Chapter 5

Classical and Bayesian Estimation in Inverse Pareto Lifetime Model using Progressively First Failure Censored Data

5.1 Introduction

The main objective of this chapter is to develop statistical inferences for the associated parameter and reliability characteristics of the IP lifetime model using the progressively first failure censored (PFFC) data from both a classical and Bayesian perspective.

Because of the severe competition in the market, product reliability is typically improving with the advancement of manufacturing technologies. Generally, in life-testing experiments, observing the failure time for all test units often takes a long time, resulting in a substantial increase in experimental time and cost. As a consequence of the time and cost constraints of the experiments, censoring is a regular phenomenon in reliability and life-testing experiments. Many researchers have investigated the Type-I censoring scheme, in which the life-testing experiment terminates when the experimental period exceeds the prescribed time, and the Type-II censoring scheme, in which the life-testing experiment terminates when the number of recorded failure units meets the intended aim. One of their limitations is that none of them allows control units to be removed during the experiment. It may be required to remove test units in some circumstances. For example, in certain exceptional instances, the unit failure is beyond the control of the experimenters and might be triggered by unforeseen laboratory equipment damage. It's also possible to remove test units from the experiment on purpose to free up laboratory equipment and supplies for other projects, as well as save time and money. Because of such limitations, Cohen (1963) introduced progressive censoring in the literature, which allows the adaptability to remove the test units before they fails from the ongoing experiment. For more details one may refer Balakrishnan and Aggarwala (2000) and Balakrishnan and Cramer (2014).

It may not always be able to fulfill the test's time and cost constraints. As a result, distinct censoring schemes have been introduced one after the other to boost the efficiency of testing procedures. When testing materials are inexpensive, we may conduct the test by putting m groups with k items within each group of n individual items. During this procedure, the first failures in each group are recorded, and the assessment will not be completed until all groups have experienced the first failure. Such a situation of the testing plan was proposed by Balasooriya (1995) called a first-failure censoring scheme.

Furthermore, Wu and Kuş (2009) suggested a novel censoring plan by combining progressive and first failure censoring schemes, known as the progressive first-failure censoring scheme (PFFCS) and data collected by using this scheme is termed as progressively first failure censored (PFFC) data. This censoring scheme bears some special cases to other censoring schemes, due to its compatible features with other censoring plans, this censoring scheme has gained a lot of coverage in literature under multiple scenarios. For example, the estimation of SSR for GIE lifetime model is studied by Krishna et al. (2017), Kayal et al. (2019) developed inferences on Chen lifetime model, Bi et al. (2020) studied bathtub shaped lifetime model for reliability estimation, the statistical inferences for inverse power Lomax lifetime model is discussed by Shi and Shi (2021), estimation of SSR for generalized Maxwell lifetime model is discussed by Saini et al. (2021a), the estimation of multicomponent SSR for Bur Type XII lifetime model is discussed by Saini et al. (2021b) etc.

Practically, the PFFCS is defined as follows: Assume that in a real-life testing experiment, n classes of individuals are being tested at the same time, each with k test units, and they are entirely independent to one another. During the experiment, when the first failure unit, say $X^{G}_{1:m:n:k}$ occurs, the group it belongs to, as well as any G_1 live groups from remaining live $n-1$ groups are randomly discarded from the experiment. Similarly, at the second failure unit, say $X^{G}_{2:m:n:k}$, the group it belongs to, as well as any G_2 live groups in the remaining live $n-2-G_1$ groups, are excluded from the experiment at random. This process is continued until the mth failed unit, say $X^{G}_{m:m:n:k}$ occurs, at that point all remaining G_m live groups are removed from the experiment. Here m and $\underset{\sim}{G} = (G_1, G_2, \ldots, G_m)$ are the prefixed number of failures and censoring schemes, respectively, in such a way that $n = m + \sum\limits_{j=1}^{m} G_j$. Then $X^{G}_{1:m:n:k} < X^{G}_{2:m:n:k} < \cdots < X^{G}_{m:m:n:k}$ are recorded as PFFC ordered sample with prefixed censoring schemes $\underset{\sim}{G} = (G_1, G_2, \ldots, G_m)$. To further demonstrate this censoring scheme, Figure 5.1 depicts the PFFC sample generation procedure. It's worth noting that the PFFCS has the following special cases:

(a) It reduces to complete sample case, when $k = 1, n = m$ and $G_j = 0;\ j = 1,2,\ldots m$.

(b) It become conventional type-II censoring plan, if $k = 1$ and $G_j = (0,\ n-m);\ j = 1,2,\ldots m-1$.

(c) It becomes progressively-II censoring plan, when $k = 1$.

(d) It reduces to first-failure censoring plan, when $G_j = 0;\ j = 1,2,\ldots n$.

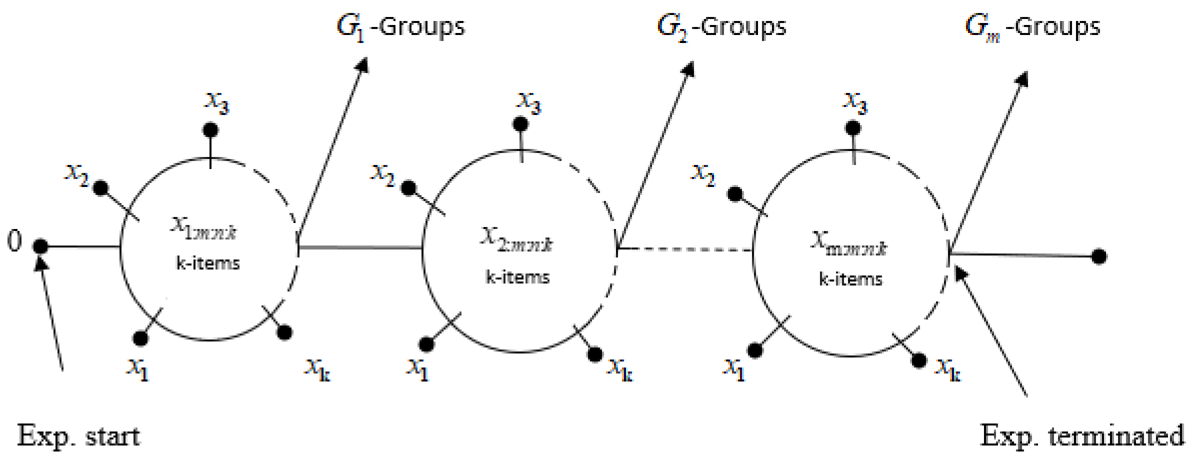


FIGURE 5.1: Schematic diagram of PFFCS.

Suppose the lifetimes of $n \times k$ test units are put on a life testing experiment following a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$, then the joint pdf of $X^G_{1:m:n:k}, X^G_{2:m:n:k}, \ldots, X^G_{m:m:n:k}$ is expressed as, see, (Wu and Kuş, 2009)

$$L\left(x^G_{1:\ m:\ n:\ k},\ x^G_{2:\ m:\ n:\ k},\ \ldots, x^G_{m:\ m:\ n:\ k}\right) = Ak^m \prod_{j=1}^{m} f_X(x^G_{j:\ m:\ n:\ k})\{1 - F_X(x^G_{j:\ m:\ n:\ k})\}^{k(G_j+1)-1},$$
$$0 < x^G_{j:\ m:\ n:\ k} < \infty; \quad \forall j = 1,2,\ldots m \tag{5.1}$$

where, $A = \ n(n-G_1-1)\ (n-G_1-G_2-2)\ \ldots\ (n-G_1-G_2-\ \ldots\ -G_{m-1}\ -m+1)$.

This chapter is organized as follows: The IP lifetime model based on PFFC data is discussed in Section 5.2. In Section 5.3 is devoted to derive MLEs of parameter and reliability characteristics. Also, derived asymptotic and bootstrap CIs for the associated model parameter. The Bayes estimators of parameter and reliability characteristics under SELF using three approximation techniques, namely TK approximation, importance sampling (IS) and M-H algorithm are discussed in Section 5.4. Also, we derived HPD credible interval for the associated model parameter. Section 5.5 presents the numerical computations using Monte Carlo simulations.

For demonstration purpose, a real data analysis is presented in Section 5.6. Finally, a concluding remarks are presented in Section 5.7.

5.2 The Model

The IP lifetime model has already been discussed in the following Chapters 2 and 4 under random censoring and progressive censoring schemes, respectively. Here, we also describe pdf, cdf and reliability characteristics of IP lifetime model for quick response under consideration of this chapter. The pdf and cdf of IP lifetime model with parameter θ, respectively, are given by

$$f_X(x;\theta) = \frac{\theta x^{\theta-1}}{(1+x)^{\theta+1}} \quad ;\, \theta > 0,\, x > 0, \tag{5.2}$$

and

$$F_X(x) = \left(\frac{x}{1+x}\right)^{\theta}; \quad \theta \geq 0, \;\; x > 0. \tag{5.3}$$

Also, the corresponding reliability (or survival) and hazard (pr failure rate) functions of IP lifetime model, respectively, are given by

$$R(x;\theta) = 1 - \left(\frac{x}{1+x}\right)^{\theta} \quad ;\, \theta > 0,\, x > 0, \tag{5.4}$$

and

$$h(x;\theta) = \frac{\theta x^{\theta-1}}{(1+x)^{\theta+1}\left[1-\left(\frac{x}{1+x}\right)^{\theta}\right]} \quad ;\, \theta > 0, x > 0. \tag{5.5}$$

As the moment of IP lifetime model does not in closed form, therefore, we consider median time to system failure (MdTSF) and given as

$$MdTSF = \frac{1}{2^{1/\theta}-1} \;\; ;\, \theta > 0. \tag{5.6}$$

5.3 Classical Estimation

In case of classical estimation method, the associated parameter and reliability characteristics are estimated by using ML estimation, asymptotic confidence and bootstrap confidence intervals methods.

