CLASSICAL AND BAYESIAN ESTIMATION METHODS FOR CENSORED DATA

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Dedicated to my Parents

DECLARATION

(As required under clause I2 of Ordinance IIA of the Central University of Haryana)

This is to certify that the material embodied in the present work, entitled ***"Classical and Bayesian Estimation Methods for Censored Data"***, is based on my original research work. The research work was carried out under the supervision of **Dr. Kapil Kumar,** Assistant Professor, Department of Statistics, Central University of Haryana, Mahendergarh. This work has not been submitted, in part or full, for any other diploma or degree of any University/Institution Deemed to be University and College/Institution of National Importance. References from other works have been duly cited at the relevant places.

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Abstract

The life testing experiments are carried out to obtain the lifetime data on patients for survival analysis and to study the reliability of electrical, electronic and mechanical systems, information theory, artificial intelligence, etc. It is challenging to obtain lifetime data on all individuals or products due to breakages, time limits, and expense restrictions. Thus, the experiments are terminated before they are completed. In those situations, we get the censored data. There are various types of censoring schemes utilized in the literature for different life testing situations.

This thesis deals with the classical and Bayesian estimation methods for the censored data. We consider three distinct censoring schemes, namely, random censoring, progressive censoring, and progressive first-failure censoring schemes in this thesis. Random censoring is a popular censoring scheme in which the censoring time is set at random rather than being predetermined, and it commonly arises in survival analysis and clinical trials. The random censoring scheme is an extension of the Type-I censoring scheme in which failure and censoring times both are taken as random variables. We consider two distinct lifetime models, namely, inverse Pareto and inverse Weibull lifetime models based on randomly censored data and developed statistical inferences for the associated model parameters and reliability characteristics from both the classical and Bayesian estimation perspectives in Chapter 2 and Chapter 3, respectively.

The progressive censoring scheme is another popular censoring scheme that allows the removal of experimental units during the experiment. Because of its flexibility, the progressive censoring scheme has several applications in a variety of disciplines and received considerable attention in the literature. The stress-strength reliability (SSR) is considered as a measure of system performance. The system becomes out of control if the system stress exceeds its strength. The model of stress-strength has found applications in many statistical problems, including quality control, engineering statistics, medical statistics and biostatistics, among others. The estimation of SSR has received considerable attention in the statistical literature. In Chapter 4, we employed the progressive censoring scheme and developed classical and Bayesian estimators of stress-strength reliability for the inverse Pareto lifetime model.

The progressive first-failure censoring scheme is a cost and time-efficient censoring scheme and it is developed as the combination of progressive and first-failure censoring schemes. Also, this censoring scheme is viewed as a generalization of a progressive censoring scheme. We study the classical and Bayesian estimation of the parameters and reliability characteristics of the inverse Pareto lifetime model using the progressive first-failure censored data in Chapter 5.

The information theory provides a simple approach for measuring the uncertainty and reciprocal information of random variables as entropy measures. The applications of entropy are described in a variety of fields, including computer science, molecular biology, hydrology,

meteorology, and others. For example, in the study of trends in gene sequences, molecular biologists use the principle of Shannon's entropy. Shannon's entropy is the most widely used entropy in statistical and information theory. We study the classical and Bayesian estimation methods for the Shannon's entropy from the Maxwell lifetime models based on progressively first-failure censored data with different applications in Chapter 6.

The statistical software R is used for computation throughout the thesis. Finally, a complete list of references and other literature surveys are given at the end of the thesis as a bibliography.

List of Research Papers

Published

1. Kumar, K. and **Kumar, I.** (2019). Estimation in inverse Weibull distribution based on randomly censored data. Statistica, 79(1):47-74. (Scopus Indexed Journal).

2. Kumar, K. and **Kumar, I.** (2020). Parameter estimation for inverse Pareto distribution with randomly censored life time data. International Journal of Agricultural and Statistical Sciences, 16(1): 419-430. (Scopus Indexed Journal).

3. **Kumar, I.** and Kumar, K. (2021). On estimation of $P(V < U)$ for inverse Pareto distribution under progressively censored data. International Journal of System Assurance Engineering and Management, DOI: *https://doi.org/10.1007/s13198-021-01193-w*, Springer. (Scopus Indexed Journal).

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Presented in Conferences

1. **Kumar, I.** and Kumar, K. (2018). Estimation of Parameters of Inverse Weibull Distribution with Randomly Censored Samples. International Conference on Recent Trends of Computing in Mathematics, Statistics and Information Technologies (RTCMSIT-2018) during March, 09-11, 2018 at Bundelkhand University, Jhansi, U.P., India.

2. **Kumar, I.** and Kumar, K. (2018). Estimation in Inverted Pareto Distribution with Application of Leukemia Data. International Conference on Emerging Innovation in Statistics & Operations Research (EISOR-2018) during December 27-30, 2018 at Maharshi Dayanand University, Rohtak, Haryana, India.

3. **Kumar, I.** and Kumar, K. (2021). Estimation of Stress-Strength Reliability of Inverse Pareto Distribution using Progressively Censored Samples. International Conference (Virtual Mode) on Emerging Trends in Statistics and Data Science in Conjunction with $40th$ Annual Convention of ISPS during September 7-10, 2021, jointly organized by Cochin University of Science & Technology, Cochin, M.D. University, Rohtak, University of Kerala, Trivandrum, Bharathiar University, Coimbatore, The Madura College (Autonomous), Madurai, India.

4. **Kumar, I.** and Kumar, K. (2021). Classical Estimation in Inverse Pareto Lifetime Model under Progressive First Failure Censoring. 7th International E-Conference on Advances in Statistics (ICAS), during October 15-17, 2021, organized by Izmir University of Economics, Turkey.

Contents

Chapter 1

Introduction and Basic Terminology

1.1 Introduction

The quality of the products or items is an important task for almost all companies or manufacturers. For example, individual components, objects, products, things, etc. must be monitored and improved by manufacturers. In particular, the lifespan of products is an important quality attribute that manufacturers must examine. The quality of the items or products are directly proportional to their lifespan, which can be monitored by an experimenter using a reliability experiment in which n identical objects or items are put to the test at the same time. One significant problem is that monitoring the failure times of all test units or objects is not always feasible. There are many real-life situations, where the experimenter has to remove some test units or items from the test unintentionally or intentionally due to breakage of test units or time restrictions or findings etc. As a result, censored samples are more appropriate rather than complete samples in life testing experiments. If only $m(< n)$ of the n test units or items placed on life test are detected before the experiment ends, the sample is said to be a censored sample. For modeling such type of data, different censoring schemes are investigated by experimenters in the literature, some of them are as follows:

(a) Type-I or time censoring

(b) Type-II or failure censoring

(c) Random censoring

(d) Hybrid censoring

(e) Progressive censoring

(f) First failure censoring

(g) Progressive first failure censoring.

Out of these censoring schemes, we have employed only the following three censoring schemes in this thesis:

1.1.1 Random Censoring

Random censoring is a natural phenomena of life testing experiments in which the test units or items under study are lost or destroyed before its complete failure. This censoring scheme was first introduced by Gilbert (1962). Type-I censoring is a particular type of random censoring that occurs at a specified time point, say $t = t_0$, see (Lawless, 2003, p. 55). Generally, in survival analysis or clinical trials, this type of censoring commonly occurs in one of the following forms: patients do not complete their treatment and leave before the trial is completed. Random censoring has recently gained popularity in survival analysis and clinical studies. In Chapters 2 and 3 of this thesis, we will go through this censoring scheme in greater depth.

1.1.2 Progressive Censoring

Progressive censoring (PC) is a censoring approach by which test units or items during the life test can be removed or withdrawn from the test at predetermined or random assessment times. Initially in the literature, the progressive Type-II censoring was given by Herd (1956) named as ‘multiple censoring’. Later on the PC scheme was inroduced by Cohen (1963) as ‘progressive Type-II censoring scheme’ in the literature. For more details about PC scheme and their applications one may refer Balakrishnan and Aggarwala (2000) & Balakrishnan and Cramer (2014). The estimation of stress-strength reliability (SSR) based on this censoring scheme in greater depth will be discussed in Chapter 4.

1.1.3 Progressive First Failure Censoring

If the lifespan of goods or objects is very long, then the testing duration of an experiment becomes too long. For such a situation Johnson (1964) introduced the first failure censoring (FFC) scheme and further updated by Balasooriya (1995) that offers a cost-effective and time-saving testing plan for life tests. This censoring scheme allows experimenters to test $k \times n$ test units by testing n groups each containing k test units and then runs all the tests simultaneously until the first failure is observed in each group. Although the FFC scheme is cost-effective and time-saving, it does not enable intermittent removal of units or objects throughout the tests. However, the PC scheme enables removals of units or objects throughout the test. Due to the cost-effectiveness and time-saving properties of the FFC scheme and intermittent removal property of PC scheme, Wu and Kuş (2009) combined them and introduced a more efficient life testing plan called progressive first failure censoring scheme (PFFCS). In Chapter 5 and Chapter 6, we will discuss this censoring scheme in further depth in connection with information theory and survival analysis.

1.2 Estimation Methods

To make any inferences about desire parameters in the statistical theory, estimation methods play important role. In parametric inferences, the mathematical form of the probability distributions are known to us except for a few arbitrary constants associated with the model called as parameters, and our primary concern is to estimate the associated parameters. The classical and Bayesian estimation methods are two popular estimation methods in statistical theory.

1.2.1 Classical Estimation Methods

The classical estimation methods assume the availability of a sample from a specified population, and statistical inference can be developed to have the best long-run performance. In this method, the sample observations are random but the parameters are assumed to be unknown constants. The information about unknown parameters are gathered from the randomness of the sample observations and it is utilized to draw inferences about the unknown parameters. Various classical point estimation methods are widely used in the literature. These methods include the following: method of moments (MM), method of maximum likelihood (ML), method of percentile (MP), method of least square (MLS), method of weighted least square (WLS), method of maximum product spacing's (MPS), method of Anderson-Darling (AD), method of right Anderson-Darling (RAD), method of Cramer-Von-Misses (CVM), etc. Among these methods, the method ML is the most popular and commonly used estimation method in statistical theory. This procedure has numerous advantages and its properties can be utilized in various cases. For more details one may refer (Casella and Berger, 2002, pp. 315-323), (Rohatgi and Saleh, 2015, pp. 388-399). We have employed the ML estimation method in the case of the classical estimation method. Also, we have used the asymptotic confidence interval estimation method for the associated model parameters based on the method of ML. Also, we have employed a well-known resampling technique as the bootstrap method for interval estimations and discussed them in Chapter 5 and Chapter 6.

1.2.2 Bayesian Estimation Method

Bayesian analysis is used in a variety of fields, including science, engineering, medicine, sports, etc. The Bayesian estimation method is based on the prior belief that all the associated model parameters are random variables, allowing previous information to be taken into account. The prior information is used to construct the posterior distribution of the parameter of interest, which is based on the data on lifetimes. This posterior distribution is used to make numerous conclusions about the lifetime parameters and the foundation of Bayes inference.

Suppose that n units are place on a life test and it is assumed that their recorded lifetimes $X_1, X_2, \ldots, X_n$ form a random sample of size n with a population having probability density function (pdf) $f(x|\theta)$, where, θ is a real valued unknown parameter and lies in the parameter space Θ. We also assume that θ is a random variable with pdf $g(\theta)$, which is known as the prior distribution of θ. Thus, the joint pdf of $\{X_1, X_2, \ldots, X_n, \theta\}$ is given by

$$J(\underset{\sim}{x}, \theta) = \prod_{i=1}^{n} f(x_i|\theta)\; g(\theta) = L(\underset{\sim}{x}, \theta) g(\theta), \tag{1.1}$$

where, $L(\underset{\sim}{x}|\theta)$ is the likelihood function. Then, the marginal pdf of $\{X_1, X_2, \ldots, X_n\}$ is given by

$$P(\underset{\sim}{x}) = \int_{\Theta} J(\underset{\sim}{x}, \theta) d\theta. \tag{1.2}$$

Therefore, using the Bayes theorem, the posterior distribution of θ is given by

$$\pi(\theta|\underset{\sim}{x}) = \frac{J(\underset{\sim}{x}, \theta)}{P(\underset{\sim}{x})} = \frac{L(\underset{\sim}{x}|\theta) g(\theta)}{\int_{\Theta} L(\underset{\sim}{x}|\theta) g(\theta) d\theta}. \tag{1.3}$$

1.2.2.1 Bayesian Approximation Techniques

The posterior mean of any function of the parameters, say $\phi(\theta)$ using posterior distribution in (1.3) is given by

$$E\left[\phi(\theta)|\underset{\sim}{x}\right] = \frac{\int\limits_{\theta\, \in\, \Theta} \phi(\theta) e^{l(\theta;\, \underset{\sim}{x})} g(\theta) d\theta}{\int\limits_{\theta\, \in\, \Theta} \phi(\theta) e^{l(\theta;\, \underset{\sim}{x})} g(\theta) d\theta}, \tag{1.4}$$

where, $l(\theta, \underset{\sim}{x}) = \ln L(\underset{\sim}{x}|\theta)$. From the above posterior mean in equation (1.4), we can say that the posterior mean is in the form of the ratio of two integrals for which the closed-form or solutions may or may not be available. Thus, in general, Bayes estimators are often obtained as a ratio of two integrals for which the closed-form solutions may or may not be available. If closed-form solutions are not available, we require some appropriate approximation techniques to get the approximate Bayes estimators. There are several approximation techniques available in the literature. For example, Lindely and Tierney-Kadane (TK) approximations are used to compute only point estimates, for more details see, Lindley (1980), Sinha (1986) and Tierney and Kadane (1986). Apart from these approximation techniques, a general class of approximation techniques are known as Markov Chain Monte Carlo (MCMC) methods which are used to make inferences based on the posterior samples. These approximation techniques provide both point and interval estimates. Gelfand and Smith (2000), Chen et al. (2000), Robert and Casella (2004), and Gelman et al. (2013) provide comprehensive explanations of MCMC techniques and their applications. In this thesis, we employ TK approximation and MCMC techniques for Bayesian computations.

1.2.2.2 Prior Distributions

A parameter's prior distribution is the probability distribution that expresses the parameter's uncertainty before the current data are observed. There are various forms of priors in the literature, some of them are as follows:

Non-informative Prior: If a prior has very little or no impact on the parameter's posterior distribution, it is said to be non-informative prior.

Informative Prior: If a prior has an effect on the parameter's posterior distribution, it is said to be informative.

Improper Prior: A prior is said to be improper prior if it has infinite density function, i.e. $\int\limits_{\Theta} g(\theta) d\theta = \infty$. For example, $g(\theta) = \frac{1}{c};\quad \theta > 0$ is an improper prior.

Proper Prior: A prior is said to be proper prior if it has finite probability function, i.e. $\int_{\Theta} g(\theta)d\theta < \infty$. For example $g(\theta) = 1$.

Conjugate Prior: If the prior and posterior distributions belong to the same family of distributions, the prior is said to be a conjugate prior for that family. As a result, the posterior distribution displays the prior distribution in shape.

1.2.2.3 Loss Function

Suppose that the estimator d estimates the unknown parameter θ of pdf $f(x|\theta)$. If the true value of the unknown parameter θ is approximated by d, the loss sustained is indicated by $L(d, \theta)$, where $L(d, \theta) = 0$ for $d = \theta$. There are various forms of loss functions in the literature, including squared error loss function (SELF), precautionary loss function (PLF), entropy loss function (ELF), generalised entropy loss function (GELF), and LINEX loss function, among others. Table 1.1 presents the loss functions with their corresponding Bayes estimators.

TABLE 1.1: Loss functions and corresponding Bayes estimators.

Notation	Loss Function	Bayes Estimator
SELF	$\left(\hat{\theta} - \theta\right)^2$	$E(\theta\|\underset{\sim}{x})$
PLF	$\frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}$	$\sqrt{E(\theta^2\|\underset{\sim}{x})}$
ELF	$\left[\frac{\hat{\theta}}{\theta} - \ln\left(\frac{\hat{\theta}}{\theta}\right) - 1\right]$	$\left[E(\theta^{-1}\|\underset{\sim}{x})\right]^{-1}$
GELF	$a\left[\left(\frac{\hat{\theta}}{\theta}\right)^q - q\ln\left(\frac{\hat{\theta}}{\theta}\right) - 1\right]$	$E\left[\theta^{-q}\|\underset{\sim}{x}\right]^{-\frac{1}{q}}$
LINEX	$a\left[e^{b(\hat{\theta}-\theta)} - b(\hat{\theta} - \theta) - 1\right]$	$-\frac{1}{b}\ln\left[E(e^{-b\theta}\|\underset{\sim}{x})\right]$

1.3 Goodness of Fit Test and Model Comparison Criteria

Obtaining information about the population from which a sample is selected is a significant challenge in statistical theory. A statistical model's goodness of fit defines how well it fits a collection of data. The disparity between actual values and predicted values from a model under consideration is often summarised by the goodness of fit measures. In statistical hypothesis testing, such measures can be utilized. Also, the model comparison tests are used to check the performance of the considered model among others. The goodness of fit and model comparison may be done in a variety of ways. The following are some of them:

1.3.1 Kolmogorov-Smirnov Test

The Kolmogorov-Smirnov (KS) test is one of the well known non-parametric goodness-of-fit test. KS test measures the distance between the observed and expected distribution functions. To perform the two-sided goodness of fit test for testing

$$H_0: F(x) = F_0(x) \quad \text{vs} \quad H_1: F(x) \neq F_0(x). \tag{1.5}$$

The KS test statistic is given by

$$D_n = \sup_x \mid F_n(x) - F_0(x) \mid,$$

where, $F_n(x)$ and $F_0(x)$ are the observed and expected distribution functions, respectively. We reject the H_0 if the computed D_n is larger than the critical value, else it may be accepted (see (Conover, 1972, pp. 309–314)).

1.3.2 Anderson-Darling Test

In literature, the Anderson–Darling (AD) goodness-of-fit test was introduced by Anderson and Darling (1954) based on the difference between the observed and expected distribution functions, but here the difference is measured in terms of the square instead of the absolute value used in the KS test. To employ the two-sided AD goodness-of-fit test for testing (1.5). The AD test statistic is given by

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln F_0(x_{i:n}) - \ln \bar{F}(x_{n-j+1:n})\right]^2,$$

where, $F_n(x)$ and $F_0(x)$ are the observed and expected distribution functions, respectively. We reject the H_0 if the computed AD statistic A^2 is larger than the critical value, else it may be accepted. For more details one may refer (Gibbons and Chakraborti, 2011, pp. 137–138).

1.3.3 Maximum Likelihood Criterion

The maximum likelihood criterion (MLC) is one of the model selection criteria. It measures the correctness of an estimated statistical model. Based on the specified complete or censored data, different competing models can be ranked according to their maximum likelihood value or equivalently minimal log-likelihood value, with the one having the lowest maximum likelihood value being the best.

