Chapter 2

Bayesian Inference for Weibull Poisson Distribution Under Censored Data Using Expectation Maximization Algorithm *

2.1 Introduction

Statistical literature have numerous distributions for modeling life-time data. Due to the enormous use of the Poisson family distribution, we consider a very flexible Weibull Poisson Distribution (WPD). It is one of the recent compounding of two most greeted probability distributions i.e., Weibull and zero truncated Poisson distribution. This distribution was pioneered by Lu and Shi (2012). The CDF of WPD with (α, β, λ) is

$$F(x) = \frac{e^{\lambda e^{-\beta x^{\alpha}}} - e^{\lambda}}{1 - e^{\lambda}}; \quad \alpha > 0, \lambda > 0, \beta > 0, x > 0. \tag{2.1}$$

*Part of this chapter has been published in reputed peer-reviewed journals with indexing SCI, SCIE, SCOPUS, see Pathak et al. (2020b).

The PDF is given by

$$f(x) = \frac{\alpha\beta\lambda e^{-\lambda}}{1-e^{-\lambda}} e^{-\beta x^{\alpha}} x^{\alpha-1} e^{\lambda e^{-\beta x^{\alpha}}}; \quad \alpha > 0, \lambda > 0, \beta > 0, x > 0, \tag{2.2}$$

where, shape parameter α and scale parameters β of WPD, while λ is the rate parameter of zero truncated Poisson distribution. This distribution has an edge over other Poisson-based distributions like Poisson-gamma, Poisson-log normal etc in the sense that it covers all types of failure rates encountered in life testing experiments, see Gonzales-Barron and Butler (2011). We may note here a typical feature of life testing experiments is censoring because, situations do arise when items/ units are lost or removed from the experiment while they are alive; i.e., quite often, it is very much difficult to get failure times of all the items/units put on test experiments owing to various restriction related to time, cost and other resources. Type-I censoring takes place when experimental time is fixed and hence number of failures become random. While type-II censoring occurs when the number of failures is fixed, but experimental time remain random. Even under these conditions, some items/ units may drop out of the experiment randomly due to some unknown causes, which are beyond the control of the experimenter. For example, consider that a medical experiment starts with n patients but after the death of first patient, some patients who are alive leave the experiment and go for treatment elsewhere. Similarly, after death of second patient a few more are leave and the process continues till predetermined number of failure (say $m < n$) are recorded. It may be assumed here that at each stage participating patient may independently decide to leave the experiment with probability p. Thus the number of patients who leave the experiment at a specified stage will follow binomial distribution with probability p. It may be argued at this stage that probability p may vary at each stage. But sake of simplicity, we shall assume that p is same at each stages. Collecting information in this way results to a censored sample and the sampling technique used is called as PT-II CBRs. The mathematical formulation of PT-II CBRs is presented in next Section. For details, one can see Balakrishnan and Sandhu (1995), Balakrishnan and Aggarwala (2000).

In last few decades, parameter estimation for Weibull lifetime models based on progressive Type-II, PT-II CBRs and optimal progressive censoring schemes are studied by several authors

((Balasooriya et al., 2000), Tse et al. (2000), (Tang et al., 2003), Ng et al. (2004) etc.). Estimation of inverse Weibull parameters have been discussed by Sultan et al. (2014). Also, in last few years and for other lifetime models by Soliman et al. (2015), Singh et al. (2014), Kumar et al. (2015), Kumar et al. (2018), Kumar et al. (2019a), Kumar et al. (2019b) etc. But, it seems as if no attempt has been made to develop estimators for the parameters of WPD under PT-II CBRs; although estimation of parameters under classical set up has also been attempted by Lu and Shi (2012).

Therefore, in this chapter we propose to develop an estimation procedure to obtain the ML Estimators (using EM algorithm) and Bayes estimators for parameters of WPD under symmetric and asymmetric loss function when sample is obtained by the use of PT-II CBRs. An important feature of this chapter is to develop the required mathematics for PT-II CBRs, EM algorithm along with its application to the bladder cancer patients data (remission time in months).

2.2 Classical and Bayesian Estimation Under PT-II CBRs

In this section, we follow the PT-II CBRs discussed in Chapter-1, Subsection 1.11.2. For details see. Viveros and Balakrishnan (1994) and Ng et al. (2004). Following Cohen (1963) for fixed removals, say $R_1 = r_1, R_2 = r_2, R_3 = r_3, \cdots, R_m = r_m$, the conditional likelihood function can be written as,

$$L(\alpha,\beta,\lambda;x|R=r) = c\prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{r_i}; \qquad -\infty < x_1 < \ldots < x_m < \infty, \tag{2.3}$$

$n,\ m\ \varepsilon$ N, $1 \leq i \leq m$ and $c = \prod_{i=1}^{m} \gamma_i$ where $\gamma_i = \sum_{j=1}^{m} (r_j + 1)$. Substituting $f(x_i)$ and $F(x_i)$ from (2.1) and (2.2) into (2.3), we have

$$L(\alpha,\beta,\lambda;x|R=r) = c\prod_{i=1}^{m} \frac{\alpha\beta\lambda x_i^{\alpha-1}}{1-e^{-\lambda}} e^{-\lambda-\beta x_i^{\alpha}+\lambda e^{-\beta x_i^{\alpha}}} \left\{\frac{1-e^{\lambda e^{-\beta x_i^{\alpha}}}}{1-e^{\lambda}}\right\}^{r_i}. \tag{2.4}$$

As mentioned earlier, in the experiment removal of the number of items/units is random and independent of each other, therefore

$$p(R_1 = r_1; p) = \binom{n-m}{r_1} p^{r_1}(1-p)^{n-m-r_1} \tag{2.5}$$

and for $i = 2, 3, ..., m-1$

$$p(R_i; p) = p(R_i = r_i | R_{i-1} = r_{i-1}, ...R_1 = r_1)$$

$$= \binom{n-m-\sum_{l=0}^{i-1} r_l}{r_i} p^{r_i}(1-p)^{n-m-\sum_{l=0}^{i-1} r_l}. \tag{2.6}$$

Hence, likelihood function can be written as

$$L(\alpha, \beta, \lambda, p; x) = L(\alpha, \beta, \lambda; x | R = r) p(R = r; p) \tag{2.7}$$

where,

$$\begin{aligned} p(R = r; p) = p(R_1 = r_1) p(R_2 = r_2 | R_1 = r_1) p(R_3 = r_3 | R_2 = r_2, R_1 = r_1)... \\ p(R_{m-1} = r_{m-1} | R_{m-2} = r_{m-2}, ...R_1 = r_1). \end{aligned} \tag{2.8}$$

Substituting from Equation (2.5) and (2.6) into (2.8), we have

$$p(R = r; p) = \frac{(n-m)! p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i) r_i}}{(n-m-\sum_{l=1}^{i-1} r_l)! \prod_{i=1}^{m-1} r_i!}, \tag{2.9}$$

now using Equation (2.4), (2.7) and (2.9), the complete likelihood can be expressed in the following form,

$$L(\alpha, \beta, \lambda, p; x) = \Phi L_1(\alpha, \beta, \lambda) L_2(p)$$

where,

$$\Phi = \frac{c(n-m)!}{(n-m-\sum_{l=1}^{i-1} r_l)! \prod_{i=1}^{m-1} r_i!},$$

$$L_1(\alpha,\beta,\lambda;x|R=r) = c\prod_{i=1}^{m} \frac{\alpha\beta\lambda x_i^{\alpha-1}}{1-e^{-\lambda}} e^{-\lambda-\beta x_i^{\alpha}+\lambda e^{-\beta x_i^{\alpha}}} \left\{ \frac{1-e^{\lambda e^{-\beta x_i^{\alpha}}}}{1-e^{\lambda}} \right\}^{r_i}, \tag{2.10}$$

$$L_2(p) = p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}.$$

Now, ML Estimates of α, β and λ are computed by maximizing L_1 and MLE of p by maximizing L_2. Taking log of both sides to Equation (2.10), we get

$$l_1(\alpha,\beta,\lambda) = \ln(L_1(\alpha,\beta,\lambda)) = m\ln\alpha + m\ln\beta + m\ln\lambda + (\alpha-1)\sum_{i=1}^{m}\ln x_i - m\lambda - \beta\sum_{i=1}^{m} x_i^{\alpha} - m\ln(1-e^{-\lambda}) + \lambda\sum_{i=1}^{m} e^{-\beta x_i^{\alpha}} + \sum_{i=1}^{m} r_i \left(\ln(e^{\lambda e^{-\beta x_i^{\alpha}}} - 1) - \ln(e^{\lambda} - 1) \right). \tag{2.11}$$

Differentiating the Equation (2.11) with respect to parameter α, β and λ and equating to zero, we obtain following three normal equations. A simultaneous solution of these provide ML Estimates of the parameters.

$$\frac{\partial l_1(\alpha,\beta,\lambda)}{\partial\alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m}\ln x_i - \beta\sum_{i=1}^{m} x_i^{\alpha}\ln x_i - \lambda\beta\sum_{i=1}^{m} e^{-\beta x_i^{\alpha}}(x_i^{\alpha}\ln x_i) + \sum_{i=1}^{m} r_i \left[\frac{\lambda e^{-\beta x_i^{\alpha}} e^{\lambda e^{-\beta x_i^{\alpha}}}}{1-e^{\lambda e^{-\beta x_i^{\alpha}}}} \beta x_i^{\alpha}\ln x_i \right] = 0, \tag{2.12}$$

$$\frac{\partial l_1(\alpha,\beta,\lambda)}{\partial\beta} = \frac{m}{\beta} - \sum_{i=1}^{m} x_i^{\alpha} - \lambda x_i^{\alpha}\sum_{i=1}^{m} e^{-\beta x_i^{\alpha}} + \sum_{i=1}^{m} r_i \left[\frac{\lambda e^{-\beta x_i^{\alpha}} e^{\lambda e^{-\beta x_i^{\alpha}}}}{1-e^{\lambda e^{-\beta x_i^{\alpha}}}} x_i^{\alpha} \right] = 0, \tag{2.13}$$

$$\frac{\partial l_1(\alpha,\beta,\lambda)}{\partial\lambda} = \frac{m}{\lambda} - m - \sum_{i=1}^{m} e^{-\beta x_i^{\alpha}} - \frac{me^{-\lambda}}{1-e^{-\lambda}} - \sum_{i=1}^{m} r_i \left[\frac{e^{\lambda e^{-\beta x_i^{\alpha}} - \beta x_i^{\alpha}}}{1-e^{\lambda e^{-\beta x_i^{\alpha}}}} - \frac{me^{\lambda}}{1-e^{\lambda}} \right] = 0. \tag{2.14}$$

Unfortunately, Equation (2.12), (2.13) and (2.14) can not be analytically solved simultaneously. Hence we propose the use of numerical iterative procedure, namely i.e. NR method for solving these. The numerical procedure used here for obtaining the iteration function and

the choice of initial guesses is based on maximum absolute row sum norms, which has been discussed by Jain et al. (2003). The EM algorithm has been proposed in this chapter to get the ML estimates of parameter α, β and λ, also discussed in Chapter 1, Subsection 1.10.1. Let Z_{ik} be the unobserved observation for the k^{th} items/ units moved out of the experiment at the time of observing i^{th} removal at time $X_i; i = 1,2,...,m$ and $k = 1,2,....,r_i$. Thus, the observed X_i's and Z_{ik}'s form the complete data. Hence the complete likelihood is

$$L(\alpha,\beta,\lambda) = \prod_{i=1}^{m}\left[\frac{\alpha\beta\lambda x_i^{\alpha-1}}{1-e^{-\lambda}} e^{-\lambda-\beta x_i^{\alpha}+\lambda e^{-\beta x_i^{\alpha}}} \prod_{k=1}^{r_i} \frac{\alpha\beta\lambda z_{ik}^{\alpha-1}}{1-e^{-\lambda}} e^{-\lambda-\beta z_{ik}^{\alpha}+\lambda e^{-\beta z_{ik}^{\alpha}}}\right].$$

The log-L function is

$$\begin{aligned}\ln L(\alpha,\beta,\lambda) = {} & n\ln(\alpha)+n\ln(\beta)+n\ln(\lambda)-n\lambda-n\ln\left(1-e^{-\lambda}\right) \\ & +(\alpha-1)\sum_{i=1}^{m}\ln x_i-\beta\sum_{i=1}^{m}x_i^{\alpha}+\lambda\sum_{i=1}^{m}e^{-\beta x_i^{\alpha}} \\ & +(\alpha-1)\sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln z_{ik}-\beta\sum_{i=1}^{m}\sum_{k=1}^{r_i}z_{ik}^{\alpha}+\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}e^{-\beta z_{ik}^{\alpha}}.\end{aligned} \tag{2.15}$$

