}(.)$	0.02094	0.02096	0.02092	0.45988	0.47151	0.45826	0.01574	0.01575	0.01573	0.05846	0.05898	0.05839
	$\hat{R}_{EBG3}(.)$	0.02101	0.02103	0.02099	0.45930	0.47093	0.45769	0.01578	0.01579	0.01577	0.05844	0.05896	0.05836
(40,16)	$\hat{R}_{ML}$	0.01141			0.34261			0.00932			0.05222		
	$\hat{R}_G(.)$	0.01102	0.01103	0.01101	0.29521	0.30274	0.29378	0.00905	0.00906	0.00905	0.04902	0.04956	0.04891
	$\hat{R}_{EBG1}(.)$	0.01102	0.01103	0.01101	0.29500	0.30253	0.29358	0.00906	0.00906	0.00905	0.04900	0.04954	0.04890
	$\hat{R}_{EBG2}(.)$	0.01101	0.01102	0.01100	0.29502	0.30255	0.29360	0.00904	0.00905	0.00903	0.04900	0.04954	0.04890
	$\hat{R}_{EBG3}(.)$	0.01104	0.01105	0.01103	0.29502	0.30255	0.29360	0.00908	0.00909	0.00908	0.04900	0.04954	0.04890
(40,12)	$\hat{R}_{ML}$	0.04016			0.80713			0.02656			0.06948		
	$\hat{R}_G(.)$	0.03874	0.03878	0.03871	0.71685	0.73706	0.71493	0.02582	0.02584	0.02581	0.06731	0.06783	0.06726
	$\hat{R}_{EBG1}(.)$	0.03876	0.03880	0.03873	0.71702	0.73724	0.71510	0.02583	0.02586	0.02582	0.06731	0.06783	0.06726
	$\hat{R}_{EBG2}(.)$	0.03878	0.03882	0.03875	0.71724	0.73746	0.71531	0.02584	0.02586	0.02582	0.06732	0.06783	0.06727
	$\hat{R}_{EBG3}(.)$	0.03875	0.03879	0.03872	0.71687	0.73708	0.71494	0.02583	0.02586	0.02582	0.06731	0.06783	0.06726

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Table 3.5 – *Continued from previous page*

		GELF						LINEX					
(n,m)		$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$
(40,8)	$\hat{R}_{ML}$	0.07113			1.30595			0.03994			0.07721		
	$\hat{R}_G(.)$	0.06910	0.06919	0.06906	1.18680	1.23216	1.18428	0.03911	0.03914	0.03909	0.07582	0.07637	0.07579
	$\hat{R}_{EBG1}(.)$	0.06915	0.06924	0.06910	1.18667	1.23203	1.18414	0.03913	0.03917	0.03911	0.07582	0.07637	0.07579
	$\hat{R}_{EBG2}(.)$	0.06926	0.06935	0.06922	1.18637	1.23172	1.18385	0.03915	0.03919	0.03913	0.07582	0.07637	0.07578
	$\hat{R}_{EBG3}(.)$	0.06905	0.06914	0.06900	1.18704	1.23241	1.18452	0.03911	0.03915	0.03909	0.07582	0.07638	0.07579

TABLE 3.6: PT-II CBRs under different censoring schemes (n,m) with fixed $n = 48,\ p = 0.05$ for the survival time of multipal myeloma patients.

(n,m)	$(48,28)$		$(48,24)$		$(48,19)$		$(48,14)$		$(48,10)$	
i	x_i	R_i	x_i	R_i	x_i	R_i	x_i	R_i	x_i	R_i
1	1	2	1	1	1	1	1	3	1	2
2	3	1	1	1	1	1	4	4	3	7
3	4	2	4	4	4	4	5	3	6	2
4	5	1	5	2	5	3	8	2	10	2
5	6	2	7	3	8	2	10	0	10	3
6	8	2	10	0	10	2	10	2	12	1
7	10	0	10	0	11	1	12	3	14	3
8	10	1	10	1	12	4	15	2	16	2
9	11	0	12	0	16	2	16	0	18	1
10	12	0	12	0	18	2	17	3	18	15
11	12	2	13	0	18	1	18	1		
12	15	1	14	0	24	0	24	0		
13	16	0	15	2	36	1	36	2		
14	16	0	16	0	40	0	40	9		
15	17	1	17	1	40	1				
16	18	0	18	3	51	0				
17	18	0	24	1	52	0				
18	18	0	40	0	56	0				
19	23	0	40	0	65	4				
20	24	2	40	0						
21	40	0	50	0						
22	40	1	51	1						
23	51	0	56	0						
24	52	0	65	4						
25	56	0								
26	65	0								
27	66	0								
28	76	2								

TABLE 3.7: Bayesian and E-Bayesian estimates of θ, λ under SELF, GELF and LINEX loss function for the survival time of multipal myeloma patients in presence of PT-II CBRs under different censoring schemes (n,m) with fixed $p = 0.05, c = 3, \gamma = 2,$ & $\delta = 1.5$.

(n,m)		$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD
(48,28)	$B(.)$	4.97790	4.97770	4.97729	3.35291	(0.19974,9.69114)	3.36920	3.36657	3.36389	4.94544	(0.16009,6.86592)
	$EBE_1(.)$	4.99195	4.99175	4.99135			3.37457	3.37194	3.36925		
	$EBE_2(.)$	4.97349	4.97329	4.97289		(4.92289,5.03292)	3.37011	3.36747	3.36479		(3.21202,3.53936)
	$EBE_3(.)$	5.01041	5.00980	5.00980			3.37904	3.37371	3.37371		
(48,24)	$B(.)$	4.64519	4.64508	4.64489	3.54530	(1.47364,7.77677)	3.49896	3.49756	3.49603	4.62521	(1.01918,6.07142)
	$EBE_1(.)$	4.63970	4.63959	4.63940			3.50395	3.50254	3.50100		
	$EBE_2(.)$	4.63885	4.63874	4.63855		(4.60337,4.68164)	3.48095	3.47956	3.47803		(3.37855,3.62834)
	$EBE_3(.)$	4.64054	4.64025	4.64025			3.52694	3.52398	3.52398		
(48,19)	$B(.)$	4.83437	4.83429	4.83415	4.03527	(1.94823,7.677)	3.97063	3.96931	3.96749	4.81261	(1.47684,6.5937)
	$EBE_1(.)$	4.83475	4.83468	4.83454			3.96887	3.96755	3.96573		
	$EBE_2(.)$	4.78439	4.78432	4.78417		(4.79965,4.86622)	3.96895	3.96763	3.96581		(3.84129,4.08844)
	$EBE_3(.)$	4.88512	4.88490	4.88490			3.96879	3.96565	3.96565		
(48,14)	$B(.)$	0.22757	0.22757	0.22757	8.01567	(4.14627,11.88508)	8.09522	8.09380	8.08830	0.22673	(0.05972,0.39374)
	$EBE_1(.)$	0.22742	0.22742	0.22742			8.09056	8.08915	8.08365		
	$EBE_2(.)$	0.22766	0.22766	0.22766		(0.22549,0.22949)	8.15649	8.15506	8.14952		(7.90785,8.28341)
	$EBE_3(.)$	0.22718	0.22718	0.22718			8.02463	8.01778	8.01778		
(48,10)	$B(.)$	0.15889	0.15888	0.15889	6.73828	(2.87992,10.59664)	6.79265	6.79095	6.78571	0.15790	(0.02777,0.28804)

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Table 3.7 – *Continued from previous page*

(n,m)		$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD
	$EBE_1(.)$	0.15885	0.15885	0.15885			6.79288	6.79117	6.78594		
	$EBE_2(.)$	0.15861	0.15860	0.15861		(0.15741,0.16051)	6.80870	6.80699	6.80175		(6.60566,6.98023)
	$EBE_3(.)$	0.15910	0.15910	0.15910			6.77705	6.77013	6.77013		

TABLE 3.8: Bayesian and E-Bayesian estimates of θ,λ under SELF, GELF and LINEX loss function for the survival time of multipal myeloma patients in presence of PT-II CBRs under different censoring schemes (n,m) with fixed $p=0.05, c=4, \gamma=3$ & $\delta=1.5$.

(n,m)		$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD
(48,28)	$B(.)$	4.97873	4.97852	4.97810	3.35291	(0.19974,9.69114)	3.36747	3.36480	3.36208	4.94544	(0.16009,6.86592)
	$EBE_1(.)$	4.98501	4.98479	4.98437			3.36658	3.36390	3.36119		
	$EBE_2(.)$	4.93788	4.93767	4.93725		(4.92452,5.03615)	3.37748	3.37480	3.37207		(3.20363,3.53404)
	$EBE_3(.)$	5.03213	5.03149	5.03149			3.35567	3.35030	3.35030		
(48,24)	$B(.)$	4.64605	4.64595	4.64577	3.54530	(1.47364,7.77677)	3.49849	3.49711	3.49561	4.62521	(1.01918,6.07142)
	$EBE_1(.)$	4.65379	4.65369	4.65351			3.49355	3.49218	3.49068		
	$EBE_2(.)$	4.64542	4.64532	4.64514		(4.60973,4.68632)	3.49260	3.49123	3.48973		(3.37319,3.61746)
	$EBE_3(.)$	4.66216	4.66187	4.66187			3.49450	3.49163	3.49163		
(48,19)	$B(.)$	4.83558	4.83549	4.83532	4.03527	(1.94823,7.677)	3.97225	3.97100	3.96928	4.81261	(1.47684,6.5937)
	$EBE_1(.)$	4.81898	4.81889	4.81872			3.97034	3.96909	3.96737		
	$EBE_2(.)$	4.81865	4.81856	4.81839		(4.79998,4.86935)	3.99570	3.99444	3.99271		(3.84933,4.09073)
	$EBE_3(.)$	4.81931	4.81905	4.81905			3.94497	3.94203	3.94203		
(48,14)	$B(.)$	0.22752	0.22752	0.22752	8.01567	(4.14627,11.88508)	8.09561	8.09418	8.08862	0.22673	(0.05972,0.39374)
	$EBE_1(.)$	0.22766	0.22766	0.22766			8.09541	8.09398	8.08842		
	$EBE_2(.)$	0.22578	0.22577	0.22578		(0.22556,0.22958)	8.07359	8.07216	8.06661		(7.89329,8.26809)
	$EBE_3(.)$	0.22955	0.22955	0.22955			8.11724	8.11022	8.11022		
(48,10)	$B(.)$	0.15880	0.15880	0.15880	6.73828	(2.87992,10.59664)	6.79146	6.78971	6.78434	0.15790	(0.02777,0.28804)

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Table 3.8 – *Continued from previous page*

(n,m)		$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD
	$EBE_1(.)$	0.15867	0.15866	0.15867			6.79212	6.79037	6.78499		
	$EBE_2(.)$	0.15887	0.15886	0.15887		(0.15732,0.16045)	6.71064	6.70891	6.70360		(6.5913,6.96652)
	$EBE_3(.)$	0.15846	0.15846	0.15846			6.87359	6.86638	6.86638		

TABLE 3.9: Bayesian and E-Bayesian estimates of θ, λ under SELF, GELF and LINEX loss function for the survival time of multipal myeloma patients in presence of PT-II CBRs under different censoring schemes (n,m) with fixed $p = 0.05, c = 3, \gamma = 2,$ & $\delta = -1.5$.

(n,m)		$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD
(48,28)	$B(.)$	4.97763	4.97767	4.97825	3.35291	(0.19974,9.69114)	3.36879	3.36932	3.37415	4.94544	(0.16009,6.86592)
	$EBE_1(.)$	4.98289	4.98293	4.98351			3.37279	3.37332	3.37815		
	$EBE_2(.)$	4.94617	4.94622	4.94679		(4.92041,5.03161)	3.35562	3.35615	3.36096		(3.20648,3.54029)
	$EBE_3(.)$	5.01961	5.02023	5.02023			3.38995	3.39535	3.39535		
		5.01961	5.02023	5.02023							
(48,24)	$B(.)$	4.64486	4.64488	4.64515	3.54530	(1.47364,7.77677)	3.49790	3.49819	3.50091	4.62521	(1.01918,6.07142)
	$EBE_1(.)$	4.65256	4.65258	4.65285			3.50019	3.50047	3.50320		
	$EBE_2(.)$	4.66275	4.66277	4.66303		(4.60584,4.68041)	3.52939	3.52968	3.53243		(3.37244,3.62082)
	$EBE_3(.)$	4.64237	4.64266	4.64266			3.47098	3.47397	3.47397		
(48,19)	$B(.)$	4.83565	4.83567	4.83588	4.03527	(1.94823,7.677)	3.96978	3.97003	3.97272	4.81261	(1.47684,6.5937)
	$EBE_1(.)$	4.83040	4.83042	4.83063			3.97198	3.97222	3.97492		
	$EBE_2(.)$	4.86059	4.86061	4.86082		(4.80269,4.87026)	3.98614	3.98639	3.98909		(3.8522,4.09438)
	$EBE_3(.)$	4.80021	4.80044	4.80044			3.95781	3.96074	3.96074		
(48,14)	$B(.)$	0.22756	0.22756	0.22756	8.01567	(4.14627,11.88508)	8.09161	8.09189	8.09825	0.22673	(0.05972,0.39374)
	$EBE_1(.)$	0.22775	0.22775	0.22775			8.09047	8.09075	8.09711		
	$EBE_2(.)$	0.22947	0.22947	0.22947		(0.22543,0.22953)	8.06009	8.06036	8.06670		(7.90251,8.26657)
	$EBE_3(.)$	0.22603	0.22603	0.22603			8.12086	8.12751	8.12751		
(48,10)	$B(.)$	0.15884	0.15884	0.15884	6.73828	(2.87992,10.59664)	6.79120	6.79155	6.79820	0.15790	(0.02777,0.28804)

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Table 3.9 – *Continued from previous page*

(n,m)		$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD
	$EBE_1(.)$	0.15866	0.15866	0.15866			6.79359	6.79394	6.80059		
	$EBE_2(.)$	0.15726	0.15726	0.15726		(0.15747,0.16041)	6.77578	6.77613	6.78276		(6.60755,6.98574)
	$EBE_3(.)$	0.16005	0.16005	0.16005			6.81140	6.81841	6.81841		

TABLE 3.10: Bayesian and E-Bayesian estimates of θ, λ under SELF, GELF and LINEX loss function for the survival time of multipal myeloma patients in presence of PT-II CBRs under different censoring schemes (n,m) with fixed $p = 0.05, c = 4, \gamma = 3$ & $\delta = -1.5$.

(n,m)		$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD
(48,28)	$B(.)$	4.97886	4.97891	4.97951	3.35291	(0.19974,9.69114)	3.37059	3.37115	3.37622	4.94544	(0.16009,6.86592)
	$EBE_1(.)$	4.97689	4.97694	4.97754			3.36549	3.36605	3.37111		
	$EBE_2(.)$	4.93876	4.93880	4.93940		(4.92215,5.03355)	3.34502	3.34557	3.35061		(3.19464,3.53052)
	$EBE_3(.)$	5.01503	5.01568	5.01568			3.38597	3.39162	3.39162		
(48,24)	$B(.)$	4.64567	4.64569	4.64595	3.54530	(1.47364,7.77677)	3.50011	3.50039	3.50304	4.62521	(1.01918,6.07142)
	$EBE_1(.)$	4.64750	4.64752	4.64778			3.49784	3.49812	3.50076		
	$EBE_2(.)$	4.67659	4.67661	4.67688		(4.61008,4.68416)	3.49251	3.49279	3.49543		(3.38304,3.62501)
	$EBE_3(.)$	4.61841	4.61869	4.61869			3.50317	3.50609	3.50609		
(48,19)	$B(.)$	4.83464	4.83466	4.83488	4.03527	(1.94823,7.677)	3.97320	3.97345	3.97618	4.81261	(1.47684,6.5937)
	$EBE_1(.)$	4.84130	4.84131	4.84153			3.98278	3.98303	3.98577		
	$EBE_2(.)$	4.82344	4.82345	4.82367		(4.80415,4.87216)	3.98320	3.98345	3.98618		(3.84751,4.09427)
	$EBE_3(.)$	4.85916	4.85939	4.85939			3.98237	3.98535	3.98535		
(48,14)	$B(.)$	0.22754	0.22754	0.22754	8.01567	(4.14627,11.88508)	8.09879	8.09908	8.10582	0.22673	(0.05972,0.39374)
	$EBE_1(.)$	0.22724	0.22724	0.22724			8.09821	8.09850	8.10524		
	$EBE_2(.)$	0.22582	0.22583	0.22583		(0.22542,0.2294)	8.09724	8.09753	8.10426		(7.90938,8.28999)
	$EBE_3(.)$	0.22865	0.22865	0.22865			8.09919	8.10621	8.10621		
(48,10)	$B(.)$	0.15888	0.15888	0.15888	6.73828	(2.87992,10.59664)	6.79080	6.79115	6.79804	0.15790	(0.02777,0.28804)

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Table 3.10 – *Continued from previous page*

(n,m)		$\hat{\theta}_S$	$\hat{\theta}_G$	$\hat{\theta}_L$	$\hat{\theta}_{ML}$	CI/HPD	$\hat{\lambda}_S$	$\hat{\lambda}_G$	$\hat{\lambda}_L$	$\hat{\lambda}_{ML}$	CI/HPD
	$EBE_1(.)$	0.15884	0.15885	0.15884			6.79710	6.79746	6.80436		
	$EBE_2(.)$	0.15960	0.15960	0.15960		(0.15735,0.1605)	6.84537	6.84573	6.85267		(6.59068,6.97344)
	$EBE_3(.)$	0.15809	0.15809	0.15809			6.74884	6.75604	6.75604		

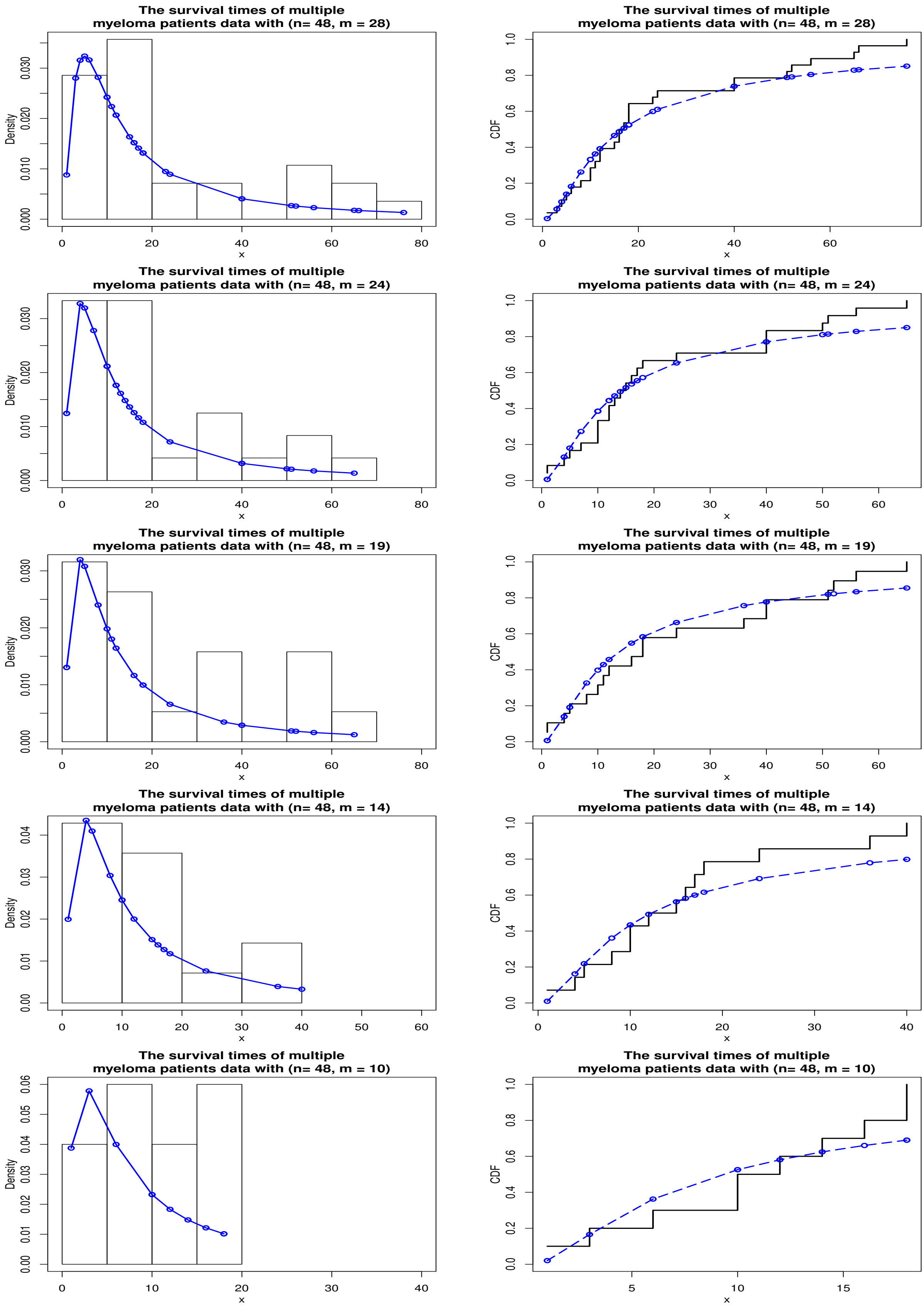

FIGURE 3.4: In the left column is PDF plot and right column is CDF plot for different scheme of the survival time of multipal myeloma patients data.

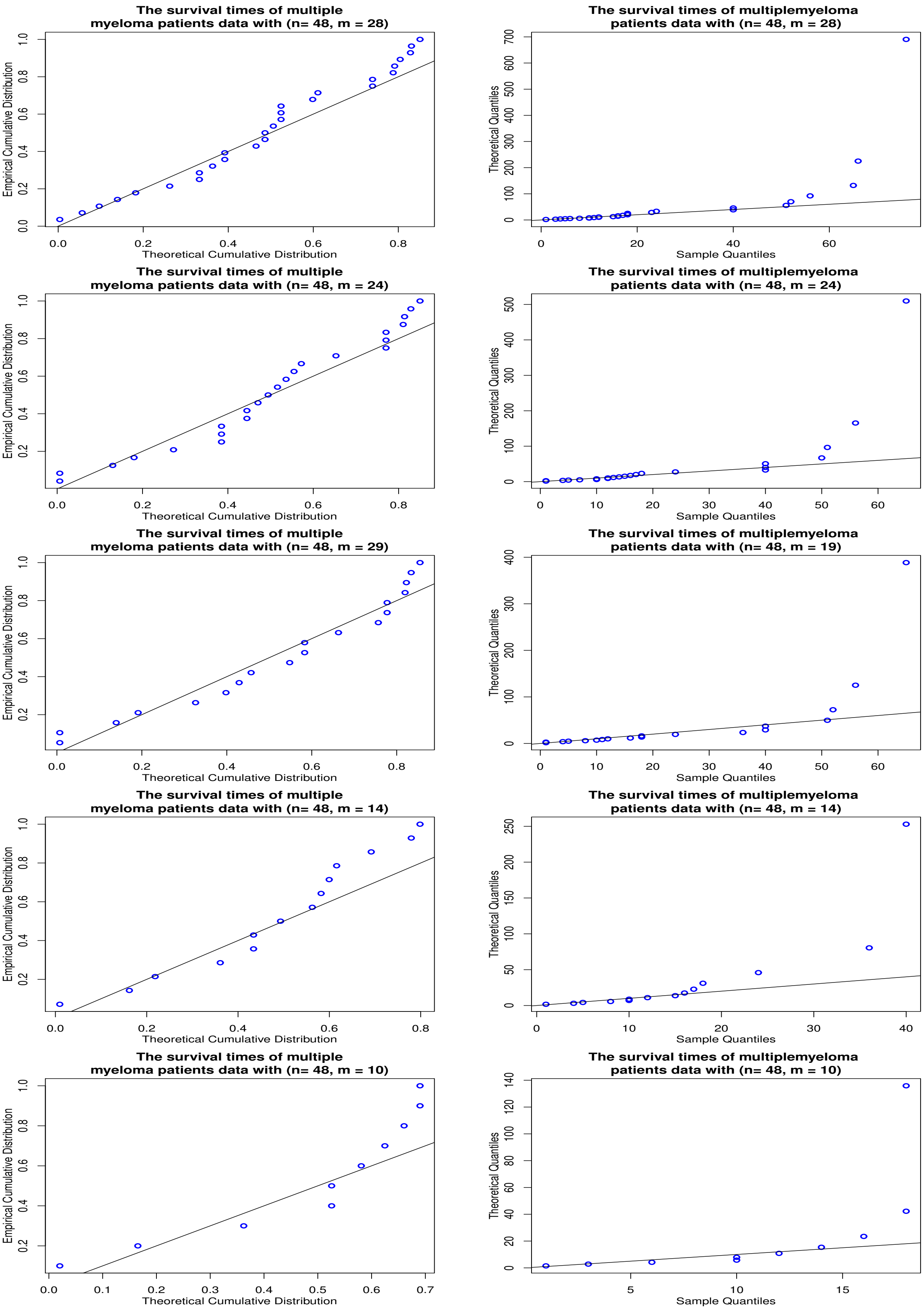

FIGURE 3.5: In the left column is P-P plot and right column is Q-Q plot for different scheme of the survival time of multipal myeloma patients data.