1.3.4 Akaike's Information Criterion

The Akaike's information criterion (AIC) was introduced by Akaike (1974). AIC measures the potentiality of estimated statistical models. For a given data set, different statistical models can be ranked according to their AIC, with the one having the lowest AIC being the best. The following formula is used to determine the AIC:

$$\mathrm{AIC} = 2k - 2\log(L), \tag{1.6}$$

where, L is the maximum value of the likelihood function for the estimated model, and k is the number of parameters in the model.

1.3.5 Bayesian Information Criterion

In literature, the Bayesian information criterion (BIC) was proposed by Schwarz (1978) for model selection criteria among a class of statistical models with different number of parameters, say k. Many competing models for a given data set of size n can be ranked according to their BIC, with the model with the lowest BIC being the best, similar to how AIC works. BIC has a greater penalty for extra factors than AIC. The following formula is used to get the BIC:

$$\mathrm{BIC} \ = k\log(n) - 2\log(L) \tag{1.7}$$

1.3.6 Kaplan-Meier Estimator

The product-limit estimator, commonly known as the Kaplan–Meier (KM) estimator, estimates the survival of lifetime data. In literature, the KM estimator was proposed by Kaplan and Meier (1958). A KM survival function estimator curve is a series of horizontal steps with decreasing amplitude that approximate the real survival function for that population. The KM estimator curve has the benefit of being able to handle censored data. The KM estimator curve is identical to the empirical survival function when there is no censoring. For more details one may refer (Lawless, 2003, p. 80).

Several competing models can be rated based on how near the curves of their predicted survival functions are to the KM estimator's curve. The KM estimator is defined as follows:

$$\hat{S}(t) = \prod_{y_i \leq t} \left(1 - \frac{1}{n_i}\right)^{d_i},$$

where, n_i= Number of surviving units at time y_i, and

$$d_i = \begin{cases} 1 & \text{failed/uncensored test units} \\ 0 & \text{censored test units.} \end{cases}$$

1.4 Thesis at a Glance

This thesis consists of six chapters. Chapter 1 is completely introductory in nature with brief discussions of life testing experiments, censoring schemes, classical and Bayesian estimation theories. Also, it presents some useful goodness of fit tests and model selection criteria.

In Chapter 2, we develop classical and Bayesian estimates of the associated model parameters of the randomly censored inverse Pareto (IP) lifetime model. To assess the performance of different estimators, the numerical computations are performed through a Monte Carlo (MC) simulation study. We investigate two randomly censored real data sets based on two different types of cancer illnesses to illustrate the feasibility of the considered model and techniques.

In Chapter 3, we focus on the inverse Weibull (IW) lifetime model with random censoring. The ML and Bayesian estimation approaches for the parameter and reliability characteristics of the IW lifetime model are developed using randomly censored data. The expected time on the test (ETT) is also computed for randomly censored data. Multiple values of real parameters are utilized in the simulation study to investigate the behaviour of these estimators. Finally, a randomly censored real data example is given for demonstration purposes.

Chapter 4 deals with the SSR for the IP lifetime model based on progressively censored data. A system's or machine's SSR is defined as a measure of performance in the context of mechanical durability. The system or machine will fail if the applied stress is greater than the strength of the system or machine at any time point. The ML estimator for SSR and the asymptotic confidence interval (ACI) are developed. The Bayes estimator and HPD credible interval for SSR are obtained using non-informative and informative priors under the GELF. An MC simulation study is used to compare the proposed estimation methods. Finally, two pairs of real data sets are assessed for demonstration reasons.

In Chapter 5, we developed classical and Bayesian inference of associated unknown parameter and reliability characteristics of IP lifetime model based on progressively first failure censored (PFFC) data. For estimation of associated parameter and reliability characteristics, we used ML in the classical estimation process. In addition, asymptotic and bootstrap confidence intervals for the parameter are calculated. The TK approximation, importance sampling, and the Metropolis-Hasting (MH) methods are used to calculate the Bayes estimates of the associated unknown parameter and reliability characteristics. Also, we compute the parameter's HPD credible interval. Extensive numerical computations are conducted to determine the performance of various estimators developed in this chapter. Finally, a real data set is examined to demonstrate the concept and methods suggested.

In Chapter 6, we discussed a problem from information theory based on PFFC data and developed statistical inferences of Shannon's entropy for the Maxwell (MW) lifetime model. Shannon's entropy is an essential quantity that determines the amount of accessible information or the uncertainty of a random process's result. The ML estimates of associated unknown parameter and entropy are computed using the expectation-maximization (EM) algorithm. Also, based on ML estimates we constructed ACIs of parameter and entropy. In addition, we also constructed bootstrap confidence intervals. The Bayes estimators and HPD credible intervals of parameter and entropy are derived under the LINEX loss function. The performance of various estimation methods is compared by an MC simulation study. Finally, real-life data has been analyzed for illustrative purposes.

The statistical software R is used for computations throughout the thesis. Finally, a complete list of references and other literature surveys is given at the end of the thesis as the bibliography. Also, a list of research papers is presented at the end of this thesis.

Chapter 2

Statistical Inference in Inverse Pareto Lifetime Model using Randomly Censored Data*

2.1 Introduction

The main objective of this chapter is to build classical and Bayesian inferences about the model parameters of the IP lifetime model using randomly censored data.

In the survival analysis, the entire lifetime of a person or an animal is not always observable. Some lifetimes may be censored, in that case, only a part of the lifetime is recorded. Therefore, censoring is a necessary part of life testing experiments. The units in these experiments are lost or removed, resulting in incomplete information. In the literature, there are different types of censoring schemes. Type-I and Type-II censoring schemes are the most extensively used censoring techniques in reliability and life testing experiments. The censoring time or the number of censored items are prefixed in these censoring techniques. Many scholars, such as Mann et al. (1974) and Sinha (1986), have examined these censoring techniques with various lifetime models extensively.

Random censoring is a common occurrence in real-world life testing experiments. For example, patients with leukemia enter into the study simultaneously after their treatments. We aim to track them throughout their lives, but censoring can take many forms, including loss to

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follow-up (e.g., the patient may elect to relocate), drop out (e.g., inadequate side effects or an unfinished course of treatment), death from other conditions, or study layoff. That is, these random features are uncontrollable by the treatments, resulting in an independent random variable called a censoring time variable. This censoring scheme was introduced by Gilbert (1962) in literature. After that some early study on random censoring can be found in Breslow and Crowley (1974), Koziol and Green (1976), etc. Recently, several authors investigated the usefulness of random censoring in literature for different lifetime models like, Ghitany and Al-Awadhi (2002) discussed ML estimates of parameters for Burr Type XII distribution, the generalized inverted Rayleigh distribution is studied by Kumar and Garg (2014), Krishna et al. (2015) studied Maxwell distribution, the generalized inverted exponential distribution is studied by Garg et al. (2016), Krishna and Goel (2017) discussed geometric distribution, the log-logistic distribution is discussed by Kumar (2018), the Birnbaum-Saunders distribution is discussed by El-Sharkawy and Ismail (2020), EL-Sagheer et al. (2020) studied three parameters Burr Type XII distribution etc.

Mathematically, random censoring can be described as follows: suppose the failure times $X_1, X_2, \ldots, X_n$ are independent and identically distributed (iid) random variables with pdf $f_X, x > 0$ and survival function $S_X, x > 0$. Associated with these failure times, $T_1, T_2, \ldots, T_n$ are iid censoring times with pdf $f_T, t > 0$ and survival function $S_T, t > 0$. Now, suppose $X_i's$ and $T_i's$ mutually independent $\forall\ i = 1, 2, \ldots, n$. We observe failure or censored time $Y_i = \min(X_i's, T_i's)$; $i = 1, 2, \ldots, n$, and the corresponding censor indicators

$$D_i = \begin{cases} 1; \text{ if failure occurs} \\ 0; \text{ if censoring occurs.} \end{cases}$$

Some spacial cases of this censoring scheme are as follows: (i) It become complete sample case when $T_i = \infty\ \forall\ i = 1, 2, \ldots, n$. (i) It reduces to Type I censoring when $T_i = t_0\ \forall i = 1, 2, \ldots, n$, where, t_0 is the pre-fixed study period. Thus, the joint pdf of Y and D is given by

$$f_{Y,D}(y, d) = \{f_X(y)S_T(y)\}^d \{f_T(y)S_X(y)\}^{1-d}\ ;\ y > 0, d = 0, 1. \tag{2.1}$$

The marginal distribution of Y and D can be obtained as

$$f_Y(y) = f_X(y)S_T(y) + f_T(y)S_X(y),\ \ y > 0, \text{ and}$$

$$P[D = d] = p^d (1-p)^{1-d}\ \ ; d = 0, 1,$$

respectively, where, p is the probability of observing a failure and it is given by

$$p = P[X \leq T] = \int_0^\infty S_T(y) f_X(y) dy.$$

There are numerous real-life situations in survival analysis where data requires a probability distribution with both decreasing and upside-down bathtub-shaped failure rate functions. For example, a disease's mortality may reach a high after a while and then gradually drop, as shown in Kundu and Howlader (2010). During the first few days after a heart transplant, while the body adjusts to the new organ, patients face an increasing failure rate of mortality. As the patient recovers, the failure rate reduces, as seen in Collett (2015). The failure function shaped like an upside-down bathtub would be acceptable in such cases.

The one parameter IP lifetime model has both the decreasing and upside-down bathtub-shaped failure rate functions depending on the true value of the parameter. Also, it has nice closed-form expressions of the cumulative distribution function (cdf) and failure rate function, both of which are useful in reliability theory or survival analysis. However, the IP lifetime model has a very nice closed form failure rate function, it has not gained much attention in the literature. Guo and Gui (2018) studied IP lifetime model based on stress-strength reliability in the case of both classical and Bayesian approaches. The application of IP lifetime model in extreme events is studied by Dankunprasert et al. (2021), Kumar et al. (2021) developed some estimation methods for associated parameter and reliability characteristics of IP lifetime model.

The main aim of this chapter is to develop the classical and Bayesian estimation procedures for the parameters of the IP lifetime model using randomly censored data. The rest of the chapter is laid out as follows: the IP lifetime model is discussed in Section 2.2. Also, a mathematical formulation is given for random censoring with failure and censoring time distributions. Section 2.3 deals with the ML estimation and ACIs of the parameters. Section 2.4 describes the formulation of Bayes estimation procedure using MCMC methods under LINEX loss function using gamma informative priors. The HPD credible intervals for the parameters are derived using MCMC techniques. Section 2.5 deals with an MC simulation study to explore the properties of various estimates developed in this chapter. Two real datasets are analyzed for illustration purposes in Section 2.6. Finally, concluding remarks are given in Section 2.7.

2.2 The Model

If a random variable X follows the IP lifetime model with parameter α denoted by IP(α), the pdf of IP lifetime model is given by

$$f_X(x;\alpha) = \frac{\alpha x^{\alpha-1}}{(1+x)^{\alpha+1}} \quad ; \alpha > 0,\ x > 0. \tag{2.2}$$

Figure 2.1 shows the pdf of IP lifetime model for distinct values of α, say 0.25, 0.75, 1.5 and

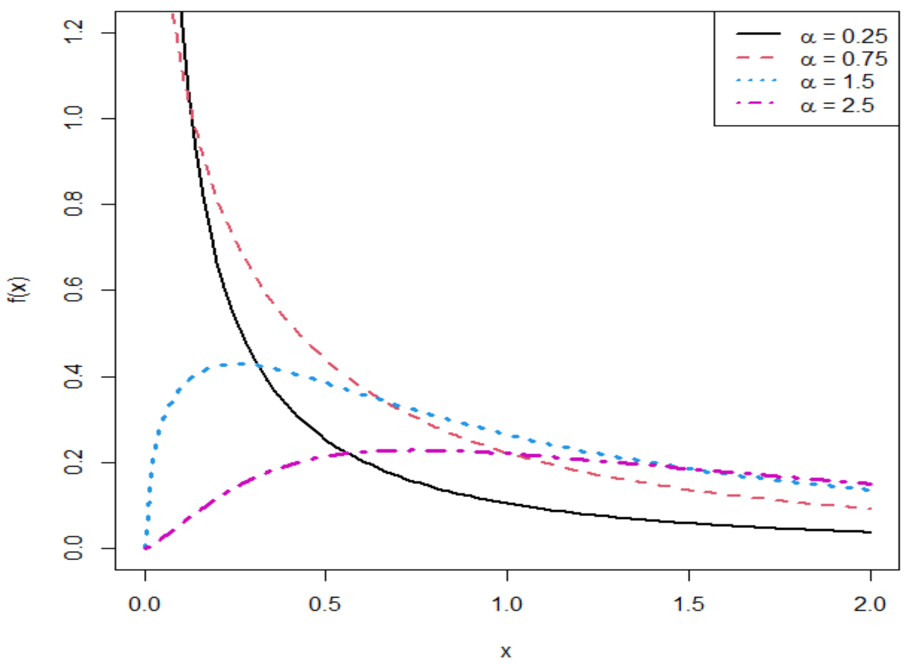

FIGURE 2.1: Plot of pdf of IPD.

2.5. Also, the corresponding cdf, survival and failure rate functions are, respectively, given by

$$F_X(x;\alpha) = \left(\frac{x}{1+x}\right)^{\alpha} \quad ; \alpha > 0,\ x > 0, \tag{2.3}$$

$$S(x;\alpha) = P(X > x) = 1 - \left(\frac{x}{1+x}\right)^{\alpha} \quad ; x > 0,\ \alpha > 0, \text{ and} \tag{2.4}$$

$$h(x;\alpha) = \frac{\alpha x^{\alpha-1}}{(1+x)^{\alpha+1}\left[1-\left(\frac{x}{1+x}\right)^{\alpha}\right]} \quad ; \alpha > 0, x > 0. \tag{2.5}$$

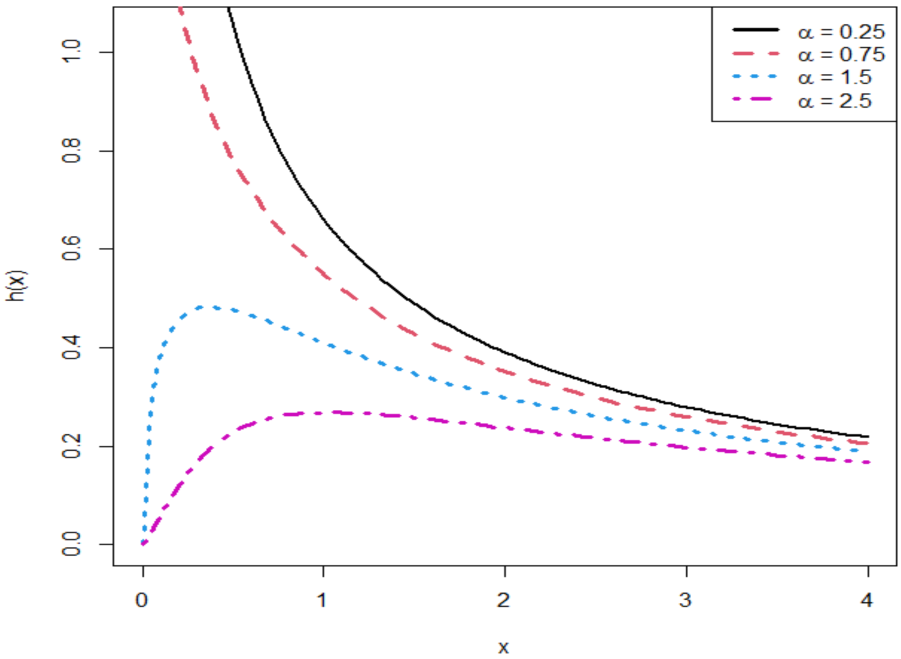

FIGURE 2.2: Plot of failure rate function of IPD.

Figure 2.2 shows the failure rate function of IP lifetime model for distinct values of α, say 0.25, 0.75, 1.5 and 2.5. From the figure 2.2, it is clear that IP lifetime model holds both decreasing and upside-down bathtub shaped failure rate functions.

Next, suppose the failure time X folow IP lifetime model with parameter α, say IP(α), and censoring time T follows IP lifetime model with parameter β, say IP(β). Then using equation (2.1), the joint pdf of randomly censored IP lifetime model is given by

$$f_{Y,D}(y,d,\alpha,\beta) = \frac{\alpha^d \beta^{1-d} y^{d(\alpha-\beta)+\beta-1}}{(1+y)^{d(\alpha-\beta)+\beta+1}} \left[1-\left(\frac{y}{1+y}\right)^{\beta}\right]^d \left[1-\left(\frac{y}{1+y}\right)^{\alpha}\right]^{1-d}; \quad y>0, \alpha>0, \beta>0, d=0,1, \tag{2.6}$$

and the probability of observing a failure is given by

$$p = \int_0^\infty S_T(y) f_X(y) dy = \frac{\beta}{\alpha+\beta}.$$

2.3 Maximum Likelihood Estimation

In this section, we derive the ML estimates, $\hat{\alpha}$ and $\hat{\beta}$ of α and β, respectively. For the observed sample $(\mathbf{y},\ \mathbf{d}) = (y_1,\ d_1), (y_2,\ d_2), \ldots (y_n,\ d_n)$ of size n. Also, compute ACIs of the parameters based on observed Fisher information matrix. The likelihood function can be written as

$$L(\mathbf{y},\ \mathbf{d},\ \alpha,\ \beta) = \prod_{i=1}^{n} \frac{\alpha^{d_i} \beta^{1-d_i} y_i^{d_i(\alpha-\beta)+\beta-1}}{(1+y_i)^{d_i(\alpha-\beta)+\beta+1}} \left[1-\left(\frac{y_i}{1+y_i}\right)^{\beta}\right]^{d_i} \left[1-\left(\frac{y_i}{1+y_i}\right)^{\alpha}\right]^{1-d_i}. \quad (2.7)$$

Thus, the log-likelihood function becomes

$$\begin{aligned} l(\alpha,\beta|\text{data}) = {} & m \ln \alpha + (n-m) \ln \beta + (\alpha-\beta) \sum_{i=1}^{n} d_i \ln y_i + (\beta-1) \sum_{i=1}^{n} \ln y_i \\ & - (\alpha-\beta) \sum_{i=1}^{n} d_i \ln(1+y_i) - (\beta+1) \sum_{i=1}^{n} \ln(1+y_i) \\ & + \sum_{i=1}^{n} d_i \ln \left[1-\left(\frac{y_i}{1+y_i}\right)^{\beta}\right] + \sum_{i=1}^{n} (1-d_i) \ln \left[1-\left(\frac{y_i}{1+y_i}\right)^{\alpha}\right], \end{aligned} \quad (2.8)$$

where, $m = \sum_{i=1}^{n} d_i$. The corresponding normal equations of the log-likelihood function obtain as follows:

$$\frac{\partial\, l(\alpha,\beta|\text{data})}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^{n} d_i \ln y_i - \sum_{i=1}^{n} d_i \ln(1+y_i) - \sum_{i=1}^{n} (1-d_i) \frac{\left(\frac{y_i}{1+y_i}\right)^{\alpha} \ln\left(\frac{y_i}{1+y_i}\right)}{\left[1-\left(\frac{y_i}{1+y_i}\right)^{\alpha}\right]} = 0 \quad (2.9)$$

$$\begin{aligned} \frac{\partial\, l(\alpha,\beta|\text{data})}{\partial \beta} = {} & \frac{n-m}{\beta} + \sum_{i=1}^{n} \ln y_i - \sum_{i=1}^{n} d_i \ln y_i + \sum_{i=1}^{n} d_i \ln(1+y_i) - \sum_{i=1}^{n} \ln(1+y_i) - \\ & \sum_{i=1}^{n} d_i \frac{\left(\frac{y_i}{1+y_i}\right)^{\beta} \ln\left(\frac{y_i}{1+y_i}\right)}{\left[1-\left(\frac{y_i}{1+y_i}\right)^{\beta}\right]} = 0 \end{aligned} \quad (2.10)$$

The ML estimates $\hat{\alpha}$ and $\hat{\beta}$ of the parameters α and β, respectively, are the solutions of the non-linear equations (2.9) and (2.10). Here, equations (2.9) and (2.10) do not have closed form solutions, any iterative method can be used to solve these equations for α and β, respectively. Here, for the computation purpose, *nlm* or *optim* or *maxLik* functions of statistical software *R* can be used.