Hence, ML estimate of the parameters are, obtained the simultaneous solution of the following three nonlinear equations

$$\begin{aligned}\frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial\alpha} = {} & \frac{n}{\alpha}-\alpha\beta\sum_{i=1}^{m}x_i^{\alpha-1}-\alpha\beta\lambda\sum_{i=1}^{m}x_i^{\alpha-1}e^{-\beta x_i^{\alpha}}+\sum_{i=1}^{m}\ln x_i \\ & -\alpha\beta\sum_{i=1}^{m}\sum_{k=1}^{r_i}z_{ik}^{\alpha-1}-\alpha\beta\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}z_{ik}^{\alpha-1}e^{-\beta z_{ik}^{\alpha}}+\sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln z_{ik}=0,\end{aligned} \tag{2.16}$$

$$\begin{aligned}\frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial\beta} = {} & \frac{n}{\beta}-\sum_{i=1}^{m}x_i^{\alpha}-\lambda\sum_{i=1}^{m}x_i^{\alpha}e^{-\beta x_i^{\alpha}}-\sum_{i=1}^{m}\sum_{k=1}^{r_i}z_{ik}^{\alpha} \\ & -\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}z_{ik}^{\alpha}e^{-\beta z_{ik}^{\alpha}}=0,\end{aligned} \tag{2.17}$$

and

$$\frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial\lambda} = \frac{n}{\lambda}-n+\frac{ne^{-\lambda}}{\left(1-e^{-\lambda}\right)}+\sum_{i=1}^{m}e^{-\beta x_i^{\alpha}}+\sum_{i=1}^{m}\sum_{k=1}^{r_i}e^{-\beta z_{ik}^{\alpha}}=0. \tag{2.18}$$

Now, to perform the EM algorithm, joint distribution of x and z can be written as

$$f(x,z;\alpha,\beta,\lambda) = P(z;\lambda) f(x|z;\alpha,\beta),$$

where,

$$P(z;\lambda) = \frac{e^{-\lambda}\lambda^z}{z!\left[1-e^{-\lambda}\right]}; \quad \lambda > 0, \quad z = 1,2,3,\cdots.$$

Since, the conditional PDF is

$$P(z|x;\alpha,\beta,\lambda) = \frac{f(x,z;\alpha,\beta,\lambda)}{f(x;\lambda)} = \alpha\beta z x^{\alpha-1} e^{-\beta z x^{\alpha}} \lambda^z \Gamma^{-1}(z+1)\left(e^{\lambda}-1\right)^{-1}; \quad z = 1,2,3,\cdots, \tag{2.19}$$

where, $\alpha > 0, \beta > 0$ and $\lambda > 0$. The E-step of EM algorithm needs the computation of the conditional expectation $(Z|X,\alpha^t,\beta^t,\lambda^t)$, where, $(\alpha^t,\beta^t,\lambda^t)$ is the current estimates of (α,β,λ). Hence from Equation (2.19), we get

$$E(z|x;\alpha^t,\beta^t,\lambda^t) = \left(1+\lambda^t e^{-\beta^t x^{\alpha^t}}\right).$$

The EM algorithm is completed with M-step, with complete data, where missing Z's are replaced by their conditional expectations $(Z|X,\alpha^t,\beta^t,\lambda^t)$. Thus, an EM iteration, takes $(\alpha^t,\beta^t,\lambda^t)$ into $\left(\alpha^{t+1},\beta^{t+1},\lambda^{t+1}\right)$ obtained from the following

$$\frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial\alpha} = \frac{n}{\alpha} - \alpha\beta\sum_{i=1}^{m} x_i^{\alpha-1} - \alpha\beta\lambda\sum_{i=1}^{m} x_i^{\alpha-1} e^{-\beta x_i^{\alpha}} + \sum_{i=1}^{m}\ln x_i$$
$$-\alpha\beta\sum_{i=1}^{m}\sum_{k=1}^{r_i}\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha-1} - \alpha\beta\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha-1} e^{-\beta\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha}}$$
$$+\sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right) = 0,$$

$$\frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{m} x_i^{\alpha} - \lambda\sum_{i=1}^{m} x_i^{\alpha} e^{-\beta x_i^{\alpha}} - \sum_{i=1}^{m}\sum_{k=1}^{r_i}\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha}$$
$$-\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha} e^{-\beta\left(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}\right)^{\alpha}} = 0,$$

and

$$\frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial \lambda} = \frac{n}{\lambda} - n + \frac{ne^{-\lambda}}{\left(1-e^{-\lambda}\right)} + \sum_{i=1}^{m} e^{-\beta x_i^{\alpha}} + \sum_{i=1}^{m}\sum_{k=1}^{r_i} e^{-\beta\left(1+\lambda^t e^{-\beta^t x_i^{\alpha t}}\right)^{\alpha}} = 0.$$

The iterative procedure obtained for EM algorithm is given below

$$\alpha^{t+1} = \frac{n}{\left\{\begin{array}{l} \alpha\beta\sum_{i=1}^{m} x_i^{\alpha-1} + \alpha\beta\lambda\sum_{i=1}^{m} x_i^{\alpha-1}e^{-\beta x_i^{\alpha}} - \sum_{i=1}^{m}\ln x_i + \alpha\beta\sum_{i=1}^{m}\sum_{k=1}^{r_i}\left(1+\lambda^t e^{-\beta^t x_i^{\alpha t}}\right)^{\alpha-1} \\ +\alpha\beta\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}\left(1+\lambda^t e^{-\beta^t x_i^{\alpha t}}\right)^{\alpha-1} e^{-\beta\left(1+\lambda^t e^{-\beta^t x_i^{\alpha t}}\right)^{\alpha}} - \sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln\left(1+\lambda^t e^{-\beta^t x_i^{\alpha t}}\right) \end{array}\right\}}$$

$$\beta^{t+1} = \frac{n}{\left\{\begin{array}{l} \sum_{i=1}^{m} x_i^{\alpha} + \lambda\sum_{i=1}^{m} x_i^{\alpha}e^{-\beta x_i^{\alpha}} + \sum_{i=1}^{m}\sum_{k=1}^{r_i}\left(1+\lambda^t e^{-\beta^t x_i^{\alpha t}}\right)^{\alpha} \\ +\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}\left(1+\lambda^t e^{-\beta^t x_i^{\alpha t}}\right)^{\alpha} e^{-\beta\left(1+\lambda^t e^{-\beta^t x_i^{\alpha t}}\right)^{\alpha}} \end{array}\right\}}$$

and

$$\lambda^{t+1} = \frac{n}{\left\{n - \frac{ne^{-\lambda}}{\left(1-e^{-\lambda}\right)} - \sum_{i=1}^{m} e^{-\beta x_i^{\alpha}} - \sum_{i=1}^{m}\sum_{k=1}^{r_i} e^{-\beta\left(1+\lambda^t e^{-\beta^t x_i^{\alpha t}}\right)^{\alpha}}\right\}}.$$

Then $\left(\alpha^{t+1},\beta^{t+1},\lambda^{t+1}\right)$ is used as the current estimates of (α,β,λ) in the next iteration. The ML estimates of (α,β,λ) can be obtained by repeating the E-step and M-step until convergence is achieved.

2.2.1 Large Sample Test Procedure

Now, we shall discuss LR method for comparing the suitability of competitive models. Note that if we take $r_i = 0$ and $n = m$ in Equation (2.16), (2.17), (2.18), these reduce to complete sample normal Equations. The observed Fisher's Information matrix is

$$J_n(\alpha,\beta,\lambda) = \begin{pmatrix} -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \alpha^2} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \beta \partial \alpha} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \beta^2} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \beta \partial \lambda} \\ -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \lambda \partial \alpha} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \lambda \partial \beta} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \lambda^2} \end{pmatrix}_{(\hat{\alpha},\hat{\beta},\hat{\lambda})}$$

where,

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \alpha^2} = \frac{n}{\alpha^2} + \sum_{i=1}^{n} \beta x_i^{\alpha} (\log(x_i))^2 (1 + \lambda e^{-\beta x_i^{\alpha}} - \beta \lambda x_i^{\alpha} e^{-\beta x_i^{\alpha}}),$$

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \alpha \partial \beta} = \frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \beta \partial \alpha} = \sum_{i=1}^{n} \beta x_i^{\alpha} \log(x_i) (1 + \lambda e^{-\beta x_i^{\alpha}} - \beta \lambda x_i^{\alpha} e^{-\beta x_i^{\alpha}}),$$

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \alpha \partial \lambda} = \frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \lambda \partial \alpha} = \sum_{i=1}^{n} \beta x_i^{\alpha} \log(x_i) e^{-\beta x_i^{\alpha}},$$

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \beta^2} = \frac{n}{\beta^2} - \lambda \sum_{i=1}^{n} (x_i^{\alpha})^2 e^{-\beta x_i^{\alpha}},$$

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \beta \partial \lambda} = \frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \lambda \partial \beta} = \sum_{i=1}^{n} x_i^{\alpha} e^{-\beta x_i^{\alpha}},$$

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial \lambda^2} = \frac{n}{\lambda^2} - n \frac{e^{\lambda}}{(1 - e^{\lambda})^2}.$$

Let $T_n(\alpha,\beta,\lambda)$ be the expectation of Fisher Information matrix, i.e.,

$$T_n(\alpha,\beta,\lambda) = E(J_n(\alpha,\beta,\lambda)) = n \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

where,

$$T_{11} = \frac{1}{\alpha^2} + \beta E \left[x_z^{\alpha} (\log(x_z))^2 (1 + \lambda e^{-\beta x_z^{\alpha}} - \beta \lambda x_z^{\alpha} e^{-\beta x_z^{\alpha}}) \right],$$

$$T_{12} = T_{21} = E \left[x_z^{\alpha} \log(x_z) (1 + \lambda e^{-\beta x_z^{\alpha}} - \beta \lambda x_z^{\alpha} e^{-\beta x_z^{\alpha}}) \right],$$

$$T_{13} = T_{31} = \beta E \left[x_z^{\alpha} \log(x_z) e^{-\beta x_z^{\alpha}} \right],$$

$$T_{22} = \frac{1}{\beta^2} - \lambda E\left[(x_z^{\alpha})^2 e^{-\beta x_z^{\alpha}}\right],$$

$$T_{23} = T_{32} = E\left[x_z^{\alpha} e^{-\beta x_z^{\alpha}}\right],$$

$$T_{33} = \frac{1}{\lambda^2} - \frac{e^{\lambda}}{(1-e^{\lambda})^2}.$$

For large n, under the usual regularity condition, we obtain that $(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ have multivariate normal distribution with attain mean (α, β, λ) and covariance matrix $T_n^{-1}(\alpha, \beta, \lambda)$. The asymptotic property of normality is useful for performing a goodness of fit test. Here, we can test the significance of the model parameters by comparing this full model with specified nested models based on the LR test. By considering null hypothesis $H_{01} : \alpha = 1$ against $H_{11} : \alpha \neq 1$ and $H_{02} : \lambda = 0$ against $H_{12} : \lambda \neq 0$, one can compare the suitability of Exponential Poisson and Weibull versus Weibull Poisson distribution respectively. The test statistic under H_{0i}, $i = 1, 2$, are

$$R_1 = -2\ln\left(\frac{L(\alpha_0, \hat{\beta}, \hat{\lambda})}{L(\hat{\alpha}, \hat{\beta}, \hat{\lambda})}\right) \quad \textit{and} \quad R_2 = -2\ln\left(\frac{L(\hat{\alpha}, \hat{\beta}, \lambda_0)}{L(\hat{\alpha}, \hat{\beta}, \hat{\lambda})}\right),$$

respectively, which are asymptotically distributed as χ^2 with degrees of freedom equal to the respective dimension of the parameter space under the null hypothesis.

2.2.2 Bayesian Estimation Under PT-II CBRs

To obtain the Bayes estimator of α, β and λ, we assume that these are independently distributed prior pdfs for α and λ are chosen by using Jeffery's method i.e., log of the parameters are uniformly distributed; resulting to the following distributions:

$$g_1(\alpha) \propto \frac{1}{\alpha}; \qquad \alpha > 0. \tag{2.20}$$

$$g_2(\lambda) \propto \frac{1}{\lambda}; \qquad \lambda > 0. \tag{2.21}$$

Keeping in mind the wide coverage of variety of prior beliefs, we have chosen gamma distribution given below as prior distribution; see for details, Nassar and Eissa (2005), Box and Tiao (2011).

$$g_3(\beta) \propto e^{-a\beta}\beta^{b-1}; \qquad a>0, b>0, \tag{2.22}$$

where, gamma distribution have scale parameter a and shape parameter b. Thus the posterior distribution of α, β and λ can easily be obtained as

$$\pi(\alpha,\beta,\lambda|x,r) \propto \frac{\alpha^{m-1}\lambda^{m-1}\beta^{m+b-1}e^{-m\lambda-\beta\sum_{i=1}^{m}x_i^{\alpha}-a\beta+\lambda\sum_{i=1}^{m}e^{-\beta x_i^{\alpha}}}}{(1-e^{\lambda})^m \prod_{i=1}^{m}x_i^{\alpha-1}\left[\frac{1-e^{\lambda e^{-\beta x_i^{\alpha}}}}{1-e^{-\lambda}}\right]^{r_i}},$$

and the respective marginal posterior pdfs of α, β and λ can be computed from the following

$$\pi_1(\alpha|x,r) = \int_0^{\infty}\int_0^{\infty}\pi(\alpha,\beta,\lambda|x,r)d\beta\, d\lambda,$$

$$\pi_2(\beta|x,r) = \int_0^{\infty}\int_0^{\infty}\pi(\alpha,\beta,\lambda|x,r)d\alpha\, d\lambda,$$

and

$$\pi_3(\lambda|x,r) = \int_0^{\infty}\int_0^{\infty}\pi(\alpha,\beta,\lambda|x,r)d\alpha\, d\beta.$$