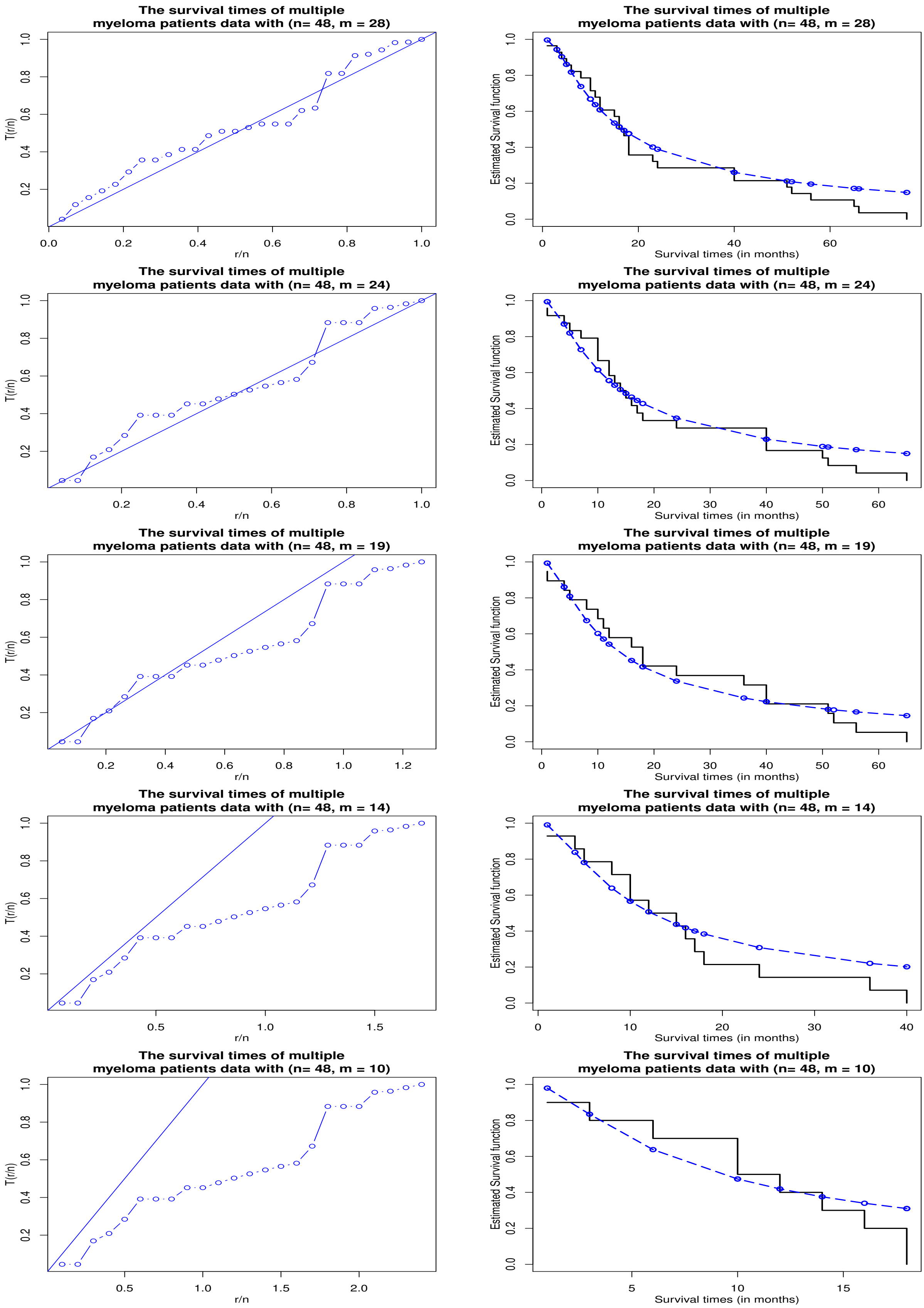

FIGURE 3.6: Left panel is TTT plot and right panel is KM plot for different scheme of the survival time of multipal myeloma patients data.

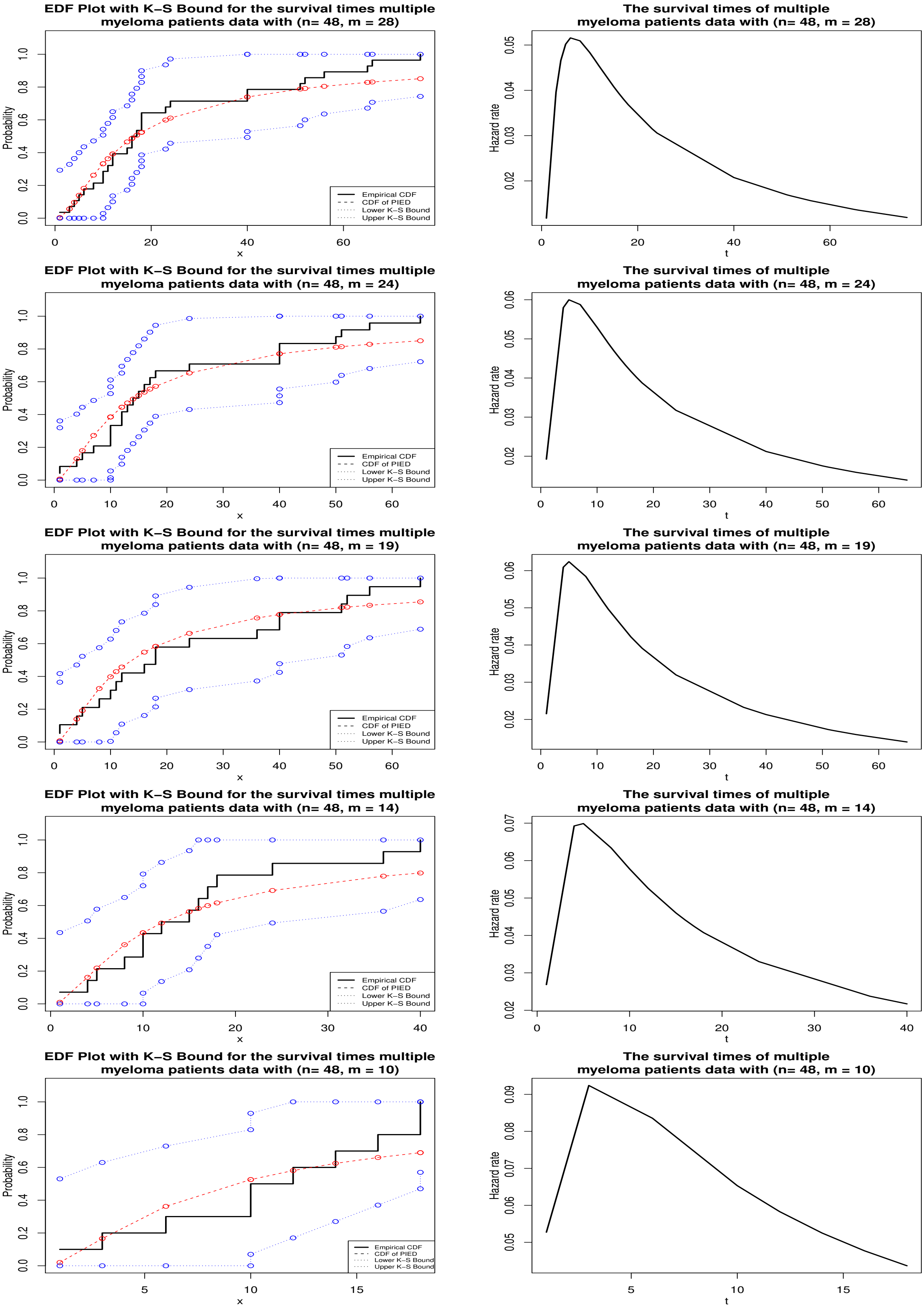

FIGURE 3.7: Left column is K-S plot and and right column is hazard plot for different scheme of the survival time of multipal myeloma patients data.

TABLE 3.11: Quantiles and estimate of λ, θ are obtained for fixed value of $\delta = 0.1$.

n		$\hat{\lambda}$	$\hat{\theta}$	$\xi_{0.05}$	$\xi_{0.25}$	$\xi_{0.50}$	$\xi_{0.75}$	$\xi_{0.95}$	Sample
48	MLE	2.575	3.872	2.213	6.144	13.497	34.210	197.341	113,52,6,40,10,7,
	SELF	2.578	4.172	2.412	6.647	14.538	36.756	211.737	66,10,10,14,16,4,
	GELF	2.476	3.036	1.669	4.659	10.346	26.403	152.919	65,5,11,10,15,5,76,
	LINEX	2.567	3.998	2.287	6.329	13.879	35.140	202.589	56,88,24,51,4,40,
	EBS1	2.574	4.164	2.403	6.624	14.488	36.633	211.031	8,18,5,16,50,40,1,
	EBS2	2.586	4.176	2.423	6.675	14.599	36.909	212.614	36,5,10,91,18,
	EBS3	2.568	4.152	2.390	6.590	14.417	36.457	210.029	1,18,6,1,23,
	EBG1	2.475	3.031	1.666	4.650	10.327	26.355	152.641	15,18,12,
	EBG2	2.484	3.040	1.676	4.678	10.389	26.510	153.536	12,17,3
	EBG3	2.467	3.022	1.656	4.622	10.265	26.199	151.748	
	EBL1	2.566	3.991	2.281	6.316	13.851	35.070	202.193	
	EBL2	2.574	4.003	2.297	6.356	13.937	35.286	203.426	
	EBL3	2.557	3.979	2.266	6.275	13.764	34.855	200.964	
28	MLE	2.677	4.608	2.825	7.677	16.684	42.036	241.688	1,3,4,5,6,8,10,
	SELF	2.778	7.756	5.712	14.125	29.698	73.548	418.925	10,11,12,12,15,
	GELF	2.194	5.033	2.584	6.921	14.953	37.557	215.565	16,16,17,18,18,
	LINEX	2.722	6.965	4.880	12.295	26.009	64.615	368.676	18,23,24,40,
	EBS1	2.782	7.767	5.729	14.165	29.782	73.751	420.075	6,40,51,52,
	EBS2	2.798	7.818	5.811	14.351	30.160	74.676	425.298	56,65,66,7
	EBS3	2.761	7.715	5.638	13.953	29.347	72.688	414.062	
	EBG1	2.194	5.040	2.590	6.934	14.980	37.622	215.935	
	EBG2	2.209	5.073	2.629	7.032	15.184	38.125	218.795	
	EBG3	2.180	5.006	2.551	6.838	14.778	37.123	213.096	
	EBL1	2.723	6.975	4.891	12.318	26.056	64.730	369.323	
	EBL2	2.742	7.021	4.966	12.492	26.414	65.605	374.276	
	EBL3	2.705	6.928	4.817	12.145	25.701	63.860	364.403	
24	MLE	2.749	3.828	2.333	6.483	14.252	36.136	208.496	1,1,4,5,7,10,
	SELF	2.736	4.305	2.658	7.294	15.921	40.210	231.492	10,10,12,12,13,
	GELF	2.563	3.045	1.732	4.836	10.738	27.401	158.688	14,15,16,18,24,
	LINEX	2.717	4.105	2.495	6.888	15.079	38.146	219.809	40,17,18,24,
	EBS1	2.730	4.294	2.643	7.257	15.844	40.019	230.404	40,40,40,50,
	EBS2	2.734	4.301	2.653	7.282	15.897	40.148	231.143	51,56,65

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Table 3.11 – *Continued from previous page*

n		$\hat{\lambda}$	$\hat{\theta}$	$\xi_{0.05}$	$\xi_{0.25}$	$\xi_{0.50}$	$\xi_{0.75}$	$\xi_{0.95}$	Sample
	EBS3	2.730	4.288	2.639	7.245	15.820	39.958	230.063	
	EBG1	2.559	3.038	1.726	4.818	10.698	27.301	158.114	
	EBG2	2.561	3.043	1.730	4.829	10.722	27.361	158.461	
	EBG3	2.557	3.033	1.722	4.807	10.674	27.240	157.768	
	EBL1	2.713	4.095	2.484	6.860	15.020	37.998	218.970	
	EBL2	2.715	4.101	2.491	6.877	15.056	38.088	219.479	
	EBL3	2.711	4.088	2.478	6.842	14.984	37.909	218.461	
19	MLE	2.849	3.543	2.226	6.214	13.714	34.855	201.382	1,1,4,5,8,10,
	SELF	2.847	3.851	2.432	6.755	14.845	37.633	217.109	11,12,16,18,18,
	GELF	2.684	2.734	1.659	4.601	10.242	26.193	151.899	24,36,40,40,51,
	LINEX	2.828	3.692	2.307	6.427	14.155	35.931	207.449	52,56,65
	EBS1	2.834	3.833	2.409	6.692	14.710	37.296	215.186	
	EBS2	2.845	3.841	2.423	6.732	14.795	37.510	216.410	
	EBS3	2.835	3.826	2.404	6.680	14.685	37.234	214.833	
	EBG1	2.677	2.721	1.648	4.571	10.176	26.025	150.933	
	EBG2	2.682	2.727	1.654	4.587	10.212	26.115	151.454	
	EBG3	2.672	2.716	1.643	4.555	10.140	25.934	150.414	
	EBL1	2.820	3.675	2.290	6.380	14.055	35.683	206.037	
	EBL2	2.826	3.682	2.299	6.404	14.108	35.815	206.787	
	EBL3	2.815	3.667	2.281	6.355	14.003	35.553	205.289	
14	MLE	2.276	3.984	2.020	5.592	12.264	31.055	179.050	1,4,5,8,10,
	SELF	2.561	10.365	7.509	17.838	37.001	90.985	516.226	10,12,15,16,17,
	GELF	1.870	6.712	3.196	8.107	17.188	42.750	244.072	18,24,36,40
	LINEX	2.495	8.967	6.142	14.860	31.017	76.521	434.950	
	EBS1	2.563	10.375	7.525	17.874	37.074	91.163	517.232	
	EBS2	2.547	10.323	7.433	17.664	36.647	90.121	511.352	
	EBS3	2.575	10.428	7.605	18.054	37.440	92.053	522.250	
	EBG1	1.870	6.719	3.201	8.117	17.207	42.797	244.336	
	EBG2	1.860	6.684	3.162	8.027	17.023	42.345	241.775	
	EBG3	1.880	6.753	3.239	8.207	17.393	43.252	246.911	
	EBL1	2.495	8.976	6.150	14.877	31.051	76.605	435.422	
	EBL2	2.481	8.930	6.079	14.714	30.719	75.794	430.844	
	EBL3	2.509	9.022	6.222	15.041	31.386	77.420	440.024	

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Table 3.11 – *Continued from previous page*

n		$\hat{\lambda}$	$\hat{\theta}$	$\xi_{0.05}$	$\xi_{0.25}$	$\xi_{0.50}$	$\xi_{0.75}$	$\xi_{0.95}$	Sample
10	MLE	2.112	3.184	1.486	4.154	9.210	23.476	135.872	1,3,6,10,10,
	SELF	3.135	14.758	13.819	31.786	65.178	159.276	900.562	12,14,16,
	GELF	2.125	9.659	5.727	13.720	28.543	70.295	399.177	18,18
	LINEX	3.008	12.085	10.559	24.687	50.923	124.845	707.167	
	EBS1	3.138	14.771	13.845	31.843	65.294	159.558	902.151	
	EBS2	3.124	14.711	13.719	31.561	64.723	158.171	894.339	
	EBS3	3.147	14.831	13.947	32.068	65.748	160.658	908.341	
	EBG1	2.125	9.667	5.733	13.733	28.568	70.355	399.516	
	EBG2	2.117	9.628	5.684	13.622	28.343	69.806	396.418	
	EBG3	2.133	9.707	5.782	13.844	28.795	70.907	402.626	
	EBL1	3.008	12.095	10.569	24.709	50.968	124.953	707.768	
	EBL2	2.997	12.046	10.481	24.511	50.567	123.979	702.280	
	EBL3	3.019	12.144	10.658	24.908	51.371	125.930	713.277	

TABLE 3.12: Quantiles and estimate of λ, θ are obtained for fixed value of $\delta = -0.1$.

n		$\hat{\lambda}$	$\hat{\theta}$	$\xi_{0.05}$	$\xi_{0.25}$	$\xi_{0.50}$	$\xi_{0.75}$	$\xi_{0.95}$	Sample
48	GELF	2.498	3.139	1.735	4.848	10.753	27.420	158.731	13,52,6,40,10,7,
	LINEX	2.699	6.182	4.148	10.684	22.769	56.783	324.659	66,10,10,14,16,4,
	EBG1	2.501	3.146	1.740	4.863	10.786	27.502	159.198	65,5,11,10,15,5,76,
	EBG2	2.511	3.157	1.753	4.898	10.863	27.695	160.311	56,88,24,51,4,40,8,
	EBG3	2.491	3.135	1.728	4.827	10.709	27.309	158.089	18,5,16,50,40,1,36,
	EBL1	2.702	6.195	4.164	10.721	22.845	56.967	325.702	5,10,91,18,1,18,6,
	EBL2	2.713	6.216	4.199	10.805	23.019	57.395	328.125	1,23,15,18,12,
	EBL3	2.692	6.173	4.128	10.638	22.672	56.542	323.288	12,17,3
28	GELF	2.201	5.432	2.860	7.553	16.233	40.662	233.044	1,3,4,5,6,8,10,
	LINEX	3.370	19.748	20.483	46.299	94.313	229.634	1295.705	10,11,12,12,15,
	EBG1	2.202	5.435	2.864	7.561	16.251	40.705	233.287	16,16,17,18,18,
	EBG2	2.219	5.482	2.917	7.689	16.517	41.359	237.000	18,23,24,40,
	EBG3	2.186	5.389	2.811	7.434	15.987	40.056	229.605	40,51,52,56,
	EBL1	3.371	19.762	20.506	46.349	94.414	229.878	1297.075	65,66,76
	EBL2	3.396	19.930	20.851	47.109	95.947	233.589	1317.951	
	EBL3	3.346	19.593	20.164	45.594	92.893	226.196	1276.366	

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Table 3.12 – *Continued from previous page*

n		$\hat{\lambda}$	$\hat{\theta}$	$\xi_{0.05}$	$\xi_{0.25}$	$\xi_{0.50}$	$\xi_{0.75}$	$\xi_{0.95}$	Sample
24	GELF	2.607	3.175	1.830	5.113	11.339	28.905	167.298	1,1,4,5,7,10,10,
	LINEX	2.933	6.654	4.958	12.596	26.720	66.476	379.591	10,12,12,13,
	EBG1	2.609	3.180	1.834	5.124	11.361	28.961	167.617	14,15,16,17,
	EBG2	2.609	3.183	1.836	5.130	11.374	28.993	167.804	18,24,40,40,
	EBG3	2.608	3.177	1.831	5.118	11.348	28.928	167.430	40,50,51,
	EBL1	2.935	6.665	4.971	12.625	26.779	66.619	380.397	56,65
	EBL2	2.936	6.671	4.979	12.642	26.813	66.703	380.866	
	EBL3	2.934	6.659	4.964	12.609	26.745	66.536	379.928	
19	GELF	2.708	2.967	1.789	4.989	11.086	28.305	163.987	1,1,4,5,8,10,
	LINEX	3.061	5.747	4.278	11.177	23.933	59.832	342.548	11,12,16,18,
	EBG1	2.710	2.967	1.791	4.993	11.096	28.330	164.128	18,24,36,40,
	EBG2	2.694	2.949	1.771	4.937	10.972	28.017	162.328	40,51,52,
	EBG3	2.726	2.986	1.811	5.050	11.221	28.645	165.940	56,65
	EBL1	3.063	5.748	4.282	11.187	23.954	59.885	342.853	
	EBL2	3.046	5.712	4.223	11.046	23.663	59.168	338.787	
	EBL3	3.081	5.783	4.342	11.329	24.248	60.606	346.943	
14	GELF	1.916	7.331	3.668	9.157	19.312	47.905	273.105	1,4,5,8,10,10,
	LINEX	3.391	34.354	37.163	82.321	166.354	403.218	2269.300	12,15,16,17,
	EBG1	1.917	7.334	3.670	9.163	19.325	47.936	273.277	18,24,36,
	EBG2	1.918	7.338	3.676	9.175	19.351	47.999	273.639	40
	EBG3	1.915	7.329	3.665	9.150	19.299	47.872	272.916	
	EBL1	3.392	34.366	37.187	82.373	166.460	403.473	2270.734	
	EBL2	3.394	34.388	37.238	82.483	166.681	404.008	2273.741	
	EBL3	3.389	34.344	37.137	82.263	166.238	402.938	2267.730	
10	GELF	2.261	10.173	6.484	15.437	32.045	78.830	447.361	1,3,6,10,10,
	LINEX	4.958	57.511	92.676	203.184	408.862	988.624	5556.204	12,14,16,
	EBG1	2.266	10.212	6.527	15.532	32.238	79.298	449.999	18,18
	EBG2	2.258	10.187	6.487	15.440	32.051	78.840	447.413	
	EBG3	2.274	10.236	6.568	15.625	32.427	79.758	452.593	
	EBL1	4.968	57.733	93.231	204.390	411.280	994.457	5588.938	
	EBL2	4.951	57.595	92.690	203.211	408.912	988.741	5556.842	

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Table 3.12 – *Continued from previous page*

n		$\hat{\lambda}$	$\hat{\theta}$	$\xi_{0.05}$	$\xi_{0.25}$	$\xi_{0.50}$	$\xi_{0.75}$	$\xi_{0.95}$	Sample
	EBL3	4.984	57.871	93.774	205.573	413.654	1000.188	5621.124	

Chapter 4

Empirical Bayesian Estimation for Kumaraswamy Distribution Using Informative Prior *

4.1 Introduction

In the previous chapter, we have discussed the procedure for obtaining the classical, Bayesian and E-Bayesian estimation under PT-II CBRs. In this chapter, we are introducing the Empirical Bayes estimator of the Kumaraswamy distribution (KD). It is one of the simplest distribution in the sense of being parsimonious in parameter. It is applicable to many natural phenomena whose outcomes have lower and upper bound, such as the height of individuals, age of person, scores obtained on a test, atmospheric temperatures, hydro logical data such as daily rain fall, daily stream flow etc see Kumaraswamy (1980). The PDF and CDF of KD (α, λ) are given by

$$f(x;\alpha,\lambda) = \alpha\lambda x^{\alpha-1}(1-x^{\alpha})^{\lambda-1}; \qquad x>0, \quad \alpha>0, \quad \lambda>0, \tag{4.1}$$

*Part of this chapter has been published in reputed peer-reviewed journals with indexing EBSCO Discovery Service, see Kumar et al. (2019b).

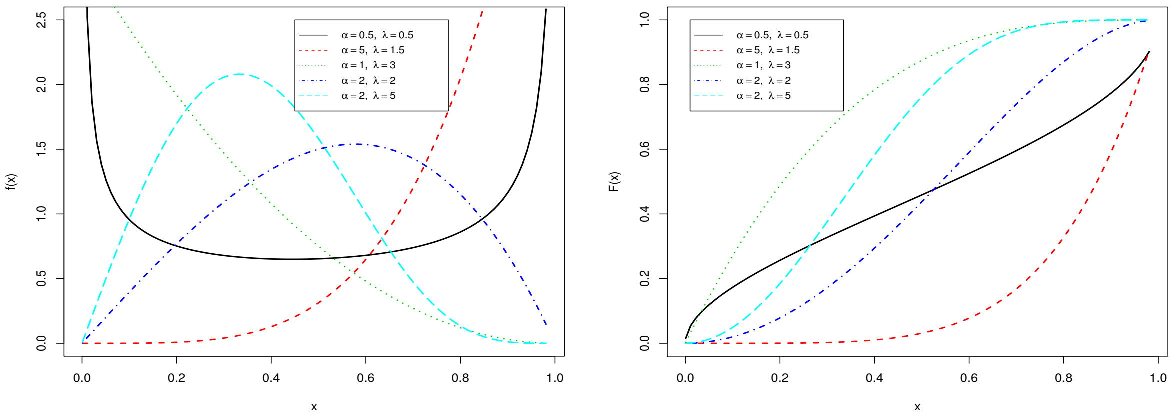

FIGURE 4.1: PDFs: left panel, CDFs: right panel of KD for different values of α and λ.

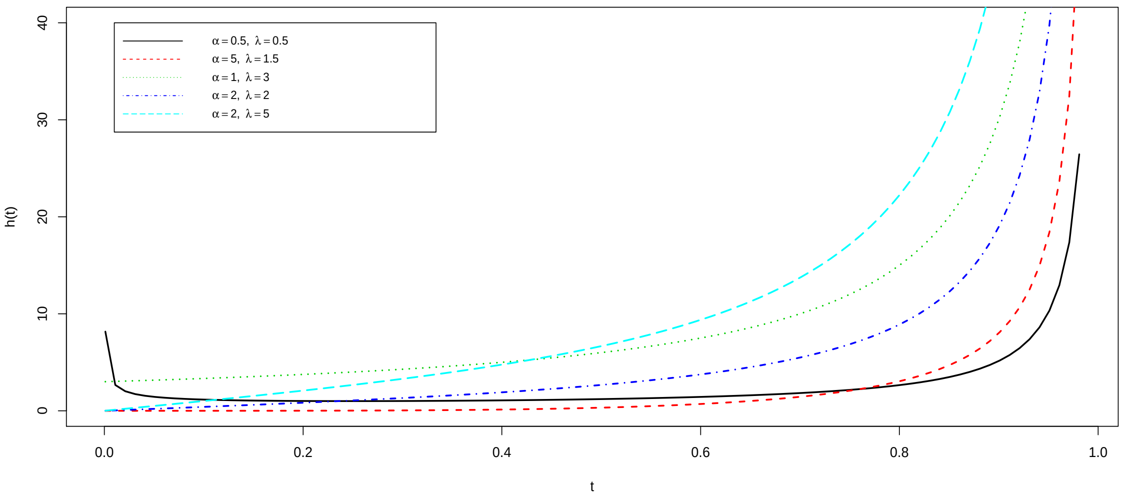

FIGURE 4.2: HF of KD for different values of α and λ.

and

$$F(x;\alpha,\lambda) = 1-(1-x^{\alpha})^{\lambda}; \qquad x>0, \quad \alpha>0, \quad \lambda>0, \tag{4.2}$$

respectively; where α and λ are shape parameters. Figure (4.1) shows PDF and CDF for $\alpha = 0.5, 5, 1, 2, 2$ and $\lambda = 0.5, 1, 3, 2, 5$ respectively.