2.3.1 Asymptotic Confidence Intervals

As the ML estimates of the unknown model parameters are not in closed form, driving the exact distributions of the ML estimates is difficult. As a result, we build the ACIs of the parameters based on the observed Fisher information matrix using the asymptotic distribution of ML estimates. Let $\hat{\theta} = (\hat{\alpha},\ \hat{\beta})$, be the MLE of $\theta = (\alpha,\ \beta)$, the observed Fisher information matrix is given by

$$I(\hat{\theta}) = \begin{bmatrix} -\frac{\partial^2 l(\alpha,\beta|\text{data})}{\partial\alpha^2} & -\frac{\partial^2 l(\alpha,\beta|\text{data})}{\partial\alpha\partial\beta} \\ -\frac{\partial^2 l(\alpha,\beta|\text{data})}{\partial\beta\partial\alpha} & -\frac{\partial^2 l(\alpha,\beta|\text{data})}{\partial\beta^2} \end{bmatrix}_{\theta=\hat{\theta}}$$

where,

$$\frac{\partial^2 l(\alpha,\beta|\text{data})}{\partial\alpha^2} = -\frac{m}{\alpha^2} - \sum_{i=1}^{n}(1-d_i)\frac{\left(\frac{y_i}{1+y_i}\right)^{\alpha}\left(\ln\left(\frac{y_i}{1+y_i}\right)\right)^2}{\left[1-\left(\frac{y_i}{1+y_i}\right)^{\alpha}\right]^2},$$

$$\frac{\partial^2 l(\alpha,\beta|\text{data})}{\partial\beta^2} = -\frac{n-m}{\beta^2} - \sum_{i=1}^{n} d_i\frac{\left(\frac{y_i}{1+y_i}\right)^{\beta}\left(\ln\left(\frac{y_i}{1+y_i}\right)\right)^2}{\left[1-\left(\frac{y_i}{1+y_i}\right)^{\beta}\right]^2},$$

$$\frac{\partial^2 l(\alpha,\beta|\text{data})}{\partial\alpha\partial\beta} = \frac{\partial^2 l(\alpha,\beta|\text{data})}{\partial\beta\partial\alpha} = 0.$$

The asymptotic distribution of ML estimates $\hat{\theta}$ follows a bivariate normal distribution i.e. $\hat{\theta} \sim N(\theta, I^{-1}(\hat{\theta}))$, see, Lawless (2003). Consequently, two sided equal tailed $100(1-\xi)\%$ ACIs of parameters α and β are given by

$$\left(\hat{\alpha} \pm z_{\xi/2}\sqrt{\hat{V}ar(\hat{\alpha})}\right) \ \text{and}\ \left(\hat{\beta} \pm z_{\xi/2}\sqrt{\hat{V}ar(\hat{\beta})}\right),$$

respectively. Here, $\hat{V}ar(\hat{\alpha})$ and $\hat{V}ar(\hat{\beta})$ are diagonal elements of the observed Fisher information matrix $I^{-1}(\hat{\theta})$ and $z_{\xi/2}$ is the upper $(\xi/2)^{th}$ percentile of the standard normal distribution N(0,1). Also, the coverage probability (CPs) for the parameters are given by

$$CP_{\alpha} = \left[\left|\frac{\hat{\alpha}-\alpha}{\sqrt{\hat{V}ar(\hat{\alpha})}}\right| \le z_{\xi/2}\right] \ \text{and}\ CP_{\beta} = \left[\left|\frac{\hat{\beta}-\beta}{\sqrt{\hat{V}ar(\hat{\beta})}}\right| \le z_{\xi/2}\right].$$

2.4 Bayesian Estimation

Here, we discussed the Bayes estimators of unknown parameters associated with the model in (2.6) under the LINEX loss function. In decision theory, a suitable loss function must be given in order to get the optimal decision. For this purpose, the squared error loss function (SELF) is commonly employed loss function in the literature. This loss function is appropriate, when overestimation and underestimation of equal magnitude have the same effects. When the true loss is not symmetric in terms of overestimation and underestimation, asymmetric loss functions are employed to characterise the implications of various losses. Varian (1975) introduced an asymmetric loss function for the first time known as LINEX loss function and it is given as follows:

$$L(\phi,\hat{\phi}) = e^{k(\hat{\phi}-\phi)} - k(\hat{\phi}-\phi) - 1, \tag{2.11}$$

where, $\hat{\phi}$ is an estimate of parameter ϕ, $k \neq 0$ is the known loss parameter. The sign and magnitude of the loss parameter k reflects the direction and degree of asymmetry, respectively. When k is positive, the over estimation is more serious than under estimation and the situation is reverse when k is negative. The LINEX loss function reduces to SELF when magnitude of k tends to zero, see, Zellner (1986). Under the LINEX loss function, the Bayes estimate of ϕ is given as follows

$$\hat{\phi}_{Bayes} = -\frac{1}{k}\ln E[e^{-k\phi}|\text{data}],$$

where, $E[e^{-k\phi}|\text{data}]$ is the posterior expectation which exist and finite. Further, we assume the prior belief of the unknown parameters α and β follows gamma distributions with the following pdfs:

$$g_1(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)}\alpha^{a_1-1}e^{-b_1\alpha} \; ; \alpha,\ a_1,\ b_1 > 0,$$
$$g_2(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)}\beta^{a_2-1}e^{-b_2\beta} \; ; \beta,\ a_2,\ b_2 > 0, \text{ respectively.}$$

Thus, the joint prior distribution of α and β can be written as

$$g(\alpha,\beta) \propto \alpha^{a_1-1}e^{-b_1\alpha}\beta^{a_2-1}e^{-b_2\beta},\ a_1,\ b_1, a_2,\ b_2 > 0. \tag{2.12}$$

The assumption of the piece-wise independent gamma priors is quite reasonable. It is noted that the non-informative priors are the special cases of independent gamma priors when hyper-parameters $a_1 = b_1 = a_2 = b_2 = 0$. Based on the observed randomly censored data, likelihood function in (2.7) and joint prior distribution of $(\alpha,\ \beta)$ in (2.12), the joint posterior distribution of α and β is given by

$$\pi(\alpha,\beta|\text{data}) = \frac{L(\text{data}|\alpha,\beta)g(\alpha,\beta)}{\int_0^\infty\int_0^\infty L(\text{data}|\alpha,\beta)g(\alpha,\beta)d\alpha d\beta}$$

$$\pi(\alpha,\beta|\text{data}) \propto \alpha^{m+a_1-1} e^{-\alpha\left[b_1-\sum\limits_{i=1}^{n} d_i \ln\left(\frac{y_i}{1+y_i}\right)\right]} \prod_{i=1}^{n}\left[1-\left(\frac{y_i}{1+y_i}\right)^{\alpha}\right]^{1-d_i}$$
$$\times \beta^{n-m+a_2-1} e^{-\beta\left[b_2-\sum\limits_{i=1}^{n} (1-d_i) \ln\left(\frac{y_i}{1+y_i}\right)\right]} \prod_{i=1}^{n}\left[1-\left(\frac{y_i}{1+y_i}\right)^{\beta}\right]^{d_i} \tag{2.13}$$

From the joint posterior distribution of α and β given in equation (2.13), we observe that the posterior distributions of α and β are independent. Thus the marginal posterior distribution of α given data **(y, d)** is obtained as

$$\pi_1(\alpha \mid \text{data}) \propto \alpha^{m+a_1-1} e^{-\alpha\left[b_1-\sum\limits_{i=1}^{n} d_i \ln\left(\frac{y_i}{1+y_i}\right)\right]} \prod_{i=1}^{n}\left[1-\left(\frac{y_i}{1+y_i}\right)^{\alpha}\right]^{(1-d_i)} \quad ; \alpha > 0 \tag{2.14}$$

Similarly, the marginal posterior distribution of β given data $(\mathbf{y},\ \mathbf{d})$ is obtained as

$$\pi_2(\beta \mid \text{data}) \propto \beta^{n-m+a_2-1} e^{-\beta\left[b_2-\sum\limits_{i=1}^{n} (1-d_i) \ln\left(\frac{y_i}{1+y_i}\right)\right]} \prod_{i=1}^{n}\left[1-\left(\frac{y_i}{1+y_i}\right)^{\beta}\right]^{d_i} \quad ; \beta > 0 \tag{2.15}$$

Thus, the expectations of any function of α say $\phi_1(\alpha)$ and β say $\phi_2(\beta)$, respectively, are given by

$$E[\phi_1(\alpha) \mid \text{data}] = \int_0^{\infty} \phi_1(\alpha)\pi_1(\alpha \mid \text{data})d\alpha \tag{2.16}$$

$$\text{and} \quad E[\phi_2(\beta) \mid \text{data}] = \int_0^{\infty} \phi_2(\beta)\pi_1(\beta \mid \text{data})d\beta. \tag{2.17}$$

From the above equation (2.16) and (2.17), we observe that the closed form solutions are not available. The above integrals can be solved numerically. Here, we use Markov Chain Monte Carlo (MCMC) techniques like, the Metropolis-Hastings (M-H) algorithm to derive the Bayes estimates of the parameters α and β, respectively.

2.4.1 MCMC Technique

Here, we use MCMC techniques to generate sequences of samples from the marginal posterior distributions of the parameters. The M-H algorithm is used to obtain sample based Bayes estimates of the unknown parameters. For more details about MCMC and M-H algorithm techniques, one may refer, Gelman et al. (2013), Robert and Casella (2004), Metropolis et al.

(1953), Hastings (1970). The marginal posterior distributions of the parameters α and β in equations (2.14) and (2.15), respectively, are not well known distributions. Therefore random numbers from these distributions can be generated by using M-H algorithm. The following steps are used to generate random numbers from the marginal posterior distribution in (2.14):

Step 1: Begin with an initial guess. $\alpha^{(0)}$.
Step 2: From the proposed density $\delta(\alpha^{(j)} \mid \alpha^{(j-1)})$, create a candidate point $\alpha_c^{(j)}$.
Step 3: Generate u using the Uniform (0,1) distribution.
Step 4: Obtain $z(\alpha_c^{(j)} \mid \alpha^{(j-1)}) = \min\left\{\frac{\pi_1(\alpha_c^{(j)}|\text{data})\delta(\alpha^{(j)}|\alpha^{(j-1)})}{\pi_1(\alpha^{(j-1)}|\text{data})\delta(\alpha_c^{(j)}|\alpha^{(j-1)})}, 1\right\}$
Step 5: If $u \leq z$ set $\alpha^{(j)} = \alpha_c^{(j)}$ with acceptance probability z otherwise $\alpha^{(j)} = \alpha^{(j-1)}$.
Step 6: To acquire the parameter sequence of α as $\left(\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(M)}\right)$, repeat steps 2-5 for $j = 1, 2, \ldots, M,$.

Here, we consider proposal density as a normal distribution. The ML estimates and variance of ML estimates from posterior distribution of α are considered as mean and variance of the proposal normal distribution, see, (Ntzoufras, 2009, pp. 44-45). To get an independent sample from the stationary distribution of the Markov chain, which is generally the posterior distribution, we discard first M_0, $\alpha^{(j)}$'s ; $j = 1, 2, \ldots, M_0$, where, M_0 $(< M)$ is the burn-in-period. Now, the approximate posterior mean of $\phi_1(\alpha)$ using M-H algorithm is obtained as

$$\hat{\phi}_{1MH}(\alpha) = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \phi_1(\alpha^{(j)}).$$

Similarly, the approximate posterior mean of $\phi_2(\beta)$ using M-H algorithm is obtained as

$$\hat{\phi}_{2MH}(\beta) = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \phi_2(\beta^{(j)}).$$

Therefore, the Bayes estimates of the parameters α and β under LINEX loss function using M-H algorithm are, respectively, given by

$$\hat{\alpha}_{MH} = -\frac{1}{k}\ln(\hat{\phi}_{1MH}(\alpha)) \text{ and } \hat{\beta}_{MH} = -\frac{1}{k}\ln(\hat{\phi}_{2MH}(\beta)).$$

2.4.2 HPD Credible Intervals

Here, we compute the HPD credible intervals of the parameters α and β using the generated MCMC samples. Let $\alpha_{(1)} < \alpha_{(2)} < \cdots < \alpha_{(M-M_0)}$ denote the ordered values of $\alpha^{(M_0+1)}$, $\alpha^{(M_0+2)}, \ldots, \alpha^{(M)}$. Then, using the algorithm proposed by Chen and Shao (1999), the $100(1-\xi)\%$, where $0 < \xi < 1$, HPD credible interval for α is given by $(\alpha_{(j)}, \alpha_{(j+[(1-\xi)(M-M_0)])})$, where j is chosen such that

$$\alpha_{(j+[(1-\xi)(M-M_0)])} - \alpha_{(j)} = \min_{1 \le i \le (M-M_0)} \left(\alpha_{(i+[(1-\xi)(M-M_0)])} - \alpha_{(i)}\right); j = 1, 2, \ldots, M - M_0,$$

where, $[x]$ is the largest integer less than or equal to x. Similarly, we can construct the $100(1-\xi)\%$ HPD credible interval for β.

2.5 Numerical Computations

Here, we perform a MC simulation study to examine the different estimators created in the preceding sections. The simulation study considers six distinct sample sizes $n = 30, 40, 50, 60, 70, 80$ for different combinations of true parameters (α, β) = (0.75, 1.5) and (1.5, 0.75), respectively. The unknown parameter α and β are estimated using ML and Bayes estimation methods in each cases. The hyper-parameters (a_1, b_1, a_2, b_2) = (3, 2, 3, 4) and (3, 4, 3, 2) are taken into account for gamma informative priors (Prior 1) in Bayesian calculations such that the prior means precisely the same as the true values of the parameters. In case of non-informative priors (Prior 0), the hyper-parameters are taken as $a_1 = b_1 = a_2 = b_2 = 0.0001$. Two distinct values of loss parameter k = -1 and 1 are taken for LINEX loss function. For MCMC technique, $M = 10,000$ sequence of parameter samples are drawn from posterior distribution and $M_0 = 1,000$ taken as burn-in-period. The 95% ACIs based on OFI matrix and HPD credible intervals based on MCMC technique are computed. The entire procedure is replicated by 1000 times. The average estimates (AE) and their associated mean squared error (MSE) are estimated for various estimators. Also determined the average length (AL) and coverage probabilities (CP) of 95% ACI and HPD credible intervals. Tables 2.1, 2.2, and 2.3 describe the findings of the MC simulation study.

These observations can lead to the following conclusions: In almost every case, as sample size grows, AEs become closer to the real value of the parameters, while MSEs go lower. Similarly, when the sample size grows, the ALs of interval estimates shrink, demonstrating the estimators' asymptotic behaviour. CPs achieve the required levels of confidence in almost every case. Bayes estimators perform more effectively in the case of Prior 1 than Prior 0 or ML estimators in terms of biases. On average, HPD credible intervals are shorter AL than ACIs. When some prior information about parameters is provided or non-informative priors are used, we suggest Bayes estimators. ML estimators can also be utilised for rapid results in other situations.

TABLE 2.1: ML and Bayes estimates of α, for different values of α.

					$\hat{\alpha}_{\text{Bayes}}$							
					$k=-1$				$k=1$			
			$\hat{\alpha}_{\text{MLE}}$		Prior 0		Prior 1		Prior 0		Prior 1	
α	n	m	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
0.75	30	21	0.7758	0.0233	0.7876	0.0256	0.7859	0.0195	0.7712	0.0217	0.7662	0.0170
	40	24	0.7780	0.0170	0.7863	0.0183	0.7692	0.0143	0.7686	0.0171	0.7590	0.0139
	50	35	0.7679	0.0134	0.7745	0.0142	0.7679	0.0113	0.7598	0.0122	0.7554	0.0105
	60	42	0.7681	0.0112	0.7736	0.0118	0.7686	0.0096	0.7608	0.0105	0.7586	0.0096
	70	43	0.7666	0.0095	0.7712	0.0099	0.7656	0.0093	0.7556	0.0088	0.7525	0.0073
	80	57	0.7593	0.0079	0.7632	0.0081	0.7611	0.0075	0.7590	0.0076	0.7578	0.0069
1.5	30	12	1.5911	0.1553	1.6783	0.2422	1.6187	0.1337	1.5483	0.1320	1.5240	0.0842
	40	16	1.5688	0.0971	1.6268	0.1288	1.5951	0.0947	1.5343	0.0869	1.5220	0.0687
	50	13	1.5510	0.0762	1.5949	0.0942	1.5977	0.0743	1.5146	0.0635	1.5217	0.0539
	60	18	1.5479	0.0655	1.5838	0.0783	1.5622	0.0586	1.5233	0.0555	1.5170	0.0468
	70	24	1.5432	0.0548	1.5730	0.0637	1.5785	0.0585	1.5168	0.0442	1.5092	0.0394
	80	23	1.5217	0.0409	1.5467	0.0460	1.5540	0.0444	1.5226	0.0408	1.5170	0.0421

TABLE 2.2: ML and Bayes estimates of β, for different values of β.

