Chapter 6

Statistical Inference of Shannon's Entropy from Maxwell Lifetime Model using Progressively First Failure Censored Data

6.1 Introduction

In this chapter, we consider a problem from information theory and proposed to developed statistical inference of Shannon's entropy for the Maxwell lifetime model based on PFFC data. The construction and application of the PFFCS have already been explored in depth in Chapter 5. Information theory provides a simple approach for measuring the uncertainty and reciprocal information of random variables as entropy measures. The applications of entropy are described in a variety of fields, including computer science, molecular biology, hydrology, meteorology, and others. For example, in the study of trends in gene sequences, molecular biologists use the principle of Shannon's entropy. For more details, one may refer Cover (1999), which contains an excellent monograph on the information theory and implications of the concept of entropy in various disciplines. Shannon's entropy is the most widely used entropy in statistical and information theory, and it was introduced by Shannon (1948). Let X be a random variable with pdf $f(.)$, the Shannon's entropy of X is expressed as follow

$$H(f) = E\left[-\ln\ f(X)\right] = -\int_{-\infty}^{\infty} f(x)\ln\ \left(f(x)\right)\ dx. \tag{6.1}$$

Recently, some attempts have been made by many scholars in parametric statistical inferences to measure the entropy for different lifetime models based on complete as well as censored data. For example, entropy for several shifted exponential populations is studied by Kayal and

Kumar (2013), Cho et al. (2014) discussed entropy for Rayleigh lifetime model under doubly generalized Type-II hybrid censoring scheme, based on generalized progressive hybrid censoring Liu and Gui (2019) estimated entropy, Du et al. (2018) established a statistical inference of information entropy for the log-logistic lifetime model based on progressively Type-I interval censored data, Yu et al. (2019) studied Shannon's entropy for the inverse Weibull lifetime model using PFFC data, Rajesh and Sunoj (2021) discussed Shannon's entropy based on length-bias and Type-I censoring, Hassan and Zaky (2021) developed Bayes estimate of entropy for the Lomax lifetime model based on record data, Shakhatreh et al. (2021) discussed differential entropy for the Weibull lifetime model in the case of objective Bayesian, and references were cited therein. The main objective of this chapter is to develop classical and Bayesian inferences for the associated parameter and Shannon's entropy of the Maxwell lifetime model using PFFC data. The Maxwell-Boltzmann distribution was first proposed by James Clerk Maxwell and Ludwig Boltzmann in late 1800 as a distribution of velocities in a gas at a given temperature, for more details, see, Bekker and Roux (2005). The Maxwell-Boltzmann distribution, popularly known as the Maxwell (MW) lifetime model, is widely used in the fields of chemistry and physics for a variety of purposes. Many basic properties of gases, such as pressure and diffusion, are explained by the MW lifetime model. The MW lifetime model has recently gained popularity as a well-known lifetime model in the literature. This lifetime model has been widely investigated by various researchers for modelling several lifetime data scenarios. For example, Krishna and Malik (2009) studied the MW lifetime model under Type-II censored data, Krishna and Malik (2012) discussed the MW lifetime model under progressive censoring, Krishna et al. (2015) discussed the MW lifetime model under randomly censored data, Tomer and Panwar (2015) established estimation procedures for the MW lifetime model using Type-I progressive hybrid censoring, and Bayesian analysis for MW lifetime model is discussed by Panwar and Tomer (2019), etc. The remainder of this chapter is as follows: Section 6.2 deals with the model description. Classical estimation methods such as ML, ACIs, and bootstrap CIs methods are developed in Section 6.3. Section 6.4, devoted to Bayesian estimation methods using TK approximation and MCMC techniques. Extensive numerical simulations are done in Section 6.5 to demonstrate the influence of numerous estimators created in this chapter. The application of the considered methodology is examined by a real data analysis in Section 6.6. Finally, in Section 6.7, there are some concluding remarks.

6.2 The Model

Let X be random variable following MW lifetime model with parameter λ i.e. $X \sim \text{MW}(\lambda)$, the pdf, cdf and failure rate (or hazard) function, respectively, are given by

$$f(x;\lambda) = \frac{4}{\sqrt{\pi}}\frac{1}{\lambda^{3/2}}x^2 e^{-\frac{x^2}{\lambda}}; \quad 0 < x < \infty, \lambda > 0, \tag{6.2}$$

$$F(x;\lambda) = \Gamma\left(\frac{x^2}{\lambda}, \frac{3}{2}\right); \quad 0 \leq x < \infty, \lambda > 0, \tag{6.3}$$

$$\text{and,} \quad h(x) = \frac{4}{\sqrt{\pi}\lambda^{3/2}}\frac{x^2 e^{-x^2\lambda}}{[1-\Gamma(x^2/\lambda, 3/2)]}; \quad x > 0,\ \lambda > 0. \tag{6.4}$$

where, $\Gamma(t,b) = \frac{1}{\Gamma(b)}\int\limits_0^\infty e^{-x}x^{b-1}dx$ is the incomplete gamma ratio. The failure rate of the MW is increasing, see Krishna and Malik (2012). Now using equations (6.2) and (6.1), the Shannon's entropy is given by

$$\begin{aligned}
H(f) = E\left[-\ln f(x)\right] &= -\int\limits_0^\infty f(x)\ln f(x)dx \\
&= -\int\limits_0^\infty f(x)\left\{\ln 4 - \frac{1}{2}\ln\pi - \frac{3}{2}\ln\lambda + 2\ln x - \frac{x^2}{\lambda}\right\}dx \\
&= -A\int\limits_0^\infty f(x)dx - 2\int\limits_0^\infty \ln x\, f(x)dx + \frac{1}{\lambda}\int\limits_0^\infty x^2 f(x)dx \\
&= -A - \frac{8}{\sqrt{\pi}}\frac{1}{\lambda^{3/2}}\int\limits_0^\infty x^2\ln x e^{-x^2/\lambda}dx + \frac{4}{\sqrt{\pi}}\frac{1}{\lambda^{5/2}}\int\limits_0^\infty x^4 e^{-x^2/\lambda}dx
\end{aligned}$$

$$H(f) = \frac{1}{2}\ln\lambda + \gamma + \frac{1}{2}\ln\pi - \frac{1}{2} \simeq H(\lambda) \quad \text{(say)}, \tag{6.5}$$

where, $A = \ln 4 - \frac{1}{2}\ln\pi - \frac{3}{2}\ln\lambda$, $\int\limits_0^\infty f(x)dx = 1$, and γ is a Euler–Mascheroni constant.

6.3 Classical Estimation

In this part, we use the expectation-maximization (EM) approach to create ML estimates of the related parameter λ and entropy $H(\lambda)$. Based on ML estimates we constructed ACIs of λ and

$H(\lambda)$. Also, we construct the bootstrap CIs for λ and $H(\lambda)$.

6.3.1 Maximum Likelihood Estimation

Let $x_{j:m:n:k};\quad j=1,2,\ldots m$ be the PFFC sample drown from MW(λ), with presumed censoring plans $\underset{\sim}{G}$ and the number of groups n, each group having k individuals with effective sample size m. Then, the likelihood function using equations (6.2), (6.3) and (5.1) is given by

$$L(\underset{\sim}{x};\lambda)=Ak^m\left(\frac{4}{\sqrt{\pi}}\right)^m\lambda^{-\frac{3m}{2}}\exp\left\{-\frac{1}{\lambda}\sum_{j=1}^{m}x_j^2\right\}\prod_{j=1}^{m}x_j^2\left[1-\Gamma\left(\frac{x_j^2}{\lambda},\frac{3}{2}\right)\right]^{k(G_j+1)-1}, \tag{6.6}$$

where, $A=n(n-G_1-1)(n-G_1-G_2-2)\ldots(n-G_1-G_2-\cdots-G_{m-1}-m+1)$.

To begin, assume that the observed and censored data are represented by $\underset{\sim}{X}=(x_{1:m:n:k},x_{2:m:n:k},\ldots,x_{m:m:n:k})$, and $\underset{\sim}{Z}=\left(z_{11},\ldots,z_{1[k(G_1+1)-1]},\ldots,z_{m1},\ldots,z_{m[k(G_m+1)-1]}\right)$, respectively. The combined forms of complete sample is given by $\underset{\sim}{Y}=(\underset{\sim}{X},\underset{\sim}{Z})$. After ignoring the additive constant, we have log-likelihood function

$$L_c(\underset{\sim}{Y};\lambda)=-\frac{3nk}{2}\ln\lambda-\frac{1}{\lambda}\sum_{i=1}^{m}x_i^2-\frac{1}{\lambda}\sum_{j=1}^{k(G_i+1)-1}Z_{ij}^2. \tag{6.7}$$

We must compute the pseudo log-likelihood function for the E-step. It can be calculated from complete sample by replacing any Z_{ij} function, such as $\eta(Z_{ij})$, with $E\left[\eta(Z_{ij}|Z_{ij}>x_i)\right]$. As a result, the pseudo log-likelihood function is given as follows

$$L_c(\underset{\sim}{Y};\lambda)=-\frac{3nk}{2}\ln\,\lambda-\frac{1}{\lambda}\sum_{i=1}^{m}x_i^2-\frac{1}{\lambda}\sum_{j=1}^{k(G_i+1)-1}E\left[Z_{ik}^2|Z_{ij>w_i}\right]. \tag{6.8}$$

For given $X_i=x$, the conditional distribution of Z_{ij}, follows a truncated MW lifetime model with left truncation at x_i. That is,

$$f(Z_{ij}|x_i,\lambda)=\frac{f(Z_{ij},\lambda)}{1-F(x_i,\lambda)};\quad Z_{ij}>x_i\;\; i=\;1,2,\;\ldots\;,m\;\text{ and }\;j=\;1,2,\;\ldots\;,[k(G_i+1)-1]. \tag{6.9}$$

The conditional expectation in (6.9) can be defined in the following way:

$$A(c,\lambda)\;=\;E\left[\,Z_{ij}^2|Z_{ij}>c\right]=\frac{3\lambda}{2[1-F(x_i,\lambda)]}\left[1-\Gamma\left(\frac{c^2}{\lambda},\frac{3}{2}\right)\right]. \tag{6.10}$$

The M-step now implies trying to maximize of the pseudo log-likelihood function, with the appropriate value of (6.8) being replaced. If the rth stage estimate of λ is $\lambda^{(r)}$, the $(r+1)^{th}$ stage estimate $\lambda^{(r+1)}$ can be estimated by maximizing the following equation

$$L_c^*(\underset{\sim}{W};\lambda) = -\frac{3nk}{2}\,\ln\,\lambda - \frac{1}{\lambda}\,\sum_{i=1}^{m}[k(G_i+1)-1]\,A(x_i,\lambda^{(r)}). \tag{6.11}$$

The following equations are used to compute $\lambda^{(r+1)}$:

$$h(\lambda) = \lambda, \tag{6.12}$$

$$\text{where,}\quad h(\lambda) = \frac{2\left[\sum_{i=1}^{m} x_j + \sum_{i=1}^{m}[k(G_i+1)-1]A(x_i,\lambda^{(r)})\right]}{3nk}. \tag{6.13}$$

Then, in the next iteration, $\lambda^{(r+1)}$ is utilized as the new true value of λ. The ML estimate of λ is computed by replacing the E-step and M-step until convergence, given the starting value $\lambda^{(0)}$ of unknown parameter. Using the invariance features of ML estimates, the ML estimate of entropy $H(\lambda)$ is computed by simply plugin $\hat{\lambda}$ in (6.5. Thus the ML estimate of entropy is given by

$$\hat{H}(\hat{\lambda}) = \frac{1}{2}\ln\hat{\lambda} + \gamma + \frac{1}{2}\ln\pi - \frac{1}{2}.$$