The reliability function (i.e. the probability of failure after time t) and the HF for distribution Equation (4.1) are given by

$$R(t) = (1-t^{\alpha})^{\lambda},$$

and

$$h(t) = \frac{\alpha\lambda t^{\alpha-1}}{(1-t^{\alpha})}.$$

respectively. The HF $h(t)$ is shown in Figure (4.2) for $\alpha = 0.5, 5, 1, 2, 2$ and $\lambda = 0.5, 1.5, 3, 2, 5$ respectively. It may be noted here that the HF has a non-monotonic shape which decreases initially remains constant in the mid and lastly increases. For the statistical and probabilistic properties and other distributions obtained under the influence of KD for the use in life testing and reliability analysis, see Jones (2009), Lemonte (2011), Xiaohu et al. (2011), Santana et al. (2012) etc. The problem of the estimation of the parameters of KD have been discussed by Lemonte (2011) and Gholizadeh et al. (2011) etc. But it seems that the empirical Bayesian inferences have not been attempted to the extent of classical and Bayesian inferences, although it is well known that empirical Bayes is a good compromise between these two. According to Morris (1983), an empirical Bayesian inference, which is, as expected, a hybrid of frequentist and Bayesian inference.

The use of KD in life testing and reliability problems have been suggested by various authors, see Jones (2009), Lemonte (2011), Kohansal (2017), Amin (2017) etc. A general problem associated with life testing is that in most of the situation one can not wait for the failure of all the items put on test. In such situation, censoring becomes unavoidable. A number of censoring schemes are available in statistical literature. One of the popular censoring scheme under use is progressive Type-II censoring scheme. On one hand it provides flexibility because it allows the intermediate removals of the items from test and on other hand it guarantees for a minimum efficiency of the estimators by fixing the number of complete observations.

In the point estimation an important element is the loss function specification. A very popular loss function is SELF, used in estimation of parameter. Which can be appropriate on the space of minimum variance-unbiased estimation. However, main drawback of this loss function is that it have equal magnitude for o.e. and u.e., can say it is symmetric loss function. In the literature, many asymmetric loss functions are available, and one of the most frequently used asymmetric loss function is the LINEX loss function, originally it was proposed by Varian (1975) and popularized by Zellner (1986a), discussed in Chapter 1, Section (1.8).

In this chapter presented a piece of work, aims to develop the empirical Bayes estimators for an unknown shape parameter of KD based on PT-II CBR under LINEX loss function. In KD, one shape parameter known $\alpha > 1$ i.e. $\alpha = 2$ with $\lambda > 1$ (unknown) have been taken due to the distribution with one mode of the KD. For $\alpha > 1$, $\lambda > 1$, $\lim_{x\to 1} f(x;\alpha,\lambda) = 0$ and $\lim_{x\to 0} f(x;\alpha,\lambda) = 0$. Therefore, it is mathematically deal with, the characteristics of KD for different parameter values see Mitnik (2013).

4.2 Likelihood Function under PT-II CBRs

Suppose that in a life testing experiment having items put on test, the lifetime of which follow the KD. Also, we considered that the lifetime experiment perform under PT-II CBR, discussed in Chapter 1, Subsection (1.11.2). The conditional likelihood function can be written as

$$L(\alpha,\lambda;x|R=r) = c\prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{r_i}; \qquad -\infty < x_1 < \ldots < x_m < \infty, \tag{4.3}$$

where $n = m + \sum_{i=1}^{m} r_i$, $n,m \,\varepsilon\, \mathrm{N}, 1 \le i \le m$ and $c = \prod_{i=1}^{m} \gamma_i$ where $\gamma_i = \sum_{j=1}^{m} (r_j+1)$, $r_i \sim B(n-m-\sum_{l=0}^{i-1} r_l, p)$ for $i = 1,2,3,\ldots m-1$ and $r_0 = 0$ substituting $f(.)$ and $F(.)$ from Equation (4.1) and (4.2) respectively, into Equation (2.3), we have

$$L(\alpha,\lambda;x|R=r) = c\prod_{i=1}^{m} \alpha\lambda x_i^{\alpha-1}(1-x_i^{\alpha})^{\lambda-1}\left\{(1-x_i^{\alpha})^{\lambda}\right\}^{r_i}. \tag{4.4}$$

Since at the every stage the removals are independent of each other with probability p for each unit, the removals are following a binomial distribution i.e.,

$$r_i \sim B(n-m-\sum_{l=0}^{i-1} r_l, p),$$

where $i = 1, 2, 3, ...m-1$. Therefore;

$$p(R_1 = r_1; p) = \binom{n-m}{r_1} p^{r_1} (1-p)^{n-m-r_1}, \tag{4.5}$$

and for $i = 2, 3, ..., m-1$

$$p(R_i; p) = p(R_i = r_i | R_{i-1} = r_{i=1}, ...R_1 = r_1)$$

$$= \binom{n-m-\sum_{l=0}^{i-1} r_l}{r_i} p^{r_i} (1-p)^{n-m-\sum_{l=0}^{i-1} r_l}. \tag{4.6}$$

It is further assumed that $R_i s$ are independent of $X_{i:m:n}$ for all i. Thus full likelihood function can be written as:

$$L(\alpha, \lambda, p; x) = L(\alpha, \beta, \lambda; x | R = r) p(R = r; p), \tag{4.7}$$

where;

$$\begin{aligned} p(R = r; p) = p(R_1 = r_1) p(R_2 = r_2 | R_1 = r_1) p(R_3 = r_3 | R_2 = r_2, R_1 = r_1)... \\ p(R_{m-1} = r_{m-1} | R_{m-2} = r_{m-2}, ...R_1 = r_1). \end{aligned} \tag{4.8}$$

Making the substitution from the Equation (2.5) and (2.6) into Equation (2.8), we get

$$p(R = r; p) = \frac{(n-m)! p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i) r_i}}{(n-m-\sum_{l=1}^{i-1} r_l)! \prod_{i=1}^{m-1} r_i!}, \tag{4.9}$$

now using Equations (2.4), (2.7) and (2.9), the full likelihood can be represented in the following form:

$$L(\alpha, \lambda, p; x) = H L_1(\alpha, \lambda) L_2(p). \tag{4.10}$$

where

$$H = \frac{c(n-m)!}{(n-m-\sum_{l=1}^{i-1} r_l)! \prod_{i=1}^{m-1} r_i!},$$

$$L_1(\alpha,\lambda;x|R=r) = \prod_{i=1}^{m} \alpha\lambda x_i^{\alpha-1} (1-x_i^{\alpha})^{-(\lambda(-1-r_i)+1)}, \tag{4.11}$$

$$L_2(p) = p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}. \tag{4.12}$$

It may be noted here that the likelihood function is product of three terms H, L_1 and L_2 ; where H is a constant term, L_1 is function of the parameters but does not involve p and L_2 is function of p but does not involve other parameters.

4.3 Estimation of Parameters

4.3.1 Maximum Likelihood Estimator

As mentioned above, only L_1 involves the parameters, hence ML estimates of the parameters are those values which maximizes L_1, we have

$$\begin{aligned} \ln L_1(\alpha,\lambda) = m\ln(\alpha) + m\ln(\lambda) + (\alpha-1)\sum_{i=1}^{m}\ln(x_i) \\ -\sum_{i=1}^{m}(\lambda(-1-r_i)+1)\ln(1-x_i^{\alpha}) \end{aligned} \tag{4.13}$$

Thus, the likelihood equations can be obtained by differentiating the log-L function given above with respect to parameter α and λ and equating to zero; i.e., ML estimates are $\hat{\alpha}$ and $\hat{\lambda}$ of α and λ respectively, can be obtained by simultaneously solving the likelihood equations:

$$\frac{m}{\alpha} + \sum_{i=1}^{m}\ln(x_i) + \sum_{i=1}^{m}(\lambda(-1-r_i)+1)\left(x_i^{-\alpha}-1\right)^{-1}\ln(x_i) = 0, \tag{4.14}$$

and

$$\frac{m}{\lambda} - \sum_{i=1}^{m} (-1 - r_i) \ln (1 - x_i^{\alpha}) = 0. \tag{4.15}$$

The above mentioned normal equation solved simultaneously but do not provided closed form solution for the estimators. Then we opted NR method to compute the ML estimators, then we are using the invariance property to the ML estimators of the reliability function $R(t)$ and the failure rate $h(t)$ at time t can be evaluated from the following:

$$\hat{R}(t) = (1 - t^{\alpha})^{\lambda}; \qquad t > 0, \tag{4.16}$$

and

$$\hat{h}(t) = \frac{\alpha \lambda t^{\alpha - 1}}{(1 - t^{\alpha})}; \qquad t > 0. \tag{4.17}$$

4.3.2 Bayes Estimator

In this sequence, we obtain the Bayes estimator of the parameter λ, when we assume that λ has a conjugate prior density,

$$\pi(\lambda, \beta) = \beta \exp(-\beta\lambda); \qquad \lambda > 0, \quad \beta > 0. \tag{4.18}$$

That is to say, we regard random variable λ with prior density an exponential distribution $\exp(\beta)$, which is used in detail Bayesian theory, see Berger (2013). It may be noted that, the exponential family prior $\pi(\lambda, \beta)$ has been used by Nassar and Eissa (2005), Kim et al. (2011) possibly because of the fact that it is flexible enough to cover a wide range of prior believes of the experimenter. Hence, mathematical formula to evaluate the posterior distribution of λ is given below,

$$\pi(\lambda|x) = \frac{\pi(\lambda, \beta) L_1(\alpha, \lambda; x|R = r)}{\int_0^{+\infty} \pi(\lambda, \beta) L_1(\alpha, \lambda; x|R = r) d\lambda}. \tag{4.19}$$

Substituting $L_1(\alpha,\lambda;x|R=r)$ and $\pi(\lambda;\beta)$ from Equation (2.10) and (4.18), respectively, in Equation (4.19). We obtain the posterior distribution after simplification as,

$$\pi(\lambda|T) = \frac{\beta\exp(-\beta\lambda)\,c\prod_{i=1}^{m}\alpha\lambda x^{\alpha-1}(1-x^{\alpha})^{-(\lambda(-1-r_i)+1)}}{\int_0^{+\infty}\beta\exp(-\beta\lambda)\,c\prod_{i=1}^{m}\alpha\lambda x^{\alpha-1}(1-x^{\alpha})^{-(\lambda(-1-r_i)+1)}d\lambda}$$
$$= \frac{(\beta+T)^{m+1}\lambda^{m}\exp(-\lambda(\beta+T))}{\Gamma(m+1)}. \tag{4.20}$$

where $T=-\sum_{i=1}^{m}(r_i+1)\ln(1-x_i^{\alpha})$. Randomly generated posterior distribution for complete sample size 20 having $\mathbf{x}=(4.602501e-07, 9.335994E-07, 1.306320E-02, 1.351230E-02, 4.355106E-02, 9.641328E-02, 1.842315E-01, 2.266576E-01, 2.577245E-01, 4.145024E-01, 7.714380E-01, 7.891063E-01, 8.778412E-01, 8.926065E-01, 9.723090E-01, 9.963840E-01, 9.986856E-01, 9.990788E-01, 9.999941E-01, 1.000000E+00)$ are presented in Figure (4.3).

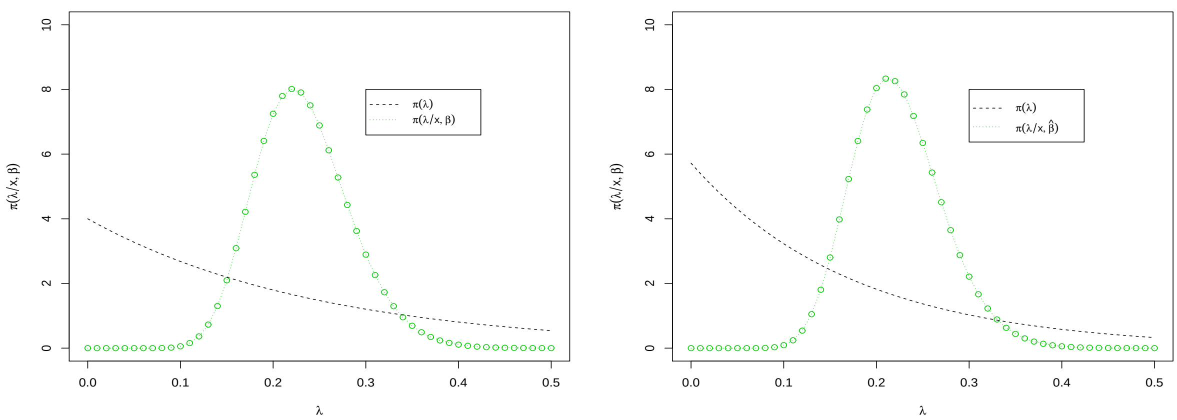

FIGURE 4.3: Informative prior $\pi(\lambda)$ and the posterior $\pi(\lambda|x,\beta)$: left panel, Informative prior $\pi(\lambda)$ and the posterior $\pi(\lambda|x,\hat{\beta})$: right panel of λ.

Note that the posterior distribution of λ is gamma distribution with parameters $(m+1)$ and $(\beta+T)$. The Bayes estimator of λ under LINEX loss function for posterior Equation (4.20) is obtained, after simplification, as

$$\hat{\lambda}_B = -\frac{1}{a}\ln\int_0^{\infty}e^{-a\lambda}\pi(\lambda|T)d\lambda = \frac{m+1}{a}\ln\left(1+\frac{a}{\beta+T}\right). \tag{4.21}$$

Similarly, the Bayes estimators of $R(t)$ and $h(t)$ at time t are obtained under LINEX loss function

$$\hat{R}_B(t) = -\frac{1}{a}\ln\int_0^\infty e^{-a(1-t^\alpha)^\lambda}\pi^*(\lambda|T)d\lambda = -\frac{1}{a}\ln\left(\sum_{s=0}^{\infty}\frac{(-a)^s}{s!}\left(1-\frac{s\ln(1-t^\alpha)}{(\beta+T)}\right)^{-(m+1)}\right), \tag{4.22}$$

and

$$\hat{h}_B(t) = -\frac{1}{a}\ln\int_0^\infty e^{\frac{-a\alpha\lambda t^{\alpha-1}}{(1-t^\alpha)}}\pi^*(\lambda|T)d\lambda = \frac{m+1}{a}\ln\left(1+\frac{a\alpha t^{\alpha-1}}{(\beta+T)(1-t^\alpha)}\right), \tag{4.23}$$

respectively.

4.3.3 Empirical Bayes Estimator

In view of this fact, Shi et al. (2005) and Yan and Gendai (2003) used the ML estimator to estimate hyper parameter of prior distribution for analyzing the Bayesian reliability quantitative indexes of cold stand by system. In Equation (4.21), the hyper parameter β is an unknown constant, so λ can not be used directly. Therefore, we make use of the ML estimator to estimate β.

$$\begin{aligned} f(x) &= \int_0^\infty f(x;\alpha,\lambda)\pi(\lambda;\beta)d\lambda \\ &= \int_0^\infty \alpha\lambda x^{\alpha-1}(1-x^\alpha)^{\lambda-1}\beta\exp(-\beta\lambda)\,d\lambda \\ &= \frac{\alpha\beta x^{\alpha-1}}{(1-x^\alpha)(\beta-\ln(1-x^\alpha))^2}, \end{aligned}$$

and

$$
\begin{aligned}
1-F(x) &= \int_x^{\infty} f(x)dx \\
&= \int_x^{\infty} \frac{\alpha\beta x^{\alpha-1}}{(1-x^{\alpha})(\beta-\ln(1-x^{\alpha}))^2}dx \\
&= \frac{\beta}{\beta-\ln(1-x^{\alpha})}.
\end{aligned}
$$

Hence, Equation (2.3) can be expressed as

$$
L(\alpha,\lambda;x|R=r) = c\prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{r_i} \tag{4.24}
$$

substituting $f(x)$ and $F(x)$ in to (4.24)

$$
L(\alpha,\lambda;x|R=r) = c\prod_{i=1}^{m} \frac{\alpha\beta x^{\alpha-1}}{(1-x^{\alpha})(\beta-\ln(1-x^{\alpha}))^2}\left(\frac{\beta}{\beta-\ln(1-x^{\alpha})}\right)^{r_i},
$$

$$
\begin{aligned}
\ln L(\alpha,\lambda;x|R=r) = \ln c + m\ln\alpha + m\ln\beta + (\alpha-1)\sum_{i=1}^{m}\ln x - \sum_{i=1}^{m}\ln(1-x^{\alpha}) \\
+\sum_{i=1}^{m} r_i \ln\beta - \sum_{i=1}^{m}(r_i+2)\ln(\beta-(1-x^{\alpha})),
\end{aligned}
$$

$$
\frac{\partial \ln L(\alpha,\lambda;x|R=r)}{\partial\beta} = \frac{m}{\beta} + \sum_{i=1}^{m} r_i\left(\frac{1}{\beta} - \frac{1}{(\beta-\ln(1-x^{\alpha}))}\right) - 2\sum_{i=1}^{m}\frac{1}{(\beta-\ln(1-x^{\alpha}))}.
$$

Now, we have considered,

$$
k_1(\beta) = \frac{m}{\beta} + \sum_{i=1}^{m} r_i\left(\frac{1}{\beta} - \frac{1}{(\beta-\ln(1-x^{\alpha}))}\right), \quad k_2(\beta) = 2\sum_{i=1}^{m}\frac{1}{(\beta-\ln(1-x^{\alpha}))}.
$$

Using iterative numerical computing method to obtain the ML estimate of β. We just draw a conclusion that $k_1(\beta) = k_2(\beta)$ has a root i.e, $\hat{\beta}$, numerically solved through R software. Since the empirical Bayes estimate of λ is

$$\hat{\lambda}_E = \frac{m+1}{a} \ln\left(1 + \frac{a}{\hat{\beta}+T}\right) \tag{4.25}$$

where β is replaced by $\hat{\beta}$ in Equation (4.21). Substituting $\hat{\beta}$ in Equation (4.22), the empirical Bayes estimation of $\hat{R}(t)$ is obtained

$$\hat{R}_E(t) = -\frac{1}{a} \ln\left(\sum_{s=0}^{\infty} \frac{(-a)^s}{s!} \left(1 - \frac{s\ln(1-t^{\alpha})}{(\hat{\beta}+T)}\right)^{-(m+1)}\right). \tag{4.26}$$

Similarly, the empirical Bayes estimation of $\hat{h}(t)$ is given as

$$\hat{h}_E(t) = \frac{m+1}{a} \ln\left(1 + \frac{a\alpha t^{\alpha-1}}{(\hat{\beta}+T)(1-t^{\alpha})}\right) \tag{4.27}$$

As it has been mentioned earlier, using $(R_i = r_i = 0; i = 1, \cdots, m-1)$ in Equation (2.3) and Equation (4.24) and proceed to above subsequent equations, we can get the Bayes and empirical estimators $\hat{\lambda}_{B_2}, \hat{\lambda}_{E_2}$, $\hat{R}_{B_2}(t), \hat{R}_{E_2}(t)$ and $\hat{h}_{B_2}(t), \hat{h}_{E_2}(t)$ of $\lambda, R(t), h(t)$ for Type-II censoring at time t, respectively. For the assessment of the above equations, we numerically calculate through R software.

4.4 Monte Carlo Simulation Study and Comparison of Estimators

An analytical study of the behavior of the estimators are not possible. Therefore, we make a study based on simulated results and hence, we need to simulate PT-II CBR samples from KD. The algorithm proposed by Balakrishnan and Sandhu (1995) have been used for simulation of samples, Since, we simulate PT-II CBR from specified KD and propose the use of following algorithm

i. Specify the value of n.

ii. Specify the value of m.

iii. Specify the value of parameters α, λ and p.

iv. Generate random number r_i from $B\left(n-m-\sum_{l=0}^{i-1} r_l, p\right)$, for $i = 1,2,3,\cdots,m-1$.

v. Set r_m according to the following relation.

vi. $r_m = \begin{cases} n-m-\sum_{l=1}^{m-1} r_l & \text{if } n-m-\sum_{l=1}^{m-1} r_l > 0 \\ 0 & \text{otherwise} \end{cases}$

vii. Generate m independent $U(0,1)$ random variables $W_1, W_2, \cdots, W_m$.

viii. For given values of the progressive type-II censoring scheme $r_i (i = 1,2,\cdots,m)$ set $V_i = W_i^{1/(i+r_m+\cdot+r_{m-i+1})} (i = 1,2,\cdots,m)$.

ix. Set $U_i = 1 - V_m V_{m-1} \cdots V_{m-i+1} (i = 1,2,\cdots,m)$, then $U_1, U_2, \cdots, U_m$ are PT-II CBR samples of size m from $U(0,1)$.

x. Finally, for given values of parameters α and λ, set $x_i = F^{-1}(U) (i = 1,2,\cdots,m)$. Then $(x_1, x_2, \cdots, x_m)$ is the required PT-II CBR sample of size m from the KD.

Comparison of Estimators

Here, we compare the different estimators obtained through PT-II CBR and Type-II censored samples. The comparison of the risks (average loss over sample space) under LINEX loss function. The estimators $\hat{\lambda}_B, \hat{\lambda}_{B_2}, \hat{\lambda}_E, \hat{\lambda}_{E_2}; \hat{R}_B(t), \hat{R}_{B_2}(t), \hat{R}_E(t), \hat{R}_{E_2}(t)$ and $\hat{h}_B(t), \hat{h}_{B_2}(t), \hat{h}_E(t), \hat{h}_{E_2}(t)$ of $\lambda, R(t)$ and $h(t)$ are respective Bayes and empirical Bayes estimators for PT-II CBR and Type-II censoring samples under LINEX loss function, respectively. Through MC simulation obtained the risks of the estimators of 1000 samples. Here, we note that the risks of the estimators are function of $n, m, a, \alpha, \lambda, \beta$ and t. The choice of hyper parameters of the prior distribution of λ can be taken in such a way that if we consider any two independent information as prior mean and variance of λ, then, $(\mu = 1/\beta, \sigma^2 = 1/\beta^2)$ whereas μ is considered as true values of the parameter λ for different confidence in terms of smaller and larger variances. On the basis of this information, the hyper parameter of λ can be easily evaluated from this relation, $(\beta = \mu/\sigma^2)$.

In order to consider the variation of these values, we obtained the simulated risks for $n = 20[10]90, m = 10[10]80, t = 0.2, \alpha = 2$(known), $\lambda = 2 = \mu$ (say prior mean of λ), $\sigma^2 = (1,3)$ (say prior variance of λ), since $\beta = (2/1, 2/3)$, $a = \pm 1.5$. We use the symbol R_L to denote the risk under LINEX loss function, and the simulated risks under LINEX loss functions are given in Tables $(4.1 - 4.2)$. Table (4.1) present the risks of estimators for PT-II CBR. The next Table (4.2) show the risks of estimators for Type-II censored samples. From Table (4.1), we can observe that for PT-II CBR, the risk of the estimators of $\hat{\lambda}_E$ and $\hat{h}_E(t)$ under LINEX loss function is the least (for both small and large prior variances i.e. $\sigma^2 = 1,3$) for both $a = +1.5$ (when o.e. is more serious than u.e.) and $a = -1.5$ (when u.e. is more serious than o.e.). But the risk of the estimators of $\hat{R}_B(t)$ under PT-II CBR is minimum for LINEX loss function with $a = \pm 1.5$ for small and large prior variances. Due to the change in the value of n and m (effective sample size), the risks of the estimators change, but follow a particular trend. Further, the risk of the estimator $\hat{\lambda}_E$ and $\hat{h}_E(t)$ under LINEX loss function was found to be least always. It is also observed that as the failure proportion (m/n) increases, the magnitude of the risk of the estimator $\hat{\lambda}_E$ and $\hat{h}_E(t)$ decreases. However, the magnitude of the risk of the estimator $\hat{R}_B(t)$ increases as failure proportion increases.

From Table (4.2), we can observe that for Type-II censoring, the risk of the estimators of $\hat{\lambda}_{E_2}, \hat{h}_{E_2}(t)$ and $\hat{R}_{B_2}(t)$ have also the least (for both small and large prior variances) for $a = \pm 1.5$ under LINEX loss function. When the change in the value of (n,m) with respective for small and large prior variances, the risks of the estimators change, they have follow a similar trend as discuss above in Table (4.1). But, the risk of the estimators at $\hat{\lambda}_{E_2}, \hat{h}_{E_2}(t)$ and $\hat{R}_{B_2}(t)$ were also found to be the least always. From Tables $(4.1 - 4.2)$, it can be seen that the behavior of the risks of the estimators under PT-II CBR is more similar to that of the estimators under

Type-II censoring. The risks were found to be least for the empirical Bayes estimators $\hat{\lambda}_E, \hat{\lambda}_{E_2}$ and $\hat{h}_E(t), \hat{h}_{E_2}(t)$ of λ and $h(t)$ with an informative prior $\Gamma(1,\beta)$ respectively. Therefore, we propose that empirical Bayesian estimator of parameter and reliability characteristics can use planning of the experiment. Hence, the reliability practitioners can save much time and cost of the experiment.