					$\hat{\beta}_{\text{Bayes}}$							
					$k=-1$				$k=1$			
			$\hat{\beta}_{\text{MLE}}$		Prior 0		Prior 1		Prior 0		Prior 1	
β	n	m	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
1.5	30	21	1.5976	0.1560	1.6876	0.2703	1.6361	0.1497	1.5349	0.1200	1.5407	0.0883
	40	24	1.5686	0.0993	1.6266	0.1308	1.5989	0.0930	1.5389	0.0887	1.5172	0.0665
	50	35	1.5429	0.0709	1.5862	0.0870	1.5848	0.0721	1.5272	0.0726	1.5086	0.0510
	60	42	1.5369	0.0620	1.5721	0.0735	1.5747	0.0611	1.5246	0.0544	1.5076	0.0485
	70	43	1.5391	0.0490	1.5688	0.0573	1.5556	0.0496	1.5187	0.0512	1.5215	0.0422
	80	57	1.5294	0.0422	1.5547	0.0479	1.5532	0.0443	1.5196	0.0399	1.5238	0.0388
0.75	30	12	0.7762	0.0210	0.7881	0.0231	0.7853	0.0198	0.7666	0.0206	0.7664	0.0182
	40	16	0.7661	0.0157	0.7744	0.0167	0.7845	0.0170	0.7653	0.0155	0.7657	0.0146
	50	13	0.7687	0.0140	0.7755	0.0148	0.7650	0.0117	0.7627	0.0132	0.7558	0.0097
	60	18	0.7651	0.0105	0.7706	0.0110	0.7722	0.0103	0.7575	0.0103	0.7584	0.0101
	70	24	0.7607	0.0087	0.7651	0.0091	0.7627	0.0079	0.7594	0.0089	0.7556	0.0079
	80	23	0.7564	0.0077	0.7604	0.0079	0.7687	0.0076	0.7577	0.0080	0.7602	0.0070

TABLE 2.3: The AL and CPs of 95% ACIs and HPD credible intervals for the different values parameters (α, β).

					$\hat{\alpha}_{\text{Bayes}}$						$\hat{\beta}_{Bayes}$			
			$\hat{\alpha}_{\text{MLE}}$		Prior 0		Prior 1		$\hat{\beta}_{\text{MLE}}$		Prior 0		Prior 1	
(α,β)	n	m	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
(0.75,1.5)	30	21	0.5638	0.954	0.5521	0.943	0.5256	0.955	1.3605	0.955	1.5308	0.976	1.3958	0.978
	40	24	0.4895	0.957	0.4790	0.950	0.4527	0.956	1.1435	0.958	1.2874	0.979	1.2102	0.983
	50	35	0.4319	0.953	0.4233	0.948	0.4078	0.956	1.0008	0.948	1.1279	0.970	1.0864	0.983
	60	42	0.3944	0.951	0.3862	0.942	0.3745	0.958	0.9093	0.955	1.0225	0.972	0.9904	0.968
	70	43	0.3643	0.949	0.3573	0.943	0.3468	0.937	0.8419	0.956	0.9482	0.972	0.9135	0.976
	80	57	0.3374	0.953	0.3308	0.942	0.3237	0.956	0.7817	0.952	0.8801	0.970	0.8588	0.971
(1.5,0.75)	30	12	1.3515	0.954	1.3268	0.948	1.2024	0.962	0.5643	0.967	0.6316	0.987	0.6000	0.982
	40	16	1.1462	0.961	1.1269	0.951	1.0514	0.947	0.4818	0.955	0.5397	0.971	0.5267	0.973
	50	13	1.0084	0.941	0.9914	0.933	0.9586	0.968	0.4324	0.940	0.4837	0.964	0.4629	0.973
	60	18	0.9176	0.956	0.9014	0.943	0.8603	0.964	0.3928	0.947	0.4397	0.969	0.4300	0.975
	70	24	0.8452	0.956	0.8323	0.951	0.8155	0.945	0.3614	0.949	0.4040	0.974	0.3942	0.973
	80	23	0.7773	0.95	0.7656	0.944	0.7489	0.962	0.3361	0.956	0.3762	0.974	0.3734	0.973

2.6 Real Data Analysis

With the help of two real datasets, we illustrate the estimation procedures discussed in the previous sections. Here, we consider two real datasets, namely leukemia patients' data (Data I) and Hodgkin's disease patients' data (Data II). These datasets are reported in (Lawless, 2003, pp. 139). Data I depicts the remission periods (in weeks) of a group of 30 leukemia patients who all got the same therapy. Data II considered the survival times (in months) of 15 patients with Hodgkin's disease who were treated with nitrogen mustards and received heavy prior therapy. Data I and Data II, respectively, are given below:

Data I: 1, 1, 2, 4, 4, 6, 6, 6, 7, 8, 9, 9, 10, 12, 13, 14, 18, 19, 24, 26, 29, 31+, 42, 45+, 50+, 57, 60, 71+, 85+, 91.
Data II: 1.05, 2.92, 3.61, 4.20, 4.49, 6.72, 7.31, 9.08, 9.11, 14.49+, 16.85, 18.82+, 26.59+, 30.26+, 41.34+.
The observations with + sign are censored times.

Before going further, we fit Data I and Data II to randomly censored IP lifetime and compare its fitting with some well-known lifetime models, namely, inverse exponential (IE) and generalized inverted exponential (GIE) lifetime models in case of random censoring. The pdfs of the competitive lifetime models are as follows:

$$\begin{aligned}\text{IE:}\;\; f(x,\theta) &= \frac{\theta}{x^2}e^{-\theta/x} \quad x>0,\theta>0,\\ \text{GIE:}\;\; f(x,\alpha,\theta) &= \frac{\alpha\theta}{x^2}e^{-\theta/x}\left(1-e^{-\theta/x}\right)^{\alpha-1} \quad x>0,\alpha,\theta>0.\end{aligned}$$

We compute ML estimates of the unknown parameters along with some useful measure of goodness-of-fit tests and model comparison criteria for both datasets, namely, the negative log-likelihood -ln$L$, the AIC defined by AIC $=2\times k-2\times \ln L$, proposed by Akaike (1974) and Bayesian information criterion (BIC) defined by $BIC = k\times \ln(n)-2\times \ln L$, proposed by Schwarz (1978), where k is the number of associated parameters in the model, n is the number of data points in the given datasets, L is the maximised value of the likelihood function for the estimated model and the Kolmogorov-Smirnov (KS) statistics with its p-values. The best lifetime model corresponds to the lowest –ln$L$, AIC, BIC, and KS statistic and the highest p-value. The KS statistic with its p-values are obtained using *ks.test* function in statistical software R, see R Core Team (2021). The results of the ML estimates and measures of goodness-of-fit tests are reported in Table 2.4 and 2.5, respectively. From these results, we observed that the performance of the randomly censored IP lifetime model is the best choice for the considered datasets.

Moreover, ML and Bayes estimates with their corresponding 95% ACIs and HPD credible intervals of the unknown parameters associated with randomly censored IP lifetime model corresponding to the above real datasets (Data I and Data II) are computed and reported in Table 2.6. The Bayes estimates are computed using non-informative priors under SELF with the help of the MCMC technique. For the M-H algorithm, we generate Markov chain $M = 10,000$ from the posterior distribution. We also examine the convergence of their stationary distributions using graphical diagnostic tools such as trace and histogram plots with Gaussian kernel density plots, as shown in Figures 2.3 and 2.4. The trace plots indicate a random scatter and show the fine mixing of the chains. The histogram plots of the generated MCMC samples show that the marginal posterior distributions of the parameters are almost symmetrical i.e. we can take the mean as the best estimate for the parameters. These plots are hallmarks of rapid MCMC convergence. From these results, we see that ML and Bayes estimates of parameters based on MCMC techniques are quite closed.

TABLE 2.4: Summary fit of the leukemia patients data (Data I).

Models	MLE	$-\ln L$	AIC	BIC	KS-Test	
					KS-Statistic	p-value
$X \sim \mathrm{IE}(\alpha)$ $T \sim \mathrm{IE}(\beta)$	$\hat{\alpha} = 6.4343$ $\hat{\beta} = 76.3664$	139.8547	283.7094	286.5118	0.1755	0.3137
$X \sim \mathrm{IP}(\alpha)$ $T \sim \mathrm{IP}(\beta)$	$\hat{\alpha} = 7.863$ $\hat{\beta} = 77.3696$	137.7025	279.4049	282.2073	0.1371	0.6256
$X \sim \mathrm{GIE}(\alpha, \beta)$ $T \sim \mathrm{GIE}(\alpha, \lambda)$	$\hat{\alpha} = 0.6619$ $\hat{\beta} = 4.7952$ $\hat{\lambda} = 63.1718$	138.2794	282.5587	286.7623	0.1599	0.4271

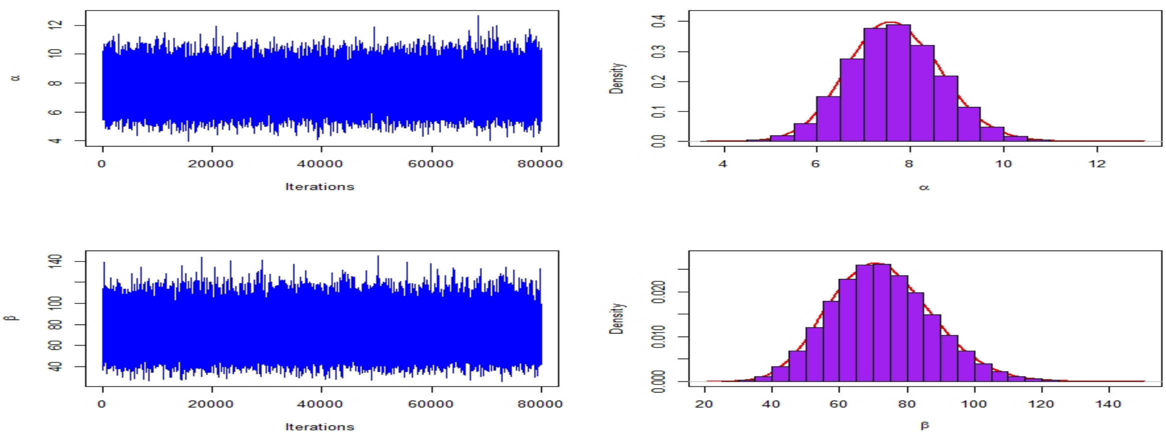

FIGURE 2.3: MCMC plot of leukemia patients data (Data I)

TABLE 2.5: Summary fit of the Hodgkin's disease patient data (Data II).

Models	MLE	$-\ln L$	AIC	BIC	KS-Test	
					KS-Statistic	p-value
$X \sim \mathrm{IE}(\alpha)$ $T \sim \mathrm{IE}(\beta)$	$\hat{\alpha} = 5.5533$ $\hat{\beta} = 27.2313$	60.0676	124.1352	125.5513	0.1303	0.9323
$X \sim \mathrm{IP}(\alpha)$ $T \sim \mathrm{IP}(\beta)$	$\hat{\alpha} = 6.6481$ $\hat{\beta} = 28.1105$	59.7491	123.4982	124.9143	0.0966	0.9965
$X \sim \mathrm{GIE}(\alpha,\beta)$ $T \sim \mathrm{GIE}(\alpha,\lambda)$	$\hat{\alpha} = 0.9172$ $\hat{\beta} = 5.2465$ $\hat{\lambda} = 26.1409$	60.0400	126.0800	128.2041	0.1157	0.9740

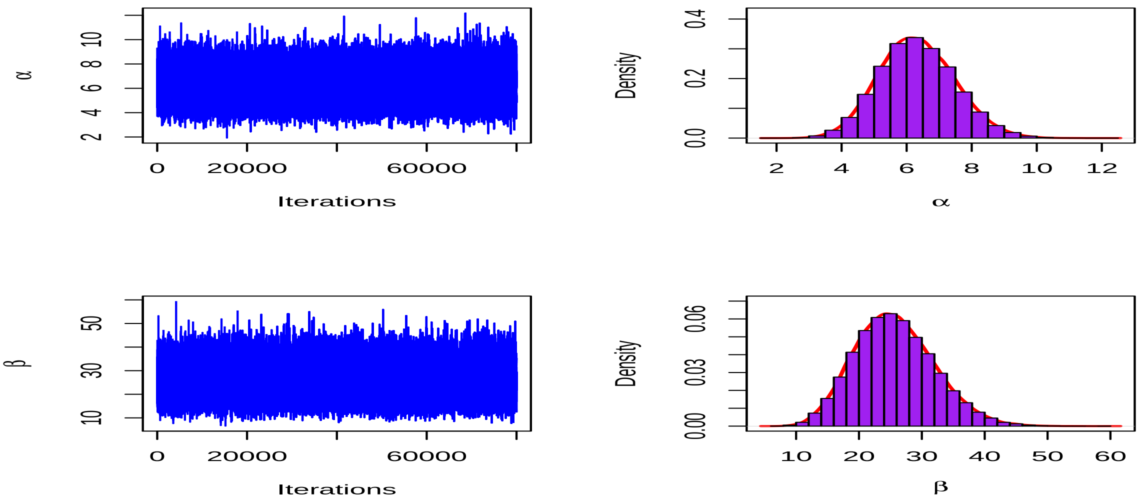

FIGURE 2.4: MCMC plot of Hodgkin's disease data (Data II)

TABLE 2.6: The ML, Bayes estimates and 95% asymptotic and HPD credible intervals of the unknown parameters corresponding to Data I and Data II, respectively.

Datasets	Parameters	MLE	95% CI	Bayes estimates	95% HPD CI
Data I	α	7.863	(5.0484, 10.6775)	7.6672	(5.7135, 9.598)
	β	77.3696	(31.6325, 123.1067)	72.6036	(40.328, 106.8078)
Data II	α	6.6481	(3.2792, 10.017)	6.3191	(4.0975, 8.6322)
	β	28.1105	(8.3706, 47.8505)	25.6675	(12.6852, 40.3103)

2.7 Concluding Remarks

The classical and Bayesian estimation techniques for the parameters of the IP lifetime model using randomly censored data were discussed in this chapter. The ML estimators and their corresponding ACIs based on the observed Fisher information matrix of the unknown parameters were derived. MCMC methods were used to approximate Bayes estimates of the parameters under the LINEX loss function. A comprehensive Monte Carlo simulation study was conducted to evaluate the performance of different estimators, and the results show that ML estimates may be employed easily with acceptable results. For more efficient estimators, the Bayesian estimation method with available prior information or convenient non-informative priors in the absence of prior information is appropriate and recommended.

Chapter 3

Statistical Inference in Inverse Weibull Lifetime Model using Randomly Censored Data *

3.1 Introduction

The main objective of this chapter is to develop classical and Bayesian inferences about the associated model parameters and reliability characteristics of the inverse Weibull (IW) lifetime model using randomly censored data. The concept of random censoring has previously been thoroughly explored in Chapter 2.

The Weibull lifetime model is the most popular and widely used lifetime model in reliability and life testing experiments due to its flexible probability density and failure rate functions. The Weibull lifetime model can have an increasing, decreasing, or constant failure rate depending upon the values of its shape parameter. However, given lifetime data with a non-monotone failure rate pattern, the Weibull lifetime model does not provide a good parametric fit. This motivates authors to investigate other, more realistic lifetime models. The failure rate function of the IW lifetime model is either unimodal or decreasing depending on the shape parameter. There are a variety of real-life instances where data shows a non-monotone, unimodal failure rate, such as cancer patients' remission times, wind speed data, rainfall data, and so on. As a result, if an empirical investigation indicates that the underlying distribution's failure rate function has a unimodal form, the IW lifetime model may be utilised to examine such data sets.

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Recently, the IW lifetime model was studied by several researchers in different disciplines, for example: Kundu and Howlader (2010) studied Bayesian inferences and prediction of the IW lifetime model for type II censored data, Sultan et al. (2014) discussed Bayesian and ML estimation methods of the IW lifetime model parameters under progressive type II censoring, Akgül et al. (2016) used IW lifetime model for the wind speed data, Akgül and Şenoğlu (2018) compared different estimation methods for rainfall data fitted on IW lifetime model, Krishna et al. (2019) studied stress-strength reliability of IW lifetime model under progressive first failure censoring, Basheer et al. (2021) studied IW lifetime model for E-Bayesian and Hierarchical estimation procedures, the multi-component stress-strength for IW lifetime model is discussed by Jana and Bera (2022), the IW lifetime model under Type-I hybrid censoring is discussed by Kazemi and Azizpoor (2021) and reference cited therein.

The rest of the chapter is structured as follows: In Section 3.2, the IW lifetime model based on a randomly censored sample is discussed. We derive ML estimates of the parameters and reliability characteristics in Section 3.3. Based on the expected Fisher information (EFI) matrix, ACIs with corresponding CPs of the unknown parameters are also computed. The expected test time (ETT) of the experiment is discussed in Section 3.4, which is based on randomly censored data from IW lifetime model. In Section 3.5, Bayes estimators of the parameters and reliability characteristics under SELF with gamma informative and non-informative priors using TK approximation and MCMC techniques are obtained. The HPD credible intervals for the parameters based on MCMC techniques are also developed. Section 3.6 deals with an MC simulation study to compare the performance of the estimators developed in this chapter. In Section 3.7 findings are illustrated by a randomly censored real data set. Finally, a concluding remark is appeared in Section 3.8.