Now, let us consider that the very much popular symmetric loss function i.e., SELF has equal weight to the o.e. and u.e. of the same magnitude. Also, consider the asymmetric loss function i.e. GELF has unequal weight to the o.e. is more serious than u.e. and vice versa. The SELF and GELF are discussed in Chapter 1, Subsection 1.8. The expressions for the Bayes estimators of the parameters α, β and λ, denoted by $\hat{\alpha}_G, \hat{\beta}_G$ and $\hat{\lambda}_G$ respectively, are given below

$$\hat{\alpha}_G = \left[\int_0^{\infty}\alpha^{-\delta}\pi_1(\alpha|x,r)d\alpha\right]^{-\frac{1}{\delta}}, \tag{2.23}$$

$$\hat{\beta}_G = \left[\int_0^{\infty}\beta^{-\delta}\pi_2(\beta|x,r)d\beta\right]^{-\frac{1}{\delta}}, \tag{2.24}$$

and

$$\hat{\lambda}_G = \left[\int_0^\infty \lambda^{-\delta}\pi_3(\lambda|x,r)d\lambda\right]^{-\frac{1}{\delta}}. \tag{2.25}$$

It may be noted that the integrals in Equation (2.23), (2.24) and (2.25) can not be reduced to closed forms. Hence, numerical computational techniques are suggested for their calculations following Tierney (1994). Who has suggested the use of well-known technique namely MCMC technique in which the samples are generated from posterior distribution by Gibbs sampler via M-H algorithms. The samples thus obtained are then used to evaluate the Bayes estimates under SELF and GELF. It may be noted that Gibbs sampler uses to generate samples from full conditionals to generate samples posterior distribution and for details Gelman et al. (2013). Full conditional posterior distributions of the parameters α, β, and λ can be written in the following form:

$$\pi_1^*(\alpha|\beta,\lambda,x,r) \propto \alpha^{m-1} e^{-\beta\sum_{i=1}^m x_i^\alpha + \lambda\sum_{i=1}^m e^{-\beta x_i^\alpha}} \prod_{i=1}^m x_i^{\alpha-1}\{1-e^{\lambda e^{-\beta x_i^\alpha}}\}^{r_i}, \tag{2.26}$$

$$\pi_2^*(\beta|\alpha,\lambda,x,r) \propto \beta^{m+b-1} e^{-\beta\sum_{i=1}^m x_i^\alpha - a\beta + \lambda\sum_{i=1}^m e^{-\beta x_i^\alpha}} \prod_{i=1}^m \{1-e^{\lambda e^{-\beta x_i^\alpha}}\}^{r_i}, \tag{2.27}$$

and

$$\pi_3^*(\lambda|\alpha,\beta,x,r) \propto \frac{\lambda^{m-1} e^{-m\lambda + \lambda\sum_{i=1}^m e^{-\beta x_i^\alpha}}}{(1-e^{-\lambda})^m} \prod_{i=1}^m \left\{\frac{1-e^{\lambda e^{-\beta x_i^\alpha}}}{1-e^{\lambda}}\right\}^{r_i}. \tag{2.28}$$

The Bayes estimators of parameter α, β and λ are evaluated from the required sample of Equation (2.26), (2.27) and (2.28), generated by using MCMC procedure. The algorithm used for obtaining Bayes estimates and HPD credible intervals is given below:

I. Set α_0, β_0 and λ_0 be the initial guess of α, β and λ.

II. Set $i = 1$.

III. Generate α_i from $\pi_1^*(\alpha|\beta_{i-1},\lambda_{i-1},x,r)$, β_i from $\pi_2^*(\beta|\lambda_{i-1},\alpha_{i-1},x,r)$ and λ_i from $\pi_3^*(\lambda|\alpha_{i-1},\beta_{i-1},x,r)$ respectively.

IV. Repeat steps 2-3, N times.

V. Obtain the Bayes estimates of α, β and λ under GELF as $\left[E(\alpha^{-\delta}|x,r)\right]^{-\frac{1}{\delta}} = \left[\frac{1}{N-N_0}\sum_{i=1}^{N-N_0}\alpha_i^{-\delta}\right]^{-\frac{1}{\delta}}$, $\left[E(\beta^{-\delta}|x,r)\right]^{-\frac{1}{\delta}} = \left[\frac{1}{N-N_0}\sum_{i=1}^{N-N_0}\beta_i^{-\delta}\right]^{-\frac{1}{\delta}}$ and $\left[E(\lambda^{-\delta}|x,r)\right]^{-\frac{1}{\delta}} = \left[\frac{1}{N-N_0}\sum_{i=1}^{N-N_0}\lambda_i^{-\delta}\right]^{-\frac{1}{\delta}}$, where N_0 is the burn in period. Substituting $\delta = -1$ in step V, we get Bayes estimates of α, β and λ under SELF.

VI. For computing the highest posterior density (HPD) credible interval of α, β and λ. We order the MCMC sample values α, β and λ (say $\alpha_1,\alpha_2,\alpha_3,\cdots,\alpha_N$ as $\alpha_{(1)},\alpha_{(2)},\alpha_{(3)},\cdots,\alpha_{(N)}$, $\beta_1,\beta_2,\beta_3,\cdots,\beta_N$ as $\beta_{(1)},\beta_{(2)},\beta_{(3)},\cdots,\beta_{(N)}$ and $\lambda_1,\lambda_2,\lambda_3,\cdots,\lambda_N$ as $\lambda_{(1)},\lambda_{(2)},\lambda_{(3)},\cdots,\lambda_{(N)}$). Then construct all the 100(1-Ψ)% credible intervals of α, β and λ, say $\{(\alpha_{(1)},\alpha_{N[(1-\Psi)]+1}),\cdots,(\alpha_{[N\Psi]},\alpha_N)\}$, $\{(\beta_{(1)},\beta_{N[(1-\Psi)]+1}),\cdots,(\beta_{[N\Psi]},\beta_N)\}$ & $\{(\lambda_{(1)},\lambda_{[N(1-\Psi)]+1}),\cdots,(\lambda_{[N\Psi]},\lambda_N)\}$ respectively. Where $[\eta]$ mentioned the largest integer less than or equal to η. Therefore, the HPD credible interval of α, β and λ is that interval which has the shortest length.

2.3 Bayes Prediction

In this Section, we have derived an expression for one sample Bayes prediction, if the experimenter is interested to know the lifetimes of the $(n-m)$ removed surviving units on the basis of observed sample. Let $Y_s = X_{m+s}, m < s \leq n$, represents the failure lifetime of the remaining units, then conditional distribution of $Y_{(s)}^{th}$ order statistics given PT-II CBRs sample $\mathbf{x}$ is given by, see Singh et al. (2013b)

$$f\left(y_{(s)}|x_{(m)},\alpha,\beta,\lambda\right) = \frac{(n-m)!\left[1-F\left(y_{(s)}\right)\right]^{n-m-s}}{(s-1)!(n-m-s)!\left[1-F\left(x_{(m)}\right)\right]^{n-m}}\left[F\left(y_{(s)}\right)-F\left(x_{(m)}\right)\right]^{s-1} f\left(y_{(s)}\right). \tag{2.29}$$

Substituting Equation (2.1) and Equation (2.2) in (2.29), we have

$$f\left(y_{(s)}|x_{(m)},\alpha,\beta,\lambda\right) = \alpha\beta y_{(s)}^{\alpha-1}\zeta(y_{(s)})log\left(\zeta(y_{(s)})\right)\frac{(n-m)!}{(s-1)!(n-m-s)!}$$
$$\left[\frac{1-\zeta(y_{(s)})}{1-\zeta(x_{(m)})}\right]^{n-m}\left[1-\zeta(y_{(s)})\right]^{-s}\left[\zeta(x_{(m)})-\zeta(y_{(s)})\right]^{s-1},$$

where, $\zeta(z) = e^{\lambda e^{-\beta z^{\alpha}}}$. One sample Bayes predictive density of $y_{(s)}^{th}$ ordered future sample can be obtained as follows

$$f\left(y_{(s)}|\mathbf{x}\right) = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} f\left(y_{(s)}|\mathbf{x},\alpha,\beta,\lambda\right)\pi\left(\alpha,\beta,\lambda|\mathbf{x}\right)d\alpha d\beta d\lambda$$

The above equation for $f\left(y_{(s)}|\mathbf{x}\right)$ cannot be expressed in closed form and hence it cannot be evaluated analytically. Therefore, MCMC techniques is proposed to be used for obtaining the approximate solution of the above predictive density.

$\{(\alpha_i,\beta_i,\lambda_i);i=1,2,\cdots,N-N_0\}$ obtained from $\pi\left(\alpha,\beta,\lambda|\mathbf{x}\right)$ using Gibbs sampling can be utilized to obtain the consistent estimate of $f\left(y_{(s)}|\mathbf{x}\right)$. It can be obtained by

$$f\left(y_{(s)}|\mathbf{x}\right) = \frac{1}{N-N_0}\sum_{i=1}^{N-N_0} f\left(y_{(s)}|\alpha_i,\beta_i,\lambda_i\right). \tag{2.30}$$

Thus, we can obtain the two-sided $100(1-\psi)\%$ prediction interval (l,u) for future sample by solving the following two equations:

$$P\left(Y_{(s)} > u|\mathbf{x}\right) = \tfrac{\psi}{2} \text{ and } P\left(Y_{(s)} > l|\mathbf{x}\right) = 1-\tfrac{\psi}{2}.$$

We are facing difficulties to obtain the explicit solution. Therefore, we need to apply as per required numerical technique for the purpose of solution of non-linear equations. Also we opted that an alternative method is MCMC discussed by Chen and Shao (1998), in the following way: Let $\left(y_{(i:s)}\right);i=1,2,\cdots,N-N_0$ be the corresponding ordered MCMC sample of $\left(y_{i:s}\right);i=1,2,\cdots,N-N_0$ from Equation (2.30). Then, the $100(1-\psi)\%$ HPD intervals for $y_{(s)}$ is $y_{(j*:s)}$, $y_{j*+[(1-\psi)M]:s}$, where $j*$ is chosen so that

$$y_{j*+[(1-\psi)N-N_0]:s} - y_{(j*:s)} = \min_{1\le j\le N-N_0-[(1-\psi)N-N_0]}\left[y_{j*+[(1-\psi)N-N_0]:s} - y_{(j*:s)}\right].$$

For considered real data set, we calculated the mean and 95% credible intervals (predictive bounds) for future samples using one sample prediction technique. The results are summarized in Table (2.4).

2.4 Expected Experiment Time

Cost is an very effective element in an experiment that is directly related to the time of experiment. Therefore, for a proper planning of the experimentation one is always interested in knowing the expected experiment time; which can be defined PT-II CBRs

$$E[X_m] = E_R[E[X_m|R=r]] \tag{2.31}$$

$$= \sum_{r_1=0}^{g(r_1)} \sum_{r_2=0}^{g(r_2)} \ldots \sum_{r_{m-1}=0}^{g(r_{m-1})} p(R,p) E[X_{m:m:n}|R=r].$$

Where $g(r_i) = n-m-r_1-\ldots-r_{i-1}$ and $p(R=r;p)$ is given in Equation (2.9). Conditioning on R the expected experiment time is

$$E[X_m|R] = \int_0^\infty x f_{X_m}(x)dx,$$

where, $f_{X_{(m)}} = C_{m-1} f(x) \sum_{j=1}^{m} a_{j,m}(1-F(x))^{\gamma_j}, 1 \le m \le n$ and $c_{m-1} = \prod_{i=1}^{m} \gamma_i, 1 \le m \le n$ and $a_{j,m} = \prod_{i=1}^{m} \frac{1}{\gamma_i - \gamma_j}; i \ne j, 1 \le j \le m \le n$. For more details about the procedure of evaluation of conditional expectation of X_m for given R, see Balakrishnan and Aggarwala (2000), Singh et al. (2013b), Tse et al. (2000). Using the suggested procedure, expected experiment times under PT-II CBRs are computed for different combinations of m and n listed in Table (2.1). The values of p, considered here are $0.1, 0.3, 0.5, 0.7$ and 0.9 while model parameters α, β and λ are arbitrarily taken as $1, 2$ and 2 respectively. The results obtained are summarized below

TABLE 2.1: Expected Experiment time $E[X_m]$ under PT-II CBRs.

n	m	$p=0.1$	$p=0.3$	$p=0.5$	$p=0.7$	$p=0.9$
	10	0.15660	0.57392	0.86262	0.93033	0.96077
	15	0.37201	0.99414	1.09666	1.13149	1.13086
30	20	0.76695	1.23138	1.27059	1.30419	1.25337
	25	0.93638	1.35836	1.37069	1.34487	1.35232
	30	1.47650	1.45380	1.49828	1.45718	1.47108
	10	0.28018	0.73055	0.91508	0.95055	0.95818
20	15	0.71918	1.10303	1.13051	1.15929	1.14585
	20	1.27924	1.28742	1.27601	1.28292	1.28714
	3	0.08832	0.12157	0.19727	0.31295	0.43709
	4	0.13669	0.22246	0.36903	0.50305	0.58098
10	6	0.29297	0.51585	0.67842	0.75186	0.76813
	10	0.99404	0.98307	0.97925	0.98829	0.99048

Now we can obtain ratio of the expected experiment time (REET) between PT-II CBRs and the complete sampling as

$$REET = \frac{E[X_m] \ \ under \ \ PT-II \ CBRs}{E[X_n] \ \ under \ \ complete \ \ sampling}. \tag{2.32}$$

It may be noted that REET indicates the reduction in experiment time. Figure (2.1) shows REET for various values of n for $m = 10$ and different removal probability $p = 0.1, 0.3, 0.5, 0.7$ and 0.9. It can be seen from the Figure that for each values of p, the REET decreases as n increases. It may be, noted that for larger value of (> 0.5) and larger $n(> 25)$; the values of REET do not change for change in the value of p. For $p \leq 0.5$ and moderate sample size (25) larger valuers of REET is noted for smaller valuers of p.