6.3.2 Asymptotic Confidence Interval

The EM approach can be used to calculate the asymptotic variance-covariance (VC) matrix for ML estimates, see Louis (1982). For this purpose, the following concepts are used as follows:

$$\text{Observe information} \;=\; \text{Complete information} \;-\; \text{Missing infomation} \tag{6.14}$$

Now, let us define $\underset{\sim}{X}$ be observed data, $\underset{\sim}{Y}$ be complete data and I_X be the corresponding observed information, I_Y be the corresponding complete information, and $I_{Y|X}(\lambda)$ be the missing information. Then, the equation (6.14) can be expressed as follows:

$$I_X(\lambda) = I_Y(\lambda) - I_{Y|X}(\lambda). \tag{6.15}$$

The complete information I_Y is given by

$$I_Y(\lambda) = -E\left[\frac{\partial^2 L_c(Y;\lambda)}{\partial\lambda^2}\right] = \frac{3nk}{2\lambda^2}. \tag{6.16}$$

The Fisher information in one observation is supplied by x_i, which is censored at the moment of the ith failure.

$$I^i_{Y|X}(\lambda) = -E_{Z_i|x_i}\left[\frac{\partial^2 \ln f(Z_{ij}|x_i,\lambda)}{\partial\lambda^2}\right] = -\frac{3}{2}\frac{1}{\lambda} + \psi'(\lambda) + \frac{3}{\lambda^2[1-F(x_i,\lambda)]}\left[1-\Gamma\left(\frac{x_i^2}{\lambda},\frac{5}{2}\right)\right].$$

Therefore, the expected information for the conditional distribution of $Y|X$ (i.e. the missing information) is

$$I_{Y|X}(\lambda) = \sum_{i=1}^{m}[k(G_i+1)-1]I_{Y|X}(\lambda). \tag{6.17}$$

We now obtain the observed information matrix $I_X(\lambda)$ by substituting equations (6.16) and (6.17) in equation (6.14). Thus, $I_X^{-1}(\lambda) = \left[I_Y(\lambda) - I_{Y|X}(\lambda)\right]^{-1}$ can be used to derive the VC matrix of parameter λ. Thus, an approximate $(1-\alpha)100\%$ CIs for λ is obtained as $\hat{\lambda} \pm z_{\alpha/2}\sqrt{Var(\hat{\lambda})}$, here, $z_{\alpha/2}$ is the upper $(\alpha/2)^{th}$ percentile of N(0,1). Also, the coverage probability (CP) for λ is given by

$$CP_\lambda = \left[\left|\frac{\hat{\lambda}-\lambda}{\sqrt{\hat{V}ar(\hat{\lambda})}}\right| \le z_{\alpha/2}\right].$$

The delta approach is now used to construct the ACI of entropy $H(\lambda)$. Assuming $\hat{\lambda}$ is the ML estimate of unknown parameter λ, the asymptotic variance of $\hat{H}$ using the delta technique is given by (Krishnamoorthy and Lin, 2010).

$$Var(H) = [b'_C I_X^{-1}(\lambda) b_C],$$

here, $b_C = \frac{\partial H(\lambda)}{\partial \lambda} = \frac{1}{2\lambda}$.

Under moderate regularity criteria, the OFI matrix is used as a consistent estimator of the Fisher information. As a result, the observed variance of $\hat{H}$ is equal to

$$\hat{V}ar(\hat{H}) \simeq [b'_C I_X^{-1}(\lambda) b_C]_{\lambda=\hat{\lambda}}.$$

Thus $\frac{\hat{H}-H}{\sqrt{\hat{V}ar(\hat{H})}} \sim N(0,1)$. Therefore, $(1-\alpha)100\%$ ACI of H is given by $\hat{H} \pm z_{\alpha/2}\sqrt{\hat{V}ar(\hat{H})}$. Also, the coverage probability (CP) of H is given by

$$CP_H = \left[\left|\frac{\hat{H}-H}{\sqrt{\hat{V}ar(\hat{H})}}\right| \le z_{\alpha/2}\right].$$

6.3.3 Bootstrap Confidence Intervals

Using the similar concept as in (5.3.2), we construct the boot-p and boot-t CIs of associated parameter λ and entropy $H(\lambda)$ of MW lifetime model. Let $X_1, X_2, \ldots, X_m$ be a PFFC samples of effective sample size m drawn from MW(λ). Then the bootstrap procedures and algorithms will be same as we have already been discussed in subsection (5.3.2).

6.3.3.1 Boot-p Confidence Intervals

Let $\hat{\lambda}_{(j)}$ and $\hat{H}_{(j)}$; $j = 1, 2, \ldots, B$ denotes the ordered values of boot-p samples $\hat{\lambda}_j$ and $\hat{H}_j$, respectively. Thus, $(1-\alpha)100\%$ boot-p CIs of λ and H, respectively, are given by

$$\left(\hat{\lambda}^*_{[(\alpha/2)B]}, \hat{\lambda}^*_{[(1-\alpha/2)B]}\right) \text{ and } \left(\hat{H}^*_{[(\alpha/2)B]}, \hat{H}^*_{[(1-\alpha/2)B]}\right),$$

where, $[a]$ is the integral part of a.

6.3.3.2 Boot-t Confidence Intervals

Let $\left(\tau^*_{1(1)} \leq \tau_{1(2)} \leq \cdots \leq \tau_{1(B)}\right)$ and $\left(\tau^*_{1(1)} \leq \tau_{1(2)} \leq \cdots \leq \tau_{1(B)}\right)$ denotes the ordered values of boot-t samples $\tau^*_{i(j)}$ for $j = 1, 2, \ldots, B, i = 1, 2$, respectively. Thus, $(1-\alpha)100\%$ boot-t CIs for λ and $H(\lambda)$, respectively, are given by

$$\left(\hat{\lambda} - \tau_{1[(1-\alpha/2)B]}\sqrt{I_X^{-1}(\hat{\lambda})}, \hat{\lambda} - \tau_{1[(\alpha/2)B]}\sqrt{I_X^{-1}(\hat{\lambda})}\right),$$

$$\left(\hat{H} - \tau_{2[(1-\alpha/2)B]}\sqrt{I_X^{-1}(\hat{H})}, \hat{H} - \tau_{2[(\alpha/2)B]}\sqrt{I_X^{-1}(\hat{H})}\right).$$

6.4 Bayesian Estimation

In this section, we derive Bayes estimators and HPD credible intervals for the parameter λ and entropy $H(\lambda)$ using the LINEX loss function. The Bayesian approach to reliability inference necessitates the inclusion of experimental data, as well as prior belief in the parameters and technical knowledge of failure mechanisms, in the inferential methods. As a result, Bayesian methods are frequently used to small sample data, which is especially useful in the case of costly life testing studies. The inverted gamma distribution is a common natural conjugate prior density for the parameter λ of the MW lifetime model in Bayesian estimation, see (Bekker and Roux, 2005), (Chaudhary et al., 2017). Therefore, we consider the prior distribution of unknown parameter λ assumes to follow an inverted gamma distribution with the following pdf:

$$g(\lambda) \propto \frac{1}{\lambda^{a+1}} \exp(-b/\lambda); \quad \lambda > 0, a, b > 0, \tag{6.18}$$

a and b are hyper-parameters, respectively. The joint posterior distribution of λ is provided by employing the likelihood function in (6.6) and the prior distribution in (6.18).

$$\pi(\lambda|\underset{\sim}{X}) = \eta \frac{1}{\lambda^{\frac{3m}{2}+a+1}} exp\left[-\frac{1}{\lambda}\left(\sum_{i=1}^{m} x_i^2 + b\right)\right] \prod_{i=1}^{m}\left[1 - \Gamma\left(\frac{x_i^2}{\lambda}, \frac{3}{2}\right)\right]^{k(G_i+1)-1}, \tag{6.19}$$

where, η^{-1} is the normalizing constant and is given by

$$\eta^{-1} = \int_0^\infty \frac{1}{\lambda^{\frac{3m}{2}+a+1}} exp\left[-\frac{1}{\lambda}\left(\sum_{i=1}^{m} x_i^2 + b\right)\right] \prod_{i=1}^{m}\left[1 - \Gamma\left(\frac{x_i^2}{\lambda}, \frac{3}{2}\right)\right]^{k(G_i+1)-1} d\lambda.$$

We have already defined the LINEX loss function in Chapter 2. For convenience, we again define the LINEX loss function as follows:

$$L(\Delta) = e^{c\Delta} - c\Delta - 1; \quad c \neq 0, \quad \Delta = \hat{\lambda} - \lambda, \tag{6.20}$$

where c is the LINEX loss parameter. Under this loss function, the Bayes estimator of any function of the parameter λ, say $\phi(\lambda)$, is given by

$$E[\phi(\lambda)] = -\frac{1}{c} \ln \left[\frac{\int_0^\infty e^{-c\phi(\lambda)} \pi(\lambda|\underset{\sim}{X}) d\lambda}{\int_0^\infty \pi(\lambda|\underset{\sim}{X}) d\lambda}\right]. \tag{6.21}$$

As seen in the preceding equation (6.21), the Bayes estimators are in the form of a ratio of two integrals for which there is no closed form solution. It is feasible to get a numerical solution for the aforementioned integral ratio. We propose utilizing two approximation approaches to solve the aforementioned ratio of integrals: the TK and MCMC methods.

6.4.1 TK Approximation Method

According to TK approximation's method proposed by Tierney and Kadane (1986), the approximation of the posterior mean of $\phi(\lambda)$ is given by

$$E\left[\phi(\lambda)|\underset{\sim}{X}\right] = \frac{\int_0^\infty e^{n\delta_\phi^*(\lambda)}d\lambda}{\int_0^\infty e^{n\delta(\lambda)}d\lambda} \simeq \left(\frac{|\Sigma_\phi^*|}{|\Sigma|}\right)^{\frac{1}{2}} e^{n[\delta_\phi^*(\hat{\lambda}_\phi^*)-\delta(\hat{\lambda}_\phi)]} \tag{6.22}$$

where, $\delta(\lambda) = \frac{1}{n}[l(\lambda)+\rho(\lambda)]$, and $\delta^*(\lambda) = \delta(\lambda)+\frac{1}{n}\ln\phi(\lambda)$, here, $l(\lambda)$ is the log-likelihood function and $\rho(\lambda) = \ln g(\lambda)$. Also, $|\Sigma_\phi^*|$ and $|\Sigma|$ are the determinants of inverse of the negative hessian of $\delta^*(\lambda)$ and $\delta(\lambda)$ at $\hat{\lambda}_{\delta^*}$ and $\hat{\lambda}_\delta$, respectively. Also, $\hat{\lambda}_\delta$ and $\hat{\lambda}_{\delta^*}$ maximize $\delta(\lambda)$ and $\delta^*(\lambda)$, respectively. Next, we observe that

$$\delta(\lambda) = \frac{1}{n}\Bigg[-\left(\frac{3m}{2}+a+1\right)\ln\lambda - \frac{1}{\lambda}\left(\sum_{i=1}^m x_i^2+b\right)+2\sum_{i=1}^m \ln x_i + \sum_{i=1}^m [k(G_i+1)-1]\ln\left(1-\Gamma\left(\frac{x_i^2}{\lambda},\frac{3}{2}\right)\right)\Bigg]$$