3.2 The Model

The pdf and corresponding cdf of IW lifetime model, respectively, are given by

$$f(x;\alpha,\beta) = \alpha\beta x^{-(\alpha+1)} e^{-\beta x^{-\alpha}} \ ; \ \ \alpha > 0, \beta > 0, \ x > 0, \tag{3.1}$$

$$F(x;\alpha,\beta) = e^{-\beta x^{-\alpha}} \ ; \ \alpha > 0, \beta > 0, \ x > 0. \tag{3.2}$$

The survival (or reliability) and failure rate (or hazard) functions, respectively, are given by

$$S(x;\alpha,\beta) = 1 - e^{-\beta x^{-\alpha}} \ ; \ \ \alpha > 0, \beta > 0, \ x > 0. \tag{3.3}$$

and

$$h(x;\alpha,\beta) = \frac{\alpha\beta x^{-(\alpha+1)}}{e^{\beta x^{-\alpha}} - 1} \;;\;\; \alpha > 0, \beta > 0,\; x > 0, \tag{3.4}$$

where, α and β are the shape and scale parameters, respectively. Figure 3.1 shows a visualization of the failure rate function of IW lifetime model for various values of the shape parameter α and scaling parameter β. Assume that the lifetime X and censoring time T, respectively, fol-

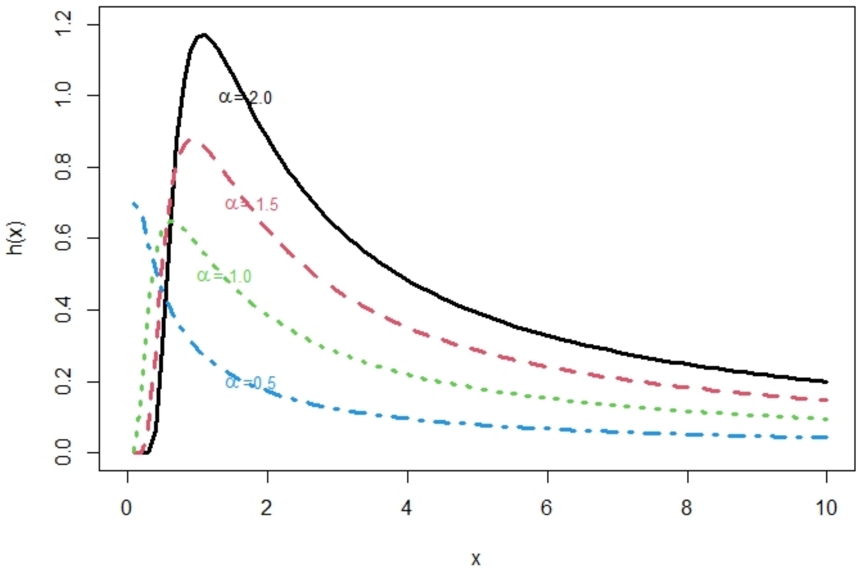

FIGURE 3.1: The plot of the failure rate function IW lifetime model with β=1.

low IW(α,β) and IW(α,λ). Then by using equation (2.6), the joint pdf of randomly censored variables (Y,D) is given by

$$f_{Y,D}(y,d,\alpha,\beta,\lambda) = \alpha\beta^d\lambda^{1-d}y^{-(\alpha+1)}e^{-y^{\alpha(\beta d+\lambda(1-d))}}(1-e^{-\lambda y^{-\alpha}})^d(1-e^{-\beta y^{-\alpha}})^{1-d} \tag{3.5}$$

and the probability of failure is obtained as

$$p = \text{P[An item fails]} = P[d=1] = P[X \leq T] = \int_0^\infty f_T(t)F_X(t)dt = \frac{\lambda}{\beta+\lambda}. \tag{3.6}$$

3.3 Maximum Likelihood Estimation

Let $(\mathbf{y},\mathbf{d}) = ((y_1,d_1),(y_2,d_2),\ldots,(y_n,d_n))$ be a randomly censored sample from model in (3.5). The likelihood function is given by

$$L(\alpha,\beta,\lambda|\mathbf{y},\mathbf{d}) = \alpha^n\beta^m\lambda^{(n-m)}\prod_{i=1}^{n} y_i^{-(\alpha+1)}e^{-\left(\beta\sum\limits_{i=1}^{n}d_iy_i^{-\alpha}+\lambda\sum\limits_{i=1}^{n}(1-d_i)y_i^{-\alpha}\right)}$$
$$\prod_{i=1}^{n}\left(1-e^{-\lambda y_i^{-\alpha}}\right)^{d_i}\prod_{i=1}^{n}\left(1-e^{-\beta y_i^{-\alpha}}\right)^{(1-d_i)}, \tag{3.7}$$

where, $m = \sum\limits_{i=1}^{n} d_i$ denotes the number of failures.
Thus, the log-likelihood function becomes

$$l(\alpha,\beta,\lambda) = n\ln\alpha + mln\beta + (n-m)\ln\lambda - (\alpha+1)S - \beta\sum_{i=1}^{n}d_iy_i^{-\alpha} - \lambda\sum_{i=1}^{n}(1-d_i)y_i^{-\alpha}$$
$$+\sum_{i=1}^{n}d_i\ln\left(1-e^{-\lambda y_i^{-\alpha}}\right)+\sum_{i=1}^{n}(1-d_i)\ln\left(1-e^{-\beta y_i^{-\alpha}}\right), \tag{3.8}$$

where, $S = \sum\limits_{i=1}^{n}\ln y_i$ denotes the log total time on test.
The corresponding normal equations are obtained as

$$\frac{\partial l(\alpha,\beta,\lambda)}{d\alpha} = \frac{m}{\beta} - S + \beta\sum_{i=1}^{n}d_iy_i^{-\alpha}\ln y_i + \lambda\sum_{i=1}^{n}(1-d_i)y_i^{-\alpha}\ln y_i$$
$$-\sum_{i=1}^{n}\frac{d_i\lambda e^{-\lambda y_i^{-\alpha}}y_i^{-\alpha}\ln y_i}{\left(1-e^{-\lambda y_i^{-\alpha}}\right)} - \sum_{i=1}^{n}\frac{(1-d_i)\beta e^{-\beta y_i^{-\alpha}}\ln y_i}{\left(1-e^{-\beta y_i^{-\alpha}}\right)} = 0 \tag{3.9}$$

$$\frac{\partial l(\alpha,\beta,\lambda)}{d\beta} = \frac{m}{\beta} - \sum_{i=1}^{n}d_iy_i^{-\alpha} + \sum_{i=1}^{n}\frac{(1-d_i)e^{-\beta y_i^{-\alpha}}y_i^{-\alpha}}{\left(1-e^{-\beta y_i^{-\alpha}}\right)} = 0 \tag{3.10}$$

$$\frac{l(\alpha,\beta,\lambda)}{d\beta} = \frac{n-m}{\lambda} - \sum_{i=1}^{n}(1-d_i)y_i^{-\alpha} + \sum_{i=1}^{n}\frac{d_ie^{-\lambda y_i^{-\alpha}}y_i^{-\alpha}}{(1-e^{-\lambda y_i^{-\alpha}})} = 0 \tag{3.11}$$

The ML estimates of α, β and λ say $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$, respectively, are the solutions of the normal equations (3.9), (3.10) and (3.11). For the solution of the system of these normal equations, some suitable iterative procedure like the Newton-Raphson method can be used. It is important to note that in this study, we used a Broyden-Fletcher-Goldfarb-Shanno (BFGS) quasi-Newton method for computation. Once we get the desired ML estimates, using the invariance property of ML estimators, see, Zehna (1966), the ML estimates of the survival and failure rate functions, respectively, are obtained as

$$\hat{S}(t) = 1 - e^{-\hat{\beta}t^{-\hat{\alpha}}}\ ;\ t > 0 \quad \text{and} \quad \hat{h}(t) = \frac{\hat{\alpha}\hat{\beta}t^{-(\hat{\alpha}+1)}}{e^{\hat{\beta}x^{-\hat{\alpha}}} - 1}\ ;\ t > 0.$$

3.3.1 Expected Fisher Information Matrix

Here, we compute the Fisher information matrix for the construction of ACIs of the unknown parameters. Zheng and Gastwirth (2001) suggested the EFI matrix for randomly censored data using the failure rate functions. The Fisher information about parameters, say θ=(α, β, λ) contained in a randomly censored sample $(\mathbf{y}, \mathbf{d})$ of size n from the model in (3.5) is given by

$$I^{Y,D}(\theta) = n \times \begin{bmatrix} I_{11}(\theta) & I_{12}(\theta) & I_{13}(\theta) \\ & I_{22}(\theta) & I_{23}(\theta) \\ & & I_{33}(\theta) \end{bmatrix}$$

where,

$$\begin{aligned} I_{11}(\theta) &= \int_0^\infty \left(\frac{\partial}{\partial\alpha}\ln h_X(x)\right)^2 f_X(x)\bar{F}_T(x)dx + \int_0^\infty \left(\frac{\partial}{\partial\alpha}\ln h_T(x)\right)^2 f_T(x)\bar{F}_X(x)dx \\ &= \alpha\beta\int_0^\infty \left(\frac{1}{\alpha} - \ln x - \frac{\beta x^{-\alpha}\ln x}{1-e^{-\beta x^{-\alpha}}}\right)^2 x^{-(\alpha+1)}e^{-\beta x^{-\alpha}}(1-e^{-\lambda x^{-\alpha}})dx \\ &+ \alpha\lambda\int_0^\infty \left(\frac{1}{\alpha} - \ln x - \frac{\lambda x^{-\alpha}\ln x}{1-e^{-\lambda x^{-\alpha}}}\right)^2 x^{-(\alpha+1)}e^{-\lambda x^{-\alpha}}(1-e^{-\beta x^{-\alpha}})dx, \end{aligned}$$

$$\begin{aligned} I_{12}(\theta) &= \int_0^\infty \left(\frac{\partial}{\partial\alpha}\ln h_X(x)\right)\left(\frac{\partial}{\partial\beta}\ln h_X(x)\right) f_X\bar{F}_T(x)dx \\ &+ \int_0^\infty \left(\frac{\partial}{\partial\alpha}\ln h_T(x)\right)\left(\frac{\partial}{\partial\beta}\ln h_T(x)\right) f_T(x)\bar{F}_X(x)dx \\ &= \alpha\beta\int_0^\infty \left(\frac{1}{\alpha} - \ln x + \frac{\beta x^{-\alpha}\ln x}{1-e^{-\beta x^{-\alpha}}}\right)\left(\frac{1}{\beta} + \frac{x^{-\alpha}}{1-e^{-\beta x^{-\alpha}}}\right) \\ &\times x^{-(\alpha+1)}e^{-\beta x^{-\alpha}}(1-e^{-\lambda x^{-\alpha}})dx, \end{aligned}$$

$$
\begin{aligned}
I_{13}(\theta) &= \int_0^\infty \left(\frac{\partial}{\partial\alpha}\ln h_X(x)\right)\left(\frac{\partial}{\partial\lambda}\ln h_X(x)\right) f_X(x)\bar{F}_T(x)dx \\
&+ \int_0^\infty \left(\frac{\partial}{\partial\alpha}\ln h_T(x)\right)\left(\frac{\partial}{\partial\lambda}\ln h_T(x)\right) f_T\bar{F}_X(x)dx \\
&= \alpha\lambda\int_0^\infty \left(\frac{1}{\alpha}-\ln x+\frac{\lambda x^{-\alpha}\ln x}{1-e^{-\lambda x^{-\alpha}}}\right)\left(\frac{1}{\lambda}-\frac{x^{-\alpha}}{1-e^{-\lambda x^{-\alpha}}}\right) \\
&\times x^{-(\alpha+1)}e^{-\lambda x^{-\alpha}}\left(1-e^{-\beta x^{-\alpha}}\right)dx,
\end{aligned}
$$

$$
\begin{aligned}
I_{22}(\theta) &= \int_0^\infty \left(\frac{\partial}{\partial\beta}\ln h_X(x)\right)^2 f_X(x)\bar{F}_T(x)dx + \int_0^\infty \left(\frac{\partial}{\partial\beta}\ln h_T(x)\right)^2 f_T(x)\bar{F}_X(x)dx \\
&= \alpha\beta\int_0^\infty \left(\frac{1}{\beta}-\frac{x^{-\alpha}}{1-e^{-\beta x^{-\alpha}}}\right)^2 x^{-(\alpha+1)}e^{-\beta x^{-\alpha}}(1-e^{-\lambda x^{-\alpha}})dx,
\end{aligned}
$$

$$
\begin{aligned}
I_{23}(\theta) &= \int_0^\infty \left(\frac{\partial}{\partial\beta}\ln h_X(x)\right)\left(\frac{\partial}{\partial\lambda}\ln h_X(x)\right) f_X(x)\bar{F}_T(x)dx \\
&+ \int_0^\infty \left(\frac{\partial}{\partial\beta}\ln h_T(x)\right)\left(\frac{\partial}{\partial\lambda}\ln h_T(x)\right) f_T(x)\bar{F}_X(x)dx = 0,
\end{aligned}
$$

$$
\begin{aligned}
I_{33}(\theta) &= \int_0^\infty \left(\frac{\partial}{\partial\lambda}\ln h_X(x)\right)^2 f_X(x)\bar{F}_T(x)dx + \int_0^\infty \left(\frac{\partial}{\partial\lambda}\ln h_T(x)\right)^2 f_T(x)\bar{F}_X(x)dx \\
&= \alpha\lambda\int_0^\infty \left(\frac{1}{\lambda}-\frac{x^{-\alpha}}{1-e^{-\lambda x^{-\alpha}}}\right)^2 x^{-(\alpha+1)}e^{-\lambda x^{-\alpha}}(1-e^{-\beta x^{-\alpha}})dx.
\end{aligned}
$$

Here, h_X and h_T are the failure rate functions of IW(α,β) and IW(α,λ), respectively. The elements of the EFI matrix $I^{Y,D}(\theta)$ need to be compute numerically.

Under some mild regularity conditions, $\hat{\theta}=(\hat{\alpha},\hat{\beta},\hat{\lambda})$ follows approximately trivariate normal distribution with mean $(\alpha,\ \beta,\ \lambda)$ and covariance matrix $\left[I^{Y,D}(\theta)\right]^{-1}$. In practice, covariance matrix $\left[I^{Y,D}(\theta)\right]^{-1}$ is estimated by observed covariance matrix $\left[I^{Y,D}(\hat{\theta})\right]^{-1}$ to obtain the required asymptotic CIs, see, Lawless (2003). Therefore, two sided equal tail 100(1-ξ)% ACIs for the parameters α, β and λ are, respectively, given by

$$(\hat{\alpha}\pm z_{\xi/2}\sqrt{\hat{V}ar(\hat{\alpha})}),\ (\hat{\beta}\pm z_{\xi/2}\sqrt{\hat{V}ar(\hat{\beta})}) \text{ and } (\hat{\lambda}\pm z_{\xi/2}\sqrt{\hat{V}ar(\hat{\lambda})}).$$

Here, $\hat{V}ar(\hat{\alpha})$, $\hat{V}ar(\hat{\beta})$ and $\hat{V}ar(\hat{\lambda})$ are diagonal elements of the observed covariance matrix $\left[I^{Y,D}(\hat{\theta})\right]^{-1}$ and $z_{\xi/2}$ is the upper $(\xi/2)^{th}$ percentile of the standard normal distribution. Also.

the CPs for the parameters α, β and λ are, respectively, given by

$$CP_\alpha = \left[\left|\frac{\hat{\alpha}-\alpha}{\sqrt{\hat{V}ar(\hat{\alpha})}}\right| \leq z_{\xi/2}\right], \quad CP_\beta = \left[\left|\frac{\hat{\beta}-\beta}{\sqrt{\hat{V}ar(\hat{\beta})}}\right| \leq z_{\xi/2}\right],$$

and

$$CP_\lambda = \left[\left|\frac{\hat{\lambda}-\alpha}{\sqrt{\hat{V}ar(\hat{\lambda})}}\right| \leq z_{\xi/2}\right].$$

3.4 Expected Time on Test

The ETT of a randomly censored life testing experiment is discussed in this section. In real life testing situations, ETT is beneficial for estimating the quantity of objects to be tested, as well as the duration and cost of the life testing experiment. ETT requires the following result:

Theorem 3.1. *In randomly censored sampling plan, the expectation of the largest order statistic* $Z = max(Y_1, Y_2, \ldots, Y_n)$ *is given by*
$E[Z] = \int\limits_0^\infty \left[1-(1-\bar{F}_X(z)\bar{F}_T(z))^n\right]dz.$

Proof. Since, $Y_i, i = 1, 2, \ldots, n$ are iid, the cdf of Z is given by

$$F_Z(z) = P[Z \leq z] = P[max(Y_1, Y_2, \ldots, Y_n) \leq z] = \{P[Y_i \leq z]\}^n \;; z > 0.$$

Note that

$$\begin{aligned} F_Y(z) &= P[Y_i \leq z] = P\big[min(X_i, T_i) \leq z\big] = 1 - P[min(X_i, T_i) > z] \\ &= 1 - P[X_i > z]P[T_i > z] = 1 - \bar{F}_X(z)\bar{F}_T(z). \end{aligned}$$

Therefore,

$$E[Z] = \int_0^\infty (1-F_Z(z))dz = \int_0^\infty \left[1-(1-\bar{F}_X(z)\bar{F}_T(z))^n\right]dz.$$

□

Now, if the failure time X follows IW(α, β) and censoring time T follows IW(α, λ), the ETT for randomly censored experiment is given by

$$ETT = \int_0^\infty \left[1 - \{1-(1-e^{-\beta z^{-\alpha}})(1-e^{-\lambda z^{-\alpha}})\}^n\right]dz \tag{3.12}$$

ETT obtained in equation (3.12) can be obtained numerically for the given values of the parameters and the sample size n. Also, the observed time on the test (OBTT) is given by

TABLE 3.1: Expected time on test (ETT) and the observed time on test (OBTT).

λ	n	$\alpha=2,\beta=0.5$			$\alpha=2,\beta=1$			$\alpha=2,\beta=2$		
		ETT	OBTT		ETT	OBTT		ETT	OBTT	
			AB	MSE		AB	MSE		AB	MSE
0.5	20	1.7601	0.0582	0.6400	2.0871	0.0833	0.5272	2.4592	0.0509	0.5224
	30	1.9620	0.0482	0.6377	2.3279	0.0573	0.6017	2.7482	0.0805	0.6903
	40	2.1175	0.0702	0.678	2.5132	0.0859	0.6609	2.9702	0.0889	0.4232
	50	2.2456	0.0912	0.8631	2.6659	0.0867	0.6270	3.1530	0.0751	0.6062
	60	2.3556	0.0718	0.9476	2.7969	0.0857	0.7214	3.3096	0.0775	0.7966
1	20	2.0871	0.0659	0.644	2.4891	0.0887	0.6800	2.9515	0.0673	0.6545
	30	2.3279	0.0891	0.619	2.7747	0.0753	0.6754	3.2921	0.0696	0.7035
	40	2.5132	0.0829	0.6797	2.9946	0.0764	0.6559	3.5542	0.0700	0.7217
	50	2.6659	0.0830	0.7519	3.1758	0.0961	0.7262	3.7702	0.0894	0.7541
	60	2.7969	0.0878	0.8634	3.3313	0.0801	0.7952	3.9554	0.0994	0.7428
2	20	2.4592	0.0191	0.6023	2.9515	0.0869	0.6880	3.5202	0.0864	0.5600
	30	2.7482	0.0935	0.7368	3.2921	0.0779	0.6381	3.9240	0.0965	0.5507
	40	2.9702	0.0819	0.7145	3.5542	0.0817	0.7594	4.2350	0.0904	0.7119
	50	3.1530	0.0836	0.7161	3.7702	0.0983	0.7037	4.4913	0.0824	0.7524
	60	3.3096	0.0680	0.7428	3.9554	0.0941	0.7269	4.7111	0.0836	0.7905

$OBTT = max(y_1, y_2, \ldots, y_n)$. We compute, the average absolute bias (AB) and mean squared error (MSE) for OBTT based on 1,000 randomly censored simulated samples from the model in (3.5). The values of ETT and AB, MSE for OBTT under randomly censored IW lifetime model for different values of the parameters and sample size n are reported in table 3.1. Table 3.1 shows that the OBTT estimates the ETT quite closely and efficiently.