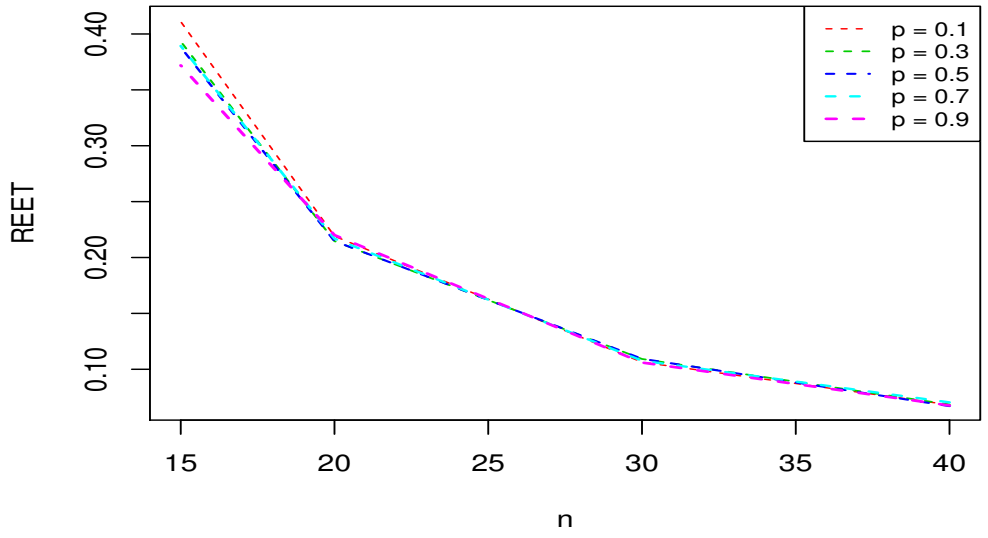

FIGURE 2.1: REET under PT-II CBRs to under complete sample.

2.5 Monte Carlo Simulation Study and Comparison of Estimators

We have seen above that proposed estimators are not obtained in the closed form; therefore, an analytical study of behavior of the estimators is not possible and we propose to study it numerically. For this purpose, we suggest the use of MCMC technique as suggested by Tierney (1994) also, for the calculation of risk (average loss over sample space) of estimators of the parameters α, β and λ. Hence, samples are generated from specified WPD and PT-II CBRs samples are obtained from these. ML estimator along with Bayes estimators under SELF and GELF are calculated. The ML estimators are denoted as; $\hat{\alpha}_M, \hat{\beta}_M$, $\hat{\lambda}_M$ where as $\hat{\alpha}_S, \hat{\beta}_S, \hat{\lambda}_S$ and $\hat{\alpha}_G, \hat{\beta}_G$, $\hat{\lambda}_G$ denote SELF and GELF estimates of the parameters α, β and λ, respectively. Similarly, $(\alpha_L^c, \alpha_U^c), (\beta_L^c, \beta_U^c), (\lambda_L^c, \lambda_U^c)$ and $(\alpha_L^h, \alpha_U^h), (\beta_L^h, \beta_U^h), (\lambda_L^h, \lambda_U^h)$ indicate $100(1-\Psi)\%$ CI and HPD credible intervals. Risk are estimated on the basis of 8000 samples. Since risk of the estimators under PT-II CBRs will be function of $n, m, p, \alpha, \beta, \lambda$, δ, a and b. The choice

of hyper parameter are made by assuming that the prior information about the parameter is available in the form of its expected value μ and its variance σ^2 reflecting the confidence in expected value. Thus a and b are calculated from equations, which can be taken in such a way that if we consider any two independent pieces of information as prior mean and variance of β are $\mu = \frac{b}{a}$ and $\sigma^2 = \frac{b}{a^2}$, where μ is taken as true values of the parameter β and smaller, moderate and large values of variances namely $0.5, 1$ and 5 which gave $(a = 4,\ b = 8)$, $(a = 2,\ b = 4)$ and $(a = 0.4,\ b = 0.8)$ respectively. We vary the effective samples size $m = 10[5]30$. The value of α, β and λ are arbitrarily taken as $1,\ 2$ and 2 respectively. The value of loss parameter δ is taken as 1.5 for o.e. to be more serious than u.e. and see Singh et al. (2011). After an extensive study of results thus obtained, conclusions are drawn regarding the behavior of the estimators. It may be mention here that the space restriction, results of various variation in the parameters are not shown. Only selected Figures are included.

2.6 Discussion of Results

We shall discuss the impact of variation of effective sample size m under PT-II CBRs, and compare the risks of all estimators of α, β and λ, obtained under GELF with the corresponding Bayes estimators under SELF and ML estimator. We observed, the risks of all the estimators of α, β and λ decrease as effective sample observations m increases. The risks of $(\hat{\alpha}_G, \hat{\beta}_G)$ and $(\hat{\alpha}_S, \hat{\beta}_S)$ are found to be close respectively to each other for all the considered situations. A similar trend is observed for $\hat{\lambda}_G$ and $\hat{\lambda}_S$ also. It is further observed that, in general, the risks of the estimators under SELF and GELF decreases, as for $\delta = +1.5$ and $\delta = -1.5$ with each prior belief of the parameter β (see Figure $(2.2 - 2.4)$). For large number of effective sample sizes, the difference between the risks of the estimators are less. The decrease in the risks is more for $\hat{\alpha}_M$ as compared to the other estimators. For almost all values of prior belief of the parameter β and δ, the risk of $\hat{\alpha}_G$ under GELF is found to be least among the considered estimators. It is also interesting to remark here that $\hat{\alpha}_G$ has the least risk under SELF. For positive values of δ,

the behavior of risks of estimators under GELF is more or less similar to the one obtained for negative δ (see Figure (2.2 – 2.4)).

Similarly, we have studied the risks of Bayes estimators β and λ respectively under SELF and GELF based on PT-II CBRs. The trend remains more or less the same as stated above under both loss functions see results in graphs, which has shown in supplementary material. Further we observed that the risk of $\hat{\beta}_G$ and $\hat{\lambda}_G$ under GELF and SELF are found to be least among the considered estimators respectively.

The Figure (2.5) shows the CI/HPD credible intervals for α. It may also noted, average CL of CI/HPD credible intervals consistently narrow down as m increases. The HPD credible intervals are better than CIs in respect of average CL. While studying the effect of large effective sample sizes m, the difference of average CL between the CIs and HPD credible intervals are negligibly small. For β and λ also, the trend of CI/HPD credible intervals, is similar to that of α. Due to space restriction, results for variations in m of CI/HPD credible intervals of β and λ are not shown here. The CI/HPD credible intervals of β and λ are given in supplementary material. Thus, we can not deny from the fact that estimates under Bayesian are more precise and accurate than ML estimates.

We also discussed the expected time to test and shown in Table (2.1), it is meaningful to comment that as the value p and m increase the expected time to test also increases. It is also observed that for fixed m, if increases the value of the sample size i.e., n, the expected time to test decreases.

2.7 An application to Bladder Cancer Data

For the application purpose, we have taken a real data set given by Lee and Wang (2003). It contains a set of remission times (in months) related to 137 cancer patients, and some patients are not present in the follow-up. The remission time in months are a subset of the data from a bladder cancer study. We have considered here a random set of 128 observations from it which

are given follow: 4.50, 32.15, 3.88, 13.80, 19.13, 4.87, 5.85, 14.24, 5.71, 7.09, 7.87, 7.59, 20.28, 5.32, 5.49, 3.02, 46.12, 2.02, 4.51, 5.17, 2.83, 9.22, 1.05, 0.20, 8.37, 3.82, 9.47, 36.66, 14.77, 26.31, 79.05, 10.06, 8.53, 2.02, 4.98, 11.98, 2.62, 4.26, 5.06, 1.76, 0.90, 11.25, 16.62, 4.40, 21.73, 10.34, 12.07, 34.26, 10.66, 6.97, 2.07, 0.51, 12.03, 0.08, 17.12, 3.36, 2.64, 1.40, 12.63, 43.01, 14.76, 2.75, 7.66, 0.81, 1.19, 7.32, 4.18, 3.36, 8.66, 1.26, 13.29, 1.46, 14.83, 6.76, 23.63, 5.62, 3.25, 18.10, 7.62, 7.63, 17.14, 25.74, 3.52, 2.87, 15.96, 17.36, 9.74, 3.31, 7.28, 1.35, 0.40, 2.26, 4.33, 9.02, 5.41, 2.69, 22.69, 6.94, 2.54, 11.79, 2.46, 7.26, 2.69, 5.34, 3.48, 8.26, 6.93, 4.23, 3.70, 0.50, 10.75, 6.54, 3.64, 5.32, 13.11, 8.65, 3.57, 5.09, 7.39, 5.41, 11.64, 2.09, 2.23, 6.25, 7.93, 4.34, 25.82, 12.02.

First of all, we checked the suitability of WPD to the above said data and compared, some specified lifetime models; Exponential Poisson (EP) and Weibull distribution. For testing the goodness of fit we used the method based on ML function, the K-S distance, the AIC, proposed by Akaike (1978), BIC proposed by Schwarz et al. (1978). The best distribution is that which has the lowest -log-L, AIC, BIC and K-S statistic and corresponding highest p values. Further, we have used a goodness of fit of distributions. We draw a Q-Q plots for the said three lifetime distribution and are shown in the Figure (2.14). A Q-Q plot shows the points $\left\{F^{-1}\left(\frac{i-0.5}{n};\hat{\Theta}_M,x_{(i)}\right)\right\},i=1,2,3,\cdots,n$, where $\hat{\Theta}_M$ is the ML estimates of the parameters of lifetime model. The values of ML estimates of the parameters of the considered lifetime models, -log-L, AIC, BIC, K-S statistic and their associated p values are reported in Table (2.2).

TABLE 2.2: The -log-L, K-S, p-value and the AIC and BIC values for the W), EP and Weibull fitted distributions.

	Estimates	-log-L	K-S	p-value	AIC	BIC
WP(α,β,λ)	(1.26853,0.01629,4.26518)	-410.189	0.046875	0.99896	826.3782	834.9343
EP(β,λ)	(0.106371,0.0000047)	-414.343	0.078125	0.82955	834.6856	843.2417
Weibull(α,β)	(1.04784,0.09389)	-414.087	0.0703125	0.90972	834.1738	842.7298

This Table shows that WPM provide better fit than EP and Weibull distribution. Further, we tested the hypothesis: $H_{01}:\alpha=1$ (Data follow Exponential Poisson) vs $H_{11}:\alpha\neq 1$ (Data follow Weibull Poisson) and $H_{02}:\lambda=0$ (Data follow Weibull) vs $H_{12}:\lambda\neq 0$ (Data follow Weibull Poisson), using the large sample test described in Subsection (2.2.1). The value of the

test statistic R_1 and R_2 are obtained as 8.30737 and 7.79551 respectively. Which reject H_{01} and H_{02}.

Now for the purpose of illustrating the method discussed in this chapter, PT-II CBR samples are generated from this data set under different schemes. The number of removals are shown in Table (2.3) under different schemes. The ML estimates of parameter α, β and λ are used to compute by EM algorithm. The initial value of parameters are chosen through contour plots of parameters, and their corresponding log-L are plotted; using R software (Figure (2.16)).

As we have no prior information about the parameter β, and we use non informative prior for which the hyper parameter of β is taken to be $(a = 0:000001; b = 0:000001)$. When implementing MCMC algorithm, the values of ML estimates are used as initial guess and CUMSUM plots are plotted, and to verified the convergence of Markov chain. Then, we evaluate Bayes estimates and HPD intervals using the formulae given in previous Section (2.3) under different censoring schemes based on Table (2.3), the Bayes estimate of parameter α, β and λ under SELF and GELF for $\delta = \pm 1.5$ are presented in Table (2.5). It may be observed from Table (2.5) that various parameter estimates, obtained using PT-II CBRs, are quite close to those obtained under complete samples.

TABLE 2.3: PT-II CBR samples under different censoring scheme $(S_{n:m})$ for fixed $n = 128, p = 0.5$.