Then, by solving the following non-linear equation, $\hat{\lambda}_\delta$ is computed:

$$\frac{\partial\delta(\lambda)}{\partial\lambda} = \frac{1}{n}\left[-\left(\frac{3m}{2}+a+1\right)\frac{1}{\lambda}+\frac{1}{\lambda^2}\left(\sum_{i=1}^m x_i^2+b\right)+\sum_{i=1}^m [k(G_i+1)-1]\psi(\lambda)\right],$$

where

$$\psi(\lambda) = \frac{\partial\ln\left[1-\Gamma\left(\frac{x_i^2}{\lambda},\frac{3}{2}\right)\right]}{\partial\lambda} = -\frac{3}{2\lambda\left[1-\Gamma\left(\frac{x_i^2}{\lambda},\frac{3}{2}\right)\right]}\left\{\frac{3}{2}\Gamma\left(\frac{x_i^2}{\lambda},\frac{5}{2}\right)-\Gamma\left(\frac{x_i^2}{\lambda},\frac{3}{2}\right)\right\}.$$

Now, obtain $|\Sigma|$ from $\Sigma^{-1} = \frac{1}{n}\left(-\frac{\partial^2\delta(\lambda)}{\partial\lambda^2}\right)$,

$$\Sigma^{-1} = -\frac{1}{n}\left[\left(\frac{3m}{2}+a+1\right)\frac{1}{\lambda^2}-\frac{2}{\lambda^3}\left(\sum_{i=1}^m x_i^2+b\right)+\sum_{i=1}^m [k(G_i+1)-1]\psi'(\lambda)\right],$$

where,

$$\psi'(\lambda) = \frac{\partial\psi(\lambda)}{\partial\lambda} = -\frac{3}{2\lambda^2}\left\{\frac{\left[\left\{1-\Gamma\left(x_i^2/\lambda,3/2\right)\right\}Q_1+Q_2\right]}{\left\{1-\Gamma\left(x_i/\lambda,3/2\right)\right\}}\right\},$$

where,

$$Q_1 = \frac{5}{2}\left[\Gamma\left(x_i^2/\lambda, 7/2\right) - 2\Gamma\left(x_i^2/\lambda, 5/2\right) + \Gamma\left(x_i^2/\lambda, 3/2\right)\right]$$

and

$$Q_2 = \frac{3}{2}\left[\Gamma\left(x_i^2/\lambda, 5/2\right) - \Gamma\left(x_i^2/\lambda, 3/2\right)\right].$$

We use $\phi(\lambda) = e^{-c\lambda}$ to compute the Bayes estimator of λ under the LINEX loss function, and the function $\delta*(\lambda)$ becomes

$$\delta^*(\lambda) = \delta(\lambda) - \frac{c\lambda}{n}.$$

Then, by solving the following non-linear equation, $\hat{\lambda}^*_{\delta^*}$ is obtained:

$$\frac{\partial\delta^*(\lambda)}{\partial\lambda} = \frac{\partial\delta(\lambda)}{\partial\lambda} - \frac{c}{n} = 0, \quad \text{and obtain} \quad |\Sigma^*| \ \text{ from } \ \Sigma_\lambda^{*-1} = -\frac{1}{n}\left(\frac{\partial^2\delta^*(\lambda)}{\partial\lambda^2}\right).$$

Under the LINEX loss function, the approximate Bayes estimator of λ is given by

$$\hat{\lambda}_{TK} = -\frac{1}{c}\ln\left[\left(\frac{|\Sigma_\lambda^*|}{|\Sigma|}\right)^{\frac{1}{2}} \exp\left\{n\left[\delta_\lambda^*(\hat{\lambda}_\delta^*) - \delta(\hat{\lambda}_\delta)\right]\right\}\right].$$

Similarly, the Bayes estimator of entropy $H(\lambda)$ is given by

$$\hat{H}_{TK} = -\frac{1}{c}\ln\left[\left(\frac{|\Sigma_H^*|}{|\Sigma|}\right)^{\frac{1}{2}} \exp\left\{n\left[\delta_H^*(\hat{H}_\delta^*) - \delta(\hat{H}_\delta)\right]\right\}\right].$$

6.4.2 MCMC Method

Here, the Bayes estimates of parameter and entropy are computed using the MCMC technique followed by M-H algorithm. We create a candidate points from a normal distribution to generate a sequence of sample from the posterior distribution of λ using data $\underset{\sim}{X}$ in (6.19). For the purpose of computation, the following steps are used:

Step 1: Begin with guess value $\lambda^{(0)}$ for λ.

Step 2: Create a candidate point $\lambda_c^{(j)}$ from the proposal density $\eta(\lambda^{(j)}|\lambda^{(j-1)})$.

Step 3: Create u from uniform $(0,1)$.

Step 4: Compute $\alpha(\lambda_c^{(j)}|\lambda^{(j-1)}) = \min\left\{\frac{\pi\left(\lambda_c^{(j)}|\underset{\sim}{w}\right)\eta\left(\lambda^{(j-1)}|\lambda_c^{(j)}\right)}{\pi\left(\lambda^{(j-1)}|\underset{\sim}{w}\right)\eta\left(\lambda_c^{(j)}|\lambda^{(j-1)}\right)}, 1\right\}$.

Step 5: If $u \leq \alpha$ set $\lambda^{(j)} = \lambda_c^{(j)}$ with acceptance probability α otherwise $\lambda^{(j)} = \lambda^{(j-1)}$.

Step 6: Compute $H^{(j)} = H(\lambda^{(j)})$ using (6.1).

Step 7: For $j = 1, 2, \ldots, M$, repeat steps 2-6 to get the sequence of the parameter λ as $(\lambda_1, \lambda_2, \ldots, \lambda_M)$ and the entropy H as $(H_1, H_2, \ldots, H_M)$, respectively.

To get an independent sample from the stationary distribution of the Markov chain, which is typically the posterior distribution, we discard first $\lambda_j^{\prime}s$ and $H_j^{\prime}s$, where, $M_0; j = 1, 2, \ldots, M_0$ is the burn-in-period. Therefore, the Bayes estimators of the parameter λ and entropy $H(\lambda)$ under the LINEX loss function, respectively, are given by

$$\hat{\lambda}_{MH} = -\frac{1}{c} \ln \left[\frac{1}{M - M_0} \sum_{j=M_0+1}^{M} e^{-c\lambda_j} \right],$$

$$\hat{H}_{MH} = -\frac{1}{c} \ln \left[\frac{1}{M - M_0} \sum_{j=M_0+1}^{M} e^{-cH(\lambda_j)} \right].$$

6.4.3 HPD Credible Interval Estimation

Using the generated MCMC samples based on the M-H algorithm, we now construct the HPD credible intervals for the parameter λ and entropy $H(\lambda)$. Let $\lambda_{(1)} < \lambda_{(2)} < \cdots < \lambda_{(M)}$ represent the ordered values of $\lambda_1, \lambda_2, \ldots, \lambda_M$. Then $(1-\alpha)100\%, \quad 0 < \alpha < 1$, HPD credible interval for λ is given by $\left(\lambda_{(j)}, \lambda_{(j+[(1-\alpha)M])}\right)$, where j is chosen such that

$$\lambda_{j+[(1-\alpha)M]} - \lambda_{(j)} = \min_{1 \le i \le \alpha M} \left(\lambda_{(i+[(1-\alpha)M])} - \lambda_{(i)}\right); \quad j = 1, 2, \ldots, M.$$

Similarly, HPD credible interval for $H(\lambda)$ is computed as $\left(H_{(j)}, H_{(j+[(1-\alpha)M])}\right)$, where j is chosen such that

$$H_{j+[(1-\alpha)M]} - H_{(j)} = \min_{1 \le i \le \alpha M} \left(H_{(i+[(1-\alpha)M])} - H_{(i)}\right); \quad j = 1, 2, \ldots, M.$$

6.5 Numerical Computations

In this section, we perform an extensive numerical computation in terms of an MC simulation to understand the impact of distinct estimators created in the previous sections. The impact of different estimators are evaluated by their average estimates (AE) and mean squared errors (MSE). The AE and MSEs of ML and Bayes estimators of the parameter and entropy are used in the MC simulation to determine the influence of different estimators established in the previous sections. The Bayes estimators of parameter and entropy are computed in the case of non-informative prior (Prior 0) and informative inverted gamma prior (Prior 1) under the LINEX loss function. Also, we obtain average lengths (AL) of 95% ACIs, bootstrap CIs, and HPD credible intervals with their corresponding coverage probabilities (CP). To obtain bootstrap CIs of the parameter λ and entropy $H(\lambda)$, we generate $B = 1000$ bootstrap samples for the prescribed sample under consideration.

For the simulation purpose, the PFFC samples are generated with several combinations of $(k,n,m,\underset{\sim}{G})$ for distinct values of associated parameter λ from MW(λ). To generate PFFC samples, we use the algorithm proposed by Balakrishnan and Sandhu (1995), with the addition that the PFFC sample $x_1,x_2,...,x_m$ can be viewed as a progressively censored sample from a population with cdf $\left[1-(1-F(x))^k\right]$, see Wu and Kuş (2009). Here, we considered two sets of parameter values $\theta = 0.75$ and $\theta = 1.5$, for which the corresponding entropy becomes 0.5057 and 0.8523, respectively. We consider group sizes $k = 3,5$, number of groups $n = 20,50$ with effective sample sizes $m = 40,80\%$ of n. For each n, we adopt three different censoring plans $\underset{\sim}{G}$ and these plans are common for each n. The different common failure plans $\underset{\sim}{G}$ for each effective sample m are as follows:

Plan 1: If $[(k,n,m),(G_1 = n-m, G_j = 0, \quad \forall \quad j = 2,3,\ldots m)]$, in this case $(n-m)$ groups are discarded from the experiment at the first failure only,

Plan 2: If $[(k,n,m),(G_j = 0, \quad \forall j = 1,2,\ldots,m-1, G_m = n-m)]$, in this case $(n-m)$ groups are discarded at mth failure, and

Plan 3: If $[(k,n=m),G_j = 0, \quad \forall j = 1,2,\ldots,m]$ this is the case of first failure censored sample.

The simplified notations are used for different combinations of censoring plans ([CS]) which are summaries in the Table 6.1. Note: the notation used in censoring schemes like $(0*7)$ denotes $(0,0,0,0,0,0,0)$ and $(4*3)$ stands for $(4,4,4)$. For Bayesian calculations of parameter

TABLE 6.1: Several combinations of censoring plans.

n	m	[CS]	Schemes	n	m	[CS]	Schemes
20	8	[1]	(12*1,0*7)	50	20	[7]	(30*1,0*19)
		[2]	(4*3,0*5)			[8]	(5*6,0*14)
		[3]	(0*7,12*1)			[9]	(0*19,30*1)
20	16	[4]	(4*1,0*15)	50	40	[10]	(10*1,0*39)
		[5]	(2*2,0*14)			[11]	(5*2,0*38)
		[6]	(0*15,4*1)			[12]	(0*39,10*1)

and entropy, the hyper-parameters (a,b) are chosen in such a way that the prior mean is exactly equal to the true values of the parameter, i.e. $\lambda = \frac{a}{b}$. Here, we consider $(a,b) = (3,4)$ and $(3,2)$ for $\theta = 0.75$ and 1.5, respectively. In case of Prior 0, hyper-parameters are taken as $(a,b) = (0.0001,0.0001)$. In order to derive Bayes estimators under LINEX loss function, we consider two choice of loss function parameter as $c = (-0.5$ and $0.5)$. Two approximation techniques such as TK approximation and M-H algorithm are used for Bayesian computations. For M-H algorithm, we generate $M = 10,000$ samples and $M_0 = 20\%$ of M is considered as burn-in period. All the simulated results for several combinations of censoring plans are summarizes in the following Tables 6.2, 6.3, 6.4, 6.5, 6.6, 6.7. These findings lead to the following conclusions:

In view of Tables 6.2, 6.3, 6.5, 6.6, this experiment has brought up some interesting observations. In almost all cases, the ML and Bayes estimates output of parameter and entropy in terms of MSEs are very adequate even for small sample sizes. MSEs are found to decrease as n or m rise. It checks the consistent behavior of the estimators. Also, the performance of Bayes estimators with Prior 1 is better than ML estimators even with Prior 0 in terms of MSEs, as Bayes estimators with Prior 1 includes some prior information. Also, Bayes estimators computed using M–H algorithm outperform the TK approximation procedure.