3.5 Bayesian Estimation

For Bayesian estimation, we use the piece-wise independent gamma priors described below for the parameters α, β, and λ:

$$g_1(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)}\alpha^{a_1-1}e^{-b_1\alpha} \; ;\alpha, a_1, b_1 > 0,$$
$$g_2(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)}\beta^{a_2-1}e^{-b_2\beta} \; ;\beta, a_2, b_2 > 0,$$
$$g_3(\lambda) = \frac{b_3^{a_3}}{\Gamma(a_3)}\lambda^{a_3-1}e^{-b_3\lambda} \; ;\lambda, a_3, b_3 > 0, \text{ respectively.}$$

Thus, the joint prior distribution of α, β and λ can be written as

$$g(\alpha,\beta,\lambda) \propto \alpha^{a_1-1}\beta^{a_2-1}\lambda^{a_3-1}e^{-(b_1\alpha+b_2\beta+b_3\lambda)} \tag{3.13}$$

The piece-wise independent gamma priors assumption is a reasonable one. These priors have been utilised by many authors on the shape and scale parameters of IW lifetime model, see, for example Singh et al. (2013), Krishna et al. (2019). It is also noted that the non-informative priors are the special cases of independent gamma priors when hyper-parameters $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0$ in (3.13).

Based on the observed randomly censored data $(\mathbf{y},\mathbf{d})$, likelihood function in (3.7) and joint prior distribution of $(\alpha,\ \beta,\ \lambda)$ in (3.13), the joint posterior distribution of α, β and λ is given by

$$\pi(\alpha,\beta,\lambda|\mathbf{y},\mathbf{d}) = \frac{L(\mathbf{y},\mathbf{d}|\alpha,\beta,\lambda)g(\alpha,\beta,\lambda)}{\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty L(\mathbf{y},\mathbf{d}|\alpha,\beta,\lambda)g(\alpha,\beta,\lambda)d\alpha d\beta d\lambda}$$

$$\begin{aligned}\pi(\alpha,\beta,\lambda|\mathbf{y},\mathbf{d}) \propto\ & \alpha^{n+a_1-1}e^{-\alpha\left(b_1+\sum\limits_{i=1}^{n}\ln y_i\right)}\beta^{m+a_2-1}e^{-\beta\left(b_2+\sum\limits_{i=1}^{n}d_i y_i^{-\alpha}\right)}\lambda^{n-m+a_3-1} \\ & e^{-\lambda\left(b_2+\sum\limits_{i=1}^{n}(1-d_i)y_i^{-\alpha}\right)}\prod_{i=1}^{n}(1-e^{-\lambda y_i^{-\alpha}})^{d_i}\prod_{i=1}^{n}(1-e^{-\beta y_i^{-\alpha}})^{1-d_i}\end{aligned} \tag{3.14}$$

Thus, the Bayes estimator of any function of α, β and λ, say, $\phi(\alpha,\beta,\lambda)$ under SELF is the posterior expectation of $\phi(\alpha,\beta,\lambda)$ and is given by

$$E[\phi(\alpha,\beta,\lambda)|\mathbf{y},\mathbf{d}] = \frac{\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty \phi(\alpha,\beta,\lambda)L(\mathbf{y},\mathbf{d}|\alpha,\beta,\lambda)g(\alpha,\beta,\lambda)d\alpha d\beta d\lambda}{\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty L(\mathbf{y},\mathbf{d}|\alpha,\beta,\lambda)g(\alpha,\beta,\lambda)d\alpha d\beta d\lambda} \tag{3.15}$$

The Bayes estimator is in the form of a ratio of two integrals for which there is no closed form solution, as shown in the accompanying equation (3.15). As a result, the following integral ratio can be solved numerically. To construct the Bayes estimators, we employ the TK approximation approach given by Tierney and Kadane (1986), as well as MCMC techniques such as the Gibbs sampling methods followed by the M-H algorithm.

3.5.1 TK Approximation Method

According to TK approximation method, the approximate Bayes estimator of $\phi(\alpha,\beta,\lambda)$ under SELF is given by

$$\hat{\phi}_{TK} = E[\phi(\alpha,\beta,\lambda)|\mathbf{y},\mathbf{d}] = \frac{\int_0^\infty\int_0^\infty\int_0^\infty e^{n\delta_\phi^*(\alpha,\beta,\lambda)}d\alpha d\beta d\lambda}{\int_0^\infty\int_0^\infty\int_0^\infty e^{n\delta(\alpha,\beta,\lambda)}d\alpha d\beta d\lambda} \tag{3.16}$$

where, $\delta(\alpha,\beta,\lambda) = \frac{1}{n}[l(\alpha,\beta,\lambda)+\rho(\alpha,\beta,\lambda)]$ and $\delta^*(\alpha,\beta,\lambda) = \left[\delta(\alpha,\beta,\lambda)+\frac{1}{n}\ln\phi(\alpha,\beta,\lambda)\right]$, here, $l(\alpha,\beta,\lambda)$ is the log-likelihood function and $\rho(\alpha,\beta,\lambda) = \ln g(\alpha,\beta,\lambda)$.
The expression (3.16) is approximated by the TK method as

$$\hat{\phi}(\alpha,\beta,\lambda) = \sqrt{\frac{|\Sigma^*|}{|\Sigma|}}e^{n[\delta_\phi^*(\hat{\alpha}_{\delta^*},\hat{\beta}_{\delta^*},\hat{\lambda}_{\delta^*})-\delta(\hat{\alpha}_\delta,\hat{\beta}_\delta,\hat{\lambda}_\delta)]}, \tag{3.17}$$

where, $|\Sigma^*|$ and $|\Sigma|$ are the determinants of inverse of negative hessian of $\delta^*(\alpha,\beta,\lambda)$ and $\delta(\alpha,\beta,\lambda)$ at $(\hat{\alpha}_{\delta^*},\hat{\beta}_{\delta^*},\hat{\lambda}_{\delta^*})$ and $(\hat{\alpha}_\delta,\hat{\beta}_\delta,\hat{\lambda}_\delta)$, respectively. Also, $(\hat{\alpha}_{\delta^*},\hat{\beta}_{\delta^*},\hat{\lambda}_{\delta^*})$ and $(\hat{\alpha}_\delta,\hat{\beta}_\delta,\hat{\lambda}_\delta)$ maximize $\delta^*(\alpha,\beta,\lambda)$ and $\delta(\alpha,\beta,\lambda)$, respectively. Next, we observe that

$$\begin{aligned}\delta(\alpha,\beta,\lambda) = \frac{1}{n}\Bigg[&(n+a_1-1)\ln\alpha+(m+a_2-1)\ln\beta+(n-m+a_3-1)\ln\lambda-(\alpha+1)S\\&-\beta\left(b_2+\sum_{i=1}^n d_i y_i^{-\alpha}\right)-\lambda\left(b_3+\sum_{i=1}^n\left(1-d_i\right)y_i^{-\alpha}\right)-b_1\alpha\\&+\sum_{i=1}^n d_i\ln\left(1-e^{-\lambda y_i^{-\alpha}}\right)+\sum_{i=1}^n(1-d_i)\ln\left(1-e^{-\beta y_i^{-\alpha}}\right)\Bigg]\end{aligned}$$

Then, $(\hat{\alpha}_\delta,\hat{\beta}_\delta,\hat{\lambda}_\delta)$ are computed by solving the following non-linear equations

$$\begin{aligned}\frac{\partial\delta}{\partial\alpha} &= \frac{n+a_1-1}{\alpha}-S+\beta\sum_{i=1}^n d_i y_i^{-\alpha}\ln y_i+\lambda\sum_{i=1}^n(1-d_i)y_i^{-\alpha}\ln y_i-b_1\\&\quad-\lambda\sum_{i=1}^n\frac{d_i e^{-\lambda y^{-\alpha}}y_i^{-\alpha}\ln y_i}{\left(1-e^{-\lambda y_i^{-\alpha}}\right)}-\beta\sum_{i=1}^n\frac{(1-d_i)e^{-\beta y_i^{-\alpha}}y_i^{-\alpha}\ln y_i}{\left(1-e^{-\beta y_i^{-\alpha}}\right)}=0\\\frac{\partial\delta}{\partial\beta} &= \frac{m+a_2-1}{\beta}-\left(b_2+\sum_{i=1}^n d_i y_i^{-\alpha}\right)+\sum_{i=1}^n\frac{(1-d_i)e^{-\beta y_i^{-\alpha}}y_i^{-\alpha}}{\left(1-e^{-\beta y_i^{-\alpha}}\right)}=0\\\frac{\partial\delta}{\partial\lambda} &= \frac{n-m+a_3-1}{\lambda}-\left(b_3+\sum_{i=1}^n(1-d_i)y_i^{-\alpha}\right)+\sum_{i=1}^n\frac{d_i e^{-\lambda y_i^{-\alpha}}y_i^{-\alpha}}{\left(1-e^{-\lambda y_i^{-\alpha}}\right)}=0\end{aligned}$$

Now, obtain $|\Sigma|$ from

$$\Sigma^{-1} = \frac{1}{n}\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix},$$

where,

$$\begin{aligned}
\delta_{11} = -\frac{\partial^2\delta}{\partial\alpha^2} &= \frac{n+a_1-1}{\alpha^2} + \beta\sum_{i=1}^{n} d_i y_i^{-\alpha}(\ln y_i)^2 + \lambda\sum_{i=1}^{n}(1-d_i)y_i^{-\alpha}(\ln y_i)^2 \\
&+ \lambda\sum_{i=1}^{n}\frac{d_i y_i^{-\alpha}(\ln y_i)^2 e^{-\lambda y_i^{-\alpha}}(e^{-\lambda y_i^{-\alpha}}+\lambda y_i^{-\alpha}-1)}{\left(1-e^{-\lambda y_i^{-\alpha}}\right)^2} \\
&+ \beta\sum_{i=1}^{n}\frac{(1-d_i)y_i^{-\alpha}(\ln y_i)^2 e^{-\beta y_i^{-\alpha}}(e^{-\beta y_i^{-\alpha}}+\beta y_i^{-\alpha}-1)}{\left(1-e^{-\beta y_i^{-\alpha}}\right)^2} \\
\delta_{12} = \delta_{21} = -\frac{\partial^2\delta}{\partial\alpha\partial\beta} &= -\sum_{i=1}^{n} d_i y_i^{-\alpha}\ln y_i - \sum_{i=1}^{n}\frac{(1-d_i)y_i^{-\alpha}\ln y_i e^{-\beta y_i^{-\alpha}}(e^{-\beta y_i^{-\alpha}}+\beta y_i^{-\alpha}-1)}{\left(1-e^{-\beta y_i^{-\alpha}}\right)^2} \\
\delta_{13} = \delta_{31} = -\frac{\partial^2\delta}{\partial\alpha\partial\lambda} &= -\sum_{i=1}^{n}(1-d_i)y_i^{-\alpha}\ln y_i - \sum_{i=1}^{n}\frac{d_i y_i^{-\alpha}\ln y_i e^{-\lambda y_i^{-\alpha}}(e^{-\lambda y_i^{-\alpha}}+\lambda y_i^{-\alpha}-1)}{\left(1-e^{-\lambda y_i^{-\alpha}}\right)^2} \\
\delta_{22} = -\frac{\partial^2\delta}{\partial\beta^2} &= \frac{m+a_2-1}{\beta^2} + \sum_{i=1}^{n}\frac{(1-d_i)y_i^{-2\alpha}e^{-\beta y_i^{-\alpha}}}{(1-e^{-\beta y_i^{-\alpha}})^2}, \quad \delta_{23} = \delta_{32} = -\frac{\partial^2\delta}{\partial\beta\partial\lambda} = 0, \\
\delta_{33} = \frac{\partial^2\delta}{\partial\lambda^2} &= \frac{n-m+a_3-1}{\lambda^2} + \sum_{i=1}^{n}\frac{d_i y_i^{-2\alpha}e^{-\lambda y_i^{-\alpha}}}{\left(1-e^{-\lambda y_i^{-\alpha}}\right)^2}
\end{aligned}$$

In order to compute the Bayes estimator of α we take $\phi(\alpha,\beta,\lambda) = \alpha$ and accordingly function $\delta^*(\alpha,\beta,\lambda)$ becomes

$$\delta^*_\alpha(\alpha,\beta,\lambda) = \delta(\alpha,\beta,\lambda) + \frac{1}{n}\ln\alpha$$

and then $(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*}, \hat{\lambda}_{\delta^*})$ are obtained as solution of the following non-linear equations

$$\frac{\partial\delta^*_\alpha}{\partial\alpha} = \frac{\partial\delta}{\partial\alpha} + \frac{1}{\alpha} = 0, \quad \frac{\partial\delta^*_\alpha}{\partial\beta} = \frac{\partial\delta}{\partial\beta} = 0, \quad \frac{\partial\delta^*_\alpha}{\partial\lambda} = \frac{\partial\delta}{\partial\lambda} = 0$$

and obtain $|\Sigma^*|$ from

$$\Sigma^{*\,-1}_\alpha = \frac{1}{n}\begin{bmatrix} \delta^*_{11} & \delta^*_{12} & \delta^*_{13} \\ \delta^*_{21} & \delta^*_{22} & \delta^*_{23} \\ \delta^*_{31} & \delta^*_{32} & \delta^*_{33} \end{bmatrix},$$

where,

$$\delta_{11}^{*}=-\frac{\partial^{2}\delta_{\alpha}^{*}}{\partial\alpha^{2}}=-\frac{\partial^{2}\delta}{\partial\alpha^{2}}+\frac{1}{\alpha^{2}},\ \delta_{12}^{*}=\delta_{12},\ \delta_{13}^{*}=\delta_{13},\ \delta_{21}^{*}=\delta_{21},\ \delta_{22}^{*}=\delta_{22},\ \delta_{23}^{*}=\delta_{23},\ \delta_{31}^{*}=\delta_{31},$$
$$\delta_{32}^{*}=\delta_{32},\ \delta_{33}^{*}=\delta_{33}.$$

Thus, the approximate Bayes estimator of α under SELF is given by

$$\hat{\alpha}_{TK}=\sqrt{\frac{|\Sigma_{\alpha}^{*}|}{|\Sigma|}}e^{n[\delta_{\alpha}^{*}(\hat{\alpha}_{\delta^{*}},\hat{\beta}_{\delta^{*}},\hat{\lambda}_{\delta^{*}})-\delta(\hat{\alpha}_{\delta},\hat{\beta}_{\delta},\hat{\lambda}_{\delta})]}$$

Similarly, we can derive the approximate Bayes estimator of β and λ as

$$\hat{\beta}_{TK}=\sqrt{\frac{|\Sigma_{\beta}^{*}|}{|\Sigma|}}e^{n[\delta_{\beta}^{*}(\hat{\alpha}_{\delta^{*}},\hat{\beta}_{\delta^{*}},\hat{\lambda}_{\delta^{*}})-\delta(\hat{\alpha}_{\delta},\hat{\beta}_{\delta},\hat{\lambda}_{\delta})]}$$

$$\hat{\lambda}_{TK}=\sqrt{\frac{|\Sigma_{\lambda}^{*}|}{|\Sigma|}}e^{n[\delta_{\lambda}^{*}(\hat{\alpha}_{\delta^{*}},\hat{\beta}_{\delta^{*}},\hat{\lambda}_{\delta^{*}})-\delta(\hat{\alpha}_{\delta},\hat{\beta}_{\delta},\hat{\lambda}_{\delta})]},$$

respectively. Next, We compute the Bayes estimator of survival function S(t).
In this case, $\phi(\alpha,\beta,\lambda)=1-e^{-\beta t^{-\alpha}}$, then

$$\delta_{\lambda}^{*}(\alpha,\beta,\lambda)=\delta(\alpha,\beta,\lambda)+\frac{1}{n}\ln\left(1-e^{-\beta t^{-\alpha}}\right).$$

Now compute $(\hat{\alpha}_{\delta^{*}},\hat{\beta}_{\delta^{*}},\hat{\lambda}_{\delta^{*}})$ by solving the following non-linear equations:

$$\frac{\partial\delta_{S(t)}^{*}}{\partial\alpha}=\frac{\partial\delta}{\partial\alpha}-\frac{\beta e^{-\beta t^{-\alpha}}t^{-\alpha}\ln t}{(1-e^{-\beta t^{-\alpha}})}=0,\ \frac{\partial\delta_{S(t)}^{*}}{\partial\beta}=\frac{\partial\delta}{\partial\beta}+\frac{e^{-\beta t^{-\alpha}}t^{-\alpha}}{(1-e^{-\beta t^{-\alpha}})}=0,\ \frac{\partial\delta_{S(t)}^{*}}{\partial\lambda}=\frac{\partial\delta}{\partial\lambda}=0$$

Now we find $\Sigma_{S(t)}^{*}$ from

$${\Sigma_{S(t)}^{*}}^{-1}=\frac{1}{n}\begin{bmatrix}\delta_{S(t)11}^{*} & \delta_{S(t)12}^{*} & \delta_{S(t)13}^{*}\\ \delta_{S(t)21}^{*} & \delta_{S(t)22}^{*} & \delta_{S(t)23}^{*}\\ \delta_{S(t)31}^{*} & \delta_{S(t)32}^{*} & \delta_{S(t)33}^{*}\end{bmatrix},$$

where,

$$\delta^*_{S(t)11} = -\frac{\partial^2 \delta^*_{S(t)}}{\partial \alpha^2} = -\frac{\partial^2 \delta}{\partial \alpha^2} - \frac{\beta e^{-\beta t^{-\alpha}} t^{-\alpha} (\ln t)^2 (e^{-\beta t^{-\alpha}} + \beta t^{-\alpha} - 1)}{(1 - e^{-\beta t^{-\alpha}})^2},$$

$$\delta^*_{S(t)12} = \delta^*_{S(t)21} = -\frac{\partial^2 \delta^*_{S(t)}}{\partial \alpha \partial \beta} = -\frac{\partial^2 \delta}{\partial \alpha \partial \beta} - \frac{e^{-\beta t^{-\alpha}} t^{-\alpha} \ln t (e^{-\beta t^{-\alpha}} + \beta t^{-\alpha} - 1)}{(1 - e^{-\beta t^{-\alpha}})^2},$$

$$\delta^*_{S(t)13} = -\frac{\partial^2 \delta^*_{S(t)}}{\partial \alpha \partial \lambda} = -\frac{\partial^2 \delta}{\partial \alpha \partial \lambda}, \; \delta^*_{S(t)22} = -\frac{\partial^2 \delta^*_{S(t)}}{\partial \beta^2} = -\frac{\partial^2 \delta}{\partial \beta^2} + \frac{t^{-2\alpha} e^{-\beta t^{-\alpha}}}{(1 - e^{-\beta t^{-\alpha}})^2},$$

$$\delta^*_{S(t)23} = \delta_{23}, \delta^*_{S(t)31} = \delta_{31}, \; \delta^*_{S(t)33} = \delta_{33} \qquad .$$