5.3.1 Maximum Likelihood Estimation

This section is devoted to derive ML estimates of the associated model parameter θ and reliability characteristics $R(t)$, $h(t)$ and $MdTSF$, respectively. Also, obtain asymptotic and bootstrap CIs of θ. Let $x^{\underset{\sim}{G}}_{i:m:n:k}$; $\quad i = 1,2,\ldots,m$, be the PFFC sample from IP lifetime model with prefixed number of failures m and censoring plan $\underset{\sim}{G} = (G_1, G_2, \ldots, G_m)$. For notation simplicity, hereafter we use $\underset{\sim}{x} = (x_1, x_2, \ldots, x_m)$ as PFFC sample. Then, using (6.6), (6.2) and (5.4), the likelihood function becomes

$$L(\underset{\sim}{x},\theta) = Ak^m\theta^m \prod_{i=1}^{m} \frac{x_i^{\theta-1}}{(1+x_i)^{\theta+1}} \left[1-\left(\frac{x_1}{1+x_i}\right)^{\theta}\right]^{k(G_i+1)-1} \tag{5.7}$$

where, $A = n(n-G_1-1)(n-G_1-G_2-2)\ldots(n-G_1-G_2-\cdots-G_{m-1}-m+1)$. The log-likelihood function is obtained as

$$l(\underset{\sim}{x},\theta) = C + m\ln\theta + \theta\sum_{i=1}^{m}\ln\left(\frac{x_i}{1+x_i}\right) + \sum_{i=1}^{m}[k(G_i+1)-1]\ln\left[1-\left(\frac{x_i}{1+x_i}\right)^{\theta}\right], \tag{5.8}$$

where, $C = \ln A + m\ln k - \sum\limits_{i=1}^{m}\ln[x_i(1+x_i)]$. The solution of the following normal equation of log-likelihood yields the ML estimate of θ,

$$\frac{\partial l(\underset{\sim}{x},\theta)}{\partial\theta} = \frac{m}{\theta} + \sum_{i=1}^{m}\ln\left(\frac{x_i}{1+x_i}\right) - \sum_{i=1}^{m}[k(G_i+1)-1]\frac{\left(\frac{x_i}{1+x_i}\right)^{\theta}\ln\left(\frac{x_i}{1+x_i}\right)}{\left[1-\left(\frac{x_i}{1+x_i}\right)^{\theta}\right]} = 0. \tag{5.9}$$

Here, the ML estimate of θ is the solution of (5.9), and because the closed form solution for (5.9) is not accessible, a suitable numerical iterative technique can be employed to compute the ML estimate of θ numerically. Once we get the ML estimate of θ say $\hat{\theta}$, then using the invariance property of ML estimation, we can obtain the ML estimates of $R(t)$, $h(t)$, and $MdTSF$, respectively.

$$\hat{R}(t) = 1-\left(\frac{t}{1+t}\right)^{\hat{\theta}}, \tag{5.10}$$

$$\hat{h}(t) = \frac{\hat{\theta}t^{\hat{\theta}-1}}{(1+t)^{\hat{\theta}+1}\left[1-\left(\frac{t}{1+t}\right)^{\hat{\theta}}\right]}. \tag{5.11}$$

$$\widehat{MdTSF} = \frac{1}{2^{1/\hat{\theta}}-1}. \tag{5.12}$$

Now, under mild regularity constraints, the MLE of θ is asymptotically normally distributed i.e. $\hat{\theta} \sim N(\theta, I^{-1}(\hat{\theta}))$, where $I(\hat{\theta})$ is the observed Fisher information,

$$I(\hat{\theta}) = E\left[-\frac{\partial^2 l(x,\theta)}{\partial \theta^2}\right]_{\theta=\hat{\theta}}, \tag{5.13}$$

with,

$$\frac{\partial^2 l(\underset{\sim}{x},\theta)}{\partial \theta^2} = -\frac{m}{\theta^2} - \sum_{i=1}^{m} [k(G_i+1)-1]\left\{\ln\left(\frac{x_i}{1+x_i}\right)\right\}^2 \frac{\left(\frac{x_i}{1+x_i}\right)^{\theta}}{\left[1-\left(\frac{x_i}{1+x_i}\right)^{\theta}\right]^2}$$

Suppose $\hat{V}ar(\hat{\theta}) = I^{-1}(\hat{\theta})$ is the observed variance of $\hat{\theta}$, the asymptotic CI of θ can be obtained as

$$\hat{\theta} \pm z_{\xi/2}\sqrt{\hat{V}ar(\hat{\theta})},$$

here, $z_{\xi/2}$ is the upper $(\xi/2)^{th}$ percentile of N(0,1). Also, the coverage probability (CP) for θ is given by

$$CP_{\theta} = \left[\left|\frac{\hat{\theta}-\theta}{\sqrt{\hat{V}ar(\hat{\theta})}}\right| \leq z_{\xi/2}\right].$$

5.3.2 Bootstrap Confidence Intervals

In literature, Efron (1979) was the first who developed the bootstrap approach. This technique employs as a simple resampling technique that permits inferential statistics to be constructed, when samples are not sufficiently large or need heavy assumptions about the underlying distribution. Later on this concept has been applied in several applications. For more details one may refer Efron (1982), Hall (1988), Davison and Hinkley (1997). In the literature, several bootstrap techniques have been developed. In this chapter, we employ two bootstrap techniques as percentile bootstrap (boot-p) and Student's t bootstrap (boot-t) based on t-statistic to construct bootstrap CIs of the associated parameter θ of IP lifetime model. In order to compute two parametric bootstrap CIs of θ, the following steps are used as follows:

5.3.2.1 Percentile Bootstrap (boot-p) Confidence Interval

Step 1: Produce a PFFC sample $x = (x_1, x_2, \ldots, x_m)$ from the IP lifetime model with a prefixed censoring scheme $\underset{\sim}{G} = (G_1, G_2, \ldots, G_m)$ and an effective sample size of m, and then compute the ML estimate $\hat{\theta}$ of θ

Step 2: Produce an independent bootstrap PFFC sample, say $\underset{\sim}{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)$ using $\hat{\theta}$.

Then, obtain the bootstrap ML estimate, say $\hat{\theta}^*$ of θ based on the generated bootstrap sample $\underset{\sim}{x}^*$.

Step 3: Replicate the Step 2, B times to generate a sequence of bootstrap ML estimates $\hat{\theta}_i^*$; $\quad i = 1,2,\ldots,B$.

Step 4: Let $\hat{\theta}^*_{(1)} \leq \hat{\theta}^*_{(2)} \leq \cdots \leq \hat{\theta}^*_{(B)}$ denote the ordered values of $\hat{\theta}_i$ for $i = 1,2,\ldots,B$. The approximate $100(1-\alpha)\%$ boot-p CI of θ is given by $\left(\hat{\theta}^*_{[(\alpha/2)\times B]}, \hat{\theta}^*_{[(1-\alpha/2)\times B]}\right)$, where $[a]$ is the integral part of a.

5.3.2.2 Student's t Bootstrap (boot-t) Confidence Interval

Step 1 and Step 2 are same as in boot-p procedure.

Step 3: Obtain the boot-t statistic $\tau^* = \frac{\hat{\theta}^*-\hat{\theta}}{\sqrt{I^{-1}(\hat{\theta}^*)}}$ for $\hat{\theta}^*$.

Step 4: Replicate steps 2-3, B times to generate a sequence of boot-t statistics τ_i^*; $\ i = 1,2,\ldots,B$.

Step 5: Suppose $\tau^*_{(1)} \leq \tau^*_{(2)} \leq \cdots \leq \tau^*_{(B)}$ be the ordered values of τ_i^* for $i = 1,2,\ldots,B$.
Thus, the approximate $100(1-\alpha)\%$ boot-t CI of θ is given by

$$\left(\hat{\theta} - \tau^*_{[(1-\alpha/2)\times B]}\sqrt{I^{-1}(\hat{\theta}^*)}, \hat{\theta} - \tau^*_{[(\alpha/2)\times B]}\sqrt{I^{-1}(\hat{\theta}^*)}\right).$$

5.4 Bayesian Estimation

This part focuses on developing Bayesian estimate methods for unknown parameters and reliability characteristics of the IP lifetime model using PFFC data under SELF. Let us consider the prior belief of an unknown parameter θ is measured to follow a gamma distribution with hyper-parameters a and b, and the corresponding pdf of prior belief is termed as

$$p(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}; \ \ \theta > 0, \ a,b > 0.$$

Therefore, by incorporating prior belief in maximum likelihood function in (5.7), the posterior distribution become

$$\pi(\theta|\underline{x}) \propto \theta^{m+a-1}\exp\left\{-\theta\left(b - \sum_{i=1}^{m}\ln\left(\frac{x_i}{1+x_i}\right)\right)\right\}\exp\left\{[k(G_i+1)-1]\ln\left[1-\left(\frac{x_i}{1+x_i}\right)^{\theta}\right]\right\}. \tag{5.14}$$

The posterior mean under SELF is the Bayes estimator of every parametric function. Here, it can be seen that the posterior distribution does not belong to any well known family of distributions, so it is quite difficult to obtain the posterior mean. In addition, the ideal posterior means are ratios of two integrals that cannot be simplified in some expressions of the closed form. In order to solve these integrals, we proposed using the following approximation methods: The TK approximation, IS, and M-H algorithm techniques.