4.5 An application to Ulcer Patients Data

Now, we extract 43 primary disease (ulcer) patients data set from Collett (2014) to show practical applicability of proposed work. It have been taken for the analysis of PT-II CBRs discussed in the context of a study based on age $\left((10^{-2}) * age\right)$ data. In order to have an idea about the associated primary disease (ulcer) patient's age failure rate, we considered, a graphical method based on TTT plot as a crude indicator see Aarset (1987). The empirical TTT is given as $T(\frac{r}{n}) = \frac{\sum_{i=1}^{r} x_{(i)} + (n-r)x_{(r)}}{\sum_{i=1}^{n} x_{(i)}}$, where $r = 1, 2, \cdots, n$ and $x_{(r)}$ is the order statistics of the sample. For this data set in Figure (4.4) shows concave TTT plots, indicating increasing failure rate functions along with Figure (4.5), (4.6) and (4.7) represent PDF/CDF plot, sample Q-Q plot and hazard plots respectively, which can be properly accommodated by KD. However, we fitted three competitive distributions, $F(x;\alpha,\lambda) = \left(1 - e^{-\lambda x}\right)^{\alpha}$, $x > 0,\ \alpha > 0, \lambda > 0$ and $F(x;\alpha,\lambda) = 1 - e^{-(\lambda x)^{\alpha}}$, $x > 0,\ \alpha > 0, \lambda > 0$ are CDFs of the EED (Exponentiated exponential distribution) and WD (Weibull distribution) respectively. Table (4.3) provides the -log-L values and the AIC, BIC and p-values for these distributions. They indicate evidence in favor of KD. The ML estimates (and their corresponding standard errors in parentheses) of the KD, EED and WD parameters are given by $\hat{\alpha} = 3.2490(0.0108917), \hat{\lambda} = 5.64104(0.03766)$; $\hat{\alpha} = 15.44731(0.12571), \hat{\lambda} = 6.531165(0.01933)$ and $\hat{\alpha} = 3.55875(0.01012), \hat{\lambda} = 1.76975(0.00186)$ respectively. But for the purpose of illustrating the method discussed in this chapter, PT-II CBR samples are generated from this data set under different schemes see Table (4.6). The box plot of different censoring schemes as well as descriptive statistics is also presented in Figure (4.8) and Table (4.7) respectively. The required numerical calculations for the considered schemes are carried out using the formula given in Section (4.3) through R software see Ihaka and Gentleman (1996). The Bayes estimates, empirical Bayes estimates of λ_B, $R_B(.)$, $h_B(.)$ and λ_E,$R_E(.)$, $h_E(.)$ under LINEX loss for $a = \pm 1.5$ are presented in Table (4.4). While, Table (4.5) shows the Bayes, empirical Bayes estimates of λ_{B_2}, $R_{B_2}(.)$, $h_{B_2}(.)$ and λ_{E_2},$R_{E_2}(.)$, $h_{E_2}(.)$ for Type-II censoring under LINEX loss for $a = \pm 1.5$. From Tables ($4.4 - 4.5$), it may also be observed that the behavior of the estimators under PT-II CBRs are more similar to that of the estimators under Type-II censoring. The estimates were found to be decreases as effective sample size increases.

4.6 Conclusion

On the basis of the previous discussion given in the above Section (4.5), we may conclude that the proposed empirical Bayes estimators $\hat{\lambda}_E, \hat{\lambda}_{E_2}$ and $\hat{h}_E(t), \hat{h}_{E_2}(t)$ are better than Bayes estimators $\hat{\lambda}_B, \hat{\lambda}_{B_2}$ and $\hat{h}_B(t), \hat{h}_{B_2}(t)$ for smaller or larger prior variance ($\sigma = 1,3$) of β with $a = \pm 1.5$. Also, we have seen that Table ($4.1 - 4.2$) under LINEX loss function for the estimators $\hat{R}_E(t)$ and $\hat{R}_{E_2}(t)$ is not always less than those of $\hat{R}_B(t)$ and $\hat{R}_{B_2}(t)$. Since the risks associated with $\hat{R}_B(t)$ and $\hat{R}_{B_2}(t)$ is smaller than the risk associated with reliability of the empirical estimators. Thus, the use of propose estimator $(\hat{\lambda}_E, \hat{R}_B(t), \hat{h}_E(t))$ and $(\hat{\lambda}_{E_2}, \hat{R}_{B_2}(t), \hat{h}_{E_2}(t))$ under PT-II CBRs and Type-II are recommended under LINEX loss function respectively.

TABLE 4.1: Risks of the estimators of λ, R and h under LINEX loss function for fixed $\alpha = 2, \lambda = 2$ and $t = 0.2$ under PT-II CBR.

σ	n	m	$R_L(\hat{\lambda}_B)$	$R_L(\hat{\lambda}_E)$	$R_L(\hat{R}_B(t))$	$R_L(\hat{R}_E(t))$	$R_L(\hat{h}_B(t))$	$R_L(\hat{h}_E(t))$	$R_L(\hat{\lambda}_B)$	$R_L(\hat{\lambda}_E)$	$R_L(\hat{R}_B(t))$	$R_L(\hat{R}_E(t))$	$R_L(\hat{h}_B(t))$	$R_L(\hat{h}_E(t))$
			$a = -1.5$						$a = +1.5$					
	20	10	2.0554	1.7067	2.2062	2.2484	7.7037	6.8461	0.7931	0.6942	0.7718	0.7807	1.5106	1.4218
	30	20	0.9801	0.7438	2.2609	2.2924	6.3574	5.7448	0.4950	0.4049	0.7837	0.7903	1.3732	1.3034
	40	30	0.5118	0.3769	2.3032	2.3312	5.4718	4.9748	0.3271	0.2584	0.7915	0.7972	1.2841	1.2241
1	50	40	0.3493	0.2581	2.3235	2.3473	5.0844	4.6820	0.2452	0.1924	0.7956	0.8004	1.2368	1.1862
	60	50	0.2605	0.1944	2.3374	2.3578	4.8334	4.4966	0.1859	0.1471	0.7993	0.8034	1.1959	1.1511
	70	60	0.2050	0.1541	2.3472	2.3651	4.6637	4.3741	0.1537	0.1233	0.8013	0.8050	1.1723	1.1329
	80	70	0.1555	0.1200	2.3601	2.3766	4.4506	4.1934	0.1224	0.0992	0.8036	0.8070	1.1463	1.1105
	90	80	0.1319	0.1014	2.3642	2.3789	4.3803	4.1522	0.1111	0.0901	0.8041	0.8071	1.1402	1.1086
	20	10	1.5727	1.5716	2.2474	2.2521	6.8021	6.7789	0.6867	0.6842	0.7798	0.7807	1.4286	1.4218
	30	20	0.7183	0.7126	2.2933	2.2982	5.7144	5.6381	0.4101	0.4049	0.7894	0.7903	1.3132	1.3034
	40	30	0.4267	0.4147	2.3207	2.3253	5.1583	5.0838	0.2659	0.2584	0.7962	0.7972	1.2343	1.2241
3	50	40	0.2565	0.2460	2.3418	2.3460	4.7584	4.6900	0.1992	0.1924	0.7995	0.8004	1.1953	1.1862
	60	50	0.1979	0.1903	2.3559	2.3599	4.5271	4.4659	0.1521	0.1471	0.8027	0.8034	1.1595	1.1511
	70	60	0.1518	0.1459	2.3650	2.3685	4.3731	4.3190	0.1274	0.1233	0.8043	0.8050	1.1405	1.1329
	80	70	0.1266	0.1220	2.3723	2.3756	4.2577	4.2091	0.1022	0.0992	0.8063	0.8070	1.1176	1.1105
	90	80	0.1040	0.1001	2.3775	2.3805	4.1727	4.1287	0.0931	0.0901	0.8065	0.8071	1.1148	1.1086

TABLE 4.2: Risks of the estimators of λ, R and h under LINEX loss function for fixed $\alpha = 2, \lambda = 2$ and $t = 0.2$ under Type-II censoring.

σ	n	m	$R_L(\hat{\lambda}_{B_2})$	$R_L(\hat{\lambda}_{E_2})$	$R_L(\hat{R}_{B_2}(t))$	$R_L(\hat{R}_{E_2}(t))$	$R_L(\hat{h}_{B_2}(t))$	$R_L(\hat{h}_{E_2}(t))$	$R_L(\hat{\lambda}_{B_2})$	$R_L(\hat{\lambda}_{E_2})$	$R_L(\hat{R}_{B_2}(t))$	$R_L(\hat{R}_{E_2}(t))$	$R_L(\hat{h}_{B_2}(t))$	$R_L(\hat{h}_{E_2}(t))$
			a=-1.5						a=+1.5					
	20	10	15.4053	15.4024	1.9869	1.9869	15.8211	15.8200	2.0154	2.0152	0.7243	0.7243	2.0365	2.0365
	30	20	14.8668	14.8624	1.9911	1.9911	15.6074	15.6056	1.9872	1.9870	0.7252	0.7252	2.0257	2.0256
	40	30	14.3394	14.3333	1.9954	1.9954	15.3939	15.3914	1.9593	1.9590	0.7262	0.7262	2.0150	2.0148
1	50	40	13.8365	13.8289	1.9996	1.9997	15.1863	15.1832	1.9301	1.9297	0.7271	0.7272	2.0037	2.0035
	60	50	13.3365	13.3276	2.0039	2.0040	14.9758	14.9720	1.9030	1.9024	0.7281	0.7281	1.9932	1.9930
	70	60	12.8660	12.8558	2.0082	2.0083	14.7738	14.7693	1.8747	1.8741	0.7290	0.7290	1.9823	1.9820
	80	70	12.4045	12.3931	2.0124	2.0125	14.5717	14.5667	1.8469	1.8462	0.7299	0.7300	1.9715	1.9712
	90	80	11.9724	11.9600	2.0166	2.0167	14.3788	14.3732	1.8170	1.8162	0.7310	0.7310	1.9599	1.9596
	20	10	15.3963	15.3956	1.9869	1.9869	15.8176	15.8173	2.0149	2.0148	0.7243	0.7243	2.0364	2.0363
	30	20	14.8698	14.8688	1.9911	1.9911	15.6086	15.6082	1.9873	1.9873	0.7252	0.7252	2.0258	2.0257
	40	30	14.3382	14.3368	1.9954	1.9954	15.3934	15.3929	1.9587	1.9586	0.7262	0.7262	2.0147	2.0147
3	50	40	13.8395	13.8379	1.9996	1.9996	15.1876	15.1870	1.9310	1.9309	0.7271	0.7271	2.0040	2.0040
	60	50	13.3360	13.3340	2.0039	2.0040	14.9756	14.9748	1.9022	1.9020	0.7281	0.7281	1.9929	1.9928
	70	60	12.8641	12.8618	2.0082	2.0082	14.7729	14.7719	1.8743	1.8742	0.7290	0.7290	1.9821	1.9821
	80	70	12.3757	12.3732	2.0127	2.0127	14.5589	14.5578	1.8446	1.8444	0.7300	0.7300	1.9706	1.9705
	90	80	11.9640	11.9613	2.0166	2.0167	14.3750	14.3738	1.8188	1.8186	0.7309	0.7309	1.9606	1.9605

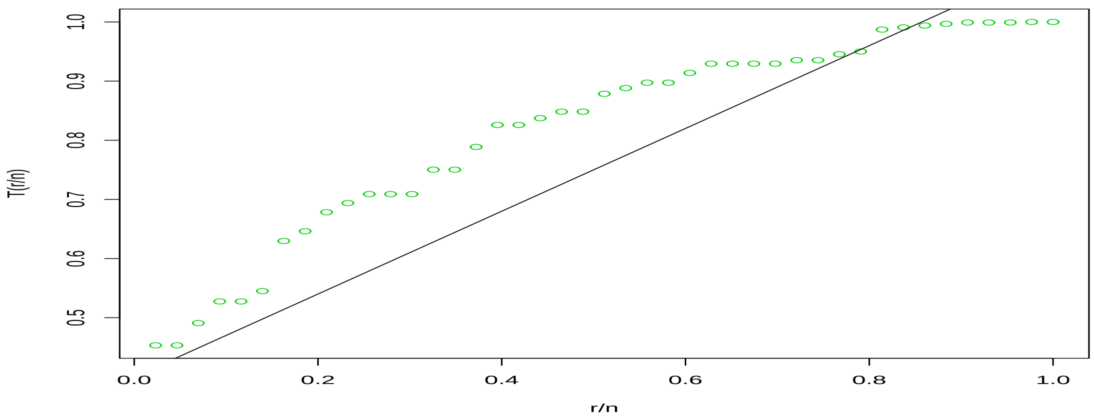

FIGURE 4.4: TTT plot for an ulcer patient with different ages $\left((10^{-2}) * age\right)$ for the primary disease.

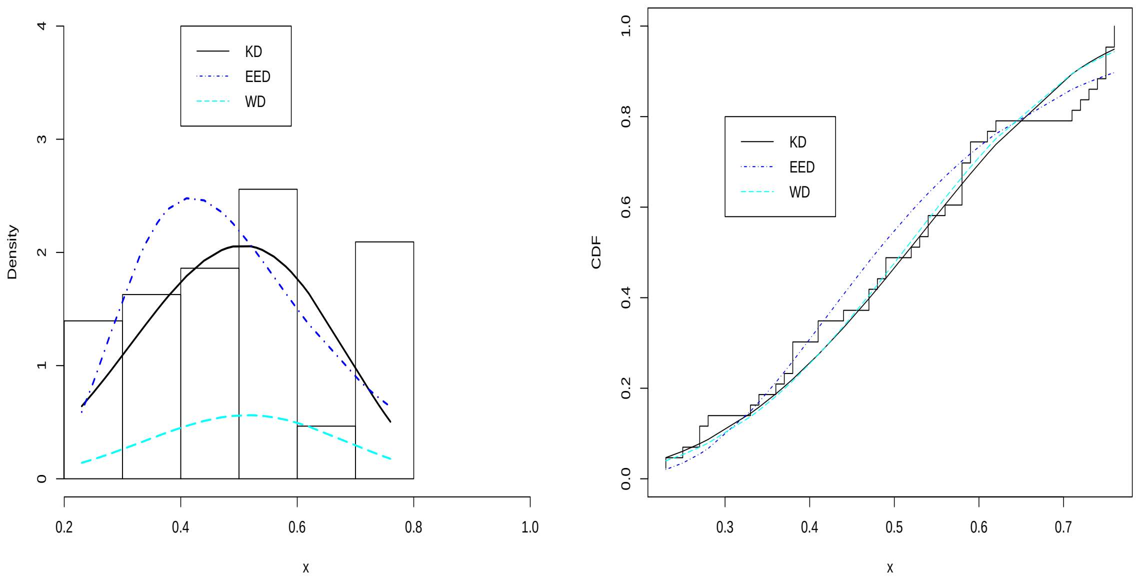

FIGURE 4.5: The PDF and CDF plots via the KD, EED and WD, for an ulcer patient with different ages $\left((10^{-2}) * age\right)$ for the primary disease. Left panel: PDF; right panel: CDF.

TABLE 4.3: The -log-L values and the AIC and BIC values for the KD, EED and WD fitted distributions.

distribution	-log-L	AIC	BIC	KS	p-value
KD	15.6765	27.35302	23.83062	0.082175	.9923
EED	18.6053	33.21062	29.68822	0.102652	.9333
WD	18.0699	32.13973	28.61733	0.086067	.9901

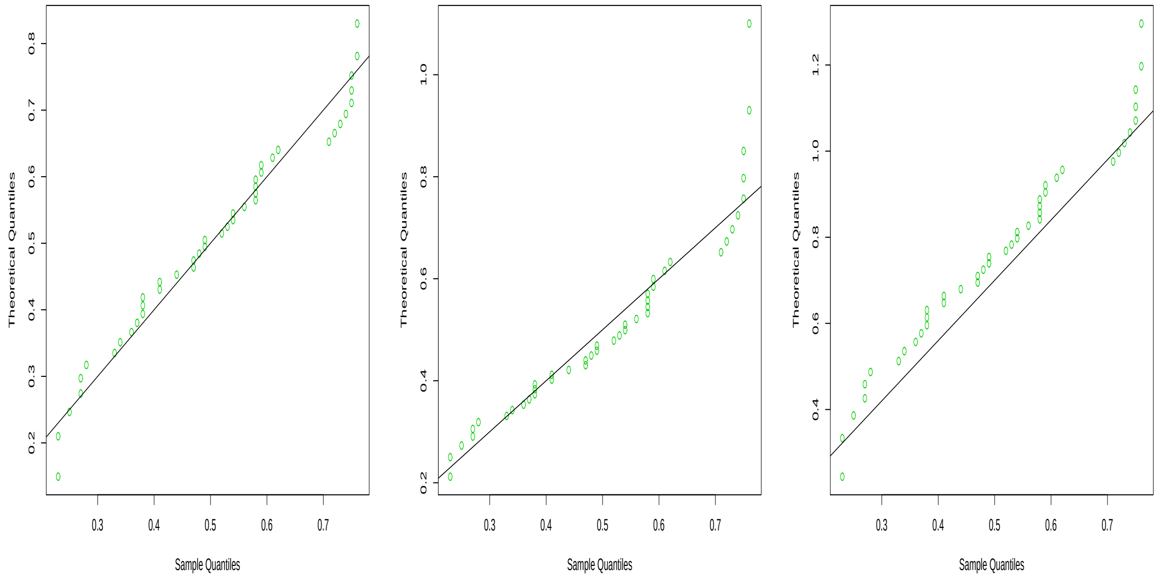

FIGURE 4.6: The sample Q-Q plots via the KD, EED and WD, for an ulcer patient with different ages $\left((10^{-2}) * age\right)$ for the primary disease. Left panel: KD; middle panel: EED; right panel:WD.

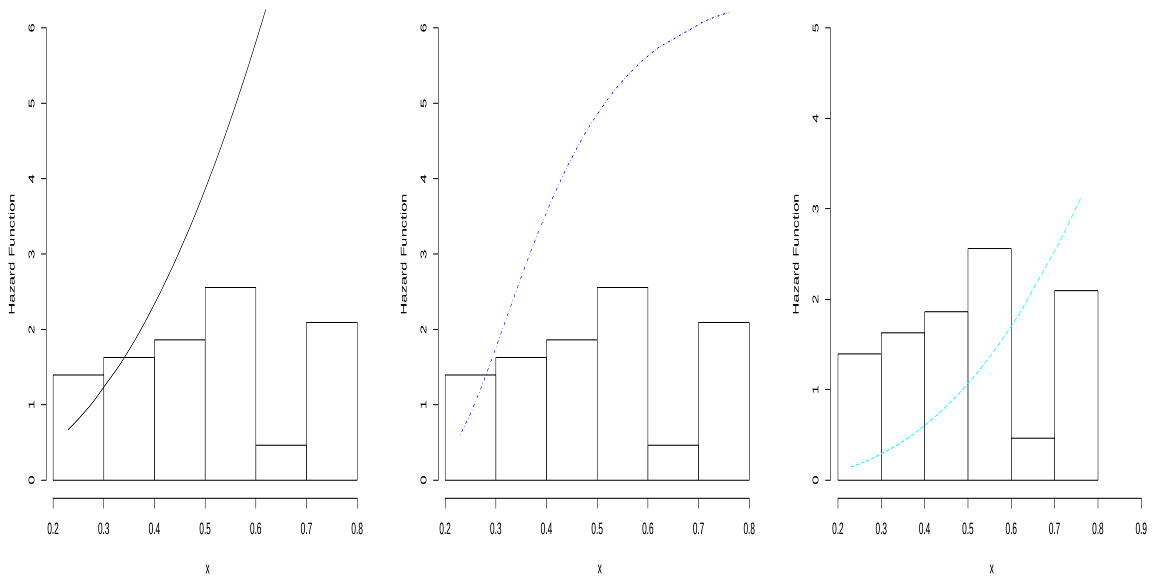

FIGURE 4.7: The hazard plots via the KD, EED and WD, for an ulcer patient with different ages $\left((10^{-2}) * age\right)$ for the primary disease. Left panel: KD; middle panel: EED; right panel:WD.

TABLE 4.4: Bayes and empirical Bayes estimates of λ, $R()$ and $h()$ under LINEX loss function for an ulcer patient with different ages $\left((10^{-2})*age\right)$ for the primary disease with fixed $n=43, p=0.5$, and $t=0.5074419$ under PT-II CBR.

Scheme	$\hat{\lambda}_B$		$\hat{\lambda}_E$		$\hat{R}_B(t)$		$\hat{R}_E(t)$		$\hat{h}_B(t)$		$\hat{h}_E(t)$	
$S_{n:m}$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$
$S_{43:20}$	22.4656	8.2216	20.7328	8.0140	0.8580	0.8568	0.8613	0.8601	2.0805	1.8104	2.0172	1.7724
$S_{43:23}$	19.8084	8.5843	18.7520	8.3980	0.8326	0.8312	0.8375	0.8361	2.4059	2.0935	2.3170	2.0321
$S_{43:25}$	15.3871	8.0207	14.7764	7.8589	0.7944	0.7924	0.7999	0.7981	2.8227	2.4187	2.7266	2.3519
$S_{43:30}$	14.1203	8.3109	13.7254	8.1758	0.7410	0.7387	0.7477	0.7453	3.4093	2.9418	3.3229	2.8555
$S_{43:33}$	10.6401	7.2122	10.3965	7.1005	0.6808	0.6777	0.6847	0.6818	3.9226	3.3225	3.8441	3.2937
$S_{43:35}$	9.8288	6.9530	9.6264	6.8518	0.6497	0.6465	0.6527	0.6494	4.1826	3.5656	4.1370	3.5207
$S_{43:38}$	7.9651	6.0887	7.8212	6.0045	0.5928	0.5893	0.5959	0.5921	4.5155	3.8749	4.5134	3.8106
$S_{43:40}$	7.7514	6.0319	7.6227	5.9539	0.5692	0.5654	0.5738	0.5702	4.7356	4.0268	4.6237	3.9718

TABLE 4.5: Bayes and empirical Bayes estimates of λ, $R()$ and $h()$ under LINEX loss function for an ulcer patient with different ages $\left((10^{-2})*age\right)$ for the primary disease with fixed $n = 43$ and $t = 0.5074419$ under Type-II censoring.

Schame	$\hat{\lambda}_{B_2}$		$\hat{\lambda}_{E_2}$		$\hat{R}_{B_2}(t)$		$\hat{R}_{E_2}(t)$		$\hat{h}_{B_2}(t)$		$\hat{h}_{E_2}(t)$	
(n,m)	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$	$a=-1.5$	$a=+1.5$
(43,20)	18.5013	7.7001	17.2407	7.4957	0.8554	0.8541	0.8578	0.8566	2.0807	1.8017	2.0223	1.7769
(43,23)	18.5814	8.3666	17.6334	8.1843	0.8277	0.8261	0.8309	0.8295	2.4480	2.1160	2.3925	2.0837
(43,25)	16.7217	8.3504	16.0359	8.1849	0.8041	0.8024	0.8055	0.8038	2.7107	2.3522	2.6790	2.3306
(43,30)	13.7657	8.1897	13.3810	8.0551	0.7434	0.7410	0.7464	0.7442	3.3630	2.8924	3.3011	2.8610
(43,33)	11.5500	7.6144	11.2843	7.4993	0.6925	0.6896	0.6953	0.6926	3.8517	3.2836	3.8030	3.2554
(43,35)	9.7266	6.9021	9.5257	6.8010	0.6490	0.6459	0.6551	0.6522	4.1296	3.5610	4.0747	3.5005
(43,38)	8.5165	6.4045	8.3633	6.3178	0.6015	0.5977	0.6065	0.6029	4.5553	3.8370	4.4561	3.7968
(43,40)	7.7646	6.0399	7.6356	5.9617	0.5673	0.5635	0.5725	0.5686	4.7640	4.0620	4.7010	3.9839

TABLE 4.6: PT-II CBR under different censoring schemes $(S_{n:m})$ for fixed $n = 43$ and $p = 0.5$ for an ulcer patient with different ages $\left((10^{-2}) * age\right)$ for the primary disease.

$S_{n:m}$	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$S_{43:20}$	X_i	0.23	0.37	0.49	0.58	0.58	0.58	0.58	0.59	0.59	0.61	0.62	0.71	0.72	0.73	0.74
	R_i	8	10	5	0	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.75	0.75	0.75	0.76	0.76										
	R_i	0	0	0	0	0										
$S_{43:23}$	X_i	0.23	0.38	0.47	0.52	0.53	0.54	0.58	0.58	0.58	0.58	0.59	0.59	0.61	0.62	0.71
	R_i	9	5	4	0	1	1	0	0	0	0	0	0	0	0	0
	X_i	0.72	0.73	0.74	0.75	0.75	0.75	0.76	0.76							
	R_i	0	0	0	0	0	0	0	0							
$S_{43:25}$	X_i	0.23	0.36	0.41	0.49	0.49	0.53	0.54	0.54	0.58	0.58	0.58	0.58	0.59	0.59	0.61
	R_i	7	5	4	0	1	0	0	1	0	0	0	0	0	0	0
	X_i	0.62	0.71	0.72	0.73	0.74	0.75	0.75	0.75	0.76	0.76					
	R_i	0	0	0	0	0	0	0	0	0	0					
$S_{43:30}$	X_i	0.23	0.38	0.41	0.47	0.47	0.48	0.49	0.49	0.52	0.53	0.54	0.54	0.56	0.58	0.58
	R_i	10	1	2	0	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.58	0.58	0.59	0.59	0.61	0.62	0.71	0.72	0.73	0.74	0.75	0.75	0.75	0.76	0.76
	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$S_{43:33}$	X_i	0.23	0.28	0.37	0.38	0.41	0.44	0.47	0.47	0.48	0.49	0.49	0.52	0.53	0.54	0.54
	R_i	4	3	1	2	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.56	0.58	0.58	0.58	0.58	0.59	0.59	0.61	0.62	0.71	0.72	0.73	0.74	0.75	0.75
	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.75	0.76	0.76												
	R_i	0	0	0												
$S_{43:35}$	X_i	0.23	0.28	0.37	0.38	0.38	0.41	0.41	0.44	0.47	0.47	0.48	0.49	0.49	0.52	0.53
	R_i	4	3	0	0	1	0	0	0	0	0	0	0	0	0	0
	X_i	0.54	0.54	0.56	0.58	0.58	0.58	0.58	0.59	0.59	0.61	0.62	0.71	0.72	0.73	0.74
	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.75	0.75	0.75	0.76	0.76										
	R_i	0	0	0	0	0										
$S_{43:38}$	X_i	0.23	0.23	0.27	0.34	0.37	0.38	0.38	0.38	0.41	0.41	0.44	0.47	0.47	0.48	0.49
	R_i	0	2	2	1	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.49	0.52	0.53	0.54	0.54	0.56	0.58	0.58	0.58	0.58	0.59	0.59	0.61	0.62	0.71
	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.72	0.73	0.74	0.75	0.75	0.75	0.76	0.76							
	R_i	0	0	0	0	0	0	0	0							
$S_{43:40}$	X_i	0.23	0.27	0.28	0.33	0.34	0.36	0.37	0.38	0.38	0.38	0.41	0.41	0.44	0.47	0.47
	R_i	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.48	0.49	0.49	0.52	0.53	0.54	0.54	0.56	0.58	0.58	0.58	0.58	0.59	0.59	0.61
	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	X_i	0.62	0.71	0.72	0.73	0.74	0.75	0.75	0.75	0.76	0.76					
	R_i	0	0	0	0	0	0	0	0	0	0					

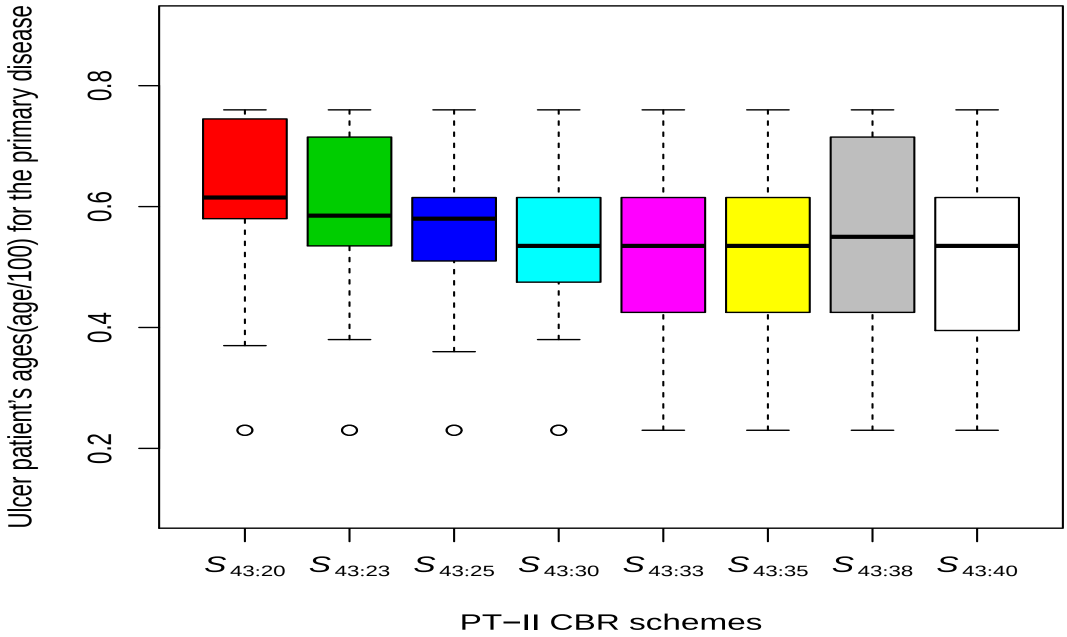

FIGURE 4.8: Box plot for PT-II CBR under different censoring schemes $S_{n:m}$ for an ulcer patient with different ages $\left((10^{-2}) * age\right)$ for the primary disease.