With the reference of Tables 6.4 and 6.7, the average lengths (AL) of ACIs, boot-p, boot-t CIs, and HPD credible intervals are shrinking with an increase in the number of failures m. It is also observed that the HPD credible intervals with Prior 1 have the smallest ALs as compared to others. It is also fair to say that all four intervals have reasonable coverage probabilities. Also, it is seen that the ALs of boot-p confidence intervals outperform as it has smaller ALs to ACI and boot-t both.

TABLE 6.2: Average ML and Bayes estimates of parameter λ, when $\lambda = 0.75$.

		$\hat{\lambda}_{ML}$		$\hat{\lambda}_{TK}$								$\hat{\lambda}_{MH}$							
				$c = -0.5$				$c = 0.5$				$c = -0.5$				$c = 0.5$			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	[CS]	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	[1]	0.7545	0.0404	0.8146	0.0529	0.9171	0.0598	0.7880	0.0442	0.8900	0.0479	0.7704	0.0092	0.8151	0.0120	0.7616	0.0085	0.8065	0.0106
	[2]	0.7538	0.0395	0.8132	0.0514	0.9142	0.0582	0.7872	0.0432	0.8877	0.0468	0.7700	0.0090	0.8140	0.0117	0.7613	0.0083	0.8055	0.0104
	[3]	0.7519	0.0365	0.8086	0.0470	0.9044	0.0532	0.7848	0.0399	0.8799	0.0432	0.7685	0.0083	0.8102	0.0107	0.7605	0.0077	0.8023	0.0096
	[4]	0.7563	0.0205	0.7885	0.0241	0.8454	0.0272	0.7762	0.0219	0.8328	0.0240	0.7627	0.0050	0.7893	0.0061	0.7579	0.0048	0.7846	0.0056
	[5]	0.7563	0.0204	0.7884	0.0240	0.8452	0.0272	0.7762	0.0219	0.8327	0.0239	0.7627	0.0050	0.7892	0.0061	0.7578	0.0048	0.7845	0.0056
	[6]	0.7558	0.0197	0.7872	0.0231	0.8424	0.0261	0.7755	0.0211	0.8303	0.0230	0.7623	0.0048	0.7881	0.0058	0.7576	0.0046	0.7835	0.0054
	[7]	0.7553	0.0166	0.7813	0.0189	0.8276	0.0211	0.7719	0.0176	0.8179	0.0190	0.7605	0.0041	0.7825	0.0048	0.7567	0.0039	0.7787	0.0045
	[8]	0.7550	0.0162	0.7807	0.0184	0.8261	0.0205	0.7714	0.0171	0.8166	0.0185	0.7603	0.0040	0.7818	0.0047	0.7565	0.0038	0.7781	0.0044
	[9]	0.7539	0.0148	0.7783	0.0167	0.8204	0.0186	0.7698	0.0157	0.8117	0.0169	0.7595	0.0037	0.7795	0.0043	0.7560	0.0035	0.7760	0.0040
	[10]	0.7518	0.0088	0.7650	0.0094	0.7895	0.0100	0.7606	0.0091	0.7850	0.0094	0.7552	0.0022	0.7671	0.0024	0.7532	0.0022	0.7651	0.0023
	[11]	0.7518	0.0088	0.7650	0.0094	0.7895	0.0100	0.7606	0.0091	0.7849	0.0094	0.7552	0.0022	0.7671	0.0024	0.7532	0.0022	0.7651	0.0023
	[12]	0.7515	0.0085	0.7644	0.0090	0.7881	0.0096	0.7602	0.0087	0.7837	0.0091	0.7550	0.0021	0.7665	0.0023	0.7531	0.0021	0.7646	0.0022
5	[1]	0.7532	0.0384	0.8115	0.0498	0.9107	0.0563	0.7864	0.0420	0.8848	0.0455	0.7695	0.0087	0.8126	0.0113	0.7610	0.0081	0.8044	0.0101
	[2]	0.7526	0.0376	0.8104	0.0486	0.9083	0.0550	0.7857	0.0411	0.8829	0.0445	0.7691	0.0085	0.8117	0.0111	0.7608	0.0079	0.8035	0.0099
	[3]	0.7508	0.0351	0.8065	0.0449	0.9000	0.0509	0.7836	0.0384	0.8763	0.0415	0.7678	0.0079	0.8084	0.0103	0.7600	0.0074	0.8007	0.0092
	[4]	0.7555	0.0194	0.7867	0.0227	0.8412	0.0256	0.7751	0.0207	0.8293	0.0226	0.7621	0.0048	0.7876	0.0057	0.7575	0.0046	0.7831	0.0053
	[5]	0.7555	0.0194	0.7866	0.0226	0.8411	0.0256	0.7751	0.0207	0.8292	0.0226	0.7621	0.0048	0.7875	0.0057	0.7575	0.0046	0.7830	0.0053
	[6]	0.7550	0.0188	0.7856	0.0219	0.8386	0.0247	0.7744	0.0200	0.8271	0.0219	0.7617	0.0046	0.7865	0.0055	0.7573	0.0044	0.7821	0.0051
	[7]	0.7547	0.0157	0.7799	0.0179	0.8242	0.0199	0.7709	0.0167	0.8150	0.0180	0.7600	0.0039	0.7811	0.0046	0.7563	0.0038	0.7774	0.0043
	[8]	0.7544	0.0154	0.7794	0.0175	0.8230	0.0194	0.7706	0.0163	0.8139	0.0176	0.7599	0.0038	0.7805	0.0044	0.7562	0.0037	0.7769	0.0042
	[9]	0.7534	0.0143	0.7773	0.0161	0.8182	0.0178	0.7692	0.0151	0.8098	0.0162	0.7591	0.0035	0.7785	0.0041	0.7558	0.0034	0.7751	0.0039
	[10]	0.7514	0.0084	0.7643	0.0089	0.7876	0.0094	0.7601	0.0086	0.7833	0.0089	0.7549	0.0021	0.7663	0.0023	0.7530	0.0020	0.7644	0.0022
	[11]	0.7514	0.0084	0.7642	0.0089	0.7876	0.0094	0.7601	0.0086	0.7833	0.0089	0.7549	0.0021	0.7663	0.0023	0.7530	0.0020	0.7644	0.0022
	[12]	0.7511	0.0081	0.7638	0.0086	0.7863	0.0091	0.7597	0.0083	0.7822	0.0086	0.7547	0.0020	0.7657	0.0022	0.7529	0.0020	0.7639	0.0021

TABLE 6.3: Average ML and Bayes estimates of entropy H, when $\lambda = 0.75$ and $H = 0.5057$.

		$\hat{\lambda}_{ML}$		$\hat{\lambda}_{TK}$								$\hat{\lambda}_{MH}$							
				$c=-0.5$				$c=0.5$				$c=-0.5$				$c=0.5$			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	[CS]	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	[1]	0.4906	0.0189	0.5126	0.0187	0.5823	0.0151	0.5033	0.0187	0.5746	0.0139	0.5073	0.0033	0.5371	0.0037	0.5036	0.0033	0.5338	0.0035
	[2]	0.4905	0.0185	0.5125	0.0183	0.5811	0.0148	0.5034	0.0183	0.5735	0.0137	0.5073	0.0032	0.5365	0.0036	0.5036	0.0032	0.5333	0.0035
	[3]	0.4905	0.0172	0.5120	0.0170	0.5768	0.0138	0.5035	0.0170	0.5696	0.0129	0.5071	0.0030	0.5347	0.0034	0.5037	0.0030	0.5317	0.0032
	[4]	0.5009	0.0091	0.5126	0.0091	0.5507	0.0083	0.5079	0.0091	0.5465	0.0079	0.5074	0.0020	0.5252	0.0021	0.5053	0.0020	0.5232	0.0021
	[5]	0.5009	0.0091	0.5126	0.0091	0.5506	0.0083	0.5079	0.0091	0.5464	0.0079	0.5074	0.0020	0.5251	0.0021	0.5053	0.0020	0.5232	0.0020
	[6]	0.5009	0.0088	0.5124	0.0088	0.5493	0.0080	0.5079	0.0088	0.5453	0.0077	0.5074	0.0019	0.5246	0.0020	0.5054	0.0019	0.5227	0.0020
	[7]	0.5020	0.0074	0.5114	0.0074	0.5424	0.0068	0.5077	0.0074	0.5390	0.0066	0.5073	0.0016	0.5219	0.0017	0.5056	0.0016	0.5203	0.0017
	[8]	0.5019	0.0072	0.5113	0.0072	0.5417	0.0067	0.5077	0.0072	0.5384	0.0064	0.5072	0.0016	0.5215	0.0017	0.5056	0.0016	0.5200	0.0017
	[9]	0.5018	0.0066	0.5110	0.0066	0.5391	0.0062	0.5076	0.0066	0.5360	0.0060	0.5071	0.0015	0.5204	0.0016	0.5056	0.0015	0.5190	0.0015
	[10]	0.5030	0.0039	0.5078	0.0039	0.5242	0.0037	0.5060	0.0039	0.5225	0.0037	0.5063	0.0009	0.5142	0.0009	0.5054	0.0009	0.5134	0.0009
	[11]	0.5030	0.0039	0.5078	0.0039	0.5242	0.0037	0.5060	0.0039	0.5224	0.0037	0.5063	0.0009	0.5142	0.0009	0.5054	0.0009	0.5134	0.0009
	[12]	0.5030	0.0038	0.5077	0.0038	0.5235	0.0036	0.5060	0.0038	0.5218	0.0035	0.5062	0.0009	0.5139	0.0009	0.5054	0.0009	0.5131	0.0009
5	[1]	0.4905	0.0180	0.5123	0.0178	0.5795	0.0144	0.5034	0.0178	0.5721	0.0134	0.5072	0.0032	0.5359	0.0036	0.5036	0.0032	0.5327	0.0034
	[2]	0.4904	0.0177	0.5122	0.0175	0.5785	0.0142	0.5035	0.0175	0.5712	0.0132	0.5072	0.0031	0.5354	0.0035	0.5036	0.0031	0.5323	0.0033
	[3]	0.4903	0.0166	0.5118	0.0164	0.5749	0.0134	0.5036	0.0164	0.5679	0.0125	0.5070	0.0029	0.5339	0.0033	0.5037	0.0029	0.5309	0.0031
	[4]	0.5008	0.0087	0.5123	0.0087	0.5488	0.0079	0.5079	0.0086	0.5448	0.0076	0.5073	0.0019	0.5243	0.0020	0.5054	0.0019	0.5225	0.0020
	[5]	0.5008	0.0086	0.5123	0.0087	0.5488	0.0079	0.5079	0.0086	0.5447	0.0076	0.5073	0.0019	0.5243	0.0020	0.5054	0.0019	0.5225	0.0020
	[6]	0.5008	0.0084	0.5122	0.0084	0.5476	0.0077	0.5079	0.0084	0.5437	0.0074	0.5073	0.0018	0.5238	0.0020	0.5054	0.0018	0.5220	0.0019
	[7]	0.5019	0.0070	0.5112	0.0070	0.5409	0.0065	0.5077	0.0070	0.5376	0.0063	0.5072	0.0016	0.5212	0.0017	0.5056	0.0016	0.5197	0.0016
	[8]	0.5018	0.0069	0.5111	0.0069	0.5403	0.0064	0.5077	0.0069	0.5371	0.0062	0.5072	0.0015	0.5209	0.0016	0.5056	0.0015	0.5194	0.0016
	[9]	0.5016	0.0064	0.5108	0.0064	0.5381	0.0060	0.5076	0.0064	0.5351	0.0058	0.5070	0.0014	0.5199	0.0015	0.5056	0.0014	0.5185	0.0015
	[10]	0.5029	0.0038	0.5077	0.0038	0.5233	0.0035	0.5060	0.0037	0.5216	0.0035	0.5062	0.0009	0.5138	0.0009	0.5054	0.0009	0.5130	0.0009
	[11]	0.5029	0.0038	0.5077	0.0037	0.5233	0.0035	0.5060	0.0037	0.5216	0.0035	0.5062	0.0009	0.5138	0.0009	0.5054	0.0009	0.5130	0.0009
	[12]	0.5029	0.0036	0.5076	0.0036	0.5227	0.0034	0.5059	0.0036	0.5211	0.0034	0.5062	0.0009	0.5135	0.0009	0.5054	0.0009	0.5128	0.0009