	$S_{128:64}$		$S_{128:77}$				$S_{128:102}$			
i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i
1	0.08	23	0.08	18	20.28	0	0.08	7	10.34	0
2	2.69	17	2.26	9	21.73	0	1.19	4	10.66	0
3	4.23	7	3.02	8	22.69	0	1.76	3	10.75	0
4	4.98	2	3.7	6	23.63	0	2.09	4	11.25	0
5	5.17	3	4.34	2	25.74	0	2.62	0	11.64	0
6	5.41	1	4.51	3	25.82	0	2.64	2	11.79	0
7	5.49	5	5.09	2	26.31	0	2.75	2	11.98	0
8	6.76	4	5.32	1	32.15	0	3.02	0	12.02	0
9	7.26	0	5.41	1	34.26	0	3.25	2	12.03	0
10	7.28	1	5.49	0	36.66	0	3.36	0	12.07	0

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Table 2.3 – *Continued from previous page*

	$S_{128:64}$		$S_{128:77}$				$S_{128:102}$			
i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i
11	7.39	1	5.62	0	43.01	0	3.48	1	12.63	0
12	7.62	0	5.71	0	46.12	0	3.57	0	13.11	0
13	7.63	0	5.85	0	79.05	0	3.64	0	13.29	0
14	7.66	0	6.25	1			3.7	1	13.8	0
15	7.87	0	6.76	0			3.88	0	14.24	0
16	7.93	0	6.93	0			4.18	0	14.76	0
17	8.26	0	6.94	0			4.23	0	14.77	0
18	8.37	0	6.97	0			4.26	0	14.83	0
19	8.53	0	7.09	0			4.33	0	15.96	0
20	8.65	0	7.26	0			4.34	0	16.62	0
21	8.66	0	7.28	0			4.4	0	17.12	0
22	9.02	0	7.32	0			4.5	0	17.14	0
23	9.22	0	7.39	0			4.51	0	17.36	0
24	9.47	0	7.59	0			4.87	0	18.1	0
25	9.74	0	7.62	0			4.98	0	19.13	0
26	10.06	0	7.63	0			5.06	0	20.28	0
27	10.34	0	7.66	0			5.09	0	21.73	0
28	10.66	0	7.87	0			5.17	0	22.69	0
29	10.75	0	7.93	0			5.32	0	23.63	0
30	11.25	0	8.26	0			5.32	0	25.74	0
31	11.64	0	8.37	0			5.34	0	25.82	0
32	11.79	0	8.53	0			5.41	0	26.31	0
33	11.98	0	8.65	0			5.41	0	32.15	0
34	12.02	0	8.66	0			5.49	0	34.26	0
35	12.03	0	9.02	0			5.62	0	36.66	0
36	12.07	0	9.22	0			5.71	0	43.01	0
37	12.63	0	9.47	0			5.85	0	46.12	0
38	13.11	0	9.74	0			6.25	0	79.05	0
39	13.29	0	10.06	0			6.54	0		
40	13.8	0	10.34	0			6.76	0		
41	14.24	0	10.66	0			6.93	0		
42	14.76	0	10.75	0			6.94	0		
43	14.77	0	11.25	0			6.97	0		
44	14.83	0	11.64	0			7.09	0		

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Table 2.3 – *Continued from previous page*

	$S_{128:64}$		$S_{128:77}$				$S_{128:102}$			
i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i
45	15.96	0	11.79	0			7.26	0		
46	16.62	0	11.98	0			7.28	0		
47	17.12	0	12.02	0			7.32	0		
48	17.14	0	12.03	0			7.39	0		
49	17.36	0	12.07	0			7.59	0		
50	18.1	0	12.63	0			7.62	0		
51	19.13	0	13.11	0			7.63	0		
52	20.28	0	13.29	0			7.66	0		
53	21.73	0	13.8	0			7.87	0		
54	22.69	0	14.24	0			7.93	0		
55	23.63	0	14.76	0			8.26	0		
56	25.74	0	14.77	0			8.37	0		
57	25.82	0	14.83	0			8.53	0		
58	26.31	0	15.96	0			8.65	0		
59	32.15	0	16.62	0			8.66	0		
60	34.26	0	17.12	0			9.02	0		
61	36.66	0	17.14	0			9.22	0		
62	43.01	0	17.36	0			9.47	0		
63	46.12	0	18.1	0			9.74	0		
64	79.05	0	19.13	0			10.06	0		

TABLE 2.4: Mean and 95 % predictive bounds for future ordered observations from the bladder cancer data set.

One sample prediction			
s	Mean	Bounds	
		l	u
1	79.04829	77.18001	80.46525
2	79.42236	78.31463	80.52509
3	79.59276	78.47721	80.69601
4	79.89351	78.78346	81.01349

TABLE 2.5: Bayes and ML estimates, CI/HPD interval for WPD parameters α, β and λ with pre-defined censoring schemes for the bladder cancer data set.

Scheme	Parameter	MLE	SELF	GELF		CI		HPD	
$S_{n:m}$				$\delta = -1.5$	$\delta = 1.5$	θ_L^C	θ_U^C	θ_L^h	θ_U^h
$S_{128:64}$	α	1.752554	1.753276	1.753308	1.753111	1.722654	1.782455	1.723427	1.78316
	β	0.001915	0.015308	0.016975	0.00012	3.52E-07	0.031815	4.53E-07	0.034268
	λ	4.664404	4.664876	4.665002	4.664246	4.569793	4.759014	4.571824	4.760873
$S_{128:77}$	α	1.690516	1.708283	1.709049	1.704437	1.550406	1.830627	1.569133	1.853984
	β	0.002777	0.080686	0.087632	0.000115	1.26E-07	0.469512	1.37E-07	0.168236
	λ	4.652349	4.671151	4.674187	4.655874	4.185615	5.119084	4.209607	5.1385
$S_{128:102}$	α	1.523263	1.511028	1.513516	1.498465	1.279468	1.767057	1.267033	1.742682
	β	0.006193	0.006015	0.006602	0.000115	4.43E-07	0.013385	8.43E-07	0.012561
	λ	4.499809	4.739747	4.982471	0.314773	1.033094	7.966523	3.946512	6.012466

2.8 Conclusion

On the basis of the discussion of results given in the previous Section, we may conclude that the proposed estimators $\hat{\alpha}_G, \hat{\beta}_G$ and $\hat{\lambda}_G$ perform better than all other considered competitive estimators, for $(\delta > 0)$ i.e., when o.e. is more serious than u.e. and for $(\delta < 0)$, when u.e. is more serious than o.e., under both the loss functions. Thus, the use of the proposed estimator $\hat{\alpha}_G, \hat{\beta}_G$ and $\hat{\lambda}_G$ are recommended under SELF and GELF. Moreover, a brief study has done on the expected experiment time by taking the various combinations of effective parameters n, p and m and it observed that on increases the value of p and m, the expected time to test increases. While, for fixed m, on increases the value of n, the expected time to test decreases. The LR test has performed the goodness of fit. The one sample Bayes prediction has also presented. Furthermore, a real data set is fitted to show the practical applicability of WPD.

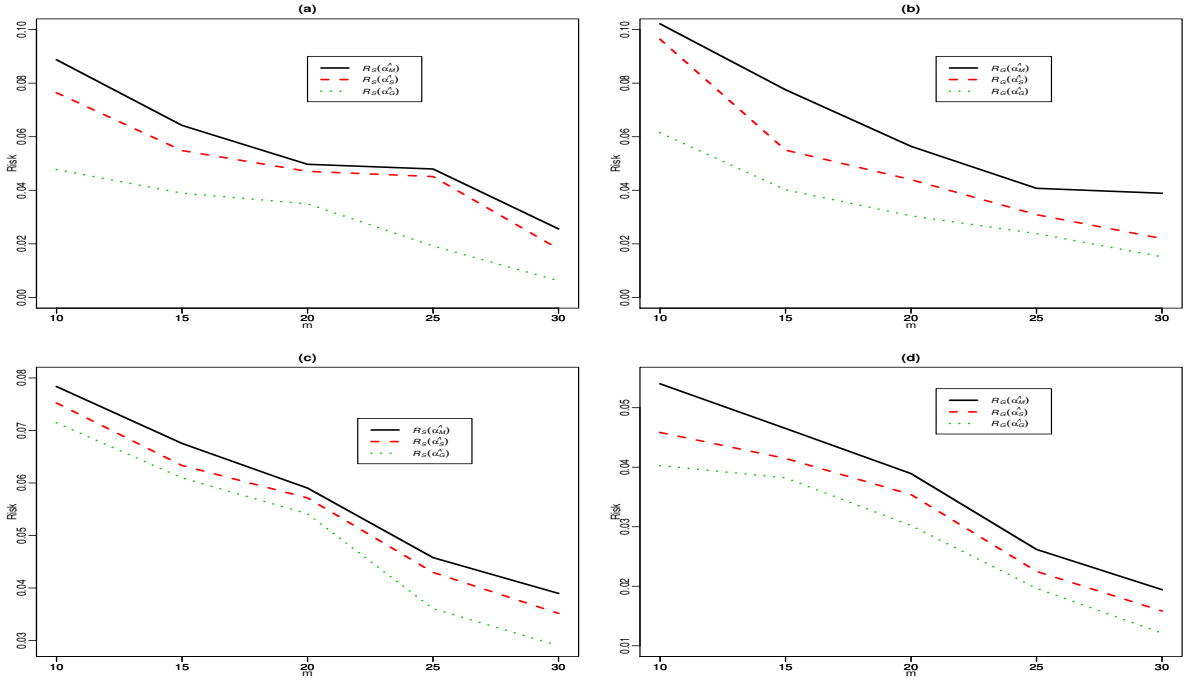

FIGURE 2.2: Risks for the estimators of parameter α for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with small prior variance, $\beta = 0.5$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

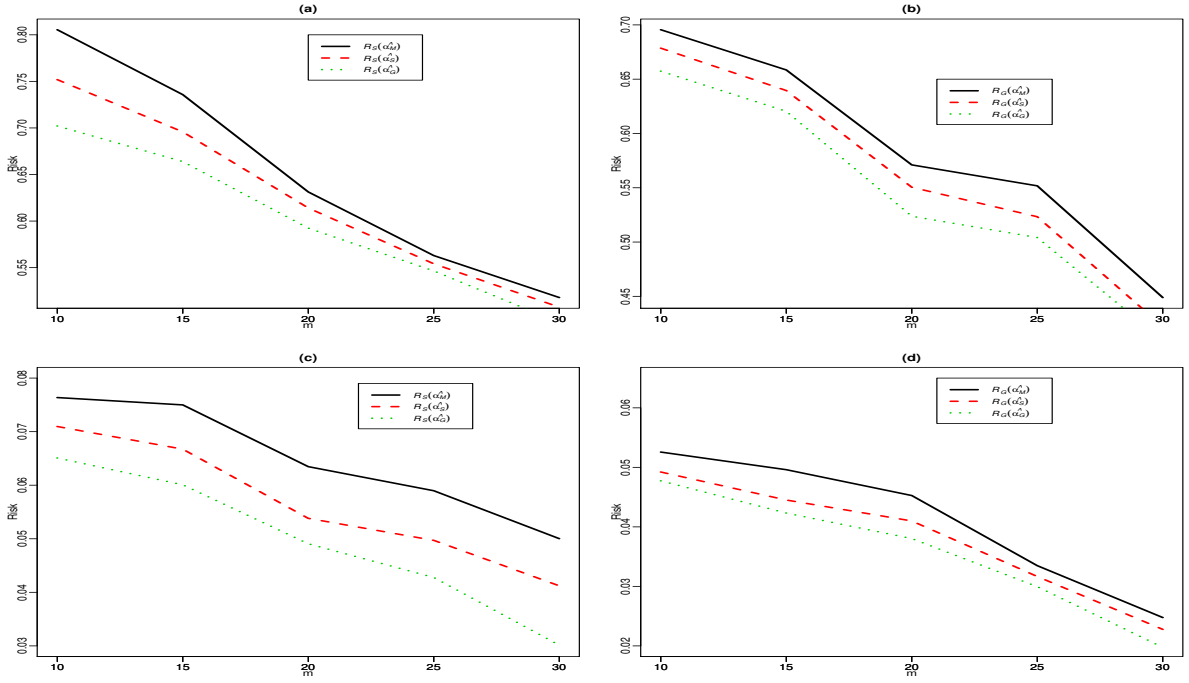

FIGURE 2.3: Risks for the estimators of parameter α for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with moderate prior variance, $\beta = 1$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

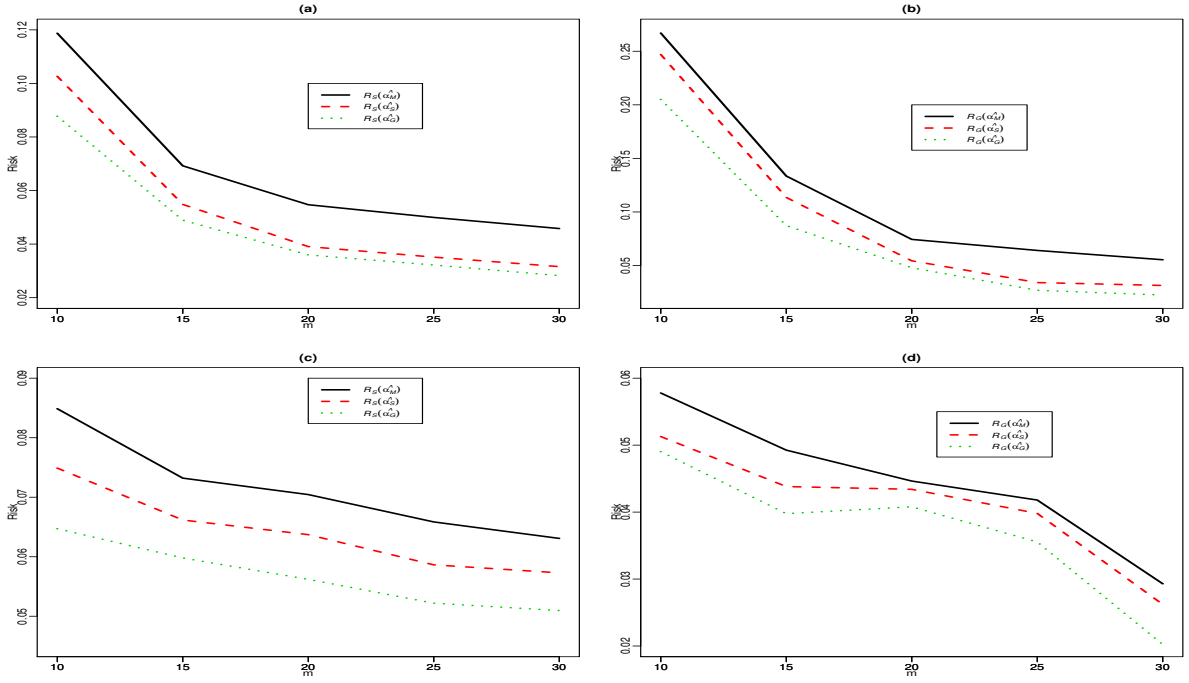

FIGURE 2.4: Risks for the estimators of parameter α for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with high prior variance, $\beta = 5$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