5.4.1 TK Approximation

Here, the TK approximation procedure is used to compute the point Bayes estimates of the parameter and reliability characteristics. For the parametric function $\phi(\theta)$, the posterior mean is given as follows:

$$J(x) = \frac{\int_0^\infty \phi(\theta)\exp\{L(\underset{\sim}{x},\theta)+\rho(\phi)\}d\theta}{\int_0^\infty \exp\{L(\underset{\sim}{x},\theta)+\rho(\phi)\}d\theta}, \tag{5.15}$$

where, $L(\underset{\sim}{x},\theta)$ is log-likelihood function and $\rho(\theta) = \ln p(\theta)$. Using TK approximation, we can write $J(x)$ as an explicit form, we have $\delta(\theta) = \frac{L(\underset{\sim}{x},\theta)+\rho(\theta)}{mk}$ and $\delta^*(\theta) = \delta(\theta) + \frac{\ln\phi(\theta)}{mk}$, and assume that $\hat{\phi}_\delta(\theta)$ and $\hat{\phi}_{\delta^*}(\theta)$ maximizes the functions $\delta(\theta)$ and $\delta^*(\theta)$, respectively. Then according to the TK approximation method $J(x)$ can be described as

$$J(x) = \left(\frac{\det(\Delta_\phi^*)}{\det(\Delta_\phi)}\right)^{\frac{1}{2}} \exp\left[mk\left\{\delta_\phi^*(\hat{\theta}_{\delta^*}) - \delta(\hat{\theta})\right\}\right]. \tag{5.16}$$

Here, we need to compute $\det(\Delta_\phi^*)$ and $\det(\Delta_\phi)$ which is the determinants of negative inverse Hessian of $\delta^*(\theta)$ and $\delta_\phi(\theta)$. By incorporating prior distribution to the log-likelihood function, the Bayes estimator of θ using the TK approximation is computed, and $\delta(\theta)$ is given as

$$\delta(\theta) = \frac{1}{mk}\left[(m+a-1)\ln\theta - \theta\left\{b - \sum_{i=1}^m \ln\left(\frac{x_i}{1+x_1}\right)\right\} + \sum_{i=1}^m [k(G_i+1)-1]\ln\left\{1-\left(\frac{x_i}{1+x_i}\right)^\theta\right\}\right]$$

Therefore, $\hat{\theta}_\delta$ is computed by solving the following non-linear equation

$$\frac{\partial\delta(\theta)}{\partial\theta} = \frac{m+a-1}{\theta} - b + \sum^m \ln\left(\frac{x_i}{1+x_i}\right) - \sum_{i=1}^m [k(G_i+1)-1]\frac{\left(\frac{x_i}{1+x_i}\right)^\theta \ln\left(\frac{x_i}{1+x_i}\right)}{\left[1-\left(\frac{x_i}{1+x_i}\right)^\theta\right]} = 0.$$

Also,

$$\frac{\partial^2 \delta(\theta)}{\partial \theta^2} = \frac{1}{mk}\left\{-\frac{m+a-1}{\theta^2} - \sum_{j=1}^{m_2}[k(G_i+1)-1]\frac{\left\{\ln\left(\frac{x_i}{1+x_i}\right)\right\}^2\left(\frac{x_i}{1+x_i}\right)^{\theta}}{\left[1-\left(\frac{x_i}{1+x_i}\right)^{\theta}\right]^2}\right\},$$

Hence,

$$\det(\Delta_\phi) = \left|\frac{\partial^2\delta(\theta)}{\partial\theta^2}\right|^{-1}_{\theta=\hat{\theta}}. \tag{5.17}$$

Since, $\delta^*(\theta)$ is the function of $\phi(\theta)$, the Bayes estimator of $\phi(\theta)$ is computed by considering

$$\delta^*(\theta) = \delta(\theta) + \frac{\ln\phi(\theta)}{mk}. \tag{5.18}$$

Now, for $\phi(\theta) = \theta$, then $\hat{\theta}^*$ is computed by solving the following equation

$$\frac{\partial\delta^*(\theta)}{\partial\theta} = \frac{\partial\delta(\theta)}{\partial\theta} + \frac{1}{mk}\frac{1}{\theta} = 0. \tag{5.19}$$

Also, using the derivative

$$\frac{\partial^2\delta^*(\theta)}{\partial\theta^2} = \frac{\partial^2\delta(\theta)}{\partial\theta^2} - \frac{1}{mk}\frac{1}{\theta^2}, \tag{5.20}$$

we get $\det(\Delta^*_\theta)$ as

$$\det(\Delta^*_\theta) = \left|\frac{\partial^2\delta^*(\theta)}{\partial\theta^2}\right|^{-1}_{\theta=\hat{\theta}^*}.$$

Thus, the Bayes estimator of θ is finally obtained by

$$\hat{\theta}_{TK} = \left(\frac{\det(\Delta^*_\theta)}{\det(\Delta_\theta)}\right)^{\frac{1}{2}} \exp\left[mk\left\{\delta^*_\theta(\hat{\theta}_{\delta^*}) - \delta(\hat{\theta}_\delta)\right\}\right].$$

Similarly, the Bayes estimators of reliability characteristics $R(t), h(t)$, and $MdTSF$, respectively are given as follows

$$\hat{R}_{TK}(t) = \left(\frac{\det(\Delta^*_{R(t)})}{\det(\Delta_{R(t)})}\right)^{\frac{1}{2}} \exp\left[mk\left\{\delta^*_{R(t)}(\hat{R}_{\delta^*(t)}) - \delta(\hat{R}_\delta(t))\right\}\right],$$

$$\hat{h}_{TK}(t) = \left(\frac{\det(\Delta^*_{h(t)})}{\det(\Delta_{h(t)})}\right)^{\frac{1}{2}} \exp\left[mk\left\{\delta^*_{h(t)}(\hat{h}_{\delta^*(t)}) - \delta(\hat{h}_\delta(t))\right\}\right]$$

and

$$\widehat{MdTSF}_{TK}(t) = \left(\frac{\det(\Delta^*_{MdTSF})}{\det(\Delta_{MdTSF})}\right)^{\frac{1}{2}} \exp\left[mk\left\{\delta^*_{MdTSF}(\widehat{MdTSF}_{\delta^*}) - \delta(\widehat{MdTSF}_{\delta})\right\}\right].$$

5.4.2 Importance Sampling Technique

The importance sampling (IS) approach is used to find the Bayes estimator of the parameter and reliability characteristics under SELF. The posterior distribution described in (5.14) can be rewritten as

$$\pi(\theta|\underset{\sim}{x}) \propto f_{GA}(\theta; m+a, S)\, U(\theta) \tag{5.21}$$

where, $S = \left[b - \sum_{i=1}^{m} \ln\left(\frac{x_i}{1+x_i}\right)\right]$ and $U(\theta) = \exp\left\{\sum_{i=1}^{m} [k(G_i+1)-1] \ln\left[1-\left(\frac{x_i}{1+x_i}\right)^{\theta}\right]\right\}$ and $f_{GA}(.; p, q)$ is a gamma density with shape p and scale q parameters, respectively. Under SELF, the Bayes estimator of $\phi(\theta)$, a function of θ, is now given by

$$\hat{\phi}_{IS}(\theta) = E[\phi(\theta)|\underset{\sim}{x}] = \frac{\int_0^{\infty} \phi(\theta)\pi(\theta|\underset{\sim}{x})d\theta}{\int_0^{\infty} \pi(\theta|\underset{\sim}{x})d\theta}. \tag{5.22}$$

Therefore, we do not need to compute the normalizing constant to approximate $\hat{\phi}IS(\theta)$ given in (5.22) using the IS techniques. The steps below are used for programming purposes:

Step 1: Produce $\theta^{(1)}$ from $f_{GA}(\theta; m+a, b+S)$.

Step 2: To obtain importance sample, repeat the above Step 1, M times, $(\theta^{(1)}), (\theta^{(2)}), \ldots, (\theta^{(M)})$.

Now we can obtain the approximate Bayes estimates of the function of parameter $\phi(\theta)$ as follows:

$$\hat{\phi}_{IS}(\theta) = \frac{\sum_{j=1}^{M} \phi(\theta^{(j)}) U(\theta^{(j)})}{\sum_{j=1}^{M} U(\theta^{(j)})}. \tag{5.23}$$

Hence, the Bayes estimates of parameter and reliability characteristics under SELF using IS method are, respectively given by

$$\hat{\theta}_{IS} = \frac{\sum_{j=1}^{M} \theta^{(j)} U(\theta^{(j)})}{\sum_{j=1}^{M} U(\theta^{(j)})}, \quad \hat{R}_{IS}(t) = \frac{\sum_{j=1}^{M} \left(1 - \left(\frac{1}{1+t}\right)^{\theta^{(j)}}\right) U(\theta^{(j)})}{\sum_{j=1}^{M} U(\theta^{(j)})}; \quad t > 0,$$

$$\hat{h}_{IS}(t) = \frac{\sum_{j=1}^{M} \frac{\theta^{(j)} t^{\theta^{(j)}-1}}{(1+t)^{\theta^{(j)}+1}\left(1-\left(\frac{t}{1+t}\right)^{\theta^{(j)}}\right)} U(\theta^{(j)})}{\sum_{j=1}^{M} U(\theta^{(j)})}; \quad t > 0, \quad \widehat{MdTSF} = \frac{\sum_{j=1}^{M} \frac{1}{2^{\frac{1}{\theta^{(j)}}}-1} U(\theta^{(j)})}{\sum_{j=1}^{M} U(\theta^{(j)})}.$$

5.4.3 Metropolis-Hastings Algorithm

Here, we consider one of the popular MCMC technique as M-H algorithm to compute the Bayes estimates of parameter and reliability characteristics. We take candidate point from a normal distribution to draw samples from the posterior distribution of $\theta|\underset{\sim}{x}$ from (5.14). For programming or computation purposes, the following steps are carried out:

Step 1: Consider initial guess value of θ say $\theta^{(0)}$.

Step 2: From the proposal density $\eta(\theta^{(j)}|\theta^{(j-1)})$, generate a candidate point $\theta_c^{(j)}$.

Step 3: Generate u using a uniform distribution $U(0,1)$.

Step 4: Obtain $A\left(\theta_c^{(j)}|\theta^{(j-1)}\right) = \min\left\{\frac{\pi\left(\theta_c^{(j)}|\underset{\sim}{x}\right)\eta\left(\theta^{(j-1)}|\theta_c^{(j)}\right)}{\pi\left(\theta^{(j-1)}|\underset{\sim}{x}\right)\eta\left(\theta_c^{(j)}|\theta^{(j-1)}\right)}, 1\right\}$.

Step 5: If $u \leq A$ set $\theta^{(j)} = \theta_c^{(j)}$ with acceptance rate A otherwise $\theta^{(j)} = \theta^{(j-1)}$.

Step 6: In order to compute the parameter sequence of θ, repeat steps 1-5, for $j = 1, 2, \ldots, M$, say $\left\{\theta^{(1)}, \theta^{(2)}, \theta^{(3)}, \ldots, \theta^{(M)}\right\}$.