TABLE 4.7: Summary of the different censoring schemes ($S_{n:m}$) for PT-II CBR.

$S_{n:m}$	Min	Q_1	Median	Mean	Q_3	Max	SD	Skewness	Kurtosis
$S_{43:20}$	0.23	0.580	0.6150	0.62450	0.74250	0.76	0.1402807	-1.1887960	0.9533382
$S_{43:23}$	0.23	0.560	0.5900	0.61170	0.73500	0.76	0.1344907	-0.9352590	0.6451634
$S_{43:25}$	0.23	0.540	0.5900	0.59960	0.73000	0.76	0.1370669	-0.7446627	0.1144300
$S_{43:30}$	0.23	0.498	0.5800	0.58370	0.71750	0.76	0.129973	-0.4230723	-0.1033113
$S_{43:33}$	0.23	0.480	0.5800	0.56360	0.71000	0.76	0.140886	-0.3267876	-0.5493007
$S_{43:35}$	0.23	0.470	0.5600	0.55400	0.66500	0.76	0.1423789	-0.1981708	-0.7473111
$S_{43:38}$	0.23	0.433	0.5500	0.54630	0.71250	0.76	0.156757	-0.2054257	-0.9473064
$S_{43:40}$	0.23	0.403	0.5350	0.52680	0.61250	0.76	0.1522546	-0.0128692	-1.0644150

Chapter 5

Bayesian Estimation of the Number of Species Using Poisson Lindley Stochastic Abundance Model

5.1 Introduction

Previous chapters are based on the lifetime problem. While this chapter deals with ecological problem to estimating the number of species are present in an organism. The problem of estimating the number of species has been discussed extensively in the biological and ecological literature (Wilson and Collins (1992), Colwell and Coddington (1994), Bunge et al. (1995)). Various approaches have been proposed like parametric and non-parametric respectively. Both of these approaches have some optimal properties. In a parametric distribution, we can fit the observed frequency counts and use the estimated parameter values to estimate the number of species see Greenwood and Yule (1920). A non-parametric approach of ML version has been given by Norris and Pollock (1998). In non-parametric, the estimators are based on the coverage of the sample and the fraction of the population. These concepts were first proposed by Chao and Lee (1992).

But authors are interested to estimate the total number of species. When the total number of species have not been caught during the experiment. The estimators have been developed in this chapter through a parametric approach when observed samples were induced a parametric model. The estimation methods needed in this chapter are based on a Poisson mixed sampling model. Because each species independently contributed as representatives of the sample according to a Poisson process. When the rate or abundance parameters for these processes are taken to be i.i.d. RV from some fixed well-known distribution, see Chao and Bunge (2002).

One parameter Lindley (1958) distribution has been used for this process. Ghitany et al. (2008) studied some properties of the one-parameter Lindley distribution. In the application part, they showed that it is more flexible and works better in modeling for different types of data than well-known exponential distribution. Now we mixed this distribution with Poisson, and get discrete Poisson Lindley distribution. For applicability of Poisson mixed distributions, authors are referring to see Sankaran (1970). Furthermore the distributions based on Poisson mixture model for species abundance problems have been study by many authors such as (Bulmer (1974), Ord and Whitmore (1986), Sichel (1986)) etc. But in estimation problem for the number of species, according to Fisher et al. (1943) and Sichel (1986), it has required a suitable Poisson mixed model for a given problem. Thus we need to Poisson mixed as well as in-truncated distributions as per the demand of the problem see, Leite et al. (2000). Bunge and Fitzpatrick (1993) shows an interesting review of the problem of estimating the number of species. Therefore, we motivated by the above study is that no attempt has been made to use Poisson Lindley distribution as a model in species problems. Therefore, in this chapter we propose to develop such an estimator and estimation procedure for the parameters. The details of the mathematical formulations are discussed in a further Section.

In the past few decades, Bayesian estimation for the number of species population parameter based on Poisson mixed models have been studied by several authors such as Lewins and Joanes (1984), Leite et al. (2000) and Barger et al. (2010), etc. Fully hierarchical and empirical Bayesian estimation of the number of species based on Poisson-Gamma mixed model has been discussed by Rodrigues et al. (2001), and for other Poisson-mixed models by Wang et al. (2007), Barger et al. (2010), etc. But, it seems as if no attempt has been made to develop Bayes estimators of the number of species based on Poisson mixed Lindley distribution. Although estimation of the number of species based on Poisson mixed models under classical set up has been attempted by Gotelli and Colwell (2011), Chao and Lee (1992), Sichel (1986), etc. Therefore, we propose to develop a Bayesian estimation procedure to obtain the estimate of the number of species (using a Lindley model as a stochastic abundance model in which the sample according to independent Poisson process i.e., Poisson Lindley). Jeffery's and Bernardo's reference priors have been obtaining and proposed the Bayes estimators of the number of species for this model. An important feature of this chapter is to develop the required mathematics for the number of species parameters and priors along with its application to biological data.

5.2 Model and Likelihood Function

In biological sampling there are S species present, it has for some time been observed. When the successive, independent and unequal samples with sizes $x_1, x_2, x_3, \cdots, x_S$ be taken from heterogeneous abundance of species. The number of individuals observed in different samples will vary in a different manner in study period $[0,t]$. The distribution of the number of observed species depends only on one parameter Poisson distribution $(t\lambda_i)$ may be easily expressed in terms of the number expected (λ_i), which is given $\frac{e^{-t\lambda}(t\lambda)^{x_i}}{x_i!}$, $i = 1,2,3,\cdots,S$. Where X_i is the variate representing the number, which has been observed in any sample. And λ_i is the parameter, which is average value of X_i, and need not be whole number. This is an extension of the Poisson process, and is provided by supposition that the values of λ are distributed as well-known Lindley distribution with density function f_η, where η is a low dimensional parameter vector. In the Lindley distribution case, $\eta = \theta$ and $f_\eta(\lambda) = f_\theta(\lambda)$ and empirical CDF is $F_\theta(\lambda)$. Thus λ must be positive, and it has followed a well known form the distribution of Lindley (θ), such that the element of frequency or probability with which it falls in any infinitesimal range.

$$F_\theta(\lambda) = F(\lambda|\theta) = 1 - \frac{(1+\theta+\lambda\theta)e^{-\lambda\theta}}{1+\theta}; \qquad \lambda > 0; \theta > 0,$$

$$f_\theta(\lambda) = f(\lambda|\theta) = \frac{\theta}{\theta+1}(1+\lambda)e^{-\lambda\theta}; \qquad \lambda > 0; \theta > 0.$$

We can only observe the number of individuals contributed to the sample by each species. When contribution is greater than 0 i.e. $X_i > 0$. The species that contribute zero individuals to the sample are unobserved. The observed data are therefore $S_j = \sum_{i=1}^{S} I(X_i = j)$ for $j \geq 1$. Thus, S_j represent the number of species that contribute j individuals to the sample. The observed number of species is $w = \sum_{j\geq 1} s_j$, and the observed number of individual is $s = \sum_{j\geq 1} js_j$, where s_j are realized values of S_j. The goal is to estimate S (or equivalently to predict s_0) based on the observed frequency counts $\{s_j : j \geq 1\}$. Without loss of generality, we can and do take $t = 1$ because the time scale does not affect any of our estimates of S.

Therefore, the marginal distribution of X_i is $p_\theta(j) = \int \frac{e^{-\lambda}\lambda^j}{j!} f(\lambda|\theta)d\lambda$ representing the zero truncated P-mixed Poisson distribution, where $f(\lambda|\theta) = \frac{\partial}{\partial\lambda}F(\lambda|\theta)$. Sankaran (1970) derived the zero truncated P-mixed Poisson Lindley distribution given below,

$$p_\theta(j) = \left(\frac{\theta}{1+\theta}\right)^2 \frac{j+\theta+2}{(\theta+1)^{j+1}}; \qquad \theta > 0; j = 0,1,2,3,\cdots, \tag{5.1}$$

when $j = 0$ then Equation (5.1) become

$$p_\theta(0) = \left(\frac{\theta}{1+\theta}\right)^2 \left(\frac{\theta+2}{\theta+1}\right); \qquad \theta > 0. \tag{5.2}$$

5.2.1 Likelihood and Information of Parameters

The likelihood can be written as

$$L(S,\theta|x) = \sum_{j\in\Delta} \prod_{j=1}^{S} p_\theta(j).$$

Where Δ is the set of $x_1, x_2, x_3, \cdots, x_S$ which correspond to the observed frequencies $(s_1, s_2, \cdots, s_S)$. Sanathanan (1972) has demonstrated that the likelihood can be written as

$$\begin{aligned} L(S,\theta|x) &= \binom{S}{w}(1-p_\theta(0))^w (p_\theta(0))^{S-w} \frac{w!}{\prod_{j\geq 1} s_j!} \prod_{j\geq 1}\left(\frac{p_\theta(j)}{1-p_\theta(0)}\right)^{s_j} \\ &= \binom{S}{w}\left(1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^w \left(\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^{S-w} \\ &\qquad \frac{w!}{\prod_{j\geq 1} s_j!} \prod_{j\geq 1}\left(\frac{\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{j+\theta+2}{(\theta+1)^{j+1}}\right)}{1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)}\right)^{s_j} \\ &= A(S,\theta)B(\theta), \end{aligned} \tag{5.3}$$

where $S \geq w$, i.e. $S - w = s_0$ is the number of unobserved species. Now, the likelihood are function of parameters S and θ in the Equation (5.3), where $\theta = (\theta_1, \theta_2, ..., \theta_m)$. Since, we consider θ is a nuisance parameter, and our interest is in estimating S. Which shows the likelihood can be factored into a binomial likelihood for w that corresponds $A(S,\theta)$, and a multinomial likelihood for the observed frequencies corresponds the $B(\theta)$. This factorization of the integrated likelihood has an important role. It was first formulated by Sanathanan (1972) who derived the asymptotic theory for the ML estimation for S and θ. Fisher Information matrix can only be found for likelihoods which are differentiable with respect to the parameters. In the species likelihood, S is discrete parameter, $S = 1, 2, ...$ This likelihood is not differentiable with respect to S. Lindsay and Roeder (1987) define information for discrete parameters using the LDS defined as

$$LDS(S) = \frac{L(S) - L(S-1)}{L(S)},$$

were $L(S)$ is the likelihood for an integer parameter S.

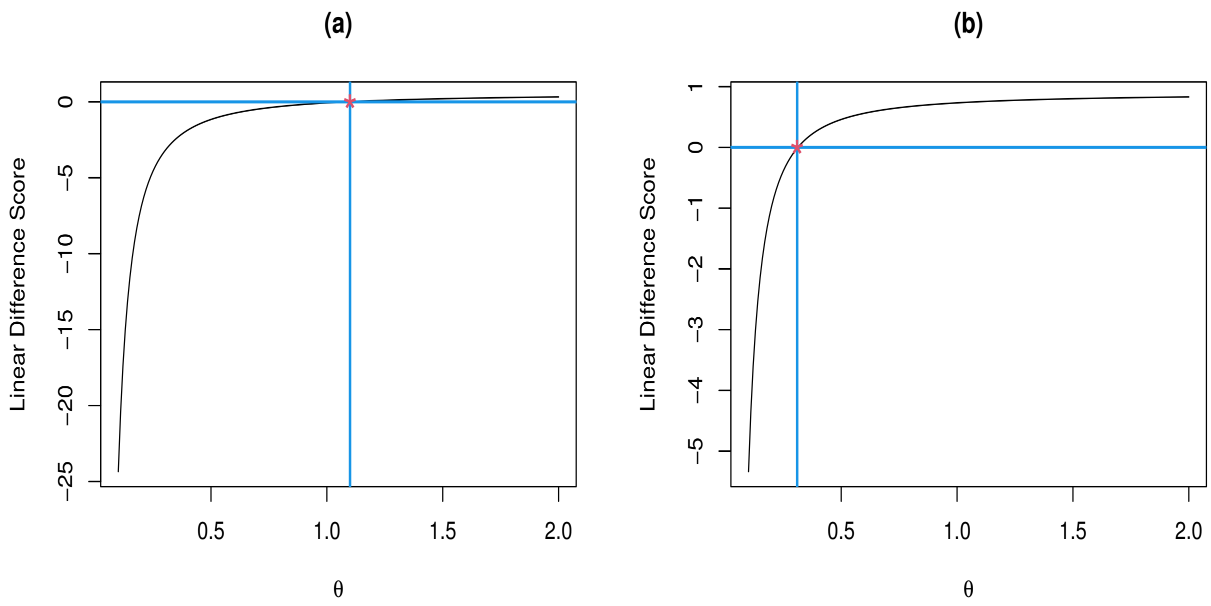

FIGURE 5.1: Plot of LDS for S with respect different θ for fixed in (a) $w = 30,\ N = 50$, and (b) $w = 45,\ N = 50$.

If $LDS(S)$ satisfies the form $LDS(S) = (Y - \mu_S)/c_S$, where μ_S and c_S are function of S and Y is random data, then $1/Var(LDS(S))$ is the information in S. In Figure (5.1), shows that $LDS(S) = 0$ then it gives the maxima of S with respective to different choice of θ. Using the method described by Lindsay and Roeder (1987) to calculate the information for S and θ, we obtain,

$$F(S,\theta) = \begin{pmatrix} \frac{1}{S}\frac{1-p_\theta(0)}{p_\theta(0)} & \left(-\frac{\partial}{\partial\theta}logp_\theta(0)\right)^T \\ -\frac{\partial}{\partial\theta}logp_\theta(0) & S(-\rho(\theta)) \end{pmatrix}.$$

Where, $\frac{\partial}{\partial\theta}p_\theta(0)$ is the column vector of partial derivatives. The $\rho(\theta) = \left(E_x\frac{\partial^2}{\partial\theta^2}logp_\theta(j)\right)$, has taken expectation with respect to p_θ. We may also observed that the diagonal elements of partitioned matrix contain elements which factor into a function of S times a function of θ. Thus we have,

$$F(S,\theta) = \begin{pmatrix} \frac{1}{S}\frac{(1+\theta)^3-\theta^2(\theta+2)}{\theta^2(\theta+1)} & -\left(\frac{2\theta^2+\theta-4}{\theta(\theta+1)(\theta+2)}\right)^T \\ -\left(\frac{2\theta^2+\theta-4}{\theta(\theta+1)(\theta+2)}\right) & S\left(\frac{2}{\theta} - \frac{2+8\theta+13\theta^2+10\theta^3+3\theta^4}{\theta(\theta+1)^5} + \psi(j,\theta)\right) \end{pmatrix}, \qquad (5.4)$$

where, $\psi(j,\theta) = \sum_{j=0}^{\infty} \frac{e^{-\theta j}(j+1)\theta^2}{(\theta+1)(j+\theta+2)^2}$.

5.3 Bayes Estimators of Parameters

In Bayesian paradigm, the parameter of interest θ and S are consider to be RV, and having their prior distribution. The selection of prior distribution is often based on the type of prior information available to us. When we have minimal or no information about the parameter then a non-informative prior should be used.

5.3.1 Bayes Estimators of Parameters Using Jeffery's Priors

The Jeffrey's prior (see Jeffreys (1946)) is one of the general rule. Using the fisher information matrix as shown above $F(S,\theta)$ in Equation (5.4). The Jeffery's prior for (S,θ) is $g_J(S,\theta)$. It based on invariance property under one to one re-parameterization. The Jeffery's prior is defined to be proportional to the square root of the Fisher information matrix. For multidimensional model, the determinant of the Fisher information is used, which preserve the invariance property. By calculating the determinant of the partitioned matrix in Equation (5.4) is,

$$\begin{aligned} g_J(S,\theta) &\propto det[F(S,\theta)]^{1/2} \\ &\propto S^{\frac{m-1}{2}} g(\theta), \end{aligned} \tag{5.5}$$

where $g(\theta)$ is some function of θ, which will depend on the dimension of the information matrix. When the dimension increases this will become complex,

$$g_J(S,\theta) \propto \left(\left(\frac{(\theta+1)^3 - \theta^2(\theta+2)}{\theta^2(\theta+2)} \right) \left(\frac{2}{\theta^2} - \frac{2+8\theta+13\theta^2+10\theta^3+3\theta^4}{\theta(\theta+1)^5} + \psi(j,\theta) \right) - \left(\frac{2\theta^2+\theta-4}{\theta(\theta+1)(\theta+2)} \right)^2 \right)^{\frac{1}{2}}. \tag{5.6}$$

Several authors have presented a general rule, and for using Jeffery's prior for an exponential family by showing that a proper posterior is produced (Jeffreys (1961), Barger and Bunge (2008) and Barger et al. (2010)). The product of two independent prior has been discussed by Jeffery. They suggest to use an idea about reasonable Jeffery's prior for integer parameter S and continuous parameter θ. Integer parameter is the interest of our study parameter. Using Bayes theorem for computing likelihood in Equation (5.3) and Jeffery's prior in Equation (5.7). We get the joint posterior distribution of $\pi_J(S,\theta)$ is,

$$\pi_J(S,\theta|x) \propto L(S,\theta|x)g_J(S,\theta),$$

$$\begin{aligned}\pi_J(S,\theta|x) \propto \binom{S}{w}\left(1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^w\left(\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^{S-w} \\ \frac{w!}{\prod_{j\geq 1} s_j!}\prod_{j\geq 1}\left(\frac{\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{j+\theta+2}{(\theta+1)^{j+1}}\right)}{1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)}\right)^{s_j} \\ \left(\left(\frac{(\theta+1)^3-\theta^2(\theta+2)}{\theta^2(\theta+2)}\right)\left(\frac{2}{\theta^2}-\frac{2+8\theta+13\theta^2+10\theta^3+3\theta^4}{\theta(\theta+1)^5}+\psi(j,\theta)\right)\right. \\ \left.-\left(\frac{2\theta^2+\theta-4}{\theta(\theta+1)(\theta+2)}\right)^2\right)^{\frac{1}{2}}. \qquad (5.7)\end{aligned}$$

Now full conditional posterior for S is

$$\begin{aligned}\pi_J(S|\theta,x) &\propto \binom{S}{w}\left(1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^{w+1}\left(\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^{S-w}, \\ &\propto NB\left(w+1,\left(1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)\right), \qquad (5.8)\end{aligned}$$

and full conditional posterior for θ is

$$\pi_J(\theta|S,x) \propto \frac{w!}{\prod_{j\geq 1} n_j!} \prod_{j\geq 1} \left(\left(\frac{\theta}{1+\theta} \right)^2 \left(\frac{j+\theta+2}{(\theta+1)^{j+1}} \right) \right)^{s_j} \left(\frac{1}{1-\left(\frac{\theta}{1+\theta}\right)^2 \left(\frac{\theta+2}{\theta+1}\right)} \right)^{w+1}$$
$$\left(\left(\frac{(\theta+1)^3-\theta^2(\theta+2)}{\theta^2(\theta+2)} \right) \left(\frac{2}{\theta^2} - \frac{2+8\theta+13\theta^2+10\theta^3+3\theta^4}{\theta(\theta+1)^5} + \psi(j,\theta) \right) \right.$$
$$\left. - \left(\frac{2\theta^2+\theta-4}{\theta(\theta+1)(\theta+2)} \right)^2 \right)^{\frac{1}{2}}. \quad (5.9)$$

Full conditional posterior for S, we can use direct sampling from negative binomial distribution with size $(w+1)$ and probability $\left(1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)$, and full conditional posterior for θ does not come in closed form then using the M-H steps uses a normal proposal distribution to get posterior samples.

5.3.2 Bayes Estimators of Parameters Using Bernardo's Reference Priors

Now, we have proposed for considering the Bernardo's reference prior. This prior is a quite general and powerful tool for obtaining automatic prior to be used in Bayesian analysis. Because of that reference prior are firstly useful with large sample but may also be helpful where the data analysis is unsure whether a sample is large. Typically the Bernardo's reference prior is the same as the Jeffery's prior in the one dimensional case, but where the parameter space Θ is bivariate or more. Non-informative prior is Bernardo's reference prior (Bernardo (1979)) based on maximizing and expected entropy (measurement of loss of information). The Bernardo's reference prior algorithm take into account (see Bernardo and Ramon (1998). It may also noted that in the standard Bayesian approach, the Bernardo's reference prior is used to obtain the joint posterior for $(S,\theta) \equiv \Theta$. In this approach, we only discuss the two groups case, where the parametric space Θ or vector is split in the parameter of interest S, and the nuisance parameter, θ under certain regularity conditions (see Bernardo (1979), Bernardo and Ramon (1998), Bernardo and Smith (2009)) for the existence of a consistent and asymptotically normal estimator of the parameters. Thus the reference prior for S, when θ is known, is used Fisher information matrix in Equation (5.4). The construction of the reference prior takes into account the order of interest of the parameters S and θ is a nuisance parameter.

Now we obtain the Bernardo's reference prior for a nuisance parameter, $m = 1$. Thus, the Fisher information matrix in Equation (5.4) will be 2×2. Let us assume $H = F^{-1}$ be the variance-covariance matrix. $(h_{11})^{-1/2} = a_0(S)b_0(\theta)$ and $(f_{22})^{1/2} = a_1(S)b_1(\theta)$ are the elements of the covariance and information matrices, respectively. The nuisance parameter θ and number of species S are independent to each other. The joint Bernardo's reference prior $g_R(S,\theta)$ will become,

$$\begin{aligned} g_R(S,\theta) &\propto (a_0(S))^{-1/2}(b_1(\theta))^{1/2} \\ &\propto S^{-1/2}(\rho(\theta))^{1/2}. \end{aligned} \tag{5.10}$$

$$g_R(S,\theta) \propto S^{-1/2}\left(-\frac{2}{\theta^2} + \frac{2+8\theta+13\theta^2+10\theta^3+3\theta^4}{\theta(\theta+1)^5} - \psi(j,\theta)\right)^{1/2}. \tag{5.11}$$

From above Equation (5.11), we may also seen the joint $g_R(S,\theta)$ factorized into a marginal distribution function of S and θ. Now, the joint posterior distribution of $\pi(S,\theta)$ is based on likelihood and Bernardo's reference prior is,

$$\pi_R(S,\theta|x) \propto L(S,\theta|x)g_R(S,\theta),$$

$$\begin{aligned} \pi_R(S,\theta|x) \propto \binom{S}{w}\left(1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^w \left(\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^{S-w} \\ \frac{w!}{\prod_{j\geqslant 1} s_j!}\prod_{j\geqslant 1}\left(\frac{\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{j+\theta+2}{(\theta+1)^{j+1}}\right)}{1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)}\right)^{s_j} \\ S^{-1/2}\left(-\frac{2}{\theta^2} + \frac{2+8\theta+13\theta^2+10\theta^3+3\theta^4}{\theta(\theta+1)^5} - \psi(j,\theta)\right)^{1/2}. \end{aligned} \tag{5.12}$$

Now full conditional posterior for S is

$$\pi_R(S|\theta,x) \propto S^{-1/2}\binom{S}{w}\left(1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^w\left(\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)\right)^{S-w}, \quad (5.13)$$

and full conditional posterior for θ is

$$\pi_R(\theta|S,x) \propto \frac{w!}{\prod_{j\geqslant 1} s_j!}\prod_{j\geqslant 1}\left(\frac{\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{j+\theta+2}{(\theta+1)^{j+1}}\right)}{1-\left(\frac{\theta}{1+\theta}\right)^2\left(\frac{\theta+2}{\theta+1}\right)}\right)^{s_j}\left(-\frac{2}{\theta^2}+\frac{2+8\theta+13\theta^2+10\theta^3+3\theta^4}{\theta(\theta+1)^5}-\psi(j,\theta)\right)^{1/2}. \quad (5.14)$$

Here, full conditional posterior distribution of S and θ are not obtainable in closed form then using the M-H steps. For posterior samples of S and θ, we used a negative binomial distribution and normal distribution as a proposal distribution for S and θ, respectively.