Thus, the Bayes estimator of $S(t)$ is given by

$$\hat{S}(t)_{TK} = \sqrt{\frac{|\Sigma^*_{S(t)}|}{|\Sigma|}} e^{n[\delta^*_{S(t)}(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*}, \hat{\lambda}_{\delta^*}) - \delta(\hat{\alpha}_{\delta}, \hat{\beta}_{\delta}, \hat{\lambda}_{\delta})]}$$

Similarly, the Bayes estimator of failure rate function is given by

$$\hat{h}(t)_{TK} = \sqrt{\frac{|\Sigma^*_{h(t)}|}{|\Sigma|}} e^{n[\delta^*_{h(t)}(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*}, \hat{\lambda}_{\delta^*}) - \delta(\hat{\alpha}_{\delta}, \hat{\beta}_{\delta}, \hat{\lambda}_{\delta})]},$$

3.5.2 Gibbs Sampling Method

To do sample-based inference, we consider adopting one of the MCMC methods as Gibbs sampling technique to produce a random sample from the joint posterior distribution. For detailed study about MCMC techniques and their applications, one may refer Robert and Casella (2004) and Gelman et al. (2013). The Gibbs sampling approach uses the full conditional posterior densities of α, *beta*, and *lambda*, respectively

$$\pi_1(\beta \mid \alpha, \mathbf{y}, \mathbf{d}) = \beta^{m+a_2-1} e^{-\beta\left(b_2 + \sum\limits_{i=1}^{n} d_i y_i^{-\alpha}\right)} \prod_{i=1}^{n} \left(1 - e^{-\beta y_i^{-\alpha}}\right)^{(1-d_i)}, \tag{3.18}$$

$$\pi_2(\lambda \mid \alpha, \mathbf{y}, \mathbf{d}) = \lambda^{n-m+a_3-1} e^{-\lambda\left(b_3 + \sum\limits_{i=1}^{n} (1-d_i) y_i^{-\alpha}\right)} \prod_{i=1}^{n} \left(1 - e^{-\beta y_i^{-\alpha}}\right)^{d_i}, \tag{3.19}$$

$$\begin{aligned} \pi_3(\alpha \mid \beta, \lambda, \mathbf{y}, \mathbf{d}) = \alpha^{n+a_1-1} & e^{-\alpha\left(b_1 + \sum\limits_{i=1}^{n} \ln y_i\right)} e^{-\left(\beta \sum\limits_{i=1}^{n} d_i y_i^{-\alpha} + \lambda \sum\limits_{i=1}^{n} (1-d_i) y_i^{-\alpha}\right)} \\ & \prod_{i=1}^{n} (1 - e^{-\lambda y_i^{-\alpha}})^{d_i} \prod_{i=1}^{n} (1 - e^{-\beta y_i^{-\alpha}})^{(1-d_i)}. \end{aligned} \tag{3.20}$$

To produce samples from the full conditional posterior densities (3.18), (3.19) and (3.20), we utilise the following algorithm:

Gibbs Sampler Algorithm

Step 1: Start with initial guess of α, β and λ say $\alpha^{(0)}$, $\beta^{(0)}$ and $\lambda^{(0)}$.
Step 2: Set $j = 1$.
Step 3: Generate $\beta^{(j)}$ from $\pi_1(\beta \mid \alpha^{(j-1)}, \mathbf{y}, \mathbf{d})$ in (3.18) using M-H algorithm with normal proposal density.
Step 4: Generate $\lambda^{(j)}$ from $\pi_2(\lambda \mid \alpha^{(j-1)}, \mathbf{y}, \mathbf{d})$ in (3.19) using M-H algorithm with normal proposal density.
Step 5: Generate $\alpha^{(j)}$ from $\pi_3(\alpha \mid \beta^{(j-1)}, \lambda^{(j-1)}, \mathbf{y}, \mathbf{d})$ in (3.20) using M-H algorithm with normal proposal density.
Step 6: Set $j = j+1$ and repeat steps 3-5 for all $j = 1, 2, \ldots, M$ to obtain MCMC samples

$\left(\alpha^{(1)}, \beta^{(1)}, \lambda^{(1)}\right), \left(\alpha^{(2)}, \beta^{(2)}, \lambda^{(2)}\right), \ldots, \left(\alpha^{(M)}, \beta^{(M)}, \lambda^{(M)}\right)$.

Now, the approximate Bayes estimator of $\phi(\alpha, \beta, \lambda)$, can be obtained as

$$\hat{\phi}_{GS}(\alpha, \beta, \lambda) = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \phi\left(\alpha^{(j)}, \beta^{(j)}, \lambda^{(j)}\right), \tag{3.21}$$

where, M_0 is the burn-in period i.e. a number of iterations in Markov chain before the stationary distribution is achieved. Thus, taking, $\phi(\alpha, \beta, \lambda) = \alpha, \beta$ and λ, the Bayes estimators of the parameters α, β and λ under SELF, respectively, are given by

$$\hat{\alpha}_{GS} = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \alpha^{(j)}, \quad \hat{\beta}_{GS} = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \beta^{(j)}, \text{ and}$$

$$\hat{\lambda}_{GS} = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \lambda^{(j)}.$$

Also, the Bayes estimators of the survival and failure rate functions, respectively, are given by

$$\hat{S}(t)_{GS} = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \left(1 - e^{-\beta^{(j)} t^{-\alpha^{(j)}}}\right) \; ; \, t > 0, \text{ and}$$

$$\hat{h}(t)_{GS} = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\frac{\alpha^{(j)}\beta^{(j)}t^{-(\alpha^{(j)}+1)}}{e^{\beta^{(j)}t^{-\alpha^{(j)}}}-1}\ ;\ t>0.$$

3.5.3 HPD Credible Intervals

Using the produced MCMC samples, we now compute the HPD credible interval of the unknown parameters. Let $\alpha_{(1)},\alpha_{(2)},\ldots,\alpha_{(M-M_0)}$ be the ordered values of $\alpha^{(M_0+1)},\alpha^{(M_0+2)},\ldots,\alpha^{(M)}$. Then 100(1-$\xi$)% of HPD credible intervals of the parameter α, is given by

$$\left(\alpha_{(j)},\ \alpha_{(j+[(1-\xi)(M-M_0)])}\right),\ \ 0<\xi<1$$

where j is chosen such that

$$\alpha_{(j+[(1-\xi)(M-M_0)])}-\alpha_{(j)} = \min_{1\le i\xi\le(M-M_0)}\left(\alpha_{(i+[(1-\xi)(M-M_0)])}-\alpha_{(i)}\right);\ j=1,2,\ldots,(M-M_0),$$

here, $[x]$ is the largest integer less than or equal to x, see, Chen and Shao (1999).

Similarly, 100(1-ξ)% HPD credible intervals for β and λ can be constructed.

3.6 Numerical Computations

In this section, we run a simulation study to compare the proposed estimators developed in the preceding sections. All of the computations were done with the statistical software R, see, R Core Team (2021). In this simulation study, we use five distinct sample sizes n= 20, 30, 40, 50, and 60. In all situations, the true value of $\lambda=1.0$ is used, as well as two distinct values of β=0.5, 1.5, and two different values of α=0.5, 2. Under SELF, non-informative as well as gamma informative priors are considered for Bayesian computation. In case of informative priors following values of hyper-parameters $(a_1,b_1,a_2,b_2,a_3,b_3)$ are taken so that prior means are exactly equal to the true values of the parameters: $(2,4,2,4,2,2),(2,4,3,2,2,2),(4,2,2,4,2,2)$ and $(4,2,3,2,2,2)$.

For each case, the ML and Bayes estimates of the unknown parameters, survival and failure rate functions are computed. The mission time $t=0.80$ is taken for survival and failure rate functions. The TK approximation and Gibbs sampling methods are used to compute the Bayes estimators of the parameters and reliability characteristics. The 95% asymptotic CIs based on expected Fisher information matrix and HPD credible intervals based on Gibbs sampling method are constructed. The integrals associated with expected Fisher information matrix are solved using the *integrate* function of software R. We take, $M = 10,000$ with burn-in period $M_0 = 2,000$ for Gibbs sampling method. The whole process was simulated 1,000 times and the average absolute biases (AB) with the corresponding mean squared errors (MSE) are computed for different estimators. Also, the average length (AL) and the coverage probabilities (CP) of 95% asymptotic confidence and HPD credible intervals are calculated. The results of the simulation study are reported in following Tables 3.2 , 3.6, 3.3, 3.7, 3.4, 3.8, 3.5, 3.9.

In simulation tables, the short notations TK stands for Tierney-Kadane method, GS stand for Gibbs sampling method, P1 for non-informative prior and P2 for gamma informative prior. From these results the following conclusions are made:

(i) The AB and MSEs of the ML and Bayes estimators of the parameters and reliability characteristics decrease as the sample size increases in all situations.

(ii) The AB and MSE decrease as the failure time parameter β increases.

(iii) In terms of both AB and MSEs, Bayes estimates outperform ML estimates as they include prior knowledge. In terms of both AB and MSEs, the Gibbs sampling approach outperforms the TK approximation method.

(iv) As the sample size n increases, the ALs of all intervals shrinks. Also, the ALs of HPD credible intervals are less than the ALs of ACIs.

(v) The CPs achieve their required confidence levels quite satisfactorily in classical estimation. However, when the true value of parameter $\alpha = 0.5$ is used in the Bayesian estimate approach, CPs only reach their nominal level. In most situations, as the true value of α and the sample size n increase, they don't hit their nominal levels.

TABLE 3.2: The ML and Bayes estimates of parameters and reliability characteristics, when α=0.5, β=0.5, λ=1, t=0.8, $S(t)$= 0.4282, $h(t)$= 0.4665.

		$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$		$\hat{S}(t)$		$\hat{h}(t)$	
n	Methods	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
	MLE	0.0856	0.0132	0.0037	0.0274	0.2759	0.3763	0.0813	0.0101	0.1097	0.0213
	TK P1	0.0844	0.0128	0.1246	0.0254	0.2721	0.1604	0.0755	0.0089	0.1078	0.0212
20	TK P2	0.0703	0.0086	0.1060	0.0178	0.2101	0.0805	0.0646	0.0064	0.0871	0.0120
	GS P1	0.0545	0.0044	0.1027	0.0183	0.2322	0.1088	0.0617	0.0061	0.0626	0.0061
	GS P2	0.0492	0.0037	0.0905	0.0136	0.1823	0.0593	0.055	0.0047	0.0536	0.0043
	MLE	0.0597	0.0060	0.0989	0.0154	0.2037	0.0764	0.0609	0.0058	0.0758	0.0096
	TK P1	0.0631	0.0067	0.1011	0.0169	0.2056	0.0755	0.0616	0.0061	0.0822	0.0114
30	TK P2	0.0529	0.0046	0.0854	0.0116	0.1788	0.0566	0.0520	0.0042	0.0654	0.0070
	GS P1	0.0476	0.0034	0.0829	0.0124	0.1812	0.0588	0.0496	0.0041	0.0543	0.0044
	GS P2	0.0454	0.0030	0.0754	0.0096	0.1596	0.0457	0.0450	0.0032	0.0497	0.0036
	MLE	0.0513	0.0045	0.0869	0.0119	0.1756	0.0534	0.0538	0.0045	0.0667	0.0075
	TK P1	0.0520	0.0048	0.0849	0.0118	0.1741	0.0527	0.0043	0.0428	0.0664	0.0080
40	TK P2	0.0465	0.0036	0.0773	0.0095	0.1591	0.0432	0.0478	0.0036	0.0595	0.0059
	GS P1	0.0482	0.0032	0.0727	0.0091	0.1561	0.0434	0.0428	0.0030	0.0543	0.0041
	GS P2	0.0457	0.0029	0.0671	0.0076	0.1432	0.0362	0.0402	0.0026	0.0495	0.0035
	MLE	0.0460	0.0036	0.0789	0.0096	0.1493	0.0370	0.0485	0.0036	0.0060	0.0551
	TK P1	0.0481	0.0039	0.0794	0.0098	0.1531	0.0390	0.0484	0.0036	0.0626	0.0063
50	TK P2	0.0424	0.0030	0.0719	0.0080	0.1382	0.0317	0.0439	0.0030	0.0050	0.0502
	GS P1	0.0479	0.0031	0.0668	0.0077	0.1376	0.0330	0.0392	0.0025	0.0523	0.0039
	GS P2	0.0456	0.0028	0.0629	0.0065	0.1255	0.0273	0.0370	0.0022	0.0035	0.2159
	MLE	0.0427	0.0029	0.0714	0.0081	0.1385	0.0321	0.0437	0.0030	0.0546	0.0048
	TK P1	0.0402	0.0026	0.0664	0.0070	0.1302	0.0283	0.0402	0.0025	0.0508	0.0041
60	TK P2	0.0396	0.0026	0.0698	0.0077	0.1308	0.0283	0.0421	0.0027	0.0513	0.0043
	GS P1	0.0459	0.0029	0.0632	0.0065	0.1188	0.0246	0.0367	0.0021	0.0514	0.0036
	GS P2	0.0483	0.0031	0.0634	0.0067	0.1199	0.0248	0.0366	0.0021	0.0539	0.0039

TABLE 3.3: ML and Bayes estimates of parameters and reliability characteristics, when α=0.5, β=1.5, λ=1, t=0.8, $S(t)$= 0.8131, $h(t)$= 0.2410.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$		$\hat{S}(t)$		$\hat{h}(t)$	
		AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
	MLE	0.0819	0.0121	0.3834	0.4124	0.2143	0.0775	0.0649	0.0065	0.0750	0.0095
	TK P1	0.0817	0.0120	0.3941	0.3528	0.2256	0.0928	0.0645	0.0063	0.0728	0.0090
20	TK P2	0.0715	0.0089	0.2898	0.1526	0.1847	0.0563	0.0532	0.0044	0.0607	0.0062
	GS P1	0.0760	0.0109	0.3318	0.2129	0.2004	0.0729	0.0615	0.0057	0.0622	0.0063
	GS P2	0.0659	0.0078	0.2558	0.1110	0.1651	0.0451	0.0507	0.0039	0.0520	0.0044
	MLE	0.0620	0.0066	0.2756	0.1413	0.1711	0.0508	0.0521	0.0043	0.0561	0.0053
	TK P1	0.0618	0.0068	0.2843	0.1541	0.1680	0.0479	0.0520	0.0042	0.0564	0.0054
30	TK P2	0.0574	0.0056	0.2381	0.0993	0.1553	0.0411	0.0456	0.0033	0.0490	0.0041
	GS P1	0.0577	0.0060	0.2527	0.1155	0.1523	0.0398	0.0492	0.0037	0.0482	0.0038
	GS P2	0.0538	0.0052	0.2157	0.0788	0.1411	0.0344	0.0435	0.0030	0.0422	0.0030
	MLE	0.0532	0.0048	0.2301	0.0948	0.1396	0.0318	0.0440	0.0030	0.0465	0.0035
	TK P1	0.0532	0.0049	0.2296	0.0918	0.1491	0.0374	0.0445	0.003	0.0481	0.0038
40	TK P2	0.0503	0.0043	0.2069	0.0738	0.1300	0.0275	0.0394	0.0024	0.0419	0.0029
	GS P1	0.0498	0.0042	0.2094	0.0740	0.1359	0.0322	0.0420	0.0027	0.0410	0.0027
	GS P2	0.0477	0.0039	0.1882	0.0596	0.1186	0.0233	0.0375	0.0022	0.0354	0.0020
	MLE	0.0464	0.0036	0.2072	0.0746	0.1244	0.0258	0.0405	0.0026	0.0430	0.0030
	TK P1	0.0453	0.0034	0.2056	0.0705	0.1309	0.0278	0.0398	0.0024	0.0421	0.0029
50	TK P2	0.0443	0.0033	0.1909	0.0621	0.1174	0.0230	0.0373	0.0022	0.0398	0.0026
	GS P1	0.0423	0.0030	0.1885	0.0584	0.1196	0.0244	0.0376	0.0021	0.0357	0.0021
	GS P2	0.0426	0.0031	0.1763	0.0523	0.1104	0.0206	0.0354	0.0019	0.0340	0.0019
	MLE	0.0432	0.0031	0.1772	0.0520	0.1185	0.0226	0.0354	0.0020	0.0370	0.0021
	TK P1	0.0405	0.0027	0.1874	0.0577	0.1135	0.0209	0.0363	0.0021	0.0381	0.0023
60	TK P2	0.0417	0.0028	0.1657	0.0452	0.1130	0.0205	0.0332	0.0017	0.0347	0.0019
	GS P1	0.0378	0.0024	0.1739	0.0493	0.1044	0.0183	0.0346	0.0019	0.0324	0.0016
	GS P2	0.0388	0.0025	0.1538	0.0389	0.1042	0.0178	0.0314	0.0015	0.0292	0.0013

TABLE 3.4: The ML and Bayes estimates of parameters and reliability characteristics, when α=2, β=0.5, λ=1, t=0.8, $S(t)$= 0.5422 , $h(t)$= 1.6493.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$		$\hat{S}(t)$		$\hat{h}(t)$	
		AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	MLE	0.3528	0.2323	0.1287	0.0273	0.2571	0.1577	0.0791	0.0101	0.4018	0.3022
	TK P1	0.3438	0.2205	0.1265	0.0268	0.2642	0.2276	0.0763	0.0093	0.3984	0.2968
	TK P2	0.2699	0.1255	0.0967	0.0149	0.2165	0.0806	0.0618	0.0060	0.2953	0.1534
	GS P1	0.2906	0.1184	0.1826	0.0514	0.2578	0.1249	0.0879	0.0115	0.3748	0.1815
	GS P2	0.2492	0.086	0.1448	0.0309	0.2149	0.0807	0.0731	0.0078	0.3102	0.1249
30	MLE	0.2529	0.1106	0.0997	0.0164	0.2059	0.0768	0.0611	0.0061	0.2933	0.1521
	TK P1	0.2489	0.1067	0.0986	0.0162	0.2081	0.0797	0.0596	0.0059	0.2920	0.1507
	TK P2	0.2103	0.0734	0.0869	0.0122	0.1788	0.0579	0.0554	0.0048	0.2396	0.0968
	GS P1	0.2946	0.1127	0.1866	0.0476	0.2455	0.1027	0.0875	0.0104	0.3945	0.1876
	GS P2	0.2686	0.0932	0.1661	0.0370	0.2178	0.0783	0.0798	0.0088	0.3611	0.1532
40	MLE	0.2164	0.0801	0.0904	0.0123	0.1784	0.0535	0.0558	0.0048	0.2562	0.1081
	TK P1	0.2136	0.0778	0.0898	0.0122	0.1796	0.0549	0.0548	0.0046	0.2551	0.1073
	TK P2	0.1858	0.0571	0.0763	0.0091	0.1545	0.0388	0.0480	0.0035	0.2106	0.0739
	GS P1	0.3059	0.1149	0.1898	0.0465	0.2436	0.0927	0.0885	0.0103	0.4102	0.1931
	GS P2	0.2871	0.1011	0.1718	0.0373	0.2183	0.0711	0.0808	0.0085	0.3831	0.1664
50	MLE	0.1839	0.0592	0.0766	0.0092	0.1531	0.0402	0.0477	0.0035	0.2164	0.0811
	TK P1	0.1822	0.0578	0.0760	0.0091	0.1533	0.0410	0.0470	0.0035	0.2159	0.0807
	TK P2	0.1625	0.0465	0.0699	0.0077	0.1368	0.0307	0.0436	0.0030	0.1892	0.0637
	GS P1	0.3188	0.1185	0.1927	0.0452	0.2367	0.0828	0.0892	0.0099	0.4249	0.1992
	GS P2	0.2999	0.1052	0.1821	0.0398	0.2148	0.0665	0.0855	0.0090	0.4044	0.1793
60	MLE	0.1677	0.0450	0.0719	0.0080	0.1349	0.0289	0.0441	0.0030	0.2006	0.0642
	TK P1	0.1662	0.0441	0.0715	0.0080	0.1349	0.0292	0.0436	0.0029	0.2001	0.0639
	TK P2	0.1458	0.0364	0.0621	0.0059	0.1270	0.0273	0.0386	0.0023	0.1715	0.0487
	GS P1	0.3255	0.1210	0.1933	0.0450	0.2272	0.0721	0.0891	0.0097	0.4310	0.2030
	GS P2	0.3104	0.1098	0.1824	0.0385	0.2277	0.0706	0.0853	0.0087	0.4125	0.1831