Using the $(M - M_0)$ observations, where M_0 is the burn-in period, we get an estimate. Hence, the approximate Bayes estimate using M-H algorithm procedure under SELF is given by

$$\hat{\phi}_{MH}(\theta) = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \phi(\theta^{(j)}).$$

Thus, using the M-H algorithm, the Bayes estimates of the parameter θ and the reliability characteristics $R(t), h(t)$, and $MdTSF$ are computed under SELF as follows:

$$\hat{\theta}_{MH} = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\theta^{(j)}, \quad \hat{R}_{MH}(t) = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\left[1-\left(\frac{t}{1+t}\right)^{\theta^{(j)}}\right],$$

$$\hat{h}_{MH}(t) = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\frac{\theta^{(j)}t^{\theta^{(j)}-1}}{(1+t)^{\theta^{(j)}+1}\left[1-\left(\frac{t}{1+t}\right)^{\theta^{(j)}}\right]}, \quad \widehat{MdTSF} = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\frac{1}{(2^{1/\theta^{(j)}}-1)}.$$

5.4.4 HPD Credible Interval

In this subsection, the HPD credible interval of θ can be obtained using generated MCMC sample. Suppose $\theta_{(1)} < \theta_{(2)} < \cdots < \theta_{(M)}$ denotes the ordered values of $\theta^{(1)},\ \theta^{(2)}, \ldots, \theta^{(M)}$. Thus, $100(1-\xi)\%$, where, $0 < \xi < 1$, HPD credible interval of θ is given by $\left(\theta_{(j)},\ \theta_{(j+[(1-\xi)M])}\right)$, where j is chosen such that

$$\theta_{(j+[(1-\xi)M])} - \theta_{(j)} = \min_{1\le i\le \xi M}\left(\theta_{(i+[(1-\xi)M])} - \theta_{(j)}\right), j = 1, 2, \ldots, M,$$

where, $[x]$ is the integer part of x.

5.5 Numerical Computations

To analyze the impact of the different estimators produced in this chapter, extensive numerical computations are done in this section. The estimators are compared with their corresponding average estimates (AE) and mean squared errors (MSE). For computations, first of all we generated PFFC samples for different combinations of (k, n, m) with prefixed censoring plans $\underset{\sim}{G}$ and distinct values of a model parameter θ. To generate PFFC samples, we use the algorithm suggested by Balakrishnan and Sandhu (1995) with some modifications in such a way that, the PFFC sample $x_1, x_2, \ldots, x_m$ can be viewed as a progressively censored sample from a population with cdf $(1-(1-F(x))^k)$, see, Wu and Kuş (2009). To see the behaviour of estimation methods, the following parameters are taken as follows: number of items within each group $k = 3, 5$, number of groups $n = 20, 30$ and prefixed number of failures $m = (80, 100)\%$ of n with prefixed censoring plans $\underset{\sim}{G}$, respectively. Also, two sets of parameter values are taken as $\theta = 0.5$ and $\theta = 1.5$, respectively. For each n, four different failure plans are adopted, and out of these, three are common for each n. The three different common failure plans are as follows:

Plan 1: $[(k,n,m),(G_1 = n-m, G_i = 0, \quad \forall \quad i = 2,3,\dots m)]$, in this case $(n-m)$ groups are discarded from the test at the first failure only,

Plan 2: $[(k,n,m),(G_i = 0, \quad \forall i = 1,2,\dots,m-1, G_m = n-m)]$, in this case $(n-m)$ groups are removed at mth failure, and

Plan 3: $[(k,n=m), G_i = 0, \quad \forall i = 1,2,\dots,m]$ this is the case of first failure censored sample.

TABLE 5.1: Several combinations of progressive censoring plans

(n,m)	CS	Plans	(n,m)	CS	Plans
(20,16)	1	(4,0*15)	(30,24)	5	(6,0*23)
	1	(1,0*4,1,0*4,1,0*4,1)		6	(2,0*11,2,0*10,2)
	3	(0*15,4)		7	(0*23,6)
(20,20)	4	(0*20)	(20,30)	8	(0*30)

The simplified notations are used for different combinations of censoring plans as shown in the Table 6.1. In addition, $t = 0.80$ (in time units) is taken as mission time to compute the reliability characteristics. The ML estimate of parameter and reliability characteristics are computed in the case of a non-Bayesian estimation process. The interval estimates of the associated model parameter θ are also computed using asymptotic and bootstrap (boot-p & boot-t) CIs, as well as their respective coverage probabilities.

Furthermore, employing an informative gamma prior, Bayes estimates of parameter and reliability characteristics are derived under SELF (Prior 1). Its related hyper-parameters (a,b) for Prior 1 are set so that the prior mean is $\theta = a/b$, i.e. $\theta = a/b$. Therefore, chosen $(a,b) = (3,2)$ and $(a,b) = (1.2,2.4)$ for $\theta = 1.5$ and 0.5, respectively. For non-informative prior (Prior 0), hyper-parameters are taken as $(a,b) \to (0,0)$. To obtain Bayes estimates, the TK approximation, IS, and M-H algorithms are utilized. $M = 10,000$ samples are generated for the IS and M-H algorithms, of which $M_0 = 2000$ is considered as the burn-in period. Also, obtained 95% HPD credible interval for the parameter θ, as well as the coverage probability.

The simulations are carried out with $N = 1000$ replications. Then, the AEs with corresponding MSEs of different estimates are computed. Suppose $\hat{\phi}_j$ is the estimate of ϕ for the jth sample, then AE $= \frac{1}{N}\sum_{j=1}^{N}\hat{\phi}_j$, MSE $= \frac{1}{N}\sum_{j=1}^{N}(\hat{\phi}_j - \phi)^2$. Also, the average lengths (AL) with corresponding coverage probabilities (CP) of 95% ACI, bootstrap (boot-p & boot-t) CI, and HPD credible intervals of parameter θ are computed. All the simulated results are summarizes in the following Tables 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, 5.9, 5.10, 5.11.

TABLE 5.2: AE and MSEs of ML and Bayes estimates of θ, when $\theta = 1.5$.

		MLE		TK				IS				MH			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	1	1.5627	0.0669	1.5705	0.0686	1.5603	0.0567	1.5604	0.0679	1.5533	0.0559	1.5703	0.0685	1.5599	0.0566
	2	1.5632	0.0642	1.5692	0.0655	1.5608	0.0534	1.5361	0.0708	1.5394	0.0557	1.5690	0.0655	1.5605	0.0534
	3	1.5690	0.0609	1.5739	0.0620	1.5499	0.0516	1.5069	0.0652	1.5053	0.0521	1.5738	0.0620	1.5499	0.0518
	4	1.5182	0.0471	1.5225	0.0475	1.5273	0.0457	1.5000	0.0516	1.5099	0.0465	1.5222	0.0475	1.5272	0.0457
	5	1.5450	0.0431	1.5500	0.0438	1.5359	0.0359	1.5022	0.0470	1.4964	0.0384	1.5501	0.0438	1.5357	0.0359
	6	1.5353	0.0384	1.5392	0.0388	1.5328	0.0346	1.4368	0.0463	1.4518	0.0405	1.5388	0.0388	1.5328	0.0345
	7	1.5292	0.0333	1.5323	0.0336	1.5445	0.0371	1.3721	0.0502	1.4054	0.0441	1.5322	0.0337	1.5444	0.0371
	8	1.5185	0.0322	1.5213	0.0324	1.5175	0.0317	1.4447	0.0368	1.4539	0.0361	1.5214	0.0324	1.5175	0.0317
5	1	1.5429	0.0462	1.5506	0.0473	1.5477	0.0422	1.4338	0.0542	1.4540	0.0456	1.5507	0.0475	1.5475	0.0422
	2	1.5369	0.0377	1.5434	0.0385	1.5471	0.0346	1.3512	0.0560	1.3999	0.0436	1.5431	0.0385	1.5475	0.0347
	3	1.5289	0.0347	1.5345	0.0353	1.5326	0.0327	1.2977	0.0730	1.3489	0.0562	1.5347	0.0353	1.5329	0.0328
	4	1.5157	0.0325	1.5206	0.0329	1.5160	0.0292	1.3527	0.0570	1.3787	0.0459	1.5205	0.0330	1.5160	0.0292
	5	1.5320	0.0291	1.5370	0.0296	1.5222	0.0250	1.2919	0.0691	1.3180	0.0572	1.5368	0.0296	1.5224	0.0250
	6	1.5268	0.0256	1.5311	0.0260	1.5277	0.0240	1.2088	0.1056	1.2498	0.0828	1.5309	0.0260	1.5280	0.0240
	7	1.5360	0.0254	1.5398	0.0258	1.5241	0.0223	1.1688	0.1292	1.1907	0.1122	1.5397	0.0258	1.5241	0.0223
	8	1.5070	0.0192	1.5102	0.0193	1.5101	0.0193	1.2180	0.0962	1.2455	0.0809	1.5101	0.0193	1.5100	0.0193

TABLE 5.3: AL and CPs of 95% ACI, Bootstrap CIs and HPD intervals of θ, when $\theta = 1.5$.

				Bootstrap				HPD			
		ACI		boot-p		boot-t		Prior 0		Prior 1	
k	CS	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
3	1	0.9357	0.948	1.0218	0.927	0.9531	0.949	1.0556	0.969	1.0127	0.979
	2	0.8930	0.940	0.9746	0.924	0.9108	0.938	1.0090	0.964	0.9726	0.975
	3	0.8750	0.941	0.9552	0.923	0.8937	0.945	0.9890	0.966	0.9431	0.970
	4	0.8414	0.952	0.8662	0.947	0.8428	0.948	0.9498	0.980	0.9238	0.968
	5	0.7600	0.942	0.8129	0.938	0.7733	0.947	0.8576	0.967	0.8300	0.979
	6	0.7192	0.946	0.7668	0.938	0.7316	0.948	0.8113	0.967	0.7912	0.968
	7	0.6961	0.952	0.7412	0.953	0.7075	0.957	0.7859	0.979	0.7762	0.965
	8	0.6873	0.944	0.6988	0.949	0.6872	0.945	0.7759	0.972	0.7580	0.965
5	1	0.7675	0.937	0.8199	0.925	0.7833	0.939	0.8693	0.964	0.8461	0.965
	2	0.7296	0.954	0.7766	0.940	0.7441	0.957	0.8266	0.977	0.8114	0.981
	3	0.7077	0.955	0.7523	0.940	0.7214	0.957	0.7995	0.976	0.7829	0.978
	4	0.6940	0.949	0.7067	0.954	0.6978	0.950	0.7849	0.974	0.7654	0.978
	5	0.6257	0.940	0.6569	0.927	0.6356	0.944	0.7071	0.967	0.6877	0.972
	6	0.5942	0.951	0.6228	0.942	0.6043	0.947	0.6704	0.970	0.6611	0.972
	7	0.5803	0.943	0.6076	0.936	0.5896	0.945	0.6554	0.960	0.6393	0.977
	8	0.5635	0.960	0.5699	0.967	0.5657	0.964	0.6358	0.979	0.6270	0.976