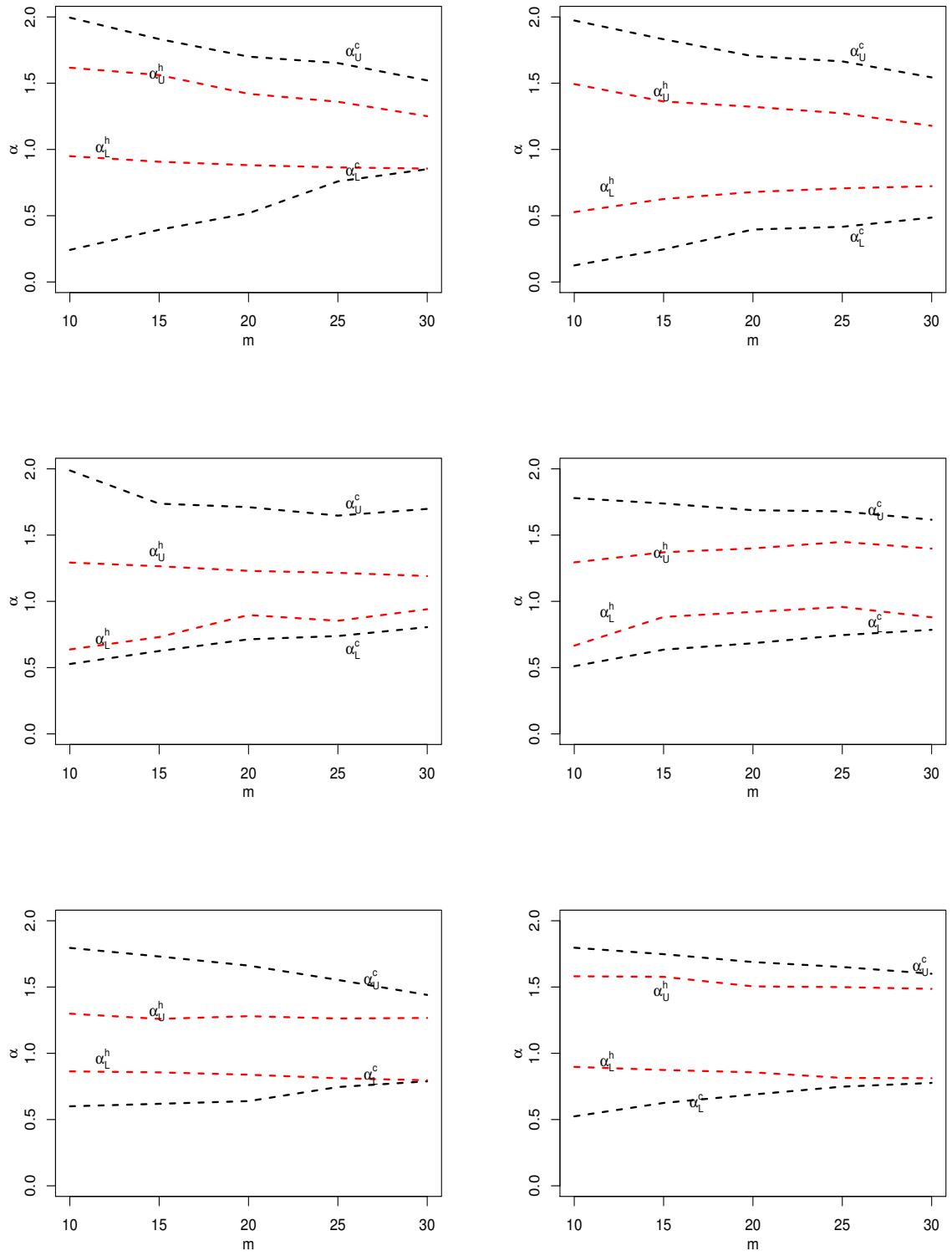

FIGURE 2.5: The CI and HPD intervals for α when prior variance is $0.5, 1$ and 5 with left panel: $\delta = 1.5$; right panel: $\delta = -1.5$, respectively.

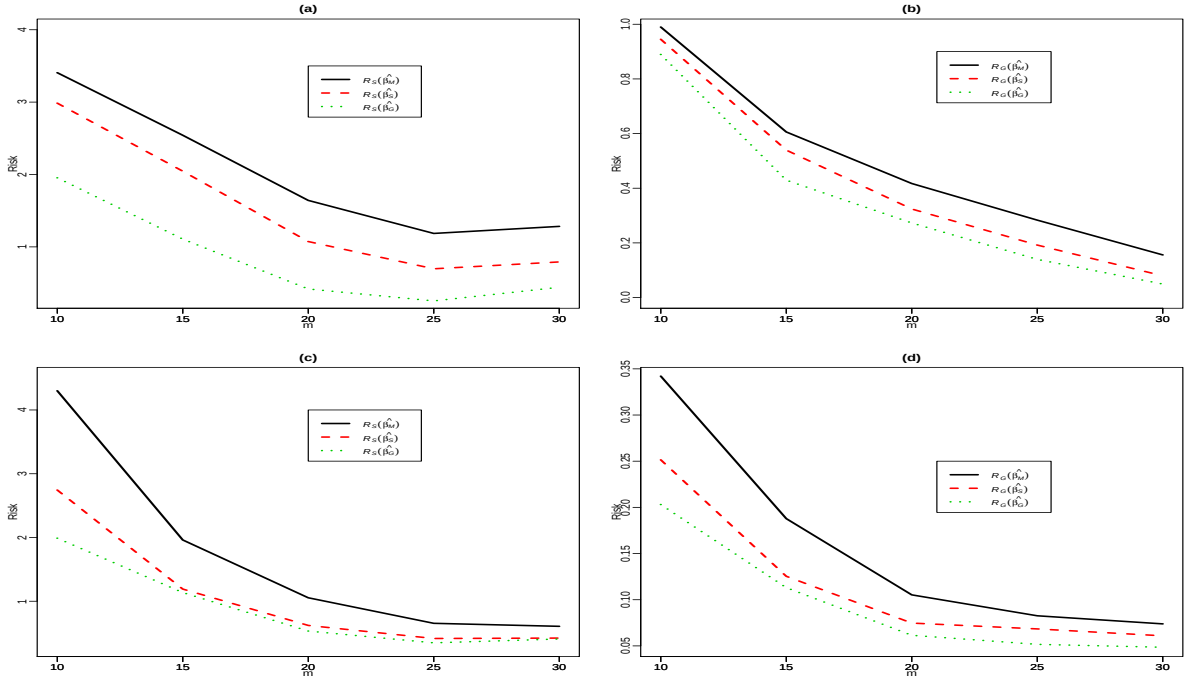

FIGURE 2.6: Risks for the estimators of parameter β for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with small prior variance, $\beta = 0.5$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

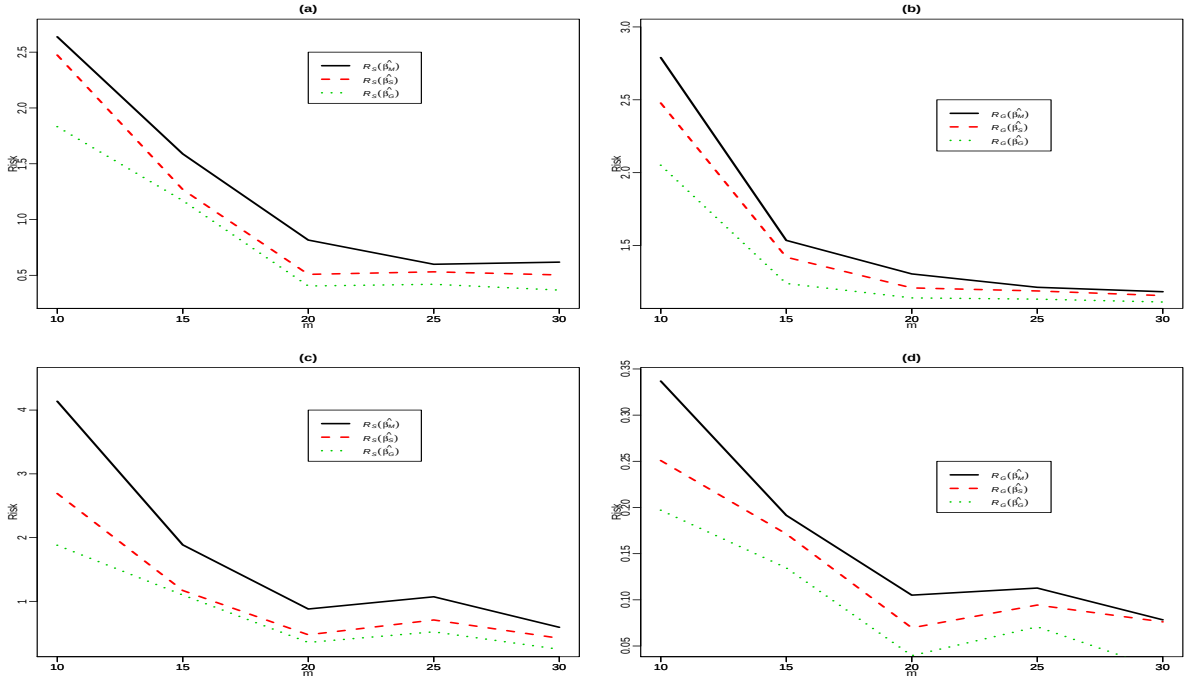

FIGURE 2.7: Risks for the estimators of parameter β for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with moderate prior variance, $\beta = 1$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

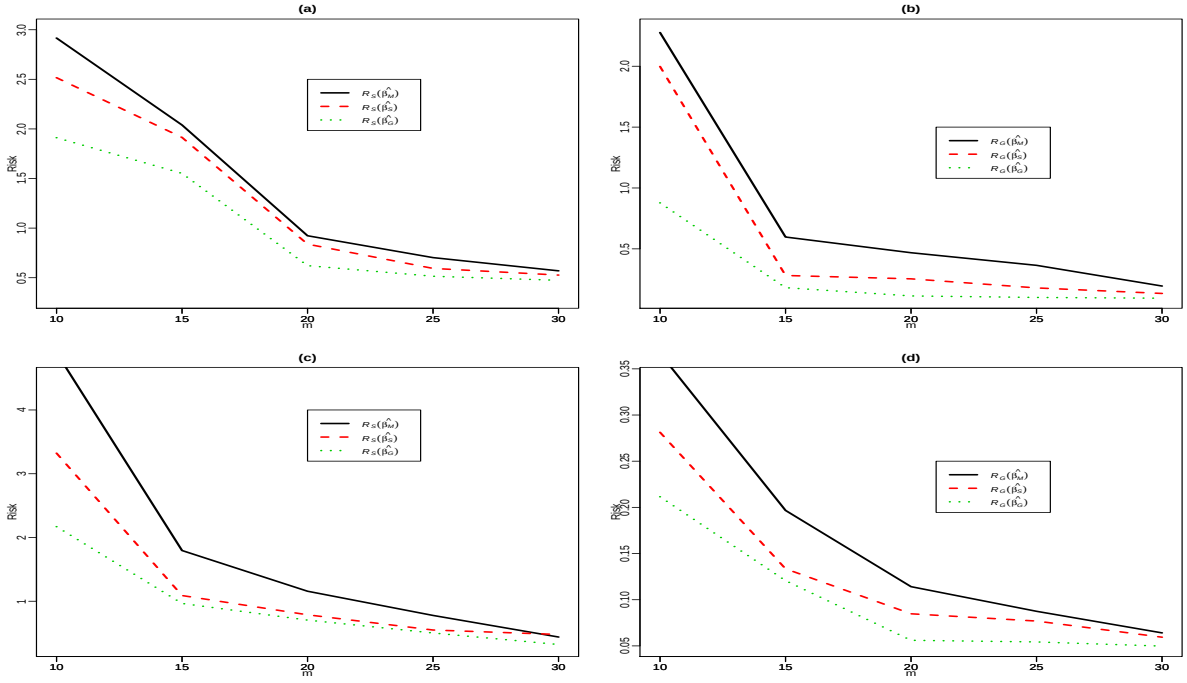

FIGURE 2.8: Risks for the estimators of parameter β for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with high prior variance, $\beta = 5$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