TABLE 6.4: The 95% ACI, bootstrap CI and HPD credible intervals of parameter λ and entropy $H(\lambda)$, when $\lambda = 0.75$ and $H = 0.5057$.

		ACI				Bootstrap								HPD							
						λ				$H(\lambda)$				λ				$H(\lambda)$			
		λ		$H(\lambda)$		Boot-p		Boot-t		Boot-p		Boot-t		Prior 0		Prior 1		Prior 0		Prior 1	
k	[CS]	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AE	MSE	AL	CP	AL	CP
3	[1]	0.7932	0.912	0.5219	0.948	0.7799	0.927	0.9120	0.952	0.5256	0.927	0.5312	0.952	0.5075	0.9550	0.5050	0.9530	0.3363	0.9540	0.3158	0.947
	[2]	0.7836	0.912	0.5163	0.947	0.7706	0.927	0.9003	0.952	0.5207	0.927	0.5258	0.952	0.5022	0.9550	0.5004	0.9520	0.3329	0.9540	0.3133	0.948
	[3]	0.7530	0.914	0.4981	0.951	0.7403	0.925	0.8678	0.951	0.5055	0.925	0.5089	0.951	0.4855	0.9550	0.4854	0.9530	0.3220	0.9540	0.3051	0.948
	[4]	0.5667	0.935	0.3746	0.948	0.5617	0.945	0.5906	0.950	0.3721	0.945	0.3722	0.950	0.3789	0.9560	0.3767	0.9520	0.2514	0.9530	0.2413	0.948
	[5]	0.5662	0.935	0.3743	0.948	0.5612	0.945	0.5900	0.950	0.3719	0.945	0.3718	0.950	0.3786	0.9560	0.3764	0.9520	0.2512	0.9530	0.2411	0.948
	[6]	0.5562	0.936	0.3680	0.945	0.5513	0.945	0.5797	0.951	0.3657	0.946	0.3658	0.951	0.3725	0.9570	0.3707	0.9520	0.2472	0.9540	0.2377	0.947
	[7]	0.5050	0.943	0.3343	0.945	0.5240	0.955	0.5420	0.954	0.3458	0.955	0.3459	0.954	0.3410	0.9570	0.3394	0.9520	0.2265	0.9530	0.2189	0.949
	[8]	0.4988	0.943	0.3304	0.945	0.5177	0.956	0.5347	0.954	0.3419	0.956	0.3416	0.954	0.3371	0.9570	0.3358	0.9520	0.2239	0.9540	0.2167	0.949
	[9]	0.4772	0.942	0.3165	0.944	0.4938	0.954	0.5112	0.953	0.3273	0.954	0.3273	0.953	0.3234	0.9560	0.3228	0.9520	0.2149	0.9520	0.2089	0.949
	[10]	0.3566	0.938	0.2372	0.941	0.3708	0.950	0.3802	0.950	0.2472	0.950	0.2472	0.950	0.2459	0.9490	0.2452	0.9470	0.1637	0.9470	0.1607	0.943
	[11]	0.3565	0.938	0.2371	0.941	0.3707	0.950	0.3801	0.950	0.2471	0.950	0.2471	0.950	0.2458	0.9490	0.2451	0.9470	0.1636	0.9470	0.1606	0.943
	[12]	0.3498	0.939	0.2327	0.942	0.3637	0.950	0.3731	0.950	0.2427	0.950	0.2427	0.950	0.2413	0.9480	0.2407	0.9470	0.1607	0.9440	0.1579	0.945
5	[1]	0.7726	0.913	0.5098	0.948	0.7598	0.925	0.8892	0.952	0.5151	0.925	0.5201	0.952	0.4962	0.9560	0.4951	0.9520	0.3290	0.9540	0.3104	0.948
	[2]	0.7646	0.912	0.5052	0.947	0.7515	0.924	0.8799	0.953	0.5107	0.924	0.5158	0.953	0.4918	0.9560	0.4913	0.9520	0.3261	0.9540	0.3083	0.948
	[3]	0.7383	0.914	0.4895	0.950	0.7256	0.923	0.8526	0.951	0.4979	0.923	0.5011	0.951	0.4772	0.9550	0.4783	0.9530	0.3167	0.9540	0.3012	0.948
	[4]	0.5517	0.936	0.3651	0.946	0.5464	0.945	0.5748	0.951	0.3628	0.945	0.3629	0.951	0.3697	0.9570	0.3681	0.9520	0.2453	0.9530	0.2362	0.949
	[5]	0.5513	0.937	0.3649	0.946	0.5460	0.945	0.5743	0.951	0.3625	0.945	0.3626	0.951	0.3694	0.9570	0.3679	0.9520	0.2452	0.9530	0.2361	0.949
	[6]	0.5424	0.936	0.3592	0.945	0.5377	0.946	0.5660	0.951	0.3576	0.946	0.3577	0.951	0.3640	0.9570	0.3628	0.9520	0.2416	0.9530	0.2331	0.95
	[7]	0.4920	0.943	0.3260	0.946	0.5094	0.953	0.5271	0.953	0.3369	0.953	0.3370	0.953	0.3328	0.9570	0.3317	0.9520	0.2210	0.9540	0.2143	0.949
	[8]	0.4870	0.943	0.3228	0.946	0.5043	0.954	0.5214	0.953	0.3336	0.954	0.3336	0.953	0.3296	0.9570	0.3287	0.9520	0.2189	0.9540	0.2125	0.949
	[9]	0.4684	0.941	0.3109	0.944	0.4840	0.952	0.5017	0.954	0.3215	0.952	0.3214	0.954	0.3178	0.9560	0.3175	0.9520	0.2112	0.9530	0.2056	0.949
	[10]	0.3474	0.938	0.2312	0.942	0.3609	0.950	0.3702	0.950	0.2409	0.950	0.2409	0.950	0.2397	0.9480	0.2392	0.9470	0.1596	0.9460	0.1569	0.945
	[11]	0.3473	0.938	0.2311	0.942	0.3608	0.950	0.3701	0.950	0.2408	0.950	0.2408	0.950	0.2396	0.9480	0.2391	0.9470	0.1596	0.9460	0.1568	0.945
	[12]	0.3413	0.940	0.2272	0.942	0.3546	0.949	0.3638	0.950	0.2368	0.949	0.2369	0.950	0.2357	0.9490	0.2353	0.9470	0.1569	0.9460	0.1544	0.945

TABLE 6.5: Average ML and Bayes estimates of λ, when $\lambda = 1.5$.

		MLE		TK								MH							
				$c=-0.5$				$c=0.5$				$c=-0.5$				$c=0.5$			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	[CS]	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	[1]	1.5091	0.1618	1.6600	0.2363	1.4669	0.1355	1.5527	0.1645	1.3987	0.1216	1.5499	0.0387	1.4760	0.0209	1.5146	0.0328	1.4495	0.0211
	[2]	1.5077	0.1579	1.6563	0.2291	1.4676	0.1330	1.5518	0.1609	1.4006	0.1196	1.5488	0.0377	1.4764	0.0206	1.5143	0.0320	1.4503	0.0208
	[3]	1.5039	0.1460	1.6446	0.2075	1.4691	0.1247	1.5486	0.1494	1.4065	0.1130	1.5453	0.0346	1.4774	0.0196	1.5131	0.0298	1.4528	0.0198
	[4]	1.5126	0.0819	1.5900	0.1015	1.4936	0.0746	1.5410	0.0841	1.4549	0.0692	1.5302	0.0207	1.4884	0.0146	1.5110	0.0189	1.4720	0.0146
	[5]	1.5124	0.0817	1.5899	0.1013	1.4936	0.0745	1.5410	0.0840	1.4550	0.0691	1.5302	0.0206	1.4884	0.0146	1.5109	0.0188	1.4720	0.0146
	[6]	1.5116	0.0789	1.5869	0.0972	1.4939	0.0722	1.5399	0.0811	1.4565	0.0671	1.5292	0.0199	1.4888	0.0142	1.5106	0.0182	1.4728	0.0142
	[7]	1.5106	0.0663	1.5725	0.0788	1.4964	0.0616	1.5347	0.0680	1.4650	0.0578	1.5250	0.0167	1.4910	0.0126	1.5095	0.0155	1.4773	0.0126
	[8]	1.5099	0.0647	1.5710	0.0767	1.4966	0.0602	1.5341	0.0663	1.4659	0.0566	1.5245	0.0163	1.4912	0.0124	1.5093	0.0152	1.4778	0.0123
	[9]	1.5078	0.0593	1.5653	0.0695	1.4970	0.0556	1.5316	0.0608	1.4688	0.0525	1.5225	0.0149	1.4919	0.0115	1.5085	0.0139	1.4795	0.0115
	[10]	1.5035	0.0353	1.5346	0.0384	1.4972	0.0340	1.5168	0.0357	1.4810	0.0329	1.5124	0.0089	1.4947	0.0077	1.5044	0.0086	1.4872	0.0077
	[11]	1.5035	0.0352	1.5346	0.0383	1.4972	0.0340	1.5168	0.0357	1.4810	0.0329	1.5124	0.0089	1.4947	0.0077	1.5044	0.0086	1.4872	0.0077
	[12]	1.5030	0.0339	1.5332	0.0368	1.4972	0.0328	1.5161	0.0344	1.4816	0.0318	1.5119	0.0085	1.4949	0.0074	1.5042	0.0082	1.4876	0.0074
5	[1]	1.5064	0.1535	1.6521	0.2210	1.5937	0.1384	1.5506	0.1566	1.4027	0.1172	1.5475	0.0365	1.5231	0.0243	1.5138	0.0312	1.4512	0.0205
	[2]	1.5051	0.1503	1.6491	0.2154	1.5920	0.1359	1.5498	0.1536	1.4043	0.1155	1.5466	0.0357	1.5227	0.0239	1.5135	0.0306	1.4519	0.0202
	[3]	1.5015	0.1403	1.6391	0.1976	1.5864	0.1278	1.5470	0.1440	1.4093	0.1098	1.5435	0.0331	1.5213	0.0226	1.5125	0.0286	1.4541	0.0193
	[4]	1.5110	0.0776	1.5857	0.0954	1.5606	0.0749	1.5394	0.0798	1.4573	0.0662	1.5288	0.0195	1.5176	0.0156	1.5104	0.0179	1.4732	0.0140
	[5]	1.5109	0.0775	1.5856	0.0952	1.5605	0.0748	1.5393	0.0797	1.4573	0.0661	1.5287	0.0195	1.5176	0.0155	1.5104	0.0179	1.4732	0.0140
	[6]	1.5100	0.0750	1.5830	0.0918	1.5589	0.0726	1.5383	0.0772	1.4586	0.0644	1.5279	0.0188	1.5171	0.0151	1.5101	0.0173	1.4739	0.0137
	[7]	1.5093	0.0630	1.5691	0.0744	1.5502	0.0613	1.5333	0.0646	1.4668	0.0553	1.5238	0.0159	1.5151	0.0132	1.5090	0.0148	1.4783	0.0121
	[8]	1.5087	0.0616	1.5679	0.0727	1.5494	0.0602	1.5328	0.0633	1.4676	0.0543	1.5234	0.0155	1.5149	0.0129	1.5089	0.0145	1.4787	0.0119
	[9]	1.5067	0.0571	1.5631	0.0667	1.5461	0.0559	1.5307	0.0587	1.4701	0.0509	1.5217	0.0143	1.5139	0.0121	1.5082	0.0134	1.4801	0.0111
	[10]	1.5028	0.0335	1.5329	0.0363	1.5246	0.0330	1.5160	0.0339	1.4819	0.0314	1.5118	0.0084	1.5078	0.0076	1.5041	0.0081	1.4878	0.0073
	[11]	1.5028	0.0334	1.5328	0.0363	1.5246	0.0330	1.5160	0.0339	1.4819	0.0314	1.5117	0.0084	1.5078	0.0076	1.5041	0.0081	1.4878	0.0073
	[12]	1.5023	0.0324	1.5317	0.0350	1.5237	0.0319	1.5154	0.0328	1.4824	0.0304	1.5113	0.0081	1.5075	0.0074	1.5039	0.0079	1.4881	0.0071