5.4 An application to Microbial Organisms Species Data

Let us consider a sample of microbial organisms species data set, taken from Barger and Bunge (2008). The data set may be assumed to be a sample from a P-mixed Poisson model having a non- monotonic HR as that of Poisson Lindley model. The data set was originally reported by Behnke et al. (2006), and it represents the classification of the organisms into species based on 18S rRNA similarity. The samples of microbes were taken from 18 meter below the water surface of the Framvaren Fjord in Norway. Diversity of these organisms is largely unknown and estimating the total number of species of microbes. Correspond the observed frequency (nonzero) and the number of species are listed as (j,s_j) : $(1,15),(2,6),(3,7),(4,2),(5,1),(6,1),(7,1),(8,1),(9,1),(12,1),(15,1),(20,1),(164,1)$. The observed number of species and observed number of individual organisms are found to be $w = 39$ and $s = 302$ respectively.

First of all, we checked the graphical method to the data set, have come from Poisson Lindley model. Figure (5.2) shows the (observed) relative frequency histogram and the postulated (or expected) relative histogram on the same graph. Which shows that Poisson Lindley model provides a satisfactory close to the agreement between two histogram appears. But there is little difference between the two histogram due to some chance fluctuation. Since, we study the chi-square test of goodness of fit. Hence, $\chi^2_{cal} = 3.70$ and $\chi^2_{tab,95\%,3} = 7.82$ then χ^2_{tab} is greater than χ^2_{cal}, so we can say that observed frequency has no significance difference between expected (hypothesized) frequency. Thus this data set has been proposed for Poisson Lindley model and compared with some well-established models, namely, Poisson and exponential-mixed Poisson model (discuss it in details Barger and Bunge (2008)). Here we used values of frequencies up to 10 selected by the criteria described therein (goodness-of-fit).

The full data includes observed frequencies greater than ten, but we only model the observed frequencies less than equal to ten. This can be interpreted as assuming the most abundant species are from a known sub population. For final estimate of the number of species were added later when observed frequencies greater than 10.

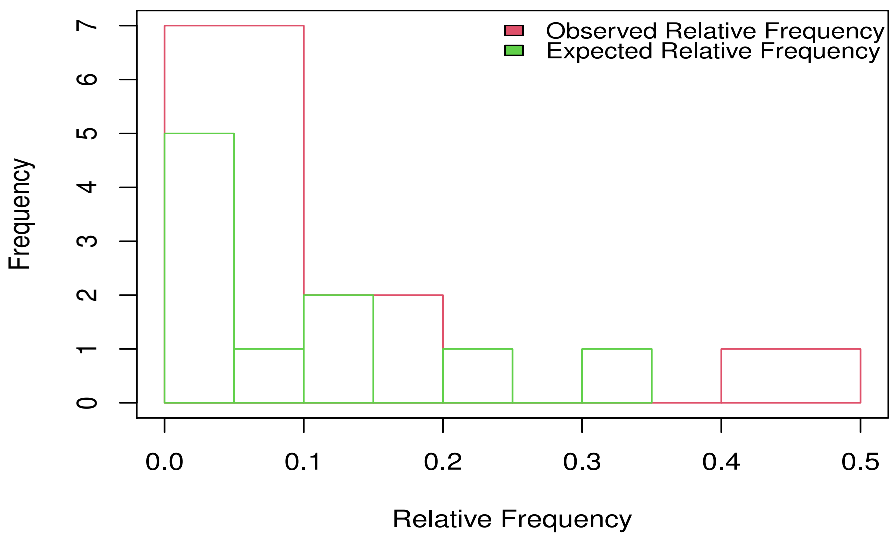

FIGURE 5.2: Observed and Expected relative frequency histogram plot of Poisson Lindley Model.

For Bayesian estimation we use MCMC sampler with M-H steps to simulate from the posterior distributions. Expression for the full conditionals posterior distribution of proposed model with Jeffery's prior is given by $\pi_J(S|\theta,x)$ and $\pi_J(\theta|S,x)$ in Equation (5.8) and (5.9), respectively and with Bernardo's reference prior is given by $\pi_R(S|\theta,x)$ and $\pi_R(\theta|S,x)$ in Equation (5.13) and (5.14), respectively. The posterior samples are taken to have an approximate effective sample size of 5000. Acceptance rates for parameters are kept below 40% and 30% for S and θ respectively.

In M-H step we use a normal proposal distribution for sampling of (nuisance) parameter θ. To obtain the sample from full conditional distribution for S in Equation (5.8) and (5.13). Figure (5.5) shows posterior simulations from each of the two models for species posterior distribution derived in Section (5.3). The proposed Poisson Lindley model parameter for Jeffrey's and Bernardo's reference priors, the posteriors are described in Subsection (5.3.1) and (5.3.2). It is well known that MCMC analysis provides reliable results only when the chains have run sufficiently large number of times and reached to the stationary distribution. In the existing literature of MCMC, a number of tools to assess the convergence of chain like mixing of chain and auto correlation are mentioned in Figure (5.3) and Figure (5.4). These Figures is enough to show that the chains in the present analysis have converged. Now, we may be mentioned here that Bayes estimators and credible intervals (with 95% confidence) have been obtained above using the MCMC procedures. The frequentist estimates for S are summarized in Table (5.1). While under Bayesian paradigm the estimate of S are summarized in Table (5.2). It has shown the posterior modes, means, median and central credible intervals. Also, we are drawn an comparison between Bayesian estimates and ML estimates; symmetric CI based on asymptotic normality, and asymptotic profile likelihood interval (described in Cormack (1992)) are included in Table (5.2). We can also see that the Bayesian estimates are always more than the ML estimates for PLJ and PLR. Further we observed that the profile likelihood intervals are comparable with credible interval estimates and the posterior mean estimates for S are also more than ML estimates. It may also notice that asymptotic 95% symmetric CI for the ML estimate in the consider Lindley model falls above the observed number of species, $w = 39$.

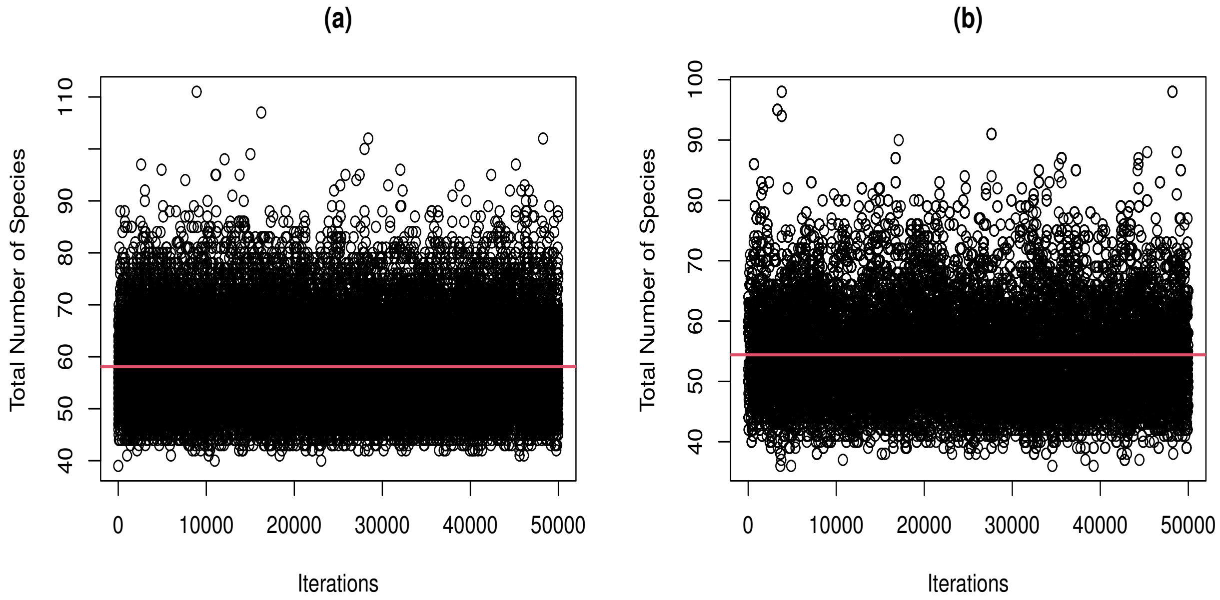

FIGURE 5.3: Trace plot of parameter S with (a) Jeffery's prior and (b) Bernardo's reference prior.

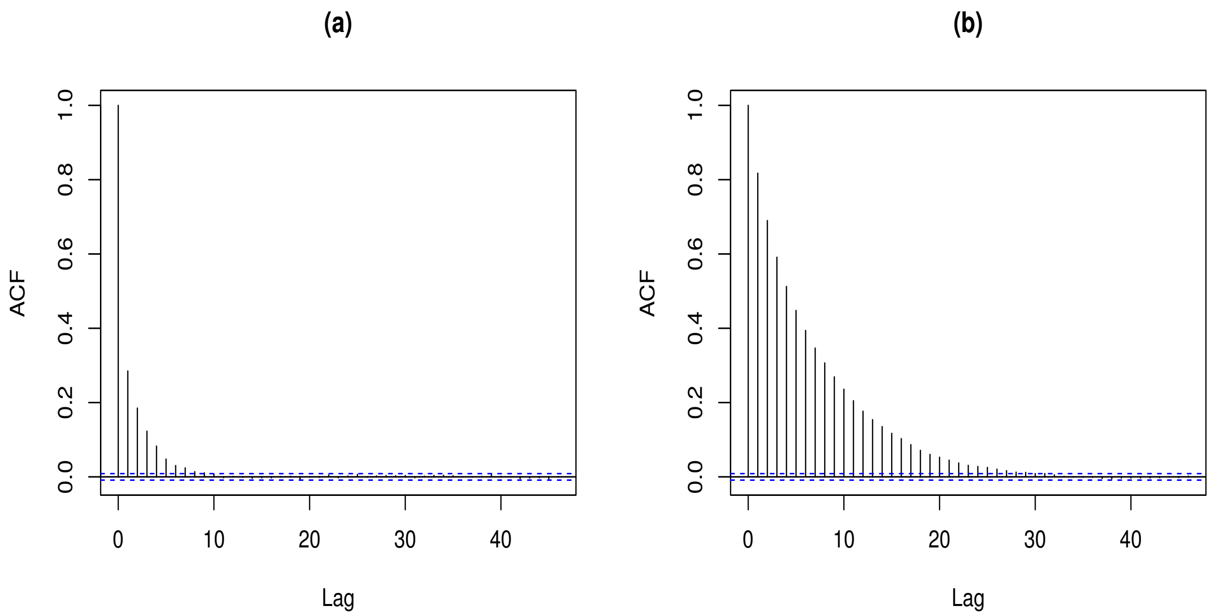

FIGURE 5.4: Auto Correlation Plot (ACF) of parameter S with (a) Jeffery's prior and (b) Bernardo's reference prior.

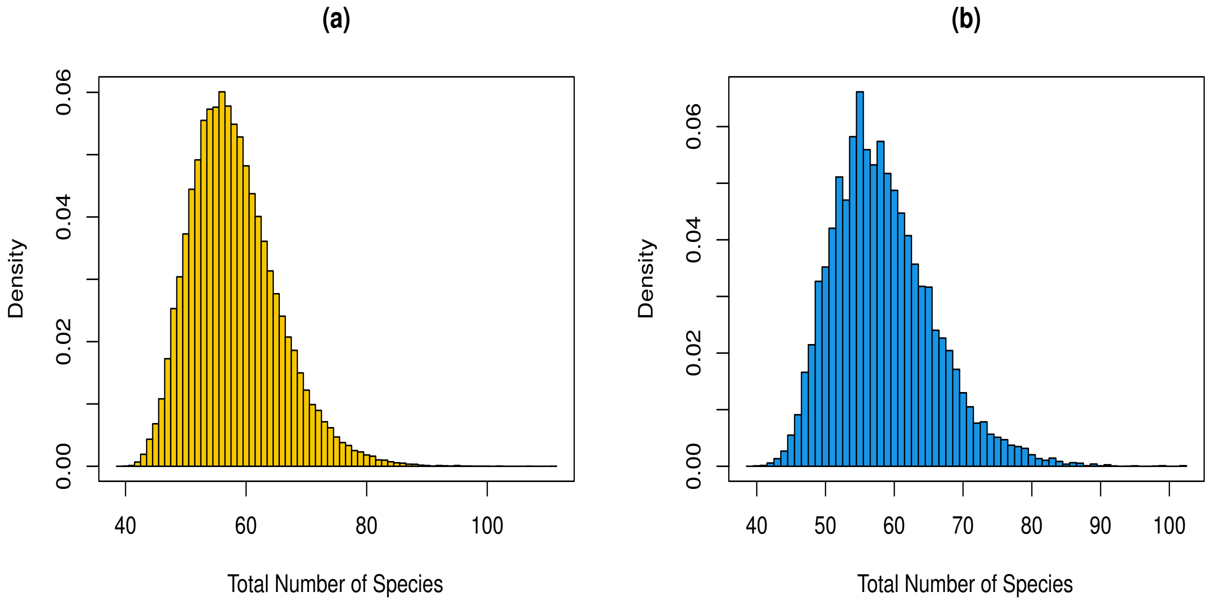

FIGURE 5.5: Posterior histogram plot of parameter S with (a) Jeffery's prior and (b) Bernardo's reference prior.

TABLE 5.1: ML estimate and 95% CI of parameter S obtained with profile likelihood θ_p and conditional likelihood θ_c.

Model	MLE	95%Confidence Interval
Poisson Lindley model	$S_p = 52.67544$	$(42.56637, 62.78451)$
	$S_c = 53.01631$	$(42.77726, 63.25537)$

TABLE 5.2: Summary statistics for posterior $\pi(S|x)$ with PLJ and PLR.

	PLJ	PLR
Mode	55	58
Mean	58.02068	58.37274
Median	57	58
95% Credible Interval	$(47, 74)$	$(47, 75)$

We next check the fit of the each model as well as the relative fit among the models. For the relative fit of models we have derived the deviance averaged over values from posterior sample for each considered model. The model deviance is defined as

$$DE(x,S,\theta) = -2\log L(S,\theta|x). \tag{5.15}$$

Now, using the posterior samples θ^i, $k = 1,2,...,N$ to obtain the average deviance, where I is the total number of posterior samples. Model deviance estimates (DE) formula is as given below,

$$\hat{DE}(x) = \frac{1}{N}\sum_{k=1}^{N} DE(x,\theta^j). \tag{5.16}$$

Table (5.3) is shown to DE of the models, lower DE shows a better fit. From Barger and Bunge (2008) considered the same data set for various models, that mentioned in the Table (5.3), such as PJ: Poisson model with Jeffrey's prior, PR: Poisson model with Bernardo's reference prior; EJ: exponential-mixed Poisson model with Jeffrey's prior and ER: exponential-mixed Poisson model with Bernardo's reference prior. Here we consider these models to compare deviance of PLJ and PLR. We obtained PLR have very minimum model DE as well as better fit for the data set.

TABLE 5.3: DIC for PJ: Poisson model with Jeffrey's prior; PR: Poisson model with Bernardo's reference prior; EJ: exponential-mixed Poisson model with Jeffrey's prior; ER:exponential-mixed Poisson model with Bernardo's reference prior; PLJ and PLR.

Model	DIC
PJ	58.05805
PR	58.06852
EJ	35.63834
ER	35.61999
PLJ	**32.5128**
PLR	**30.83205**

In this sequence, we plot the expected frequencies using posterior samples of the parameters S for PLJ and PLR. We see that for this small data set PLR fit is acceptable, see in Figure (5.6).

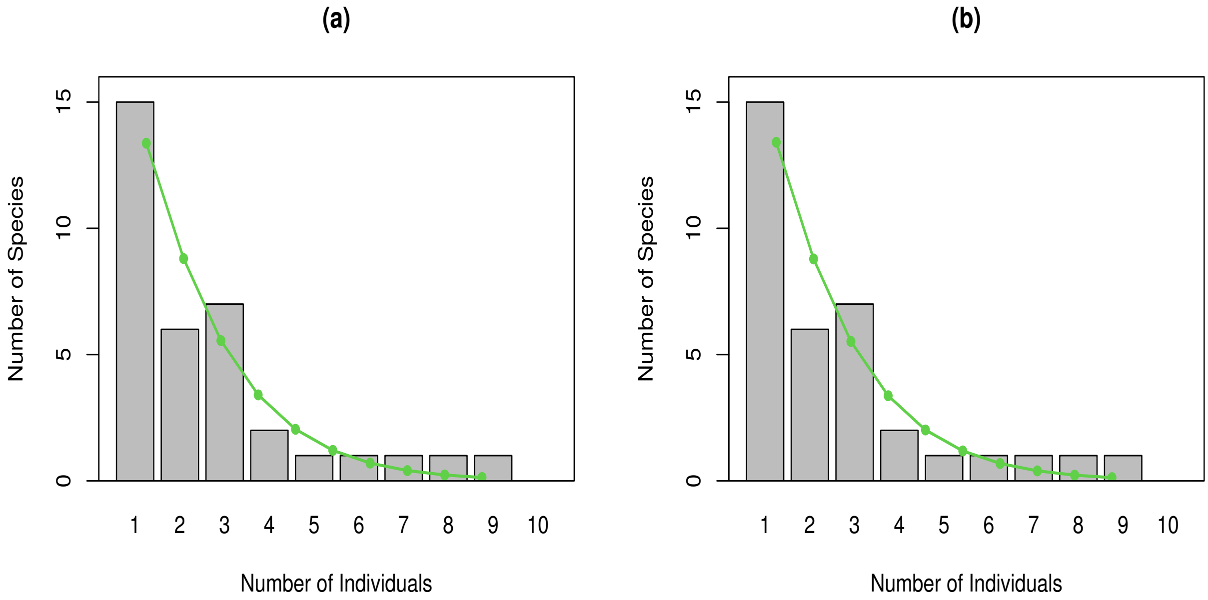

FIGURE 5.6: Expected frequency plot of S with (a) Jeffrey's prior and (b) Bernardo's reference prior.

The Jeffrey's and Bernardo's reference prior for the Poisson Lindley model give very similar results. These priors have been very minimal effect on resulting estimates. Hence, the model selection is highly influence by the final estimates. It is a very important problem for the choice of models.

5.5 Monte Carlo Simulation Study and Comparison of Estimators

We shall compute and set side by side the estimators obtained under ML and Bayesian estimators. The estimators $\hat{S}_p$, $\hat{S}_c$, $\hat{S}_J$ and $\hat{S}_R$ denotes the profile ML estimator, conditional ML estimator, Bayes estimator with Jeffery's prior and Bernardo's reference prior, respectively. Here, S stands for the total number of species as a discrete parameter and θ were abundance parameter generated from Lindley distribution as a nuisance parameter. For the stochastic abundance of the model we have stopping time t and w stands for the observed number of species in the sample experiment. The comparisons are based on the square root of average risk (expected loss over sample space) of the estimators of the parameters S, denoted by $R(S)$.

In the simulation study number of species was to be fixed to be $S = 50, 60$ and 70. The abundance parameters were generated from Lindley distribution with parameter $\theta = 0.5$ and 1.2. We considered the two stopping time i.e. $t = 0.4$ and 1.2. So we observed that the capture fraction are (C.F.=$\frac{w}{S} * 100$) lies between 70% and 98%. For each simulated data set, four estimates were reported $\hat{S}_p$, $\hat{S}_c$, $\hat{S}_J$ and $\hat{S}_R$. The estimates average risk based on the asymptotic formula for each estimator was also obtained.

We excluded those data sets for which the iterative steps for any ML estimates did not get solution or the overlap fraction was negative. (This occurred only when the capture fraction was 50%). The procedure continued until 5000 data sets had been generated. The estimates are obtained through ML method using NR iterative method for nuisance parameter. For the proposed estimator also observed the acceptance rate through M-H algorithm nearly 28% and 40%. For the 5000 generated datasets, the average estimates and their square root of average risk of parameter estimates were given in Table (5.4) and (5.5). All estimates were computed using frequencies f_j. The problem of cut off point selected did not arise because only few species were observed more than 10 times in most generated data sets. In the mentioned Table (5.4) and (5.5), we observed on various fixed frequencies j, when it was increases then coverage fraction also increases and we obtain in a trend that risk of the estimators decreases gradually. We also list the observed CI/HPD interval for the nominal 95%. Also, in this Table (5.4) and (5.5), we observed that coverage fraction increase as increases the frequencies than the observed number of species gets more closer to estimated number of species (as most of the time we got over estimate of parameter).

5.6 Conclusion

We observed that the ML estimation plays an important role in estimating the number of observed species or unobserved species. Intuitively, when there are low coverage fractions i.e. few overlaps of observed species and estimated species, we know that the true number of species is much higher than the observed. On the other hand, if the coverage fraction is high then we are likely to have seen most of the species. Based on this idea, we have proposed a consistent estimator for the number of species, under a P-mixed Poisson Lindley model. Here the model has low dimensional parameter space then computation became easy. Also, in the parameter space known as hyper parameter or nuisance parameter. For these hyper parameters we have non-informative prior or objective prior i.e. Jeffery's and Bernardo's reference prior, it can be based on one's belief. Both Jeffrey's and Bernardo's reference prior have simple forms in the case when there is only one nuisance parameter, and become increasingly complex as the dimension of the parameter space grows. For the comparison of these considered models based on model deviance criteria in Table (5.3), PLR has the lesser deviance then we can say that PLR gives a more optimum estimate of the number of species. In simulated Table (5.4) and (5.5), shows the posterior mean (estimate of number of species) as Bayes estimate under squared error loss function (see Pathak et al. (2020a)) and square root of average risk. When j increases then CF increases but the estimate of S and $R(S)$ decreases and the estimate of the number of species $\hat{S}$ is the case of o.e., i.e. $\hat{S} > w$. We observed that $R(S)$ under Bayes estimate of Bernardo's reference prior have a minimum than $R(S)$ under Bayes estimate of Jeffery's prior.

TABLE 5.4: ML estimates and Bayes estimates of number of species S and square root of average risk $R(S)$ for Poisson Lindley Model with fixed $\theta = 0.5$.