TABLE 5.4: AE and MSEs of ML and Bayes estimates of $R(t)$, when $t = 0.80$ and $R(t) = 0.7037$.

k	CS	MLE		TK				IS				MH			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
		AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	1	0.7127	0.0032	0.7094	0.0031	0.7082	0.0027	0.7076	0.0031	0.7070	0.0027	0.7093	0.0031	0.7082	0.0026
	2	0.7131	0.0031	0.7097	0.0030	0.7091	0.0025	0.7027	0.0035	0.7047	0.0027	0.7097	0.0030	0.7090	0.0025
	3	0.7148	0.0029	0.7114	0.0028	0.7068	0.0024	0.6969	0.0034	0.6974	0.0028	0.7114	0.0028	0.7068	0.0024
	4	0.7037	0.0025	0.7003	0.0024	0.7019	0.0024	0.6956	0.0028	0.6984	0.0025	0.7003	0.0024	0.7019	0.0024
	5	0.7105	0.0022	0.7082	0.0021	0.7057	0.0018	0.6979	0.0025	0.6971	0.0022	0.7083	0.0021	0.7056	0.0018
	6	0.7086	0.0019	0.7064	0.0019	0.7054	0.0017	0.6829	0.0030	0.6869	0.0025	0.7063	0.0019	0.7054	0.0017
	7	0.7076	0.0017	0.7054	0.0017	0.7081	0.0018	0.6671	0.0036	0.6757	0.0030	0.7053	0.0017	0.7081	0.0018
	8	0.7051	0.0017	0.7029	0.0017	0.7021	0.0017	0.6856	0.0023	0.6877	0.0023	0.7029	0.0017	0.7021	0.0017
5	1	0.7098	0.0023	0.7080	0.0023	0.7079	0.0021	0.6813	0.0035	0.6869	0.0028	0.7080	0.0023	0.7079	0.0021
	2	0.7090	0.0020	0.7073	0.0019	0.7087	0.0017	0.6614	0.0042	0.6744	0.0030	0.7072	0.0019	0.7088	0.0017
	3	0.7074	0.0018	0.7056	0.0018	0.7055	0.0017	0.6469	0.0057	0.6610	0.0042	0.7057	0.0018	0.7056	0.0017
	4	0.7044	0.0018	0.7026	0.0017	0.7019	0.0016	0.6617	0.0042	0.6691	0.0033	0.7025	0.0017	0.7019	0.0016
	5	0.7087	0.0015	0.7074	0.0015	0.7044	0.0013	0.6461	0.0054	0.6536	0.0043	0.7074	0.0015	0.7044	0.0013
	6	0.7077	0.0014	0.7066	0.0013	0.7060	0.0013	0.6221	0.0085	0.6345	0.0065	0.7065	0.0013	0.7060	0.0013
	7	0.7100	0.0013	0.7088	0.0013	0.7054	0.0012	0.6099	0.0107	0.6171	0.0090	0.7088	0.0013	0.7054	0.0012
	8	0.7035	0.0011	0.7023	0.0011	0.7023	0.0011	0.6254	0.0076	0.6337	0.0063	0.7023	0.0011	0.7023	0.0011

TABLE 5.5: AE and MSEs of ML and Bayes estimates of $h(t)$, when $t = 0.80$ and $h(t) = 0.4386$.

		MLE		TK				IS				MH			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	1	0.4279	0.0027	0.4287	0.0027	0.4302	0.0022	0.4304	0.0027	0.4314	0.0023	0.4286	0.0027	0.4302	0.0022
	2	0.4277	0.0026	0.4286	0.0026	0.4298	0.0021	0.4351	0.0029	0.4339	0.0023	0.4286	0.0026	0.4298	0.0021
	3	0.4263	0.0025	0.4274	0.0024	0.4319	0.0021	0.4407	0.0027	0.4407	0.0022	0.4273	0.0024	0.4319	0.0021
	4	0.4366	0.0020	0.4376	0.0020	0.4364	0.0019	0.4420	0.0022	0.4398	0.0020	0.4376	0.0020	0.4364	0.0019
	5	0.4307	0.0018	0.4313	0.0018	0.4339	0.0015	0.4408	0.0020	0.4418	0.0017	0.4312	0.0018	0.4339	0.0015
	6	0.4326	0.0016	0.4332	0.0016	0.4343	0.0014	0.4543	0.0022	0.4510	0.0019	0.4332	0.0016	0.4343	0.0014
	7	0.4337	0.0014	0.4344	0.0014	0.4319	0.0015	0.4682	0.0025	0.4609	0.0022	0.4344	0.0014	0.4319	0.0015
	8	0.4359	0.0014	0.4366	0.0014	0.4374	0.0014	0.4523	0.0017	0.4504	0.0017	0.4366	0.0014	0.4373	0.0014
5	1	0.4313	0.0019	0.4313	0.0019	0.4317	0.0017	0.4553	0.0025	0.4508	0.0021	0.4313	0.0019	0.4316	0.0017
	2	0.4323	0.0016	0.4324	0.0016	0.4313	0.0014	0.4728	0.0028	0.4620	0.0021	0.4324	0.0016	0.4312	0.0014
	3	0.4338	0.0015	0.4340	0.0015	0.4343	0.0014	0.4849	0.0038	0.4733	0.0028	0.4339	0.0015	0.4342	0.0014
	4	0.4366	0.0014	0.4369	0.0014	0.4376	0.0013	0.4725	0.0029	0.4666	0.0023	0.4368	0.0014	0.4376	0.0013
	5	0.4329	0.0013	0.4330	0.0012	0.4359	0.0011	0.4859	0.0036	0.4798	0.0029	0.4330	0.0012	0.4358	0.0011
	6	0.4339	0.0011	0.4340	0.0011	0.4346	0.0010	0.5051	0.0056	0.4954	0.0043	0.4340	0.0011	0.4345	0.0010
	7	0.4319	0.0011	0.4321	0.0011	0.4352	0.0010	0.5146	0.0069	0.5092	0.0059	0.4321	0.0011	0.4352	0.0010
	8	0.4379	0.0009	0.4381	0.0008	0.4381	0.0008	0.5027	0.0050	0.4962	0.0042	0.4381	0.0008	0.4381	0.0008

TABLE 5.6: AE and MSEs of ML and Bayes estimates of $MdTSF$, when $MdTSF = 1.7024$.

		MLE		TK				IS				MH			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	1	1.7923	0.1350	1.8042	0.1386	1.7895	0.1144	1.7897	0.1370	1.7795	0.1126	1.8039	0.1383	1.7890	0.1143
	2	1.7930	0.1295	1.8022	0.1322	1.7901	0.1079	1.7552	0.1427	1.7596	0.1123	1.8020	0.1323	1.7897	0.1079
	3	1.8011	0.1229	1.8088	0.1252	1.7746	0.1042	1.7137	0.1313	1.7112	0.1049	1.8086	0.1253	1.7747	0.1044
	4	1.7290	0.0948	1.7357	0.0957	1.7426	0.0920	1.7039	0.1038	1.7179	0.0936	1.7353	0.0956	1.7424	0.0919
	5	1.7668	0.0868	1.7745	0.0884	1.7543	0.0724	1.7066	0.0946	1.6983	0.0772	1.7746	0.0883	1.7541	0.0724
	6	1.7531	0.0773	1.7591	0.0783	1.7499	0.0696	1.6140	0.0926	1.6350	0.0811	1.7586	0.0783	1.7500	0.0696
	7	1.7443	0.0670	1.7493	0.0677	1.7665	0.0748	1.5222	0.1000	1.5693	0.0880	1.7491	0.0678	1.7664	0.0747
	8	1.7292	0.0648	1.7337	0.0652	1.7283	0.0638	1.6250	0.0736	1.6380	0.0722	1.7338	0.0652	1.7282	0.0638
5	1	1.7640	0.0930	1.7754	0.0955	1.7712	0.0851	1.6097	0.1084	1.6383	0.0914	1.7755	0.0959	1.7710	0.0851
	2	1.7553	0.0759	1.7650	0.0776	1.7702	0.0697	1.4928	0.1114	1.5615	0.0870	1.7647	0.0776	1.7707	0.0699
	3	1.7439	0.0698	1.7523	0.0711	1.7497	0.0659	1.4173	0.1450	1.4896	0.1118	1.7526	0.0712	1.7500	0.0660
	4	1.7252	0.0654	1.7326	0.0662	1.7261	0.0586	1.4950	0.1134	1.5316	0.0915	1.7325	0.0663	1.7260	0.0587
	5	1.7482	0.0586	1.7558	0.0597	1.7347	0.0502	1.4090	0.1373	1.4458	0.1137	1.7555	0.0596	1.7350	0.0503
	6	1.7409	0.0515	1.7472	0.0524	1.7424	0.0484	1.2921	0.2093	1.3497	0.1644	1.7470	0.0524	1.7428	0.0484
	7	1.7539	0.0511	1.7595	0.0519	1.7373	0.0450	1.2360	0.2559	1.2666	0.2224	1.7594	0.0520	1.7373	0.0449
	8	1.7126	0.0385	1.7175	0.0388	1.7173	0.0389	1.3049	0.1910	1.3435	0.1607	1.7174	0.0389	1.7172	0.0389

TABLE 5.7: AE and MSEs of ML and Bayes estimates of θ, when $\theta = 0.5$.