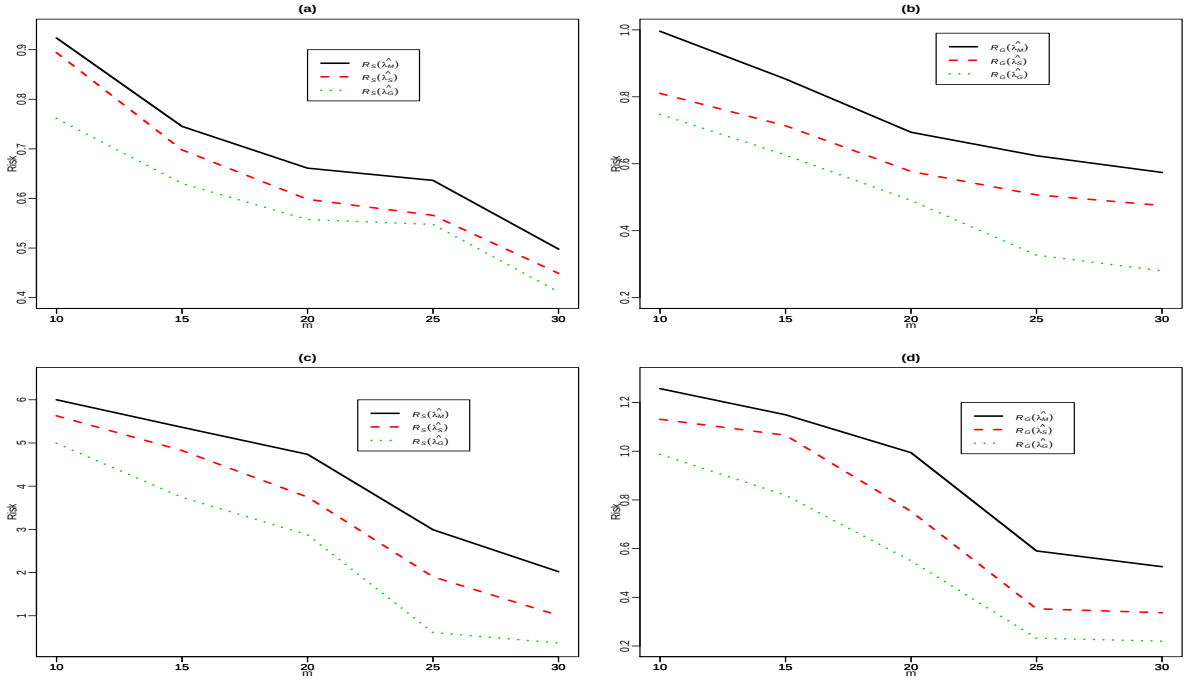

FIGURE 2.9: Risks for the estimators of parameter λ for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with small prior variance, $\beta = 0.5$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

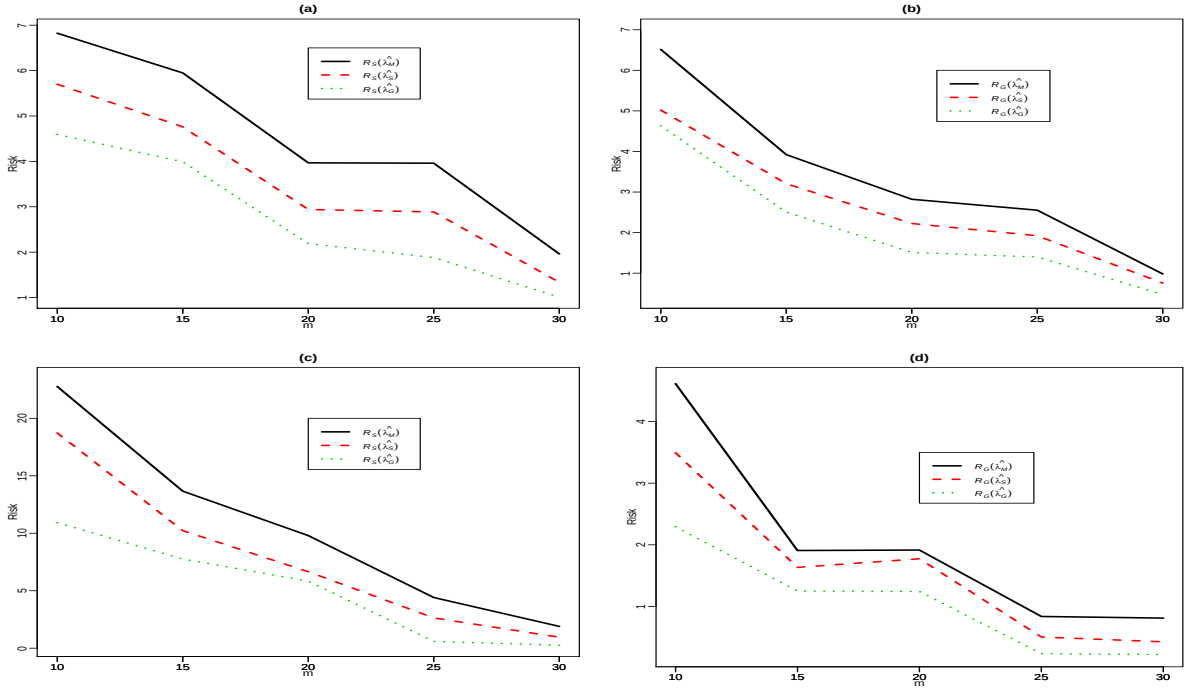

FIGURE 2.10: Risks for the estimators of parameter λ for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with moderate prior variance, $\beta = 1$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

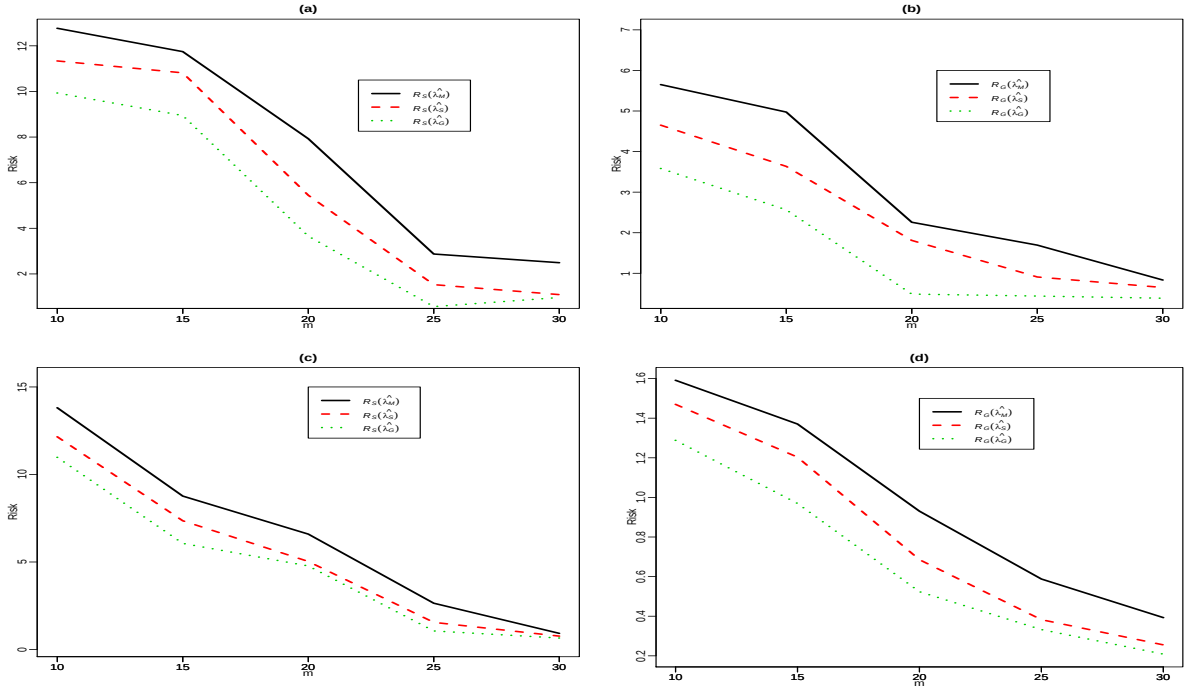

FIGURE 2.11: Risks for the estimators of parameter λ for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with high prior variance, $\beta = 5$; for panels (a) and (b) δ = 1.5; for panels (c) and (d) δ = - 1.5.

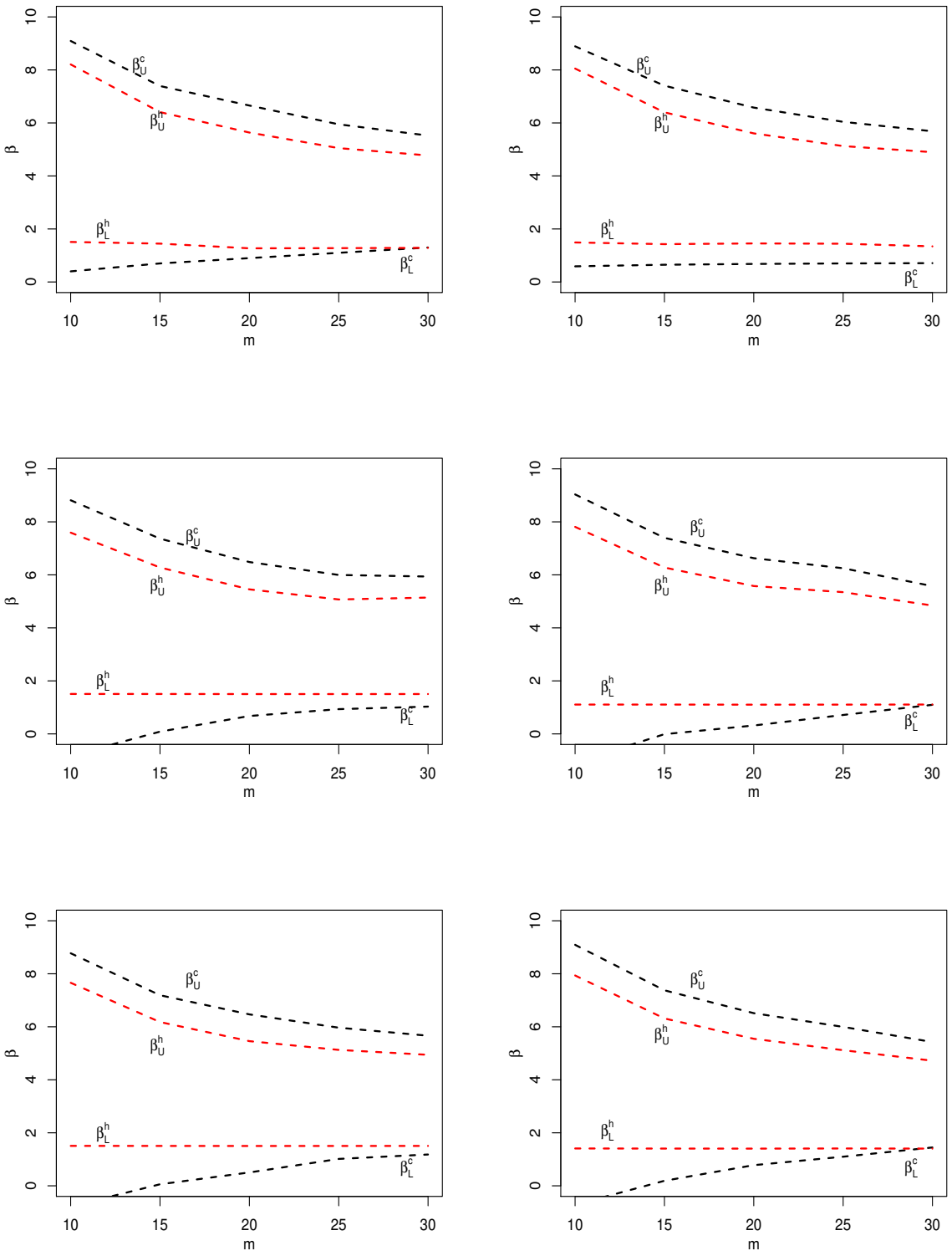

FIGURE 2.12: The CI and HPD intervals for β when prior variance is $0.5, 1$ and 5 with left panel: $\delta = 1.5$; right panel: $\delta = -1.5$, respectively.

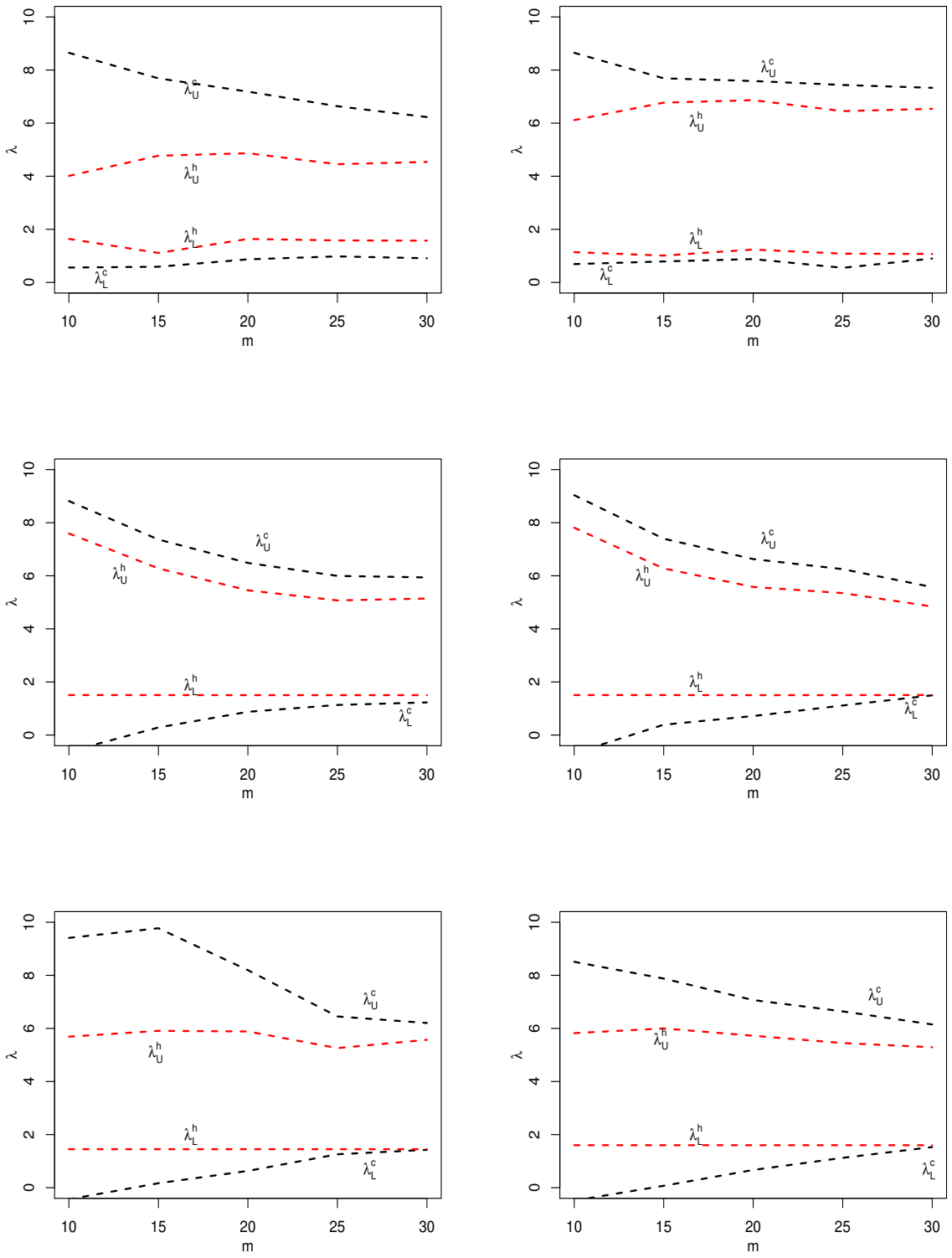

FIGURE 2.13: The CI and HPD interval for λ when prior variance is $0.5, 1$ and 5 with left panel: $\delta = 1.5$; right panel: $\delta = -1.5$, respectively.