TABLE 6.6: Average ML and Bayes estimates of entropy H, when $\lambda = 1.5$ and $H = 0.8523$.

		MLE		TK								MH							
				$c=-0.5$				$c=0.5$				$c=-0.5$				$c=0.5$			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	[CS]	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	[1]	0.8372	0.0189	0.8592	0.0187	0.8066	0.0174	0.8499	0.0187	0.7991	0.0181	0.8539	0.0033	0.8329	0.0024	0.8501	0.0033	0.8298	0.0025
	[2]	0.8371	0.0185	0.8590	0.0183	0.8074	0.0171	0.8499	0.0183	0.8000	0.0178	0.8538	0.0032	0.8333	0.0024	0.8502	0.0032	0.8302	0.0025
	[3]	0.8371	0.0172	0.8586	0.0170	0.8101	0.0160	0.8501	0.0170	0.8031	0.0166	0.8537	0.0030	0.8342	0.0023	0.8502	0.0030	0.8314	0.0024
	[4]	0.8475	0.0091	0.8592	0.0091	0.8305	0.0087	0.8545	0.0091	0.8263	0.0089	0.8540	0.0020	0.8413	0.0016	0.8519	0.0020	0.8394	0.0017
	[5]	0.8475	0.0091	0.8592	0.0091	0.8305	0.0087	0.8545	0.0091	0.8263	0.0089	0.8540	0.0020	0.8413	0.0016	0.8519	0.0020	0.8394	0.0017
	[6]	0.8475	0.0088	0.8590	0.0088	0.8312	0.0084	0.8545	0.0088	0.8272	0.0086	0.8540	0.0019	0.8416	0.0016	0.8519	0.0019	0.8398	0.0016
	[7]	0.8485	0.0074	0.8580	0.0074	0.8348	0.0071	0.8543	0.0074	0.8314	0.0073	0.8538	0.0016	0.8433	0.0014	0.8521	0.0016	0.8418	0.0014
	[8]	0.8485	0.0072	0.8579	0.0072	0.8353	0.0070	0.8543	0.0072	0.8320	0.0071	0.8538	0.0016	0.8435	0.0014	0.8521	0.0016	0.8420	0.0014
	[9]	0.8483	0.0066	0.8575	0.0066	0.8366	0.0064	0.8542	0.0066	0.8336	0.0065	0.8537	0.0015	0.8442	0.0013	0.8522	0.0015	0.8428	0.0013
	[10]	0.8496	0.0039	0.8544	0.0039	0.8425	0.0039	0.8525	0.0039	0.8407	0.0039	0.8529	0.0009	0.8472	0.0009	0.8520	0.0009	0.8463	0.0009
	[11]	0.8496	0.0039	0.8544	0.0039	0.8425	0.0039	0.8525	0.0039	0.8407	0.0039	0.8529	0.0009	0.8472	0.0009	0.8520	0.0009	0.8463	0.0009
	[12]	0.8495	0.0038	0.8543	0.0038	0.8428	0.0037	0.8525	0.0038	0.8411	0.0038	0.8528	0.0009	0.8474	0.0008	0.8520	0.0009	0.8465	0.0008
5	[1]	0.8371	0.0180	0.8589	0.0178	0.8505	0.0124	0.8500	0.0178	0.8011	0.0174	0.8538	0.0032	0.8483	0.0023	0.8502	0.0032	0.8306	0.0025
	[2]	0.8370	0.0177	0.8588	0.0175	0.8504	0.0122	0.8500	0.0175	0.8020	0.0171	0.8537	0.0031	0.8484	0.0023	0.8502	0.0031	0.8310	0.0024
	[3]	0.8369	0.0166	0.8583	0.0164	0.8503	0.0117	0.8502	0.0164	0.8046	0.0161	0.8536	0.0029	0.8485	0.0022	0.8502	0.0029	0.8319	0.0023
	[4]	0.8474	0.0087	0.8589	0.0087	0.8536	0.0072	0.8545	0.0086	0.8276	0.0085	0.8539	0.0019	0.8511	0.0016	0.8519	0.0019	0.8400	0.0016
	[5]	0.8474	0.0086	0.8589	0.0087	0.8536	0.0072	0.8545	0.0086	0.8276	0.0085	0.8539	0.0019	0.8511	0.0016	0.8519	0.0019	0.8400	0.0016
	[6]	0.8474	0.0084	0.8587	0.0084	0.8536	0.0070	0.8545	0.0084	0.8283	0.0082	0.8539	0.0018	0.8511	0.0015	0.8520	0.0018	0.8403	0.0016
	[7]	0.8485	0.0070	0.8578	0.0070	0.8535	0.0061	0.8543	0.0070	0.8325	0.0069	0.8538	0.0016	0.8515	0.0013	0.8522	0.0016	0.8422	0.0014
	[8]	0.8484	0.0069	0.8577	0.0069	0.8535	0.0060	0.8543	0.0069	0.8329	0.0068	0.8537	0.0015	0.8515	0.0013	0.8522	0.0015	0.8424	0.0014
	[9]	0.8482	0.0064	0.8574	0.0064	0.8534	0.0056	0.8542	0.0064	0.8342	0.0063	0.8536	0.0014	0.8515	0.0012	0.8521	0.0014	0.8431	0.0013
	[10]	0.8495	0.0038	0.8543	0.0038	0.8521	0.0035	0.8525	0.0037	0.8413	0.0037	0.8528	0.0009	0.8517	0.0008	0.8520	0.0009	0.8466	0.0008
	[11]	0.8495	0.0038	0.8543	0.0037	0.8521	0.0035	0.8525	0.0037	0.8413	0.0037	0.8528	0.0009	0.8517	0.0008	0.8520	0.0009	0.8466	0.0008
	[12]	0.8495	0.0036	0.8542	0.0036	0.8520	0.0034	0.8525	0.0036	0.8416	0.0036	0.8528	0.0009	0.8517	0.0008	0.8520	0.0009	0.8468	0.0008

TABLE 6.7: The 95% ACI, bootstrap CI and HPD credible intervals of parameter λ and entropy H, when $\lambda = 1.5$ and $H = 0.8523$.