		MLE		TK				IS				MH			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	1	0.5211	0.0082	0.5237	0.0084	0.5211	0.0066	0.5201	0.0088	0.5183	0.0068	0.5236	0.0084	0.5211	0.0066
	2	0.5188	0.0070	0.5208	0.0071	0.5218	0.0061	0.5099	0.0076	0.5114	0.0063	0.5206	0.0071	0.5218	0.0061
	3	0.5172	0.0065	0.5188	0.0066	0.5205	0.0058	0.4975	0.0067	0.5019	0.0059	0.5188	0.0066	0.5206	0.0058
	4	0.5105	0.0058	0.5119	0.0059	0.5116	0.0052	0.5048	0.0061	0.5056	0.0056	0.5118	0.0059	0.5116	0.0052
	5	0.5161	0.0048	0.5178	0.0049	0.5192	0.0050	0.5014	0.0053	0.5062	0.0051	0.5177	0.0049	0.5192	0.0050
	6	0.5143	0.0044	0.5156	0.0045	0.5155	0.0042	0.4805	0.0050	0.4841	0.0045	0.5155	0.0045	0.5154	0.0042
	7	0.5110	0.0038	0.5120	0.0038	0.5122	0.0036	0.4572	0.0055	0.4628	0.0051	0.5120	0.0038	0.5121	0.0036
	8	0.5038	0.0034	0.5047	0.0034	0.5064	0.0035	0.4800	0.0042	0.4819	0.0037	0.5046	0.0034	0.5063	0.0035
5	1	0.5130	0.0050	0.5155	0.0051	0.5173	0.0051	0.4747	0.0060	0.4855	0.0054	0.5156	0.0052	0.5173	0.0051
	2	0.5163	0.0045	0.5185	0.0046	0.5137	0.0041	0.4574	0.0064	0.4587	0.0058	0.5187	0.0046	0.5136	0.0041
	3	0.5137	0.0042	0.5156	0.0043	0.5117	0.0041	0.4379	0.0079	0.4436	0.0070	0.5156	0.0043	0.5116	0.0041
	4	0.5088	0.0038	0.5105	0.0039	0.5073	0.0035	0.4527	0.0062	0.4577	0.0055	0.5104	0.0039	0.5073	0.0035
	5	0.5095	0.0031	0.5111	0.0031	0.5122	0.0031	0.4312	0.0077	0.4375	0.0066	0.5110	0.0031	0.5122	0.0031
	6	0.5112	0.0031	0.5126	0.0031	0.5087	0.0028	0.4057	0.0114	0.4093	0.0105	0.5125	0.0031	0.5088	0.0028
	7	0.5084	0.0025	0.5097	0.0026	0.5076	0.0025	0.3854	0.0151	0.3910	0.0138	0.5096	0.0026	0.5076	0.0025
	8	0.5011	0.0023	0.5022	0.0023	0.5035	0.0022	0.4046	0.0111	0.4098	0.0101	0.5022	0.0023	0.5036	0.0022

TABLE 5.8: AL and CPs of 95% ACI, Bootstrap CIs, and HPD intervals of θ, when $\theta = 0.5$.

				Bootstrap				HPD			
		ACI		boot-p		boot-t		Prior 0		Prior 1	
k	CS	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
3	1	0.3120	0.931	0.3410	0.914	0.3179	0.931	0.3506	0.952	0.3433	0.969
	2	0.2963	0.943	0.3236	0.928	0.3024	0.945	0.3331	0.961	0.3282	0.976
	3	0.2884	0.932	0.3143	0.924	0.2940	0.935	0.3245	0.967	0.3206	0.973
	4	0.2830	0.943	0.2910	0.939	0.2837	0.944	0.3180	0.973	0.3129	0.972
	5	0.2540	0.946	0.2714	0.933	0.2585	0.947	0.2850	0.965	0.2826	0.960
	6	0.2410	0.940	0.2569	0.931	0.2451	0.946	0.2702	0.955	0.2673	0.974
	7	0.2326	0.954	0.2481	0.945	0.2368	0.962	0.2608	0.974	0.2582	0.977
	8	0.2280	0.945	0.2320	0.943	0.2280	0.945	0.2554	0.963	0.2542	0.976
5	1	0.2551	0.943	0.2727	0.927	0.2607	0.944	0.2871	0.962	0.2845	0.960
	2	0.2451	0.951	0.2605	0.939	0.2497	0.954	0.2759	0.970	0.2700	0.973
	3	0.2378	0.948	0.2529	0.934	0.2423	0.950	0.2673	0.967	0.2624	0.962
	4	0.2331	0.942	0.2364	0.938	0.2336	0.939	0.2621	0.969	0.2574	0.975
	5	0.2081	0.947	0.2184	0.944	0.2115	0.951	0.2338	0.967	0.2316	0.967
	6	0.1989	0.944	0.2081	0.936	0.2018	0.944	0.2227	0.967	0.2194	0.963
	7	0.1921	0.956	0.2008	0.951	0.1948	0.961	0.2158	0.969	0.2130	0.971
	8	0.1874	0.953	0.1893	0.953	0.1879	0.950	0.2097	0.969	0.2089	0.968

TABLE 5.9: AE and MSEs of ML and Bayes estimates of $R(t)$, when $t = 0.80$ and $R(t) = 0.3333$.

k	CS	MLE		TK				IS				MH			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
		AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	1	0.3430	0.0022	0.3430	0.0022	0.3420	0.0017	0.3412	0.0023	0.3406	0.0018	0.3429	0.0022	0.3420	0.0017
	2	0.3420	0.0019	0.3418	0.0019	0.3426	0.0016	0.3362	0.0020	0.3373	0.0017	0.3417	0.0019	0.3426	0.0016
	3	0.3412	0.0018	0.3409	0.0018	0.3420	0.0015	0.3299	0.0019	0.3325	0.0016	0.3409	0.0018	0.3421	0.0016
	4	0.3378	0.0016	0.3374	0.0016	0.3374	0.0014	0.3338	0.0017	0.3343	0.0016	0.3373	0.0016	0.3374	0.0014
	5	0.3410	0.0013	0.3410	0.0013	0.3418	0.0013	0.3324	0.0015	0.3350	0.0014	0.3410	0.0013	0.3418	0.0013
	6	0.3401	0.0012	0.3400	0.0012	0.3400	0.0011	0.3214	0.0015	0.3235	0.0013	0.3400	0.0012	0.3399	0.0011
	7	0.3384	0.0011	0.3382	0.0011	0.3384	0.0010	0.3088	0.0017	0.3119	0.0016	0.3382	0.0011	0.3384	0.0010
	8	0.3346	0.0010	0.3344	0.0010	0.3353	0.0010	0.3213	0.0013	0.3224	0.0011	0.3343	0.0010	0.3353	0.0010
5	1	0.3393	0.0014	0.3397	0.0014	0.3407	0.0014	0.3181	0.0018	0.3240	0.0016	0.3398	0.0014	0.3407	0.0014
	2	0.3412	0.0012	0.3415	0.0012	0.3390	0.0011	0.3087	0.0020	0.3096	0.0018	0.3416	0.0012	0.3390	0.0011
	3	0.3399	0.0012	0.3401	0.0012	0.3380	0.0011	0.2979	0.0025	0.3012	0.0022	0.3400	0.0012	0.3380	0.0011
	4	0.3373	0.0011	0.3374	0.0011	0.3358	0.0010	0.3062	0.0020	0.3091	0.0017	0.3374	0.0011	0.3357	0.0010
	5	0.3378	0.0009	0.3381	0.0009	0.3387	0.0008	0.2943	0.0025	0.2980	0.0021	0.3380	0.0009	0.3387	0.0008
	6	0.3387	0.0009	0.3389	0.0009	0.3369	0.0008	0.2798	0.0037	0.2819	0.0034	0.3389	0.0009	0.3369	0.0008
	7	0.3373	0.0007	0.3375	0.0007	0.3364	0.0007	0.2679	0.0049	0.2712	0.0045	0.3374	0.0007	0.3364	0.0007
	8	0.3335	0.0006	0.3335	0.0006	0.3343	0.0006	0.2792	0.0036	0.2822	0.0032	0.3335	0.0006	0.3343	0.0006

TABLE 5.10: AE and MSEs of ML and Bayes estimates of $h(t)$, when $t = 0.80$ and $h(t) = 0.6944$.

		MLE		TK				IS				MH			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	1	0.6885	0.0007	0.6881	0.0007	0.6887	0.0006	0.6891	0.0008	0.6895	0.0006	0.6880	0.0007	0.6887	0.0006
	2	0.6891	0.0006	0.6888	0.0006	0.6885	0.0005	0.6920	0.0007	0.6915	0.0006	0.6888	0.0006	0.6884	0.0005
	3	0.6896	0.0006	0.6894	0.0006	0.6888	0.0005	0.6956	0.0006	0.6943	0.0005	0.6894	0.0006	0.6888	0.0005
	4	0.6916	0.0005	0.6914	0.0005	0.6915	0.0005	0.6935	0.0005	0.6932	0.0005	0.6914	0.0005	0.6914	0.0005
	5	0.6898	0.0004	0.6895	0.0004	0.6891	0.0004	0.6944	0.0005	0.6929	0.0005	0.6895	0.0004	0.6891	0.0004
	6	0.6903	0.0004	0.6901	0.0004	0.6902	0.0004	0.7006	0.0005	0.6995	0.0004	0.6902	0.0004	0.6902	0.0004
	7	0.6913	0.0003	0.6912	0.0003	0.6911	0.0003	0.7076	0.0005	0.7059	0.0005	0.6912	0.0003	0.6911	0.0003
	8	0.6935	0.0003	0.6934	0.0003	0.6928	0.0003	0.7007	0.0004	0.7001	0.0003	0.6934	0.0003	0.6929	0.0003
5	1	0.6908	0.0004	0.6902	0.0005	0.6897	0.0004	0.7024	0.0005	0.6991	0.0005	0.6902	0.0005	0.6897	0.0004
	2	0.6897	0.0004	0.6893	0.0004	0.6907	0.0004	0.7076	0.0006	0.7072	0.0005	0.6892	0.0004	0.6907	0.0004
	3	0.6905	0.0004	0.6901	0.0004	0.6913	0.0004	0.7135	0.0007	0.7117	0.0006	0.6901	0.0004	0.6913	0.0004
	4	0.6920	0.0003	0.6917	0.0003	0.6926	0.0003	0.7090	0.0006	0.7075	0.0005	0.6917	0.0003	0.6926	0.0003
	5	0.6917	0.0003	0.6914	0.0003	0.6910	0.0003	0.7155	0.0007	0.7135	0.0006	0.6914	0.0003	0.6910	0.0003
	6	0.6912	0.0003	0.6909	0.0003	0.6921	0.0002	0.7233	0.0011	0.7222	0.0010	0.6909	0.0003	0.6921	0.0002
	7	0.6920	0.0002	0.6918	0.0002	0.6924	0.0002	0.7296	0.0014	0.7278	0.0013	0.6918	0.0002	0.6924	0.0002
	8	0.6942	0.0002	0.6940	0.0002	0.6936	0.0002	0.7236	0.0010	0.7220	0.0009	0.6940	0.0002	0.6936	0.0002

TABLE 5.11: AE and MSEs of ML and Bayes estimates of $MdTSF$, when $MdTSF = 0.3333$.