TABLE 3.5: The ML and Bayes estimates of parameters and reliability characteristics, when α=2, β=1.5, λ=1, t=0.8, $S(t)$= 0.9040, $h(t)$= 0.6220.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$		$\hat{S}(t)$		$\hat{h}(t)$	
		AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	MLE	0.3267	0.1896	0.3681	0.2911	0.2091	0.0725	0.0432	0.0028	0.2198	0.0790
	TK P1	0.3194	0.1812	0.3747	0.3139	0.2071	0.0902	0.0435	0.0030	0.2130	0.0756
	TK P2	0.2702	0.1226	0.2853	0.1458	0.1809	0.0552	0.0390	0.0024	0.1809	0.0538
	GS P1	0.2805	0.1381	0.2451	0.1128	0.1793	0.0574	0.0341	0.0018	0.1482	0.0352
	GS P2	0.2530	0.1118	0.1933	0.0641	0.1538	0.0390	0.0304	0.0015	0.1244	0.0252
30	MLE	0.2542	0.1134	0.2796	0.1319	0.1670	0.0451	0.0376	0.0022	0.1865	0.0551
	TK P1	0.2503	0.1095	0.2817	0.1354	0.1659	0.0446	0.0374	0.0022	0.1826	0.0537
	TK P2	0.2338	0.0939	0.2263	0.0962	0.1581	0.0417	0.0306	0.0015	0.1460	0.0360
	GS P1	0.2266	0.0878	0.1999	0.0674	0.1546	0.0394	0.0287	0.0013	0.1265	0.0255
	GS P2	0.2168	0.0814	0.1588	0.0447	0.1471	0.0360	0.0238	0.0009	0.1015	0.0169
40	MLE	0.2083	0.0718	0.2435	0.1039	0.1404	0.0313	0.0325	0.0016	0.1571	0.0390
	TK P1	0.2060	0.0700	0.2448	0.1059	0.1398	0.0311	0.0323	0.0016	0.1542	0.0378
	TK P2	0.2033	0.0698	0.2025	0.0694	0.1366	0.0312	0.0285	0.0013	0.1313	0.0271
	GS P1	0.1886	0.0597	0.1763	0.0544	0.1478	0.0342	0.0252	0.0010	0.1079	0.0185
	GS P2	0.1911	0.0612	0.1480	0.0361	0.1399	0.0316	0.0224	0.0008	0.0927	0.0133
50	MLE	0.1874	0.0584	0.2089	0.0766	0.1293	0.0274	0.0290	0.0013	0.1391	0.0319
	TK P1	0.1853	0.0569	0.2096	0.0776	0.1289	0.0272	0.0293	0.0014	0.1378	0.0314
	TK P2	0.1748	0.0489	0.1983	0.0670	0.1178	0.0221	0.0260	0.0011	0.1192	0.0228
	GS P1	0.1684	0.0470	0.1554	0.0421	0.1469	0.0333	0.0225	0.0008	0.0978	0.0154
	GS P2	0.1619	0.0426	0.1827	0.0560	0.1283	0.0259	0.0207	0.0007	0.0853	0.0114
60	MLE	0.1740	0.0479	0.1854	0.0585	0.1174	0.0224	0.0260	0.0010	0.1240	0.0242
	TK P1	0.1724	0.0469	0.1859	0.0590	0.1170	0.0223	0.0259	0.0011	0.1225	0.0238
	TK P2	0.1595	0.0419	0.1715	0.0478	0.1097	0.0190	0.0242	0.0009	0.1144	0.0205
	GS P1	0.1549	0.0387	0.1429	0.0334	0.1400	0.0295	0.0204	0.0006	0.0885	0.0119
	GS P2	0.1419	0.0333	0.1310	0.0283	0.1338	0.0263	0.0191	0.0006	0.0821	0.0105

TABLE 3.6: The AL and CPs of 95% ACIs and HPD credible intervals of parameters when α=0.5, β=0.5, λ=1.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$	
		AL	CP	AL	CP	AL	CP
	ACI	0.3671	0.935	0.5843	0.920	1.2637	0.943
20	HPD P1	0.2281	0.930	0.5834	0.932	0.3498	0.935
	HPD P2	0.2153	0.941	0.3299	0.941	0.7012	0.945
	ACI	0.2851	0.930	0.4827	0.935	0.9586	0.938
30	HPD P1	0.1784	0.938	0.2902	0.941	0.6215	0.944
	HPD P2	0.1718	0.932	0.2814	0.946	0.5878	0.952
	ACI	0.2433	0.947	0.4135	0.930	0.8181	0.938
40	HPD P1	0.1511	0.941	0.2556	0.941	0.5355	0.940
	HPD P2	0.1471	0.946	0.2452	0.945	0.5157	0.941
	ACI	0.2159	0.940	0.3718	0.943	0.7231	0.940
50	HPD P1	0.1338	0.940	0.2296	0.947	0.4791	0.940
	HPD P2	0.1312	0.948	0.2231	0.948	0.464	0.9453
	ACI	0.1954	0.949	0.3426	0.936	0.6567	0.943
60	HPD P1	0.1204	0.948	0.2123	0.948	0.4344	0.942
	HPD P2	0.1189	0.949	0.2072	0.947	0.4255	0.952

TABLE 3.7: AL and CPs of 95% ACIs and HPD credible intervals of parameters when α=0.5, β=1.5, λ=1.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$	
		AL	CP	AL	CP	AL	CP
	ACI	0.3642	0.952	1.7017	0.953	0.9861	0.929
20	HPD P1	0.2497	0.939	1.1026	0.937	0.6654	0.929
	HPD P2	0.2395	0.976	0.9716	0.945	0.6184	0.931
	ACI	0.2840	0.954	1.2835	0.943	0.8124	0.942
30	HPD P1	0.1969	0.940	0.8683	0.941	0.5399	0.942
	HPD P2	0.1909	0.952	0.8055	0.957	0.5249	0.943
	ACI	0.2418	0.935	1.1032	0.961	0.6874	0.95
40	HPD P1	0.1684	0.939	0.7378	0.944	0.4715	0.943
	HPD P2	0.1633	0.947	0.7133	0.946	0.4521	0.955
	ACI	0.2152	0.949	0.9663	0.947	0.6188	0.951
50	HPD P1	0.1486	0.942	0.6615	0.942	0.4219	0.949
	HPD P2	0.1459	0.946	0.6368	0.955	0.4111	0.952
	ACI	0.1955	0.945	0.8734	0.965	0.5618	0.936
60	HPD P1	0.1339	0.944	0.6076	0.937	0.3819	0.945
	HPD P2	0.1328	0.948	0.582	0.955	0.3756	0.955

TABLE 3.8: The AL and CPs of 95% ACIs and HPD credible intervals of parameters when α=2, β=0.5, λ=1.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$	
		AL	CP	AL	CP	AL	CP
	ACI	1.4658	0.918	0.5896	0.929	1.2104	0.933
20	HPD P1	0.8695	0.938	0.3921	0.932	0.7789	0.939
	HPD P2	0.8107	0.941	0.3579	0.935	0.7164	0.942
	ACI	1.1445	0.948	0.4825	0.928	0.9524	0.926
30	HPD P1	0.6727	0.946	0.3210	0.935	0.6383	0.936
	HPD P2	0.6475	0.952	0.3050	0.941	0.6041	0.940
	ACI	0.9762	0.939	0.4165	0.934	0.8203	0.943
40	HPD P1	0.5659	0.940	0.2717	0.936	0.5553	0.944
	HPD P2	0.5487	0.943	0.2606	0.943	0.5298	0.948
	ACI	0.8637	0.941	0.3722	0.936	0.7243	0.937
50	HPD P1	0.4904	0.943	0.2339	0.938	0.4926	0.941
	HPD P2	0.4820	0.944	0.2290	0.941	0.4724	0.952
	ACI	0.7808	0.949	0.3400	0.935	0.6517	0.947
60	HPD P1	0.4355	0.952	0.2011	0.940	0.4427	0.947
	HPD P2	0.4307	0.951	0.1989	0.943	0.4372	0.951

TABLE 3.9: The AL and CPs of 95% ACIs and HPD credible intervals of parameters when α=2, β=1.5, λ=1.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$	
		AL	CP	AL	CP	AL	CP
	ACI	1.4535	0.948	1.6405	0.955	1.0017	0.931
20	HPD P1	0.9936	0.946	1.0499	0.940	0.6852	0.936
	HPD P2	0.9293	0.949	0.9477	0.945	0.6452	0.945
	ACI	1.1502	0.957	1.2786	0.945	0.7987	0.937
30	HPD P1	0.7845	0.955	0.8503	0.929	0.5575	0.940
	HPD P2	0.7476	0.952	0.7969	0.958	0.5387	0.942
	ACI	0.9687	0.954	1.1014	0.948	0.6938	0.958
40	HPD P1	0.6611	0.945	0.7416	0.923	0.4875	0.944
	HPD P2	0.6452	0.948	0.7010	0.959	0.4722	0.948
	ACI	0.8667	0.946	0.9628	0.931	0.6200	0.943
50	HPD P1	0.5901	0.945	0.6541	0.912	0.4373	0.943
	HPD P2	0.5740	0.946	0.6338	0.949	0.4216	0.952
	ACI	0.7814	0.945	0.8769	0.934	0.5626	0.938
60	HPD P1	0.5319	0.945	0.5994	0.938	0.3976	0.940
	HPD P2	0.5196	0.951	0.5781	0.947	0.3889	0.945

3.7 Real Data Analysis

With the help of real data, we demonstrate the estimation techniques developed in this chapter. Here, we consider a secondary data set having remission times (in weeks) of a group of 30 leukemia patients who received similar treatments. This data set is reported in Lawless (2003) and observations with + sign are censored times.

1, 1, 2, 4, 4, 6, 6, 6, 7, 8, 9, 9, 10, 12, 13, 14, 18, 19, 24, 26, 29, 31+, 42, 45+, 50+, 57, 60, 71+, 85+, 91.

Before going further, we fit the randomly censored IW lifetime model and compared its fitting with some well-known lifetime models like generalized inverted exponential (GIE), gamma, and Weibull lifetime models for this data set. We calculate ML estimates of the associated unknown parameters together with some useful measure of goodness-of-fit tests, namely, the negative log-likelihood function $-\ln L$, the AIC defined by $AIC = 2\times k - 2\times \ln L$, proposed by Akaike (1974) and BIC defined by $BIC = k\times \ln(n) - 2\times \ln L$, proposed by Schwarz (1978), where k is the number of parameters in the model, n is the number of observations in the given data set, L is the maximized value of the likelihood function for the estimated model and Kolmogorov-Smirnov (KS) statistic with its p-values. The lowest $-\ln L$, AIC, BIC, and KS statistics, as well as the highest p-value, indicate the optimal lifetime model. Table 3.10 contains the results of the ML estimates as well as goodness-of-fit test measures. These findings

TABLE 3.10: Summary fit of the real data set of remission times (in weeks) of 30 leukemia patients.

Model	MLE	-lnL	AIC	BIC	KS Test	
					Statistic	p-value
$X\sim$ IW(α,β) $T\sim$ IW(α,λ)	$\hat{\alpha}$=0.7774 $\hat{\beta}$=4.9231 $\hat{\lambda}$=33.3523	137.7351	281.4701	285.6737	0.1373	0.624
$X\sim$ GIE(α,β) $T\sim$ GIE(α,λ)	$\hat{\alpha}$=0.6619 $\hat{\beta}$= 4.7952 $\hat{\lambda}$=63.1718	138.2794	282.5587	286.7623	0.1599	0.4272
$X\sim$ Weibull(α,β) $T\sim$ Weibull(α,λ)	$\hat{\alpha}$=0.9714 $\hat{\beta}$=0.0365 $\hat{\lambda}$=0.0073	140.4595	286.9191	291.1227	0.1556	0.4621
$X\sim$ gamma(α,β) $T\sim$ gamma(α,λ)	$\hat{\alpha}$=1.0441 $\hat{\beta}$=0.0346 $\hat{\lambda}$=0.0072	140.4587	286.9175	291.1211	0.1710	0.3444

demonstrate that for the data set under consideration, a randomly censored IW lifetime model is the appropriate choice. For the fitting of randomly censored data via the graphs, we also consider the Kaplan-Meier (KM) product limit estimator. The KM product-limit estimator for survival function was proposed by Kaplan and Meier (1958) and is given by

$$\hat{S}(t) = \prod_{y_i \le t} \left(1 - \frac{1}{n_i}\right)^{d_i},$$

where, n_i is the number of items survived at time y_i and $d_i = 1$ if item failed, 0 otherwise. Figure 3.2 shows the graphs of the KM estimator and the estimated survival functions of the considered models. The survival function estimate for the IW lifetime model is fairly close to that provided by the KM estimator, as seen in 3.2. As a result, the KM estimator recommends using the IW lifetime model to describe this data set. We provide the results of the estimation

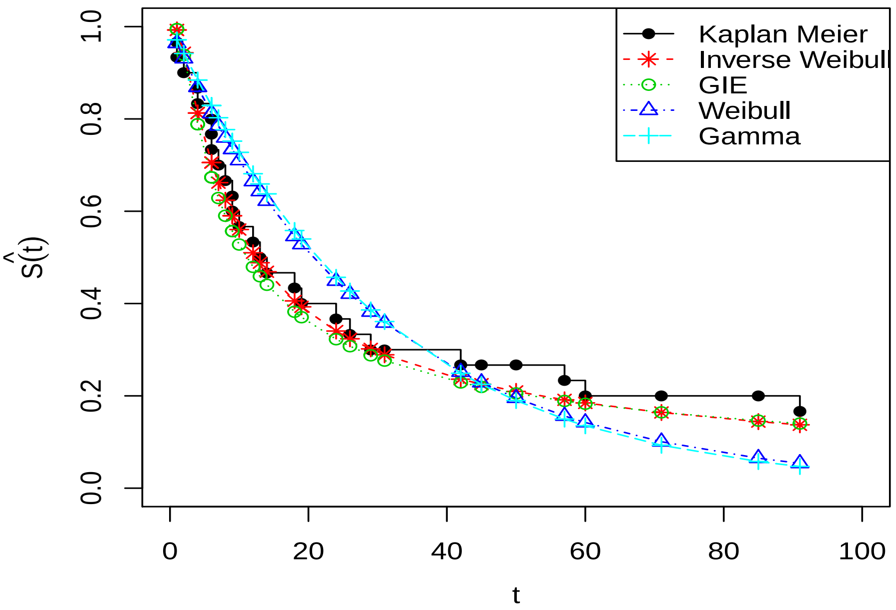

FIGURE 3.2: The plot of estimates of survival functions of considered models

techniques examined in this article based on the above real data set in Table 3.11. The unknown parameters and reliability characteristics are estimated using ML and Bayes methods. We use the median of the data as the mission time ($t = 13.5$) for reliability characteristics. The Bayes estimates are derived using non-informative priors under SELF since we don't have any prior information about the parameters. The TK approximation and Gibbs sampling procedures are used to get the Bayes estimates of parameters. We create a Markov chain using $M = 1,00,000$ for the Gibbs sampling technique using the MH algorithm. The first $M_0 = 20,000$ observations are discarded as burn-in observations, and every $10th$ observation is used as the iid observation of produced MCMC samples of α, β, and λ. We also use graphical diagnostic tools like trace, autocorrelation function (ACF), and histogram with Gaussian kernel density plots to assess the

TABLE 3.11: The ML and Bayes estimates of the parameters and reliability characteristics corresponding to the real data set of remission times (in weeks) of 30 leukemia patients.

Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{S}(t)$	$\hat{h}(t)$
MLE	0.7774	4.9231	33.3524	0.4784	0.0409
	(0.5768, 0.9781)	(2.7112, 7.1349)	(2.3046, 64.4001)		
TK	0.7759	4.9254	36.4505	0.4760	0.0410
MCMC	0.7238	4.6407	24.4085	0.5048	0.0370
	(0.6146, 0.8573)	(3.3702, 5.8947)	(22.4474, 26.2966)		

convergence of their stationary distributions. Figure 3.3 shows the trace, ACF, and histogram with Gaussian kernel plots for the parameters. For all parameters, the trace plots demonstrate a random dispersion around the mean value (represented by a solid line) and fine chain mixing. Chains exhibit very low autocorrelations, as shown by ACF plots. The marginal distributions of the parameters are remarkably symmetrical, as seen by the histogram plots of the produced MCMC samples, implying that the mean is the best estimate for the parameters. These graphs are, in fact, indicative of rapid MCMC convergence.

3.8 Concluding Remarks

The IW lifetime model is a useful lifetime model for representing the failure rate functions with unimodal behaviour. The classical and Bayesian estimation procedures for the parameters and reliability characteristics of IWD under the random censoring model were discussed in this chapter. The MLEs for the unknown parameters as well as the reliability characteristics were calculated. Based on expected Fisher information, asymptotic confidence intervals for the parameters are also calculated. ETT was computed for a randomly censored experiment. TK approximation and Gibbs sampling methods were used to approximate Bayes estimators of the parameters and reliability characteristics under