		ACI				Bootstrap								HPD							
						θ				$H(\theta)$				θ				$H(\theta)$			
		θ		$H(\theta)$		Boot-p		Boot-t		Boot-p		Boot-t		Prior 0		Prior 1		Prior 0		Prior 1	
k	[CS]	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AE	MSE	AL	CP	AL	CP
3	[1]	1.5864	0.912	0.5257	0.948	1.5597	0.927	1.8239	0.952	0.5324	0.927	0.5335	0.952	1.0150	0.9550	0.8815	0.9600	0.3363	0.9540	0.3055	0.9590
	[2]	1.5672	0.912	0.5198	0.947	1.5411	0.927	1.8007	0.952	0.5271	0.927	0.5279	0.952	1.0044	0.9550	0.8740	0.9600	0.3329	0.9540	0.3027	0.9590
	[3]	1.5059	0.914	0.5007	0.951	1.4806	0.925	1.7356	0.951	0.5105	0.925	0.5103	0.951	0.9710	0.9550	0.8506	0.9600	0.3220	0.9540	0.2941	0.9590
	[4]	1.1333	0.935	0.3746	0.948	1.1233	0.945	1.1811	0.950	0.3721	0.945	0.3721	0.950	0.7579	0.9560	0.7004	0.9580	0.2514	0.9530	0.2387	0.9580
	[5]	1.1323	0.935	0.3743	0.948	1.1224	0.945	1.1799	0.950	0.3719	0.945	0.3718	0.950	0.7573	0.9560	0.7000	0.9580	0.2512	0.9530	0.2385	0.9580
	[6]	1.1125	0.936	0.3680	0.945	1.1025	0.946	1.1595	0.951	0.3657	0.946	0.3658	0.951	0.7451	0.9570	0.6900	0.9580	0.2472	0.9540	0.2350	0.9580
	[7]	1.0100	0.943	0.3343	0.945	1.0479	0.954	1.0839	0.954	0.3458	0.955	0.3459	0.954	0.6820	0.9570	0.6394	0.9580	0.2265	0.9530	0.2170	0.9570
	[8]	0.9977	0.943	0.3304	0.944	1.0355	0.956	1.0695	0.954	0.3419	0.956	0.3416	0.954	0.6741	0.9570	0.6327	0.9580	0.2239	0.9540	0.2146	0.9580
	[9]	0.9544	0.941	0.3165	0.944	0.9876	0.954	1.0225	0.953	0.3273	0.954	0.3273	0.953	0.6469	0.9560	0.6094	0.9580	0.2149	0.9520	0.2065	0.9570
	[10]	0.7133	0.938	0.2372	0.941	0.7416	0.950	0.7605	0.950	0.2472	0.950	0.2472	0.950	0.4918	0.9490	0.4758	0.9480	0.1637	0.9470	0.1602	0.9470
	[11]	0.7130	0.938	0.2371	0.941	0.7413	0.950	0.7601	0.950	0.2471	0.950	0.2471	0.950	0.4916	0.9490	0.4756	0.9480	0.1636	0.9470	0.1601	0.9470
	[12]	0.6995	0.938	0.2327	0.942	0.7275	0.949	0.7461	0.950	0.2427	0.950	0.2427	0.950	0.4826	0.9480	0.4674	0.9470	0.1607	0.9440	0.1573	0.9460
5	[1]	1.5452	0.913	0.5129	0.948	1.5195	0.925	1.7784	0.952	0.5210	0.925	0.5219	0.952	0.9924	0.9560	0.8658	0.9600	0.3290	0.9540	0.3050	0.9570
	[2]	1.5293	0.912	0.5081	0.947	1.5031	0.925	1.7597	0.953	0.5162	0.925	0.5175	0.953	0.9836	0.9560	0.8594	0.9600	0.3261	0.9540	0.3027	0.9570
	[3]	1.4766	0.914	0.4917	0.950	1.4514	0.923	1.7052	0.951	0.5024	0.923	0.5023	0.951	0.9544	0.9550	0.8385	0.9600	0.3167	0.9540	0.2950	0.9570
	[4]	1.1034	0.936	0.3651	0.946	1.0927	0.945	1.1496	0.951	0.3628	0.945	0.3629	0.951	0.7394	0.9570	0.6853	0.9580	0.2453	0.9530	0.2345	0.9570
	[5]	1.1026	0.937	0.3649	0.946	1.0919	0.945	1.1488	0.951	0.3626	0.945	0.3627	0.951	0.7389	0.9570	0.6849	0.9580	0.2452	0.9530	0.2344	0.9570
	[6]	1.0849	0.936	0.3592	0.945	1.0753	0.946	1.1320	0.951	0.3576	0.946	0.3577	0.951	0.7280	0.9570	0.6759	0.9580	0.2416	0.9530	0.2312	0.9570
	[7]	0.9840	0.944	0.3260	0.946	1.0189	0.953	1.0542	0.953	0.3369	0.953	0.3370	0.953	0.6656	0.9570	0.6254	0.9580	0.2210	0.9540	0.2130	0.9570
	[8]	0.9739	0.943	0.3228	0.946	1.0086	0.954	1.0427	0.953	0.3336	0.954	0.3336	0.953	0.6592	0.9570	0.6199	0.9590	0.2189	0.9540	0.2111	0.9570
	[9]	0.9368	0.941	0.3109	0.944	0.9681	0.952	1.0034	0.954	0.3215	0.953	0.3214	0.954	0.6355	0.9560	0.5995	0.9580	0.2112	0.9530	0.2041	0.9560
	[10]	0.6948	0.938	0.2312	0.942	0.7218	0.950	0.7404	0.950	0.2409	0.950	0.2409	0.950	0.4794	0.9480	0.4644	0.9480	0.1596	0.9450	0.1565	0.9480
	[11]	0.6945	0.938	0.2311	0.942	0.7216	0.950	0.7401	0.950	0.2408	0.950	0.2408	0.950	0.4793	0.9480	0.4643	0.9480	0.1596	0.9460	0.1564	0.9480
	[12]	0.6826	0.940	0.2272	0.942	0.7091	0.949	0.7275	0.950	0.2368	0.949	0.2368	0.950	0.4713	0.9490	0.4569	0.9470	0.1569	0.9460	0.1539	0.9480

6.6 Real Data Application

A real data analysis is done in this portion to demonstrate the feasibility of the considered MW lifetime model and methodology used in this chapter. Here, we consider the tensile strength (in GPa) of 100 observations of carbon fibers, which are as follows:

3.70, 3.11, 4.42, 3.28, 3.75, 2.96 ,3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27 ,2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79 ,1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22 ,5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83 ,1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22 ,2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

Originally this set was reported by Nichols and Padgett (2006) and further studied by Mohammed et al. (2017) and Xie and Gui (2020), respectively. To begin, we use the scaled TTT transform to examine the behaviour of the failure rate function of the considered data set. The scaled TTT is calculated as follows:

$$\psi(r/n) = \left[\sum_{j=1}^{r} t_{(i)} + (n-r)t_r\right] \Big/ \left(\sum_{j=1}^{r} t_{(i)}\right),$$

where, $t_{(i)}, \quad i = 1,2,\ldots,n$ denotes the ith order statistic and $r = 1,2,\ldots,n$. If the plot $(r/n, \psi(r/n))$ is convex (concave), the failure rate function has a decreasing (increasing) shape. For more details one may refer Mudholkar et al. (1996). The scaled TTT plot of the considered data set is displayed in Figure 6.1. From Figure 6.1, it is clear that the considered data set follows increasing failure rate function. This empirical behavior of failure rate function is quite similar to the considered MW lifetime model.

Further, we check whether the considered data set is well fitted to the MW lifetime model or not using some goodness-of-fit tests. Here, we consider Kolmogrov-Smirnov (KS) and Anderson-Darling (AD) tests statistics along with their corresponding p-values. The KS and AD statistics with their corresponding p-values (in parenthesis) are 0.0884 (0.4145) and 0.7977 (0.4824), respectively. According to the obtained p-values, the considered model is fitted quite well for the considered real data set. Also, to see the feasibility of fitting graphically for the considered real data set, we plot empirical & fitted cdfs and probability-probability (P-P) plots of considered MW lifetime model and displayed in Figure 6.2. Figure 6.2, also suggests the considered model is well fitted with the considered real data set.

Moreover, to illustrate the methodologies adopted throughout the study, we can consider the PFFC data. After dividing the above-mentioned complete data set of sample size 100 into

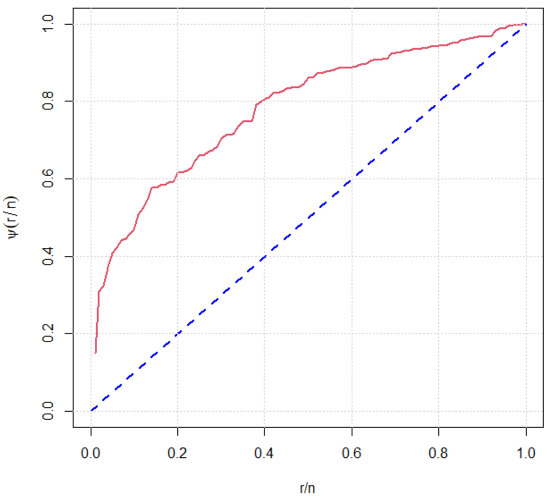


FIGURE 6.1: TTT plots under consideration of real data set.

$n = 25$ groups, each with $k = 4$ items, the FFC sample has been collected. The grouped data and the corresponding FFC samples are reported in Table 6.8. The items with "+" within each group indicate the first failure. Next, we generate six PFFC samples using 6 different combinations of censoring plans for the obtained first failure censored data in Table 6.8, with m=(40 & 80) % of n. The several censoring plans and the corresponding PFFC samples are presented in Table 6.9. We construct the ML and Bayes estimates of λ and $H(\lambda)$ for all censoring plans. We utilise non-informative priors to construct Bayes estimators since we don't have prior knowledge. Under the LINEX loss function, the Bayes estimators are calculated using TK approximation and MCMC techniques at two distinct values of the loss parameter c= (-0.5 & 0.5). For parameter λ and entropy $H(\lambda)$, we also build 95% ACI, boot-p, boot-t CIs, and HPD credible intervals. ML and Bayes estimates of the parameter and entropy are shown in Table 6.10. Also, various interval estimates of parameter and entropy are reported in Table 6.11 and 6.12, respectively. We use graphical diagnostic tools such as the trace map, boxplot, and histogram with Gaussian density plots to confirm the convergence of their stationary distributions, as illustrated in Figure 6.3. A random dispersion around the mean (shown by a thick red line) and a fine variety of parameter sequences can be seen in the trace plot. As shown by the boxplots and histograms of created samples, the posterior distribution is almost symmetric, meaning that the mean may be chosen as the best estimate of the parameter.

TABLE 6.8: Grouped real data set (Observation with "+" indicates the first failure (FF) in the group).

Groups → Items↓	1	2	3	4	5	6	7	8	9	10	11	12	13
I	3.27	2.05	3.33	1.87	2.03	3.68	2.87	2.67+	1.84	1.73	2.82	2.77	2.73
II	2.97	1.61	4.38	2.97	2.12	2.68	1.59+	2.96	0.39+	3.19	2.41+	2.17	2.88
III	3.11	3.11	1.69	1.57+	0.85+	4.90	1.89	3.09	2.17	1.57+	3.60	3.51	3.75
IV	2.03+	1.25+	1.18+	1.59	1.84	2.38+	2.43	4.20	2.35	2.93	3.22	1.08+	1.69+
FF Obs.	2.03	1.25	1.18	1.57	0.85	2.38	1.59	2.67	0.39	1.57	2.41	1.08	1.69
Ordered FF obs.	0.39	0.81	0.85	0.98	1.08	1.12	1.18	1.22	1.25	1.36	1.41	1.47	1.57
Groups→ Items↓	14	15	16	17	18	19	20	21	22	23	24	25	
I	5.56	2.81	2.55	4.70	1.36+	3.22	1.61	3.15	4.91	3.31	2.76	0.98+	
II	1.41+	3.39	2.17	2.59	2.83	1.71+	2.85	4.42	1.17	1.92	5.08	3.39	
III	2.48	3.68	3.56	3.19	2.74	3.65	1.47+	2.00+	1.12+	1.80+	3.28	2.50	
IV	2.95	0.81+	1.22+	2.56+	2.53	3.15	2.79	2.81	3.70	2.55	2.48+	3.31	
FF Obs.	1.41	0.81	1.22	2.56	1.36	1.71	1.47	2.00	1.12	1.80	2.48	0.98	
Ordered FF obs.	1.57	1.59	1.69	1.71	1.80	2.00	2.03	2.38	2.41	2.48	2.56	2.67	

TABLE 6.9: Censoring schemes and progressively first failure censored samples corresponding to considered real data set.

k	[CS]	Schemes	Progressively first failure censored samples
4	[1]	(15,0*9)	0.39 ,1.80 ,1.84 ,2.03 ,2.12 ,2.17, 2.48, 2.50, 2.73 ,2.77
	[2]	(5*3,0*7)	0.39 ,1.18, 1.57, 2.03, 2.12, 2.17, 2.48, 2.50, 2.73, 2.77
	[3]	(0*9,15)	0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.18, 1.22, 1.25, 1.36
4	[4]	(5,0*19)	0.39, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.59, 1.61, 1.69 ,1.80, 1.84, 2.03, 2.12, 2.17, 2.48, 2.50, 2.73, 2.77
	[5]	(2,3,0*18)	0.39, 0.98, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.59, 1.61, 1.69, 1.80, 1.84, 2.03, 2.12, 2.17, 2.48, 2.50, 2.73, 2.77
	[6]	(0*19,5)	0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.59, 1.61, 1.69, 1.80, 1.84, 2.03, 2.12

TABLE 6.10: ML and Bayes estimates of λ and $H(\lambda)$ under consideration of real data set for $k = 4, n = 25$, m=(40 & 80)% of n and c=(-0.5 & 0.5).