		$t = 0.4$					$t = 1.2$			
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
50	5	77.8	$\hat{S}_p$	76.2915	28.9324	(59.37,93.25)	78.36	76.5461	28.7271	(59.6793.41)
			$\hat{S}_c$	76.9169	29.5737	(59.77,94.05)		77.1656	29.3547	(60.08,94.24)
			$\hat{S}_J$	79.8932	33.4933	(54.54,107.03)		80.3515	33.0884	(54.25,108.73)
			$\hat{S}_R$	79.0514	32.3035	(54.75,107.03)		78.6280	30.6371	(54.88,108.73)
	6	85.44	$\hat{S}_p$	69.5552	20.5514	(56.57,82.53)	89.18	68.9346	19.8140	(56.91,80.95)
			$\hat{S}_c$	69.9760	20.9738	(56.85,83.11)		69.3029	20.1864	(57.15,81.4)
			$\hat{S}_J$	71.3986	22.4712	(53.54,89.7)		70.5116	21.4886	(54.33,86.79)
			$\hat{S}_R$	71.6397	22.6900	(53.95,89.7)		70.7497	21.6782	(54.77,86.79)
	7	90.92	$\hat{S}_p$	67.0534	17.9341	(56.05,78.12)	93	65.5289	16.3219	(55.39,75.66)
			$\hat{S}_c$	67.3796	18.2695	(56.21,78.55)		65.8139	16.6204	(55.58,76.04)
			$\hat{S}_J$	68.3769	19.3467	(53.86,82.91)		66.5949	17.4944	(53.75,79.59)
			$\hat{S}_R$	68.6156	19.5206	(54.27,82.91)		66.9356	17.7806	(54.18,79.59)
	8	94.26	$\hat{S}_p$	64.2468	14.8935	(54.81,73.68)	95.72	62.4558	12.8146	(53.92,70.98)
			$\hat{S}_c$	64.5011	15.1549	(54.97,74.03)		62.6723	13.0348	(54.06,71.27)
			$\hat{S}_J$	65.1451	15.8593	(53.35,76.73)		63.1699	13.5595	(52.97,73.31)
			$\hat{S}_R$	65.4924	16.1759	(53.81,76.73)		63.4992	13.8800	(53.24,73.31)

Continued on next page

Table 5.4 – *Continued from previous page*

				$t = 0.4$				$t = 1.2$		
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
	9	96.2	$\hat{S}_p$	62.0680	12.4986	(53.78,70.35)	96.84	60.7107	11.0496	(53.05,68.36)
			$\hat{S}_c$	62.2752	12.7117	(53.91,70.63)		60.8942	11.2383	(53.16,68.62)
			$\hat{S}_J$	62.7266	13.1984	(52.86,72.48)		61.2626	11.6359	(52.33,70.07)
			$\hat{S}_R$	63.0553	13.5120	(53.21,72.48)		61.6034	11.9750	(52.63,70.07)
	10	97.44	$\hat{S}_p$	60.7469	11.1974	(53.21,68.28)	98.88	58.4898	8.7739	(52.11,64.86)
			$\hat{S}_c$	60.9253	11.3820	(53.31,68.53)		58.6251	8.9134	(52.19,65.05)
			$\hat{S}_J$	61.2840	11.7706	(52.51,69.9)		58.8438	9.1489	(51.74,65.71)
			$\hat{S}_R$	61.5844	12.0653	(52.8,69.9)		59.1375	9.4437	(51.94,65.71)
60	5	77	$\hat{S}_p$	90.4434	31.8015	(72.06,108.82)	78.68	88.4186	30.0404	(71.06,105.76)
			$\hat{S}_c$	91.0633	32.4214	(72.48,109.64)		88.9895	30.6258	(71.46,106.51)
			$\hat{S}_J$	93.7993	35.3535	(65.86,123.84)		91.0347	32.6833	(65.78,117.57)
			$\hat{S}_R$	92.7634	33.9882	(66.21,123.84)		91.3674	33.2955	(65.9,117.57)
	6	83.78	$\hat{S}_p$	85.0163	26.5623	(69.91,100.11)	87.87	83.7267	24.8610	(69.95,97.49)
			$\hat{S}_c$	85.4789	27.0317	(70.23,100.72)		84.1218	25.2577	(70.23,98.01)
			$\hat{S}_J$	87.0985	28.7321	(66.04,109.02)		85.4179	26.5925	(66.64,104.24)
			$\hat{S}_R$	87.0159	28.4700	(66.13,109.02)		85.6225	26.7642	(67.32,104.24)

Continued on next page

Table 5.4 – *Continued from previous page*

		$t=0.4$					$t=1.2$			
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
	7	89.95	$\hat{S}_p$	80.5178	21.4039	(68.17,92.85)	92.5	79.8360	20.5628	(68.25,91.41)
			$\hat{S}_c$	80.8551	21.7486	(68.41,93.29)		80.1377	20.8704	(68.46,91.80)
			$\hat{S}_J$	81.8485	22.7658	(65.69,98.15)		80.9866	21.7665	(66.09,95.74)
			$\hat{S}_R$	82.1537	23.0656	(66.11,98.15)		81.3036	22.0575	(66.41,95.74)
	8	92.53	$\hat{S}_p$	77.5003	18.4388	(66.67,88.32)	95.67	76.0319	16.9688	(66.34,85.71)
			$\hat{S}_c$	77.7768	18.7250	(66.86,88.69)		76.2616	17.2124	(66.51,86.02)
			$\hat{S}_J$	78.5471	19.6118	(64.8,92.07)		76.8104	17.8450	(64.96,88.49)
			$\hat{S}_R$	78.8317	19.8316	(65.31,92.07)		77.1322	18.1022	(65.19,88.49)
	9	96.18	$\hat{S}_p$	74.4787	14.9179	(65.39,83.56)	96.82	73.7224	14.2342	(65.03,82.41)
			$\hat{S}_c$	74.6861	15.1294	(65.53,83.84)		73.9155	14.4340	(65.16,82.66)
			$\hat{S}_J$	75.1361	15.6065	(64.25,85.75)		74.3160	14.8792	(64.1,84.34)
			$\hat{S}_R$	75.4869	15.9535	(64.59,85.75)		74.6553	15.2032	(64.27,84.34)
	10	97.43	$\hat{S}_p$	72.7065	13.4414	(64.52,80.88)	98.3	70.8159	11.0848	(63.46,78.16)
			$\hat{S}_c$	72.8828	13.6272	(64.63,81.12)		70.9639	11.2367	(63.55,78.36)
			$\hat{S}_J$	73.2362	14.0305	(63.69,82.56)		71.2179	11.5070	(62.96,79.35)
			$\hat{S}_R$	73.5298	14.3036	(63.98,82.56)		71.5269	11.8238	(63.07,79.35)
70	5	73.69	$\hat{S}_p$	103.6304	36.6810	(83.29,123.96)	77.56	105.1706	37.0413	(85.57,124.76)

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Table 5.4 – *Continued from previous page*

		t = 0.4					t = 1.2			
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
			$\hat{S}_c$	104.2959	37.3353	(83.75,124.83)		105.7769	37.6502	(86.07,125.54)
			$\hat{S}_J$	107.8331	41.5143	(76.88,139.73)		108.1667	40.1847	(79.13,138.5)
			$\hat{S}_R$	106.9667	40.4973	(76.16,139.73)		108.2674	41.0500	(79.48,138.5)
	6	83.66	$\hat{S}_p$	98.9745	30.7352	(82.71,115.23)	87.49	100.2521	31.8337	(84.57,115.93)
			$\hat{S}_c$	99.4332	31.1990	(83.04,115.82)		100.6760	32.2682	(84.87,116.47)
			$\hat{S}_J$	101.1014	32.9567	(78.17,124.5)		102.1903	33.9195	(80.3,123.89)
			$\hat{S}_R$	100.9205	32.5382	(78.51,124.5)		102.2757	33.8317	(80.95,123.89)
	7	90.13	$\hat{S}_p$	95.0908	26.7498	(81.48,108.69)	90.87	94.6346	25.8961	(81.31,107.95)
			$\hat{S}_c$	95.4368	27.1127	(81.73,109.13)		94.9700	26.2487	(81.55,108.38)
			$\hat{S}_J$	96.5155	28.3390	(78.27,114.38)		96.0162	27.4287	(78.34,113.44)
			$\hat{S}_R$	96.6356	28.2402	(78.83,114.38)		96.1968	27.5364	(78.83,113.44)
	8	92.59	$\hat{S}_p$	90.9369	21.9594	(79.09,102.77)	94.51	89.0097	19.5819	(78.17,99.84)
			$\hat{S}_c$	91.2182	22.2539	(79.29,103.14)		89.2532	19.8317	(78.34,100.16)
			$\hat{S}_J$	91.9776	23.1059	(76.91,106.84)		89.8473	20.4659	(76.5,102.91)
			$\hat{S}_R$	92.3141	23.4021	(77.18,106.84)		90.1793	20.7751	(76.78,102.91)
	9	95.14	$\hat{S}_p$	87.8295	18.4818	(77.49,98.16)	96.91	85.9045	16.5534	(76.58,95.22)
			$\hat{S}_c$	88.0565	18.7156	(77.65,98.46)		86.0963	16.7532	(76.71,95.47)

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Table 5.4 – *Continued from previous page*

				$t = 0.4$				$t = 1.2$		
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
			$\hat{S}_J$	88.5855	19.2908	(75.99,100.91)		86.4736	17.1578	(75.45,97.23)
			$\hat{S}_R$	88.9589	19.6409	(76.32,100.91)		86.8590	17.5628	(75.75,97.23)
	10	98.07	$\hat{S}_p$	84.4702	14.9258	(75.86,93.07)	98.29	83.4878	14.0530	(75.27,91.71)
			$\hat{S}_c$	84.6377	15.0978	(75.97,93.29)		83.6439	14.2149	(75.37,91.91)
			$\hat{S}_J$	84.9569	15.4426	(75.03,94.6)		83.9271	14.5218	(74.6,93.08)
			$\hat{S}_R$	85.2850	15.7594	(75.32,94.6)		84.2193	14.8126	(74.68,93.08)

TABLE 5.5: ML estimates and Bayes estimates of number of species S and square root of average risk $R(S)$ for Poisson Lindley Model with fixed $\theta = 1.2$.

				$t = 0.4$				$t = 1.2$		
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
50	5	69.1	$\hat{S}_p$	74.1148	26.3888	(55.88,92.34)	78.58	73.2085	25.2695	(57.57,88.84)
			$\hat{S}_c$	74.8486	27.0927	(56.35,93.34)		73.7696	25.8190	(57.93,89.59)
			$\hat{S}_J$	77.9722	30.1572	(49.74,109.39)		76.1269	28.2345	(53.1,100.67)
			$\hat{S}_R$	76.5438	28.5623	(49.76,109.39)		75.7119	27.6897	(53.35,100.67)
	6	76.78	$\hat{S}_p$	71.8245	24.3243	(56.14,87.51)	86.12	68.9278	19.9289	(56.28,81.57)
			$\hat{S}_c$	72.4037	24.9170	(56.52,88.28)		69.3331	20.3418	(56.55,82.11)
			$\hat{S}_J$	74.8051	28.1481	(51.8,98.95)		70.6759	21.7849	(53.3,88.42)
			$\hat{S}_R$	74.5278	27.2211	(52.25,98.95)		70.8299	21.8530	(53.74,88.42)
	7	81.76	$\hat{S}_p$	69.1489	20.3952	(55.52,82.76)	91.26	65.7806	16.4045	(55.23,76.33)
			$\hat{S}_c$	69.6124	20.8574	(55.83,83.39)		66.0849	16.7135	(55.42,76.73)
			$\hat{S}_J$	71.6071	22.9607	(52.17,91.85)		66.9679	17.6509	(53.41,80.54)
			$\hat{S}_R$	71.2835	22.5005	(52.48,91.85)		67.2743	17.9216	(53.69,80.54)
	8	87.7	$\hat{S}_p$	67.9515	19.2072	(55.96,79.94)	94.78	64.0293	14.6732	(54.76,73.29)
			$\hat{S}_c$	68.3262	19.5954	(56.21,80.44)		64.2763	14.9230	(54.92,73.63)
			$\hat{S}_J$	69.6012	20.9793	(53.34,86.14)		64.8531	15.5358	(53.42,76.15)
			$\hat{S}_R$	69.7103	20.9837	(53.8,86.14)		65.2441	15.9165	(53.8,76.15)

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Table 5.5 – *Continued from previous page*

				$t = 0.4$				$t = 1.2$		
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
	9	90.24	$\hat{S}_p$	65.1419	15.9535	(54.61,75.67)	97.48	61.6460	12.1177	(53.77,69.52)
			$\hat{S}_c$	65.4504	16.2714	(54.81,76.09)		61.8358	12.3131	(53.89,69.78)
			$\hat{S}_J$	66.4124	17.3559	(52.8,80.18)		62.2131	12.7169	(52.96,71.3)
			$\hat{S}_R$	66.6825	17.5443	(53.15,80.18)		62.5770	13.0802	(53.28,71.3)
	10	93.52	$\hat{S}_p$	63.6564	14.3420	(54.29,73.02)	98.6	59.8611	10.2902	(52.89,66.83)
			$\hat{S}_c$	63.9114	14.6084	(54.45,73.36)		60.0173	10.4519	(52.98,67.04)
			$\hat{S}_J$	64.5567	15.3228	(52.86,76.07)		60.3057	10.7678	(52.3,68.02)
			$\hat{S}_R$	64.9064	15.6527	(53.07,76.07)		60.6127	11.0681	(52.62,68.02)
60	5	68	$\hat{S}_p$	91.0171	34.2648	(69.94,112.08)	76.5	89.3137	31.3091	(71.15,107.50)
			$\hat{S}_c$	91.8144	35.0148	(70.47,113.15)		89.9317	31.9256	(71.54,108.31)
			$\hat{S}_J$	98.6509	45.6611	(63.47,135.94)		93.0898	35.5656	(65.33,122.91)
			$\hat{S}_R$	94.5460	38.3863	(62.14,135.94)		91.9702	34.1541	(65.48,122.91)
	6	76.55	$\hat{S}_p$	85.6931	27.2472	(68.60,102.78)	84.88	82.9419	24.0653	(68.74,97.14)
			$\hat{S}_c$	86.2705	27.8175	(68.99,103.54)		83.3651	24.4880	(69.03,97.69)
			$\hat{S}_J$	89.3155	31.7867	(63.46,116.12)		84.9407	26.1157	(65.08,104.91)
			$\hat{S}_R$	88.1867	29.9551	(63.61,116.12)		85.0612	26.2612	(65.74,104.91)

Continued on next page

Table 5.5 – *Continued from previous page*

				$t = 0.4$				$t = 1.2$		
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
	7	83.37	$\hat{S}_p$	82.0504	23.2424	(67.73,96.36)	90.78	80.6208	21.4043	(68.43,92.80)
			$\hat{S}_c$	82.4883	23.6751	(68.03,96.94)		80.9496	21.7418	(68.66,93.23)
			$\hat{S}_J$	84.2316	25.4629	(64.08,105.27)		81.9179	22.7546	(65.95,97.8)
			$\hat{S}_R$	84.0615	25.2632	(64.47,105.27)		82.2095	23.0186	(66.3,97.8)
	8	87.92	$\hat{S}_p$	80.7663	21.6105	(67.83,93.63)	93.67	77.5658	18.8377	(66.94,88.19)
			$\hat{S}_c$	81.1308	21.9815	(68.14,94.12)		77.8341	19.1331	(67.15,88.55)
			$\hat{S}_J$	82.3833	23.3154	(65.08,99.84)		78.7200	20.6560	(65.09,92.62)
			$\hat{S}_R$	82.5667	23.4472	(65.38,99.84)		78.8093	20.0348	(65.58,92.62)
	9	90.65	$\hat{S}_p$	78.1751	19.1565	(66.72,89.62)	96.53	75.4442	16.1671	(66.13,84.75)
			$\hat{S}_c$	78.4786	19.4666	(66.93,90.02)		75.6591	16.3918	(66.28,85.03)
			$\hat{S}_J$	79.3394	20.3871	(64.73,94.09)		76.1513	16.9503	(64.88,87.15)
			$\hat{S}_R$	79.6369	20.6902	(64.89,94.09)		76.4896	17.2611	(65.2,87.15)
	10	92.53	$\hat{S}_p$	77.6222	18.3225	(66.75,88.49)	97.37	72.1140	12.5488	(64.15,80.12)
			$\hat{S}_c$	77.9000	18.6103	(66.94,88.86)		72.2842	12.7234	(64.21,80.35)
			$\hat{S}_J$	78.6466	19.4320	(64.8,92.23)		72.6064	13.0685	(63.32,81.63)
			$\hat{S}_R$	78.9632	19.7020	(65.21,92.23)		72.9243	13.3803	(63.6,81.63)
70	5	69.89	$\hat{S}_p$	106.6320	39.1589	(84.42,128.85)	75.8	103.8727	35.6500	(84.18,123.58)

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Table 5.5 – *Continued from previous page*

				$t = 0.4$				$t = 1.2$		
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
			$\hat{S}_c$	107.3826	39.9099	(84.95,129.83)		104.4938	36.2573	(84.61,124.38)
			$\hat{S}_J$	110.5150	43.7610	(76.11,146.65)		106.7120	38.5803	(77.22,136.99)
			$\hat{S}_R$	110.2608	43.8537	(76.31,146.65)		106.2939	38.0505	(77.52,136.99)
	6	78.3	$\hat{S}_p$	103.5278	35.6120	(84.56,122.52)	83.2	98.9265	30.4256	(82.54,115.34)
			$\hat{S}_c$	104.1108	36.2054	(84.96,123.25)		99.3953	30.9075	(82.85,115.94)
			$\hat{S}_J$	106.2811	38.3011	(78.23,135.21)		101.2658	32.9727	(77.97,125.06)
			$\hat{S}_R$	106.2227	38.6766	(79.07,135.21)		101.0559	32.5457	(78.44,125.06)
	7	82.63	$\hat{S}_p$	99.8923	31.1279	(83.09,116.69)	89.07	94.8172	25.6796	(81.08,108.69)
			$\hat{S}_c$	100.3756	31.6211	(83.46,117.39)		95.1733	26.0420	(81.21,109.04)
			$\hat{S}_J$	102.3696	33.7969	(78.08,127.07)		96.2523	27.1824	(77.77,114.47)
			$\hat{S}_R$	102.1939	33.4930	(78.46,127.07)		96.5055	27.3575	(78.29,114.47)
	8	87.59	$\hat{S}_p$	94.9343	26.0750	(80.78,109.09)	93.73	89.7655	20.3863	(78.56,101.07)
			$\hat{S}_c$	95.3066	26.4556	(81.042,109.57)		90.0235	20.6525	(78.76,101.39)
			$\hat{S}_J$	96.4929	27.7044	(77.43,115.52)		90.6547	21.3275	(76.52,104.35)
			$\hat{S}_R$	96.7343	27.9168	(77.95,115.52)		91.0031	21.6521	(76.85,104.35)
	9	89.54	$\hat{S}_p$	92.4444	23.4610	(79.45,105.47)	96.39	86.5609	17.2582	(76.92,96.23)
			$\hat{S}_c$	92.7732	23.7999	(79.68,105.87)		86.7631	17.4668	(77.06,96.46)

Continued on next page

Table 5.5 – *Continued from previous page*

				$t = 0.4$				$t = 1.2$		
S	j	CF		$\hat{S}$	$R(S)$	CI/HPD	CF	$\hat{S}$	$R(S)$	CI/HPD
			$\hat{S}_J$	93.8092	24.9482	(76.7,110.74)		87.1982	17.9444	(75.63,98.4)
			$\hat{S}_R$	94.0383	25.1089	(76.98,110.74)		87.5530	18.2800	(76,98.4)
	10	91.89	$\hat{S}_p$	90.2471	21.6947	(78.48,102.03)	97.47	83.9483	14.5015	(75.38,92.53)
			$\hat{S}_c$	90.5305	21.9985	(78.64,102.48)		84.1160	14.6755	(75.49,92.76)
			$\hat{S}_J$	91.4327	23.2054	(76.2,106.49)		84.4377	15.0264	(74.53,94.07)
			$\hat{S}_R$	91.5546	23.0372	(76.46,106.49)		84.7767	15.3642	(74.82,94.07)

Bibliography

Aarset, M. V. (1987). How to identify a bathtub hazard rate. *IEEE Transactions on Reliability*, 36(1):106–108.

Aggarwala, R. and Balakrishnan, N. (1998). Some properties of progressive censored order Statistics from arbitrary and Uniform distributions with applications. *Journal of Statistical Planning and Inference*, 70(1):35–49.

Ahmad, K., Fakhry, M., and Jaheen, Z. (1997). Empirical Bayes estimation of P(Y< X) and characterizations of Burr-type X model. *Journal of Statistical Planning and Inference*, 64(2):297–308.

Ahmadi, J., Doostparast, M., and Parsian, A. (2005). Estimation and prediction in a two-parameter Exponential distribution based on k-record values under linex loss function. *Communications in Statistics–Theory and Methods*, 34(4):795–805.

Akaike, H. (1978). A Bayesian analysis of the minimum AIC procedure. *Annals of the Institute of Statistical Mathematics*, 30(1):9–14.

Amin, E. A. (2017). Estimation of stress-strength reliability for Kumaraswamy Exponential distribution based on upper record values. *International Journal of Contemporary and Mathematical Sciences*, 2(2):59–71.

Antoniak, C. E. (1974). Mixtures of Dirichlet processes with applications to Bayesian nonparametric problems. *The annals of statistics*, :1152-1174.

Ashour, S. and Afify, W. (2008). Estimation of the parameters of exponentiated Weibull family with type-ii progressive interval censoring with random removals. *Applied Sciences Research*, 4(11):1428–1442.

Balakrishnan, N. (2007). Progressive censoring methodology: an appraisal. *TEST*, 16(2):211–259.

Balakrishnan, N. and Aggarwala, R. (2000). *Progressive Censoring: Theory, Methods, and Applications*. Springer Science & Business Media.

Balakrishnan, N. and Sandhu, R. A. (1995). A simple simulational algorithm for generating progressive type-II censored samples. *The American Statistician*, 49(2):229–230.

Balasooriya, U., Saw, S. L., and Gadag, V. (2000). Progressively censored reliability sampling plans for the Weibull distribution. *Technometrics*, 42(2):160–167.

Barger, K. and Bunge, J. (2008). Bayesian estimation of the number of species using noninformative priors. *Biometrical Journal: Journal of Mathematical Methods in Biosciences*, 50(6):1064–1076.

Barger, K., Bunge, J., et al. (2010). Objective Bayesian estimation for the number of species. *Bayesian Analysis*, 5(4):765–785.

Barnett, V. (1966). Evaluation of the maximum-likelihood estimator where the likelihood equation has multiple roots. *Biometrika*, 53(1/2):151–165.

Barreto-Souza, W. and Cribari-Neto, F. (2009). A generalization of the Exponential-Poisson distribution. *Statistics and Probability Letters*, 79(24):2493–2500.

Basu, A. and Ebrahimi, N. (1991). Bayesian approach to life testing and reliability estimation using asymmetric loss function. *Journal of statistical planning and inference*, 29(1-2):21–31.

Behnke, A., Bunge, J., Barger, K., Breiner, H.-W., Alla, V., and Stoeck, T. (2006). Microeukaryote community patterns along an o2/h2s gradient in a supersulfidic anoxic fjord (framvaren, norway). *Applied and environmental microbiology*, 72(5):3626–3636.

Berger, J. O. (2013). *Statistical Decision Theory and Bayesian Analysis*. Springer Science & Business Media.

Berger, J. O. and Bernardo, J. M. (1989). Estimating a product of means: Bayesian analysis with reference priors. *Journal of the American Statistical Association*, 84(405):200–207.

Berger, J. O. and Bernardo, J. M. (1992a). Ordered group reference priors with application to the multinomial problem. *Biometrika*, 79(1):25–37.

Berger, J. O. and Bernardo, J. M. (1992b). Reference priors in a variance components problem. In *Bayesian analysis in statistics and econometrics,:177-194*.

Bernardo, J. M. (1979). Reference posterior distributions for Bayesian inference. *Journal of the Royal Statistical Society: Series B (Methodological)*, 41(2):113–128.

Bernardo, J. M. and Ramon, J. M. (1998). An introduction to Bayesian reference analysis: inference on the ratio of multinomial parameters. *Journal of the Royal Statistical Society: Series D (The Statistician)*, 47(1):101–135.

Bernardo, J. M. and Smith, A. F. (2009). *Bayesian Theory*, volume 405. John Wiley & Sons.

Berry, D. A., Christensen, R., et al. (1979). Empirical Bayes estimation of a Binomial parameter via mixtures of Dirichlet processes. *The Annals of Statistics*, 7(3):558–568.

Borgella Jr, R. and Gavin, T. A. (2005). Avian community dynamics in a fragmented tropical landscape. *Ecological Applications*, 15(3):1062–1073.

Box, G. E. and Tiao, G. C. (2011). *Bayesian Inference in Statistical Analysis*, volume 40. John Wiley & Sons.

Brooks, S. (1998). Markov Chain Monte Carlo method and its application. *Journal of the royal statistical society: series D (the Statistician)*, 47(1):69–100.

Buckland, S., Goudie, I., and Borchers, D. (2000). Wildlife population assessment: past developments and future directions. *Biometrics*, 56(1):1–12.

Bulmer, M. (1974). On fitting the Poisson Lognormal distribution to species-abundance data. *Biometrics*, :101-110.

Bunge, J. and Fitzpatrick, M. (1993). Estimating the number of species: a review. *Journal of the American Statistical Association*, 88(421):364–373.

Bunge, J., Fitzpatrick, M., and Handley, J. (1995). Comparison of three estimators of the number of species. *Journal of Applied Statistics*, 22(1):45–59.

Calabria, R. and Pulcini, G. (1994). An engineering approach to Bayes estimation for the Weibull distribution. *Microelectronics Reliability*, 34(5):789–802.

Casella, G. and George, E. I. (1992). Explaining the Gibbs sampler. *The American Statistician*, 46(3):167–174.

Chao, A. and Bunge, J. (2002). Estimating the number of species in a stochastic abundance model. *Biometrics*, 58(3):531–539.

Chao, A. and Lee, S.-M. (1992). Estimating the number of classes via sample coverage. *Journal of the American statistical Association*, 87(417):210–217.

Chen, M. and Shao, Q. (1998). Monte Carlo estimation of Bayesian credible and HPD intervals. *Journal of Computation and Graphical Statistics*, 6:66–92.

Childs, A. and Balakrishnan, N. (2000). Conditional inference procedures for the Laplace distribution when the observed samples are progressively censored. *Metrika*, 52(3):253–265.

Cobabe, E. A. and Allmon, W. D. (1994). Effects of sampling on paleoecologic and taphonomic analyses in high-diversity fossil accumulations: an example from the eocene gosport sand, alabama. *Lethaia*, 27(2):167–178.

Cohen, A. C. (1963). Progressively Censored Samples in Life Testing. *Technometrics*, 5(3):327–339.

Cohen, A. C. (1965). Maximum likelihood estimation in the Weibull distribution based on complete and on censored samples. *Technometrics*, 7(4):579–588.

Cohen, A. C. and Norgaard, N. J. (1977). Progressively censored sampling in the three-parameter Gamma distribution. *Technometrics*, 19(3):333–340.

Collett, D. (2014). *Modelling Survival Data in Medical Research*, volume 3. Chapman and Hall/CRC.

Colwell, R. K. and Coddington, J. A. (1994). Estimating terrestrial Biodiversity through extrapolation. *Philosophical Transactions of the Royal Society of London. Series B: Biological Sciences*, 345(1311):101–118.

Cormack, R. (1992). Interval Estimation for Mark-Recapture Studies of Closed Populations. *Biometrics*, :567-576.

Dash, M. C. (2001). *Fundamentals of Ecology*. Tata McGraw-Hill Education.

Davis, H. T. and Feldstein, M. L. (1979). The generalized Pareto law as a model for progressively censored survival data. *Biometrika*, 66(2):299–306.

Dempster, A. P., Laird, N. M., and Rubin, D. B. (1977). Maximum likelihood from incomplete data via the EM algorithm. *Journal of the Royal Statistical Society. Series B (Methodological)*, 39(1):1–38.

Dey, D. K., Ghosh, M., and Srinivasan, C. (1986). Simultaneous estimation of parameters under entropy loss. *Journal of Statistical Planning and Inference*, 15:347–363.

Doostparast, M. (2009). A note on estimation based on record data. *Metrika*, 69(1):69–80.

El-Sagheer, R. M. (2017). E-Bayesian estimation for Rayleigh model using progressive type-II censoring data. *Journal of Statistical Theory and Applications*, 16(2):239–247.

Fisher, R. A., Corbet, A. S., and Williams, C. B. (1943). The Relation Between the Number of Species and the Number of Individuals in a Random Sample of an Animal Population. *The Journal of Animal Ecology*, :42-58.

Gauss, C. F. (1855). *Méthode des moindres carrés: Mémoires sur la combinaison des observations*. Mallet-Bachelier.

Gelfand, A. E. and Smith, A. F. (1990). Sampling-based approaches to calculating marginal densities. *Journal of the American statistical association*, 85(410):398–409.

Gelman, A., Carlin, J. B., Stern, H. S., Dunson, D. B., Vehtari, A., and Rubin, D. B. (2013). *Bayesian Data Analysis*. Chapman and Hall/CRC.

Geman, S. and Geman, D. (1984). Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images. *IEEE Transactions on pattern analysis and machine intelligence*, (6):721–741.