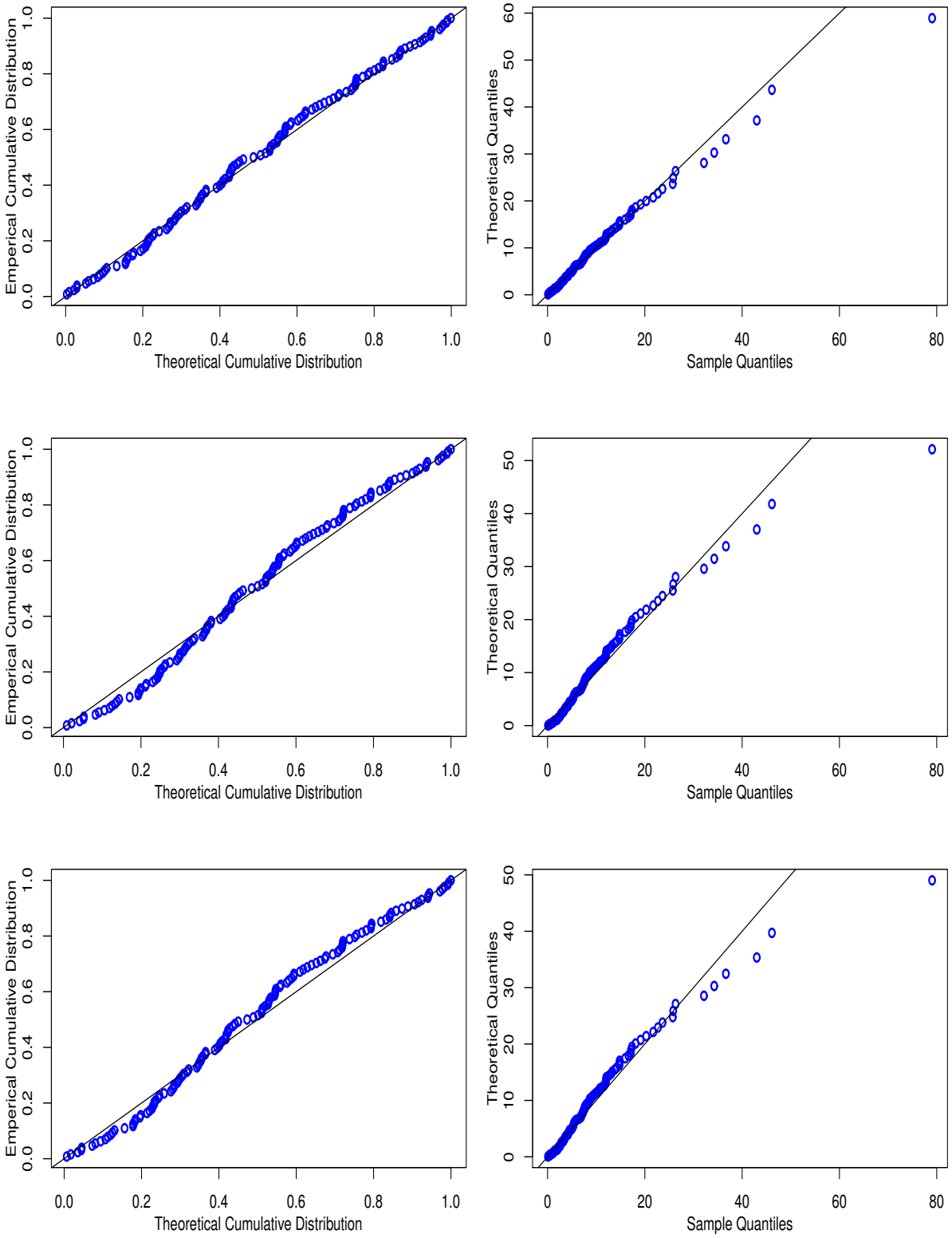

FIGURE 2.14: Top row: WP, Middle row: EP, Last row: WD shows the P-P and Q-Q plot for bladder cancer data set.

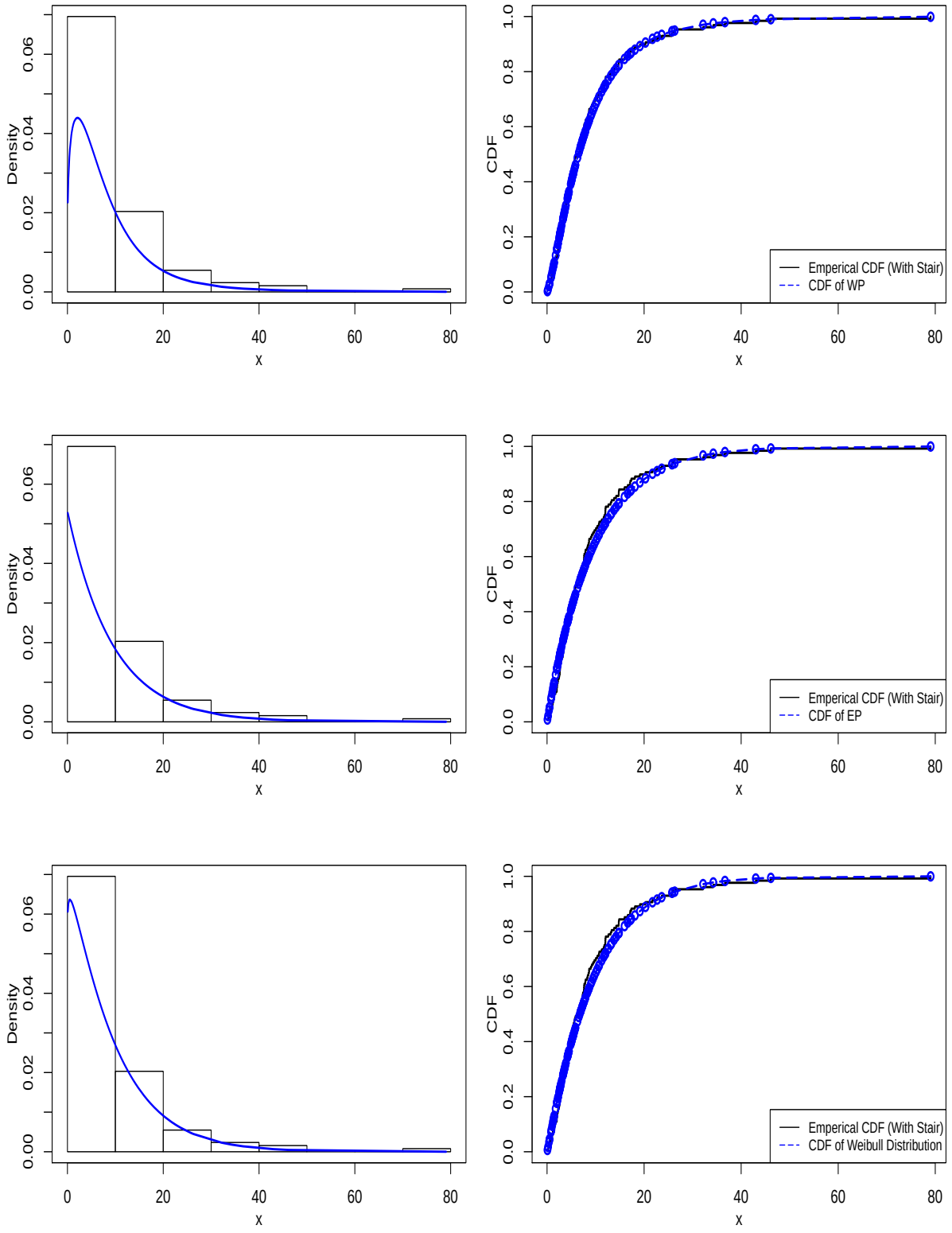

FIGURE 2.15: Top row: WP, Middle row: EP, Last row: WD shows the PDF and CDF Plot of bladder cancer data set.

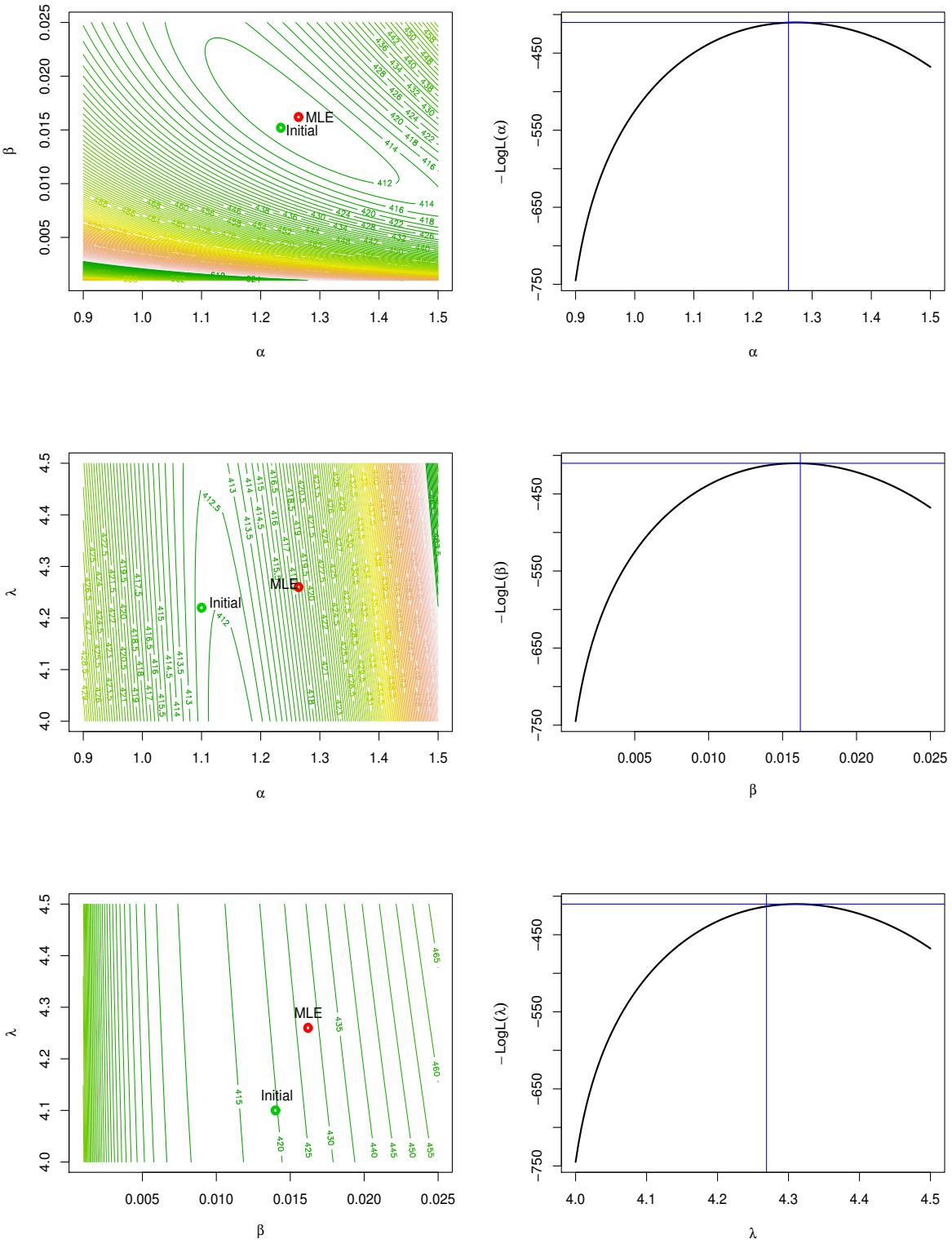

FIGURE 2.16: The Contour and -log-L plot of α, β and λ for bladder cancer data set.