		MLE		TK Bayes				M-H Bayes			
				$c=-0.5$		$c=0.5$		$c=-0.5$		$c=0.5$	
k	[CS]	$\hat{\lambda}$	$\hat{H}$	$\hat{\lambda}$	H	$\hat{\lambda}$	$\hat{H}$	$\hat{\lambda}$	$\hat{H}$	$\hat{\lambda}$	$\hat{\lambda}$
4	[1]	9.2897	1.7640	13.4153	1.7800	8.7078	1.7714	9.6163	1.7467	8.5906	1.7436
	[2]	10.6695	1.8333	11.5620	1.8493	9.9031	1.8411	11.1367	1.8167	9.8262	1.8137
	[3]	5.6674	1.5169	6.5958	1.5330	5.5463	1.5304	5.7261	1.5016	5.3784	1.4988
4	[4]	6.6806	1.5992	7.2764	1.6083	6.5572	1.6046	6.7445	1.5903	6.4575	1.5887
	[5]	6.7637	1.6054	7.3707	1.6145	6.6362	1.6107	6.8301	1.5965	6.5371	1.5949
	[6]	5.7635	1.5254	6.2046	1.5344	5.6944	1.5308	5.8011	1.5169	5.5937	1.5153

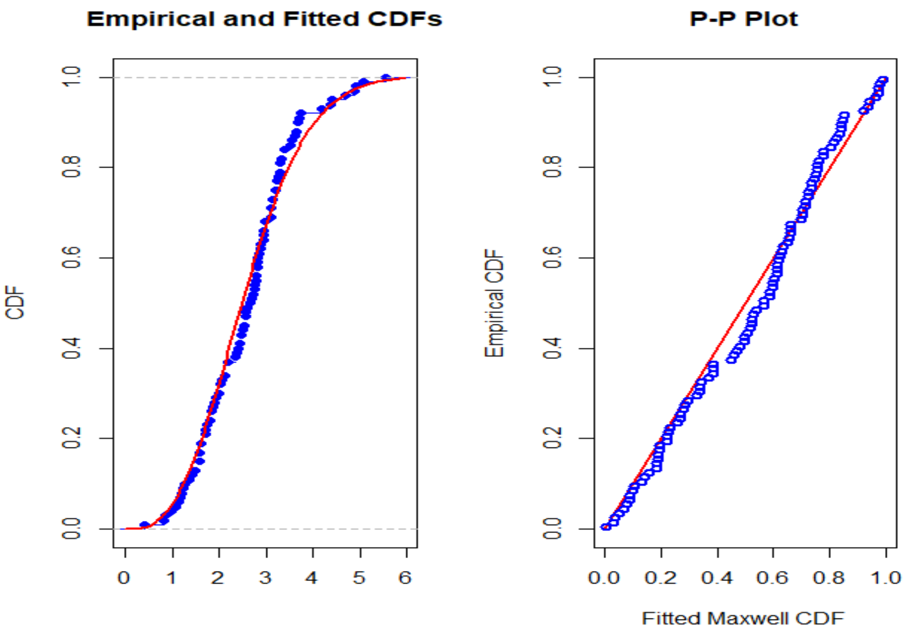


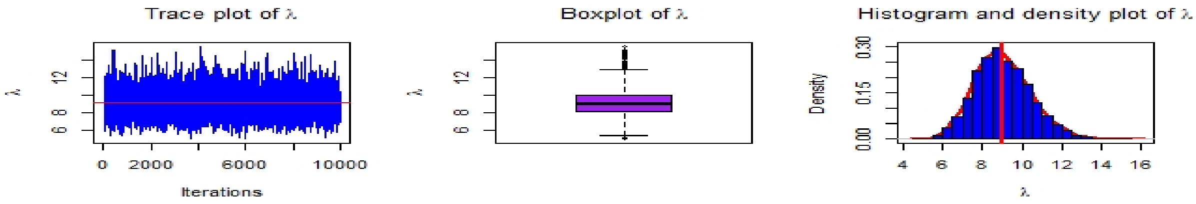
FIGURE 6.2: Empirical and fitted Maxwell distribution plots for real data.

TABLE 6.11: The 95% ACI, boot-p, boot-t confidence/HPD credible intervals of parameter λ, under consideration of real data set for $k = 4, n = 25$ m=(40 & 80)% of n and c=(-0.5 & 0.5).

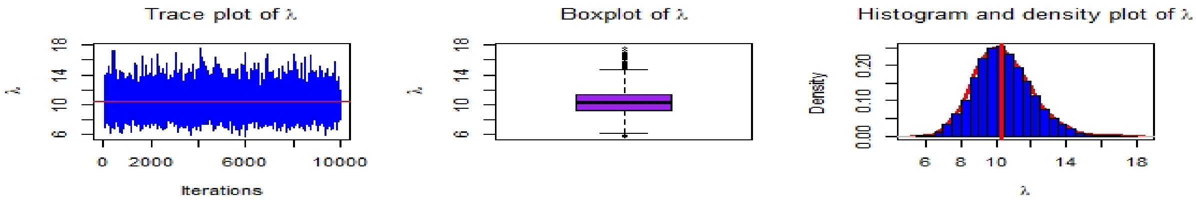
k	[CS]	ACI	Bootstrap		HPD credible intercals	
			boot-p	boot-t	c=-0.5	c=0.5
4	[1]	(4.973, 13.606)	(6.636, 12.366)	(6.213, 11.944)	(6.357, 11.872)	(6.357, 11.872)
	[2]	(5.804, 15.535)	(7.816, 13.602)	(7.736, 13.523)	(7.365, 13.587)	(7.365, 13.587)
	[3]	(3.157, 8.178)	(3.880, 8.365)	(2.970, 7.454)	(3.967, 7.187)	(3.967, 7.187)
	[4]	(4.478, 8.883)	(5.006, 8.771)	(4.590, 8.355)	(5.125, 8.058)	(5.125,8.058)
	[5]	(4.538, 8.989)	(5.044, 8.824)	(4.703, 8.483)	(5.192, 8.156)	(5.192, 8.156)
	[6]	(3.893, 7.634)	(4.064, 7.962)	(3.565, 7.463)	(4.443, 6.937)	(4.443, 6.938)

6.7 Concluding Remarks

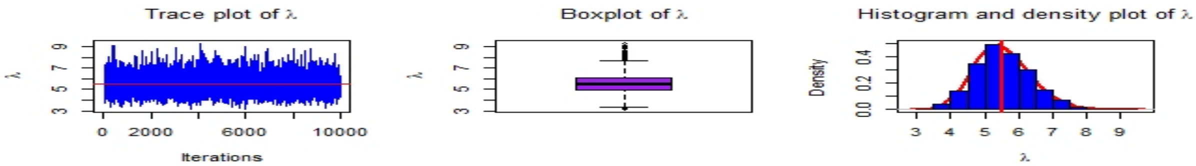
In this chapter, we developed some inferences of associated parameters and entropy of MW lifetime model based on PFFC data from both classical and Bayesian points of view. For classical estimation procedures, we applied the EM algorithm to compute ML estimates. Also, we obtained asymptotic and two bootstraps (boot-p & boot-t) confidence intervals. Further, in the case of Bayesian estimation procedures, we applied two approximation techniques such as TK approximation and M-H algorithm to approximate the Bayes estimators under LINEX loss function. In addition, we use the M-H algorithm to compute the HPD credible intervals. Moreover, a numerical computation is performed employing a Monte Carlo simulation and real data applications to know the performance of different estimators and the potentiality of



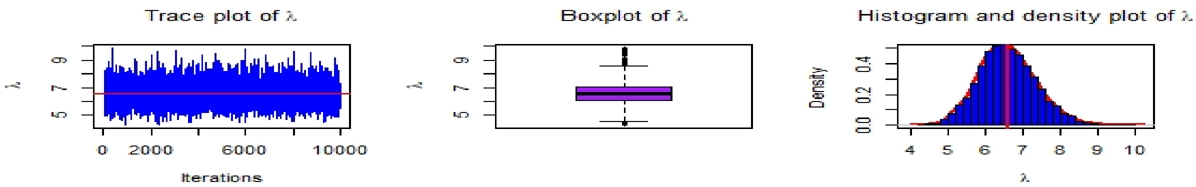
(A) Scheme 1.



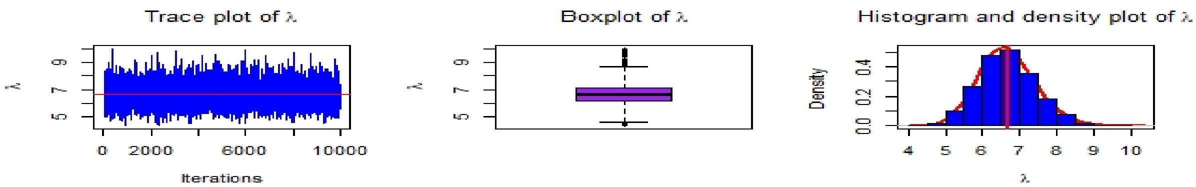
(B) Scheme 2.



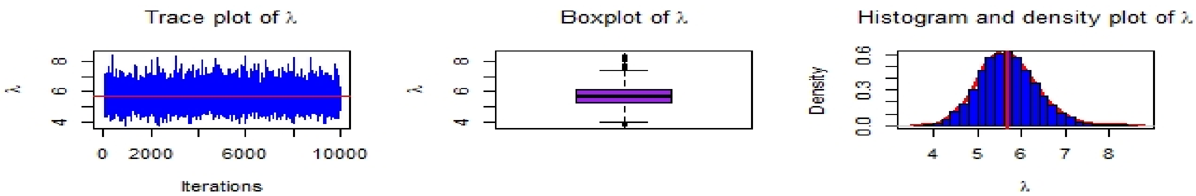
(C) Scheme 3.



(D) Scheme 4.



(E) Scheme 5.



(F) Scheme 6.

FIGURE 6.3: MCMC diagnostic plots for different censoring schemes under consideration of real data set.

TABLE 6.12: The 95% ACI, boot-p, boot-t confidence/HPD credible intervals of entropy H, under consideration of real data set for $k = 4, n = 25$ m=(40 & 80)% of n and c=(-0.5 & 0.5).

k	[CS]	ACI	Bootstrap		HPD credible intercals	
			boot-p	boot-t	c=-0.5	c=0.5
4	[1]	(1.532, 1.996)	(1.596, 1.907)	(1.621, 1.932)	(1.591, 1.900)	(1.591, 1.900)
	[2]	(1.605, 2.061)	(1.678, 1.955)	(1.712, 1.989)	(1.664, 1.968)	(1.664, 1.968)
	[3]	(1.295, 1.738)	(1.327, 1.712)	(1.322, 1.706)	(1.353, 1.648)	(1.353, 1.648)
	[4]	(1.434, 1.764)	(1.455, 1.735)	(1.463, 1.743)	(1.472, 1.698)	(1.472, 1.698)
	[5]	(1.441, 1.770)	(1.459, 1.738)	(1.472, 1.752)	(1.478, 1.704)	(1.478, 1.704)
	[6]	(1.363, 1.688)	(1.351, 1.687)	(1.364, 1.700)	(1.400, 1.623)	(1.400, 1.623)

considered methodologies and models. Finally, based on the observed data we recommend the use of Bayesian estimation of the parameter and entropy based on the MCMC method for the MW lifetime model when the prior information about the parameter is available. If prior information is not available then the ML estimates may be are preferred.