		MLE		TK				IS				MH			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	1	0.3621	0.0130	0.3675	0.0135	0.3636	0.0105	0.3630	0.0140	0.3602	0.0107	0.3674	0.0135	0.3637	0.0105
	2	0.3588	0.0111	0.3633	0.0114	0.3642	0.0097	0.3496	0.0119	0.3511	0.0098	0.3631	0.0114	0.3642	0.0097
	3	0.3568	0.0102	0.3607	0.0105	0.3624	0.0093	0.3340	0.0103	0.3390	0.0091	0.3607	0.0105	0.3625	0.0093
	4	0.3483	0.0091	0.3519	0.0092	0.3513	0.0081	0.3430	0.0095	0.3437	0.0087	0.3519	0.0093	0.3513	0.0082
	5	0.3548	0.0076	0.3584	0.0078	0.3602	0.0078	0.3378	0.0082	0.3438	0.0079	0.3584	0.0078	0.3602	0.0079
	6	0.3525	0.0069	0.3555	0.0071	0.3552	0.0067	0.3118	0.0075	0.3162	0.0067	0.3554	0.0071	0.3550	0.0067
	7	0.3482	0.0059	0.3508	0.0060	0.3508	0.0056	0.2834	0.0078	0.2900	0.0072	0.3507	0.0060	0.3508	0.0056
	8	0.3393	0.0052	0.3417	0.0053	0.3438	0.0054	0.3111	0.0061	0.3132	0.0055	0.3415	0.0053	0.3437	0.0054
5	1	0.3511	0.0079	0.3558	0.0081	0.3579	0.0080	0.3052	0.0087	0.3182	0.0081	0.3559	0.0081	0.3579	0.0080
	2	0.3550	0.0071	0.3591	0.0074	0.3530	0.0064	0.2840	0.0091	0.2855	0.0083	0.3593	0.0074	0.3528	0.0064
	3	0.3517	0.0065	0.3554	0.0067	0.3505	0.0064	0.2608	0.0109	0.2674	0.0097	0.3553	0.0067	0.3504	0.0064
	4	0.3456	0.0059	0.3489	0.0060	0.3449	0.0055	0.2782	0.0087	0.2840	0.0077	0.3488	0.0060	0.3448	0.0055
	5	0.3461	0.0048	0.3492	0.0049	0.3505	0.0048	0.2524	0.0107	0.2596	0.0093	0.3491	0.0049	0.3505	0.0048
	6	0.3482	0.0047	0.3509	0.0049	0.3460	0.0043	0.2229	0.0155	0.2268	0.0143	0.3508	0.0049	0.3461	0.0043
	7	0.3447	0.0039	0.3471	0.0040	0.3444	0.0038	0.1998	0.0202	0.2061	0.0187	0.3470	0.0040	0.3445	0.0038
	8	0.3356	0.0034	0.3378	0.0035	0.3394	0.0034	0.2213	0.0151	0.2271	0.0138	0.3377	0.0035	0.3395	0.0034

From these findings, the following interpretations are drawn: the MSEs of ML and Bayes estimates of parameter and reliability characteristics decrease as n increases in almost all cases. Also, it is seen that Bayes estimates have smaller MSEs than ML estimates in almost all cases. Also, the Bayes estimates using Prior 1 performed quite better than Prior 0, as it includes prior information. It is also observed that the MSEs are decreasing with an increasing number of individuals within each group. The ALs of asymptotic, bootstrap (boot-p, boot-t) and HPD narrow down with an increase in n in almost all cases. In the case of HPD, ALs are more narrow as compared to the asymptotic and bootstrap confidence intervals. In almost all cases, the CPs of ML and Bayes estimates of θ achieve the desired confidence coefficient.

5.6 Real Data Analysis

In this section, we analyzed a real data set as an example to illustrate the situation of life testing experiments for IP lifetime model with PFFC data. Here, we take head and neck cancer data from Efron (1988). These data are survival times (in days) treated with combined radiotherapy and chemotherapy of 45 patients suffering from head and neck cancer disease and given as follows:

12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46,58.36, 63.47, 68.46,78.26,74.47,81,43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209,249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776.

Recently, Sharma et al. (2015) and Sharma (2018) studied and fitted head neck cancer data to the inverse Lindley (IL) and generalized inverse Lindley (GIL) lifetime models, respectively. To begin, we assess the failure rate function of the data set using the scaled total time on test (TTT) transform. The scaled TTT is calculated as follows:

$$\psi(r/n) = \left[\sum_{j=1}^{r} t_{(i)} + (n-r)t_r\right] \Bigg/ \left(\sum_{j=1}^{r} t_{(i)}\right),$$

where, $t_{(i)}, \quad i = 1,2,\ldots,n$ represent the ith order statistic and $r = 1,2,\ldots,n$. If the plot $(r/n, \psi(r/n))$ is convex (concave), the failure rate function has a decreasing (increasing) shape. If it start concave and then become convex (begins convex and then becomes concave), the failure rate function is upside down bathtub shaped (bathtub shaped), respectively, for more details about scale TTT, see, Mudholkar et al. (1996). The scaled TTT plot of head-neck cancer data set is given in Figure 6.1. This Figure suggests that the head-neck cancer data set follow upside down bathtub shaped failure rate function. This empirical behaviour of failure rate function is quite similar to the considered IPD model. Further, check whether the considered real data

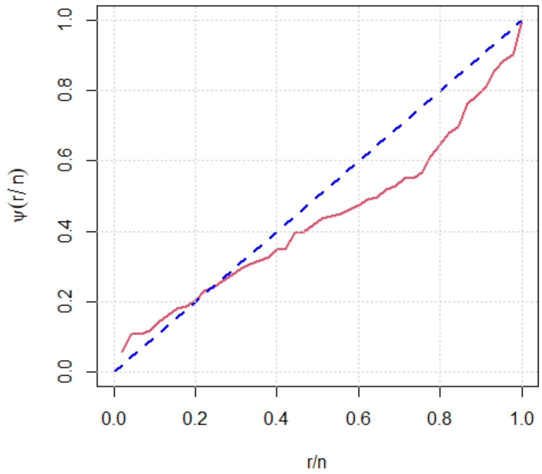


FIGURE 5.2: TTT plot for plots for head-neck cancer disease data.

set is good-fit to the IP lifetime model or not using some goodness-of-fit tests. Kolmogrov-Smirnov (KS) and Anderson-Darling (AD) goodness-of-fit tests were employed in this study, and test statistics and p-values were obtained. The ML estimates of associated parameters are also used to generate two information theoretic criteria based on the log-likelihood function, namely AIC and BIC. To assess the goodness-of-fit, test statistics and p-values are used. Then, based on the considered real data set, compare the fitting of the IP lifetime model with the IL and GIL lifetime models. The best lifetime model has the lowest AIC, BIC, $-\ln L$, KS, and AD test statistics and the greatest p-value for the KS and AD tests. The fittings of IP lifetime model and competitive models are reported in table 5.12. From Table 5.12, it is noticed that IP, IL and

TABLE 5.12: Summery of fitted models for head-neck cancer disease data.

					AD Test		KS Test	
Models	MLE	AIC	BIC	-ln L	statistic	p-value	statistic	p-value
IP	$\hat{\theta} = 76.4848$	570.9288	572.7354	284.4644	0.4095	0.8386	0.0783	0.9255
IL	$\hat{\theta} = 76.3539$	571.0640	572.8707	284.5320	0.4190	0.8290	0.0818	0.9002
GIL	$\hat{\alpha} = 1.0248$	573.0149	576.6282	284.5074	0.4153	0.8326	0.0890	0.8376
	$\hat{\beta} = 83.9189$							

GIL all models are good-fitted to the consider data set. Among all the fitted models IP lifetime model outperform as it has lower AIC, BIC, $-\ln L$, KS and AD statistics with high p-value. So choice of IP lifetime model is quite reasonable for this data set.

Furthermore, the first failure censored sample was collected by randomly arranging the considered complete data-set into $n = 15$ groups with $k = 3$ sample points inside each group. The observation with '+' sings are first failure observations in the respective groups as shown in

TABLE 5.13: First failure censored head neck cancer disease data.

Group / Items	1	2	3	4	5	6	7	8
(i)	74.47	155.00	63.47+	23.56+	173.00	47.38	12.20+	195.00
(ii)	43.00+	130.00+	194.00	119.00	58.36+	41.35	68.46	725.00
(iii)	140.00	159.00	519.00	432.00	84.00	37.00+	110.00	23.74+
Group / Items	**9**	**10**	**11**	**12**	**13**	**14**	**15**	
(i)	55.46+	339.00	133.00	209.00	94.00+	633.00	469.00	
(ii)	1776.00	817.00	281.00	112.00+	319.00	146.00	92.00	
(iii)	78.26	179.00+	25.87+	127.00	249.00	31.98+	81.00+	

Table 5.13. Consequently, the ordered first failure censored sample is given by

$$12.20, 23.56, 23.74, 25.87, 31.98, 37, 43, 55.46, 58.36, 63.47, 81, 94, 112, 130, 179.$$

Now, applying four different progressive censoring plans on the above first failure censored sample with prefixed number failure $m = 10$. The four different censoring plans and their corresponding PFFC samples are as follows:

Scheme 1: $k = 3, n = 15, m = 10, \underset{\sim}{G} = (5, 0*9)$,
$\underset{\sim}{x}$ = 12.20, 43, 55.46, 58.36, 63.47, 81, 94, 112, 130, 179.

Scheme 2: $k = 3, n = 15, m = 10, \underset{\sim}{G} = (1, 0*2, 1, 0*2, 2, 0*2, 1)$,
$\underset{\sim}{x}$ = 12.20, 23.74, 25.87, 31.98, 43, 55.46, 58.36, 94, 112, 130.