Geman, S. and Geman, D. (1993). Stochastic relaxation, Gibbs distributions and the Bayesian restoration of images. *Journal of Applied Statistics*, 20(5-6):25–62.

Ghitany, M. E., Atieh, B., and Nadarajah, S. (2008). Lindley distribution and its application. *Mathematics and computers in simulation*, 78(4):493–506.

Gholizadeh, R., Khlilpor, M., and Hsadian, M. (2011). Bayesian estimation in the Kumaraswamy distribution under progressively type-II censoring data. *International Journal of Engineering, Science and Technology*, 3(9).

Gilks, W., Richardson, S., and Spiegelhalter, D. (1996). Markov Chain Monte Carlo in practice. *Markov Chain Monte Carlo in practice. Chapman and Hall, London.*

Goel, P. K. (1983). Information measures and Bayesian hierarchical models. *Journal of the American Statistical Association*, 78(382):408–410.

Goel, P. K. and Degroot, M. H. (1981). Information about hyperparameters in hierarchical models. *Journal of the American Statistical Association*, 76(373):140–147.

Golub, G. H. and Welsch, J. H. (1969). Calculation of Gauss quadrature rules. *Mathematics of computation*, 23(106):221–230.

Gonzales-Barron, U. and Butler, F. (2011). A comparison between the discrete Poisson-Gamma and Poisson-Lognormal distributions to characterise microbial counts in foods. *Food Control*, 22(8):1279–1286.

Good, I. J. (1950). Probability and the weighing of evidence. Technical report, C. Griffin London.

Good, I. J. (1953). The population frequencies of species and the estimation of population parameters. *Biometrika*, 40(3-4):237–264.

Gotelli, N. J. and Colwell, R. K. (2011). Estimating species richness. *Biological diversity: frontiers in measurement and assessment*, 12:39–54.

Grabski, F. and Sarhan, A. (1996). Empirical Bayes estimation in the case of Exponential reliability. *Reliability Engineering & System Safety*, 53(2):105–113.

Greenwood, M. and Yule, G. U. (1920). An inquiry into the nature of frequency distributions representative of multiple happenings with particular reference to the occurrence of multiple attacks of disease or of repeated accidents. *Journal of the Royal statistical society*, 83(2):255–279.

Gupta, I. (2017). Bayesian and E-Bayesian method of estimation of parameter of Rayleigh distribution-A Bayesian approach under linex loss function. *International Journal of Statistics and Systems*, 12(4):791–796.

Han, M. (2007). E-Bayesian estimation of failure probability and its application. *Mathematical and Computer Modelling*, 45(9-10):1272–1279.

Han, M. (2011a). E-Bayesian estimation and hierarchical Eayesian estimation of failure probability. *Communications in Statistics-Theory and Methods*, 40(18):3303–3314.

Han, M. (2011b). Estimation of failure probability and its applications in lifetime data analysis. *International Journal of Quality, Statistics, and Reliability*, 2011.

Han, M. (2017a). The E-Bayesian and hierarchical Bayesian estimations for the system reliability parameter. *Communications in Statistics-Theory and Methods*, 46(4):1606–1620.

Han, M. (2017b). The E-Bayesian and hierarchical Bayesian estimations of Pareto distribution parameter under different loss functions. *Journal of Statistical Computation and Simulation*, 87(3):577–593.

Han, M. (2018). E-Bayesian estimations of the reliability and its E-posterior risk under different loss functions. *Communications in Statistics-Simulation and Computation*, 1-19.

Han, M. (2019a). E-Bayesian estimation and its E-MSE under the scaled squared error loss function, for exponential distribution as example. *Communications in Statistics-Simulation and Computation*, 48(6):1880–1890.

Han, M. (2019b). E-Bayesian estimation of the Exponentiated distribution family parameter under linex loss function. *Communications in Statistics-Theory and Methods*, 48(3):648–659.

Han, M. (2019c). E-Bayesian estimations of parameter and its evaluation standard: E-MSE (expected mean square error) under different loss functions. *Communications in Statistics-Simulation and Computation*, 1-18.

Han, M. (2020). E-Bayesian estimation and its E-posterior risk of the exponential distribution parameter based on complete and type I censored samples. *Communications in Statistics-Theory and Methods*, 49(8):1858–1872.

Hartigan, J. et al. (1964). Invariant prior distributions. *Annals of Mathematical Statistics*, 35(2):836–845.

Hastings, W. K. (1970). Monte Carlo sampling methods using Markov Chains and their applications. *Oxford University Press*.

Heyde, C. and Johnstone, I. (1979). On asymptotic posterior normality for stochastic processes. *Journal of the Royal Statistical Society: Series B (Methodological)*, 41(2):184–189.

Hildebrand, F. (1974). Chapter 7, Introduction to Numerical Analysis. *McGraw-Hill New York*.

Hong, S.-H., Bunge, J., Jeon, S.-O., and Epstein, S. S. (2006). Predicting microbial species richness. *Proceedings of the National Academy of Sciences*, 103(1):117–122.

Huang, S. R. and Wu, S. J. (2012). Bayesian estimation and prediction for Weibull model with progressive censoring. *Journal of Statistical Computation and Simulation*, 82(11):1607–1620.

Ihaka, R. and Gentleman, R. (1996). R A language for data analysis and graphics. *Journal of Computational and Graphical Statistics*, 5:299–314.

Jackman, S. (2000). Estimation and inference via Bayesian simulation: An introduction to Markov Chain Monte Carlo. *American journal of political science*, :375-404.

Jaheen, Z. F. (2004). Empirical Bayes inference for generalized Exponential distribution based on records. *Communications in Statistics-Theory and Methods*, 33(8):1851–1861.

Jaheen, Z. F. and Okasha, H. M. (2011). E-Bayesian estimation for the Burr type-XII model based on type-II censoring. *Applied Mathematical Modelling*, 35(10):4730–4737.

Jain, M., Iyengar, S., and Jain, R. (2003). Numerical Methods for Scientific and Engineering Computation. *New Age International Publishers, New Delhi*.

Jaynes, E. T. (1968). Prior probabilities. *IEEE Transactions on systems science and cybernetics*, 4(3):227–241.

Jaynes, E. T. (1983). Papers on probability, statistics and statistical physics. *Reidel*.

Jeffreys, H. (1946). An Invariant Form for the Prior Probability in Estimation Problems. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 186(1007):453–461.

Jeffreys, H. (1961). *Theory of Probability, (3rd Edition)*. OUP Oxford.

Jones, M. C. (2009). Kumaraswamy's distribution: A Beta-type distribution with some tractability advantages. *Statistical Methodology*, 6(1):70–81.

Kale, B. (1961). On the solution of the likelihood equation by iteration processes. *Biometrika*, 48(3/4):452–456.

Kale, B. (1962). On the solution of likelihood equations by iteration processes. the multiparametric case. *Biometrika*, 49(3/4):479–486.

Kamps, U. and Cramer, E. (2001). On distributions of generalized order statistics. *Statistics*, 35(3):269–280.

Kim, C. and Han, K. (2009). Estimation of the scale parameter of the Rayleigh distribution under general progressive censoring. *Journal of the Korean Statistical Society*, 38(3):239–246.

Kim, C., Jung, J., and Chung, Y. (2011). Bayesian estimation for the Exponentiated Weibull model under type-II progressive censoring. *Statistical Papers*, 52(1):53–70.

Kızılaslan, F. (2017). The E-Bayesian and hierarchical Bayesian estimations for the proportional reversed hazard rate model based on record values. *Journal of Statistical Computation and Simulation*, 87(11):2253–2273.

Klein, J. P. and Moeschberger, M. L. (2006). *Survival Analysis: Techniques for Censored and Truncated Data*. Springer Science & Business Media.

Kohansal, A. (2017). On estimation of Reliability in a multicomponent stress-strength model for Kumaraswamy Exponential distribution based on progressively censored sample. *Statistical papers*, :1-40.

Krishna, H. and Kumar, K. (2013). Reliability estimation in generalized inverted exponential distribution with progressively type-II censored sample. *Journal of Statistical Computation and Simulation*, 83(6):1007–1019.

Krishna, H. and Malik, M. (2012). Reliability estimation in Maxwell distribution with progressively type-II censored data. *Journal of Statistical Computation and Simulation*, 82(4):623–641.

Kumar, D., Singh, U., Singh, S. K., and Bhattacharyya, G. (2015). Bayesian estimation of Exponentiated Gamma parameter for progressive type-II censored data with Binomial removals. *Journal of Statistics Applications and Probability*, 4(2):265.

Kumar, M., Pathak, A., and Soni, S. (2019a). Bayesian inference for Rayleigh distribution under step-stress partially accelerated test with progressive type-II censoring with Binomial removal. *Annals of Data Science*, 6(1):117–152.

Kumar, M., Singh, S. K., and Singh, U. (2018). Bayesian inference for Poisson-inverse Exponential distribution under progressive type-II censoring with Binomial removal. *International Journal of System Assurance Engineering and Management*, 9(6):1235–1249.

Kumar, M., Singh, S. K., Singh, U., and Pathak, A. (2019b). Empirical Bayes estimator of parameter, reliability and hazard rate for kumaraswamy distribution. *Life Cycle Reliability and Safety Engineering*, 8(3):243–256.

Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes. *Journal of Hydrology*, 46(1-2):79–88.

Kundu, D. (2008). Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring. *Technometrics*, 50(2):144–154.

Kundu, D. and Howlader, H. (2010). Bayesian inference and prediction of the inverse Weibull distribution for Type-II censored data. *Computational Statistics & Data Analysis*, 54(6):1547–1558.

Kuo, L. (1986). Computations of mixtures of Dirichlet processes. *SIAM Journal on Scientific and Statistical Computing*, 7(1):60–71.

Laplace, P. d. (1812). Theorie analytique des Probabilites. *Paris: Courcier*.

Lawless, J. F. (2011). *Statistical Models and Methods for Lifetime Data*, volume 362. John Wiley & Sons.

Leamer, E. E. and Leamer, E. E. (1978). *Specification searches: Ad hoc inference with nonexperimental data*, volume 53. Wiley New York.

Lee, E. T. and Wang, J. (2003). *Statistical Methods for Survival Data Analysis*, volume 476. John Wiley & Sons.

Legendre, A. M. (1806). Nouvelles Methodes pour la determination des orbites des cometes. *Paris: Courcier*.

Lehmann, E. L. (1983). *Theory of Point Estimation*. Wiley, New York.

Leite, J. G., Rodrigues, J., and Milan, L. A. (2000). A Bayesian Analysis for Estimating the Number of Species in a Population using Nonhomogeneous Poisson Process. *Statistics & Probability Letters*, 48(2):153–161.

Lemonte, A. J. (2011). Improved point estimation for the Kumaraswamy distribution. *Journal of Statistical Computation and Simulation*, 81(12):1971–1982.

Lewins, W. A. and Joanes, D. (1984). Bayesian estimation of the number of species. *Biometrics*, :323-328.

Lindley, D. V. (1958). Fiducial distributions and Bayes' theorem. *Journal of the Royal Statistical Society. Series B (Methodological)*, :102-107.

Lindley, D. V. (1980). Approximate Bayesian methods. *Trabajos de estadística y de investigación operativa*, 31(1):223–245.

Lindley, D. V. and Smith, A. F. (1972). Bayes estimates for the linear model. *Journal of the Royal Statistical Society: Series B (Methodological)*, 34(1):1–18.

Lindsay, B. G. and Roeder, K. (1987). A Unified Treatment of Integer Parameter Models. *Journal of the American Statistical Association*, 82(399):758–764.

Little, R. J. and Rubin, D. B. (2019). *Statistical analysis with missing data*, volume 793. John Wiley & Sons.

Louzada-Neto, F., Cancho, V. G., and Barriga, G. D. C. (2011). The Poisson-Exponential distribution: a Bayesian approach. *Journal of Applied Statistics*, 38(6):1239–1248.

Lu, W. and Shi, D. (2012). A new compounding life distribution: the Weibull-Poisson distribution. *Journal of applied statistics*, 39(1):21–38.

Maritz, J. (1967). Smooth empirical Bayes estimation for continuous distributions. *Biometrika*, 54(3-4):435–450.

McLachlan, G. J. and Krishnan, T. (2007). *The EM algorithm and extensions*, volume 382. John Wiley & Sons.

Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H., and Teller, E. (1953). Equation of state calculations by fast computing machines. *The journal of chemical physics*, 21(6):1087–1092.

Metropolis, N. and Ulam, S. (1949). The Monte Carlo method. *Journal of the American statistical association*, 44(247):335–341.

Mitnik, P. A. (2013). New properties of the Kumaraswamy distribution. *Communications in Statistics - Theory and Methods*, 42(5).

Morris, C. N. (1983). Parametric Empirical Bayes inference: Theory and applications. *Journal of the American Statistical Association*, 78(381):47–55.

Nassar, M. M. and Eissa, F. H. (2005). Bayesian estimation for the Exponentiated Weibull model. *Communications in Statistics-Theory and Methods*, 33(10):2343–2362.

Nelson, W. B. (2003). *Applied Life Data Analysis*, volume 521. John Wiley & Sons.

Ng, H., Chan, P. S., and Balakrishnan, N. (2004). Optimal progressive censoring plans for the Weibull distribution. *Technometrics*, 46(4):470–481.

Norris, J. L. and Pollock, K. H. (1998). Non-Parametric MLE for Poisson Species Abundance Models Allowing for Heterogeneity Between Species. *Environmental and Ecological Statistics*, 5(4):391–402.

Okasha, H. M. (2014). E-Bayesian estimation for the Lomax distribution based on type-II censored data. *Journal of the Egyptian Mathematical Society*, 22(3):489–495.

Ord, J. K. and Whitmore, G. A. (1986). The Poisson-Inverse Gaussian disiribuiion as a model for species abundance. *Communications in Statistics-theory and Methods*, 15(3):853–871.

Pandey, B. and Rai, O. (1992). Bayesian estimation of mean and square of mean of normal distribution using linex loss function. *Communications in Statistics-Theory and Methods*, 21(12):3369–3391.

Parsian, A. and Sanjari Farsipour, N. (1993). On the admissibility and inadmissibility of estimators of scale parameters using an asymmetric loss function. *Communications in Statistics-Theory and Methods*, 22(10):2877–2901.

Pathak, A., Kumar, M., Singh, S. K., and Singh, U. (2020a). Assessing the effect of E-Bayesian inference for Poisson inverse exponential distribution parameters under different loss functions and its application. *Communications in Statistics-Theory and Methods*, :1-459.

Pathak, A., Kumar, M., Singh, S. K., and Singh, U. (2020b). Bayesian inference: Weibull Poisson model for censored data using the expectation-maximization algorithm and its application to bladder cancer data. *Journal of Applied Statistics*, :1-23.

Pensky, M. and Singh, R. (1999). Empirical Bayes estimation of reliability characteristics for an Exponential family. *Canadian Journal of Statistics*, 27(1):127–136.

Pollock, K. H. (2000). Capture-recapture models. *Journal of the American Statistical Association*, 95(449):293–296.

Raftery, A. E. and Zeh, J. E. (1998). Estimating bowhead whale population size and rate of increase from the 1993 census. *Journal of the American Statistical Association*, 93(442):451–463.

Rastogi, M. K. and Tripathi, Y. M. (2013). Inference on unknown parameters of a Burr distribution under hybrid censoring. *Statistical Papers*, 54(3):619–643.

Reyad, H. M. and Ahmed, S. O. (2016). Bayesian and E-Bayesian estimation for the Kumaraswamy distribution based on type-II censoring. *International Journal of Advanced Mathematical Sciences*, 4(1):10–17.

Robbins, H. (1955). *An empirical Bayes approach to statistics*. Office of Scientific Research, US Air Force.

Robbins, H. (1964). The empirical Bayes approach to statistical decision problems. *The Annals of Mathematical Statistics*, 35(1):1–20.

Robert, C. and Casella, G. (2013). *Monte Carlo statistical methods*. Springer Science & Business Media.

Rodrigues, J., Milan, L. A., and Leite, J. G. (2001). Hierarchical Bayesian Estimation for the Number of Species. *Biometrical Journal: Journal of Mathematical Methods in Biosciences*, 43(6):737–746.

Rohatgi, V. K. and Saleh, A. M. E. (2015). *An Introduction to Probability and Statistics*. John Wiley & Sons.

Sanathanan, L. (1972). Estimating the size of a multinomial population. *The Annals of Mathematical Statistics*, pages 142–152.

Sankaran, M. (1970). 275 note: The Discrete Poisson-Lindley Distribution. *Biometrics*, :145-149.

Santana, T. V. D., Ortega, E. M. M., and Cordeiro, G. M. (2012). The Kumaraswamy Log-Lgistic distribution. *Journal of Statistics Theory and Applications*, 11(3).

Schabe, H. (1991). Bayes estimates under asymmetric loss. *IEEE transactions on reliability*, 40(1):63–67.

Schafer, J. L. (1997). *Analysis of incomplete multivariate data*. CRC press.

Schlaifer, R. and Raiffa, H. (1961). *Applied statistical decision theory*.

Schwarz, C. J. and Seber, G. A. (1999). Estimating animal abundance: review iii. *Statistical Science*, :427-456.

Schwarz, G. et al. (1978). Estimating the dimension of a model. *The annals of statistics*, 6(2):461–464.

Seber, G. A. and Schwarz, C. J. (2002). Capture-recapture: before and after euring 2000. *Journal of Applied Statistics*, 29(1-4):5–18.

Shah, J. and Patel, M. (2011). Bayes estimation of a two-parameter Geometric distribution under multiply Type II censoring. *Journal of Quality and Reliability Engineering*, 2011.

Shaw, J. (1988). A quasirandom approach to integration in Bayesian statistics. *The Annals of Statistics*, :895-914.

Shi, Y., Shi, X., and Xu, Y. (2005). Approximate confidence limits of the reliability performances for a cold standby series system. *Journal of Applied Mathematics and Computing*, 19(1-2):439–445.

Shojaee, O., Azimi, R., and Babanezhad, M. (2012). Empirical Bayes estimators of parameter and reliability function for compound Rayleigh distribution under record data. *American Journal of Theoretical and Applied Statistics*, 1(1):12–15.

Sichel, H. (1986). Parameter estimation for a word frequency distribution based on occupancy theory. *Communications in Statistics-Theory and Methods*, 15(3):935–949.

Singh, P., Singh, S. K., and Singh, U. (2008a). Bayes estimator of inverse Gaussian parameters under general entropy loss function using lindley's approximation. *Communications in Statistics—Simulation and Computation*, 37(9):1750–1762.

Singh, R., Singh, S., Singh, U., and Singh, G. (2009). Bayes Estimator of Generalized-Exponential parameters under general entropyoss function using lindley's approximation. *Statistics in Transition*, 10(1):109–127.

Singh, R., Singh, S. K., Singh, U., and Singh, G. P. (2008b). Bayes estimator of Generalized-Exponential parameters under linex loss function using lindley's approximation. *Data Science Journal*, 7:65–75.

Singh, S., Singh, U., Kumar, M., and Vishwakarma, P. (2014). Classical and Bayesian inference for an extension of the Exponential distribution under progressive type-II censored data with Binomial removals. *Journal of Statistics Applications and Probability Letters*, 1(3):75–86.

Singh, S. K., Singh, U., and Kumar, D. (2011). Bayesian estimation of the Exponentiated Gamma parameter and reliability function under asymmetric loss function. *REVSTAT-Stat J*, 9(3):247–260.

Singh, S. K., Singh, U., and Kumar, D. (2013a). Bayes estimators of the reliability function and parameter of inverted exponential distribution using informative and non-informative priors. *Journal of Statistical computation and simulation*, 83(12):2258–2269.

Singh, S. K., Singh, U., and Kumar, M. (2016). Bayesian estimation for Poisson-Exponential model under progressive type-II censoring data with Binomial removal and its application to ovarian cancer data. *Communications in Statistics-Simulation and Computation*, 45(9):3457–3475.

Singh, S. K., Singh, U., and Sharma, V. K. (2013b). Expected total test time and Bayesian estimation for generalized Lindley distribution under progressively type-II censored sample where removals follow the Beta-Binomial probability law. *Applied Mathematics and Computation*, 222:402–419.

Sinha, S. et al. (1976). Bayesian inference about the reliability function for the Exponential distributions. *Communications in Statistics-Theory and Methods*, 5(5):471–479.

Sinha, S. K. (1986). *Reliability and Life Testing*. John Wiley & Sons, Inc.

Smith, A., Skene, A., Shaw, J., and Naylor, J. (1987). Progress with numerical and graphical methods for practical Bayesian statistics. *Journal of the Royal Statistical Society: Series D (The Statistician)*, 36(2-3):75–82.

Smith, A. F. and Roberts, G. O. (1993). Bayesian computation via the Gibbs sampler and related markov Chain Monte Carlo methods. *Journal of the Royal Statistical Society: Series B (Methodological)*, 55(1):3–23.

Smith, A. F. M. (1991). Bayesian computational methods. *Philosophical Transactions of the Royal Society of London. Series A: Physical and Engineering Sciences*, 337(1647):369–386.

Smith, K. L. and Jones, M. L. (2005). Watershed-level sampling effort requirements for determining riverine fish species composition. *Canadian Journal of Fisheries and Aquatic Sciences*, 62(7):1580–1588.

Soliman, A. A., Ellah, A. H. A., Abou-Elheggag, N. A., and El-Sagheer, R. M. (2015). Inferences using type-II progressively censored data with Binomial removals. *Arabian Journal of Mathematics*, 4(2):127–139.

Sultan, K., Alsadat, N., and Kundu, D. (2014). Bayesian and maximum likelihood estimations of the inverse Weibull parameters under progressive type-II censoring. *Journal of Statistical Computation and Simulation*, 84(10):2248–2265.

Tang, Y., Xie, M., and Goh, T. (2003). Statistical analysis of a Weibull extension model. *Communications in Statistics-Theory and Methods*, 32(5):913–928.

Tanner, M. A. (2012). *Tools for statistical inference*. Springer.

Tierney, L. (1994). Markov Chains for exploring posterior distributions. *The Annals of Statistics*, 22(4):1701–1728.

Tierney, L. and Kadane, J. B. (1986). Accurate approximations for posterior moments and marginal densities. *Journal of the american statistical association*, 81(393):82–86.

Tse, S. K., Yang, C., and Yuen, H.-K. (2000). Statistical analysis of Weibull distributed lifetime data under type-II progressive censoring with Binomial removals. *Journal of Applied Statistics*, 27(8):1033–1043.

Tse, S. K., Y. C. and Yuen, H. K. (2000). Statistical analysis of Weibull distributed lifetime data under type II progressive censoring with Binomial removals. *Journal of Applied Statistics*, 27(8):1033–1043.

Upadhyay, S. K. and Gupta, A. (2010). A Bayes analysis of modified Weibull distribution via markov Chain Monte Carlo simulation. *Journal of Statistical Computation and Simulation*, 80:241–254.

Upadhyay, S. K., V. N. and Smith, A. F. M. (2001). Bayes inference in life testing and reliability via markov Chain Monte Carlo simulation. *Sankhyā: The Indian Journal of Statistics, Series A (1961-2002)*, 63(1):15–40.

Varian, H. R. (1975). A Bayesian approach to real estate assessment. *Amsterdam: North Holland Publishing Co*, 36(1):195–208.

Villegas, C. (1977). On the representation of ignorance. *Journal of the American Statistical Association*, 72(359):651–654.

Villegas, C. (1981). Inner statistical inference II. *The Annals of Statistics*, :768-776.

Viveros, R. and Balakrishnan, N. (1994). Interval estimation of parameters of life from progressively censored data. *Technometrics*, 36(1):84–91.

Walther, B. A. and Martin, J.-L. (2001). Species richness estimation of bird communities: how to control for sampling effort? *Ibis*, 143(4):413–419.

Wang, X., He, C. Z., and Sun, D. (2007). Bayesian population estimation for small sample capture-recapture data using noninformative priors. *Journal of Statistical Planning and Inference*, 137(4):1099–1118.

Wilson, R. M. and Collins, M. F. (1992). Capture-recapture estimation with samples of size one using frequency data. *Biometrika*, 79(3):543–553.

Winkler, R. L. (1972). A decision-theoretic approach to interval estimation. *Journal of the American Statistical Association*, 67(337):187–191.

Wu, C. J. (1983). On the convergence properties of the EM algorithm. *The Annals of statistics*, :95-103.

Wu, S. J. and Chang, C. T. (2002). Parameter estimations based on Exponential progressive type II censored data with Binomial removals. *International journal of information and management sciences*, 13(3):37–46.

Xiaohu, L., Yanyan, H., and Xueyan, Z. (2011). The Kumaraswamy Binomial distribution. *Chinese Journal of Applied Probability and Statistics*, 27(5).

Yan, S. and Gendai, G. (2003). Bayes estimation for Reliability indexes of cold standby system. *Journal of North China Electric Power University China*, 30(2):96–99.

Yousefzadeh, F. (2017). E-Bayesian and hierarchical Bayesian estimations for the system reliability parameter based on asymmetric loss function. *Communications in Statistics-Theory and Methods*, 46(1):1–8.

Yuen, H.-K. and Tse, S.-K. (1996). Parameters estimation for Weibull distributed lifetimes under progressive censoring with random removeals. *Journal of Statistical Computation and Simulation*, 55(1-2):57–71.

Zeh, J. E., Ko, D., Krogman, B. D., and Sonntag, R. (1986). A multinomial model for estimating the size of a whale population from incomplete census data. *Biometrics*, pages 1–14.

Zellner, A. (1986a). Bayesian estimation and prediction using asymmetric loss functions. *Journal of the American Statistical Association*, 81(394):446–451.

Zellner, A. (1986b). On assessing prior distributions and Bayesian regression analysis with g-prior distributions. *Bayesian inference and decision techniques*.

Zellner, A. (1994). Bayesian and Non-Bayesian Estimation Using Balanced Loss Functions. In Statistical decision theory and related topics V. *Springer*, :377-390.

Zellner, A. (1996). Introduction to Bayesian inference in econometrics. *John Wiley & Sons Inc.*