Scheme 3: $k = 3, n = 15, m = 10, \underset{\sim}{G} = (0*9, 5)$,
$\underset{\sim}{x}$ = 12.20, 23.56, 23.74, 25.87, 31.98, 37, 43, 55.46, 58.36, 63.47.

Scheme 4: $k = 3, n = m = 15, \underset{\sim}{G} = (0*15)$,
$\underset{\sim}{x}$ = 12.20, 23.56, 23.74, 25.87, 31.98, 37, 43, 55.46, 58.36, 63.47, 81, 94, 112, 130, 179.

The ML and Bayes estimators of parameter and reliability characteristics under consideration of different censoring plans are obtained and reported in Table 5.14. The reliability characteristics $R(t)$ and $h(t)$ are computed at mission time t as median of the considered data. The Bayes estimates of parameter and reliability characteristics are obtained using non-informative prior as information about considered data are unavailable. For M-H algorithm and importance sampling, $M = 10,000$ Markov chains are generated and $M_0 = 2500$ are taken as burn-in-period. The 95% ACI, boot-p, boot-t CIs and HPD credible intervals are computed and tabulated in Table 5.15. For bootstrap confidence intervals each PFFC samples are replicated by $B = 1000$

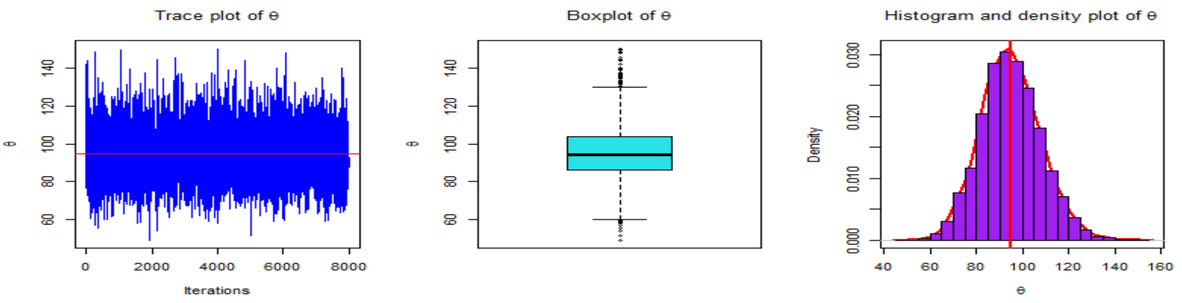
times. Also, Figure 6.3 shows the diagnostic plots of Markov chains for all censoring schemes under consideration of real data set, which verifies the convergence of stationary distributions for generation of Markov chain from posterior. The trace plot shows a random scatter about the mean and shows fine mixture of the parameter chains. The boxplots and histograms of generated samples shows the posterior distribution are almost symmetric i.e. posterior mean can be the best estimate in almost all censoring schemes under consideration of real data set.

TABLE 5.14: ML and Bayes estimates of parameter and reliability characteristics under consideration of head-neck cancer disease data for $k=3, n=15, m=10$.

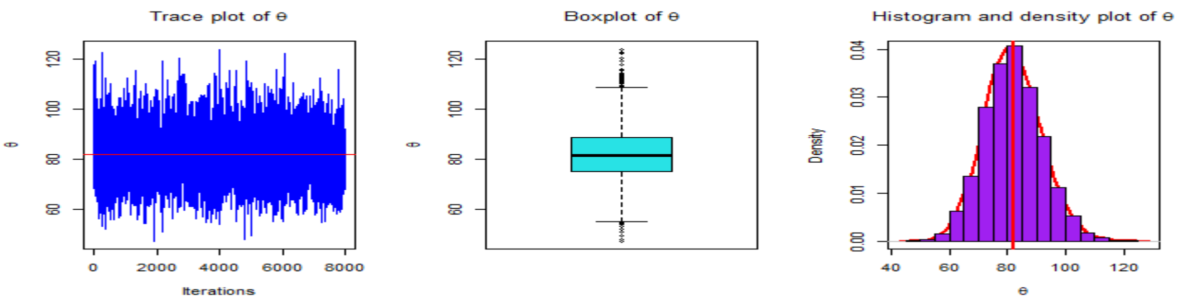
Schemes / Parameters	Scheme 1	Scheme 2	Scheme 3	Scheme 4
$\hat{\theta}$	90.9509	83.8542	74.7172	74.9301
$\hat{\theta}_{\mathrm{TK}}$	91.3190	84.2256	75.0075	75.1549
$\hat{\theta}_{\mathrm{IS}}$	87.4737	81.3659	72.6638	72.9190
$\hat{\theta}_{\mathrm{MH}}$	91.5311	83.3144	73.0104	94.0702
$\hat{R}(t)$	0.7174	0.8404	0.8479	0.7909
$\hat{R}_{\mathrm{TK}}(t)$	0.7186	0.8339	0.8417	0.7851
$\hat{R}_{\mathrm{IS}}(t)$	0.7112	0.8325	0.8367	0.8421
$\hat{R}_{\mathrm{MH}}(t)$	0.7034	0.8315	0.8398	0.7819
$\hat{h}(t)$	0.0069	0.0076	0.0085	0.0086
$\hat{h}_{\mathrm{TK}}(t)$	0.0070	0.0077	0.0086	0.0087
$\hat{h}_{\mathrm{IS}}(t)$	0.0069	0.0078	0.0088	0.0070
$\widehat{h}_{\mathrm{MH}}(t)$	0.0071	0.0079	0.0088	0.0089
$\widehat{MdTSF}$	130.7150	120.4767	107.2949	107.6020
$\widehat{MdTSF}_{\mathrm{TK}}$	131.2270	121.0074	107.7120	107.9234
$\widehat{MdTSF}_{\mathrm{IS}}$	131.5521	119.6980	104.8326	135.2153
$\widehat{MdTSF}_{\mathrm{MH}}$	125.6985	116.8870	104.3325	104.7007

TABLE 5.15: The 95% asymptotic, boot-p, boot-t confidence and HPD credible intervals of parameter θ under consideration of head-neck cancer disease data.

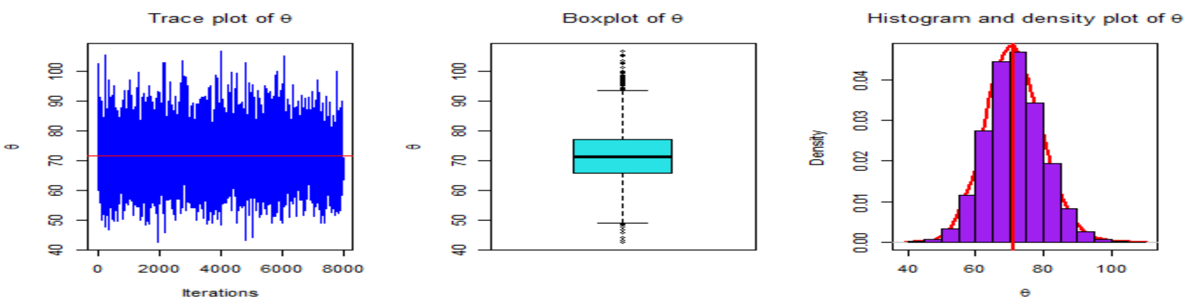
Schemes / Parameters	Scheme 1	Scheme 2	Scheme 3	Scheme 4
$\hat{\theta}_{\mathrm{ACI}}$	(56.14, 125.75)	(55.56, 112.14)	(50.49, 98.93)	(50.94, 98.91)
$\hat{\theta}_{\text{boot-p}}$	(66.13, 162.29)	(54.61, 129.40)	(45.54, 91.92)	(52.83, 111.10)
$\hat{\theta}_{\text{boot-t}}$	(44.66, 127.29)	(53.94, 129.27)	(60.82, 119.90)	(50.61, 106.11)
$\hat{\theta}_{\mathrm{HPD}}$	(45.02, 78.92)	(78.92, 83.85)	(70.21, 75.15)	(70.46, 75.35)



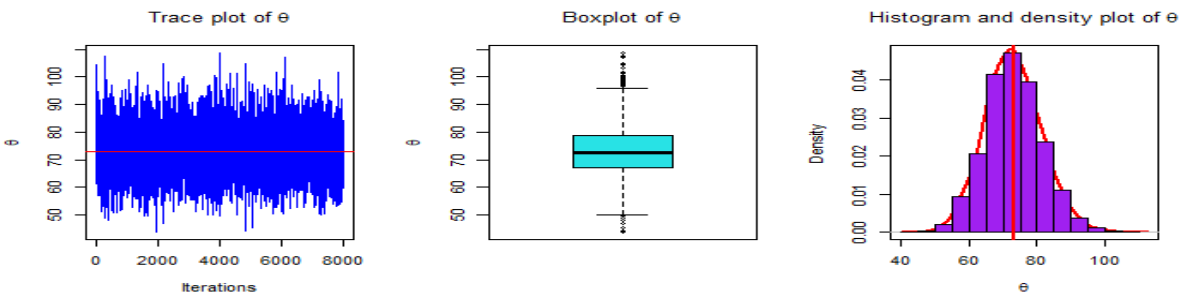
(A) Scheme 1.



(B) Scheme 2.



(C) Scheme 3.



(D) Scheme 4.

FIGURE 5.3: MCMC diagnostic plots for different censoring schemes under consideration of head-neck cancer disease data.

5.7 Concluding Remarks

In this chapter, some inference procedures about the parameter and reliability characteristics for IP lifetime under PFFC data were developed. The ML and Bayes estimates of unknown parameter and reliability characteristics were computed. For Bayesian estimation, TK approximation, importance sampling, and the M-H algorithm using non-informative and gamma informative priors under SELF were considered. Based on the asymptotic normality of ML estimates and bootstrap methods, the 95% asymptotic, boot-p, and boot-t CIs of the parameter were constructed. Also, the HPD credible interval of a parameter based on MCMC samples was computed. An extensive numerical computation was performed to determine the potentiality of different estimators developed in this chapter. A real data set was studied to determine the feasibility of the considered IP lifetime model. From the simulated results, it is observed that Bayesian estimation using MCMC followed by the M-H algorithm outperforms. Therefore, we recommend the use of this consider methodology for all practical purpose in classical as well as Bayesian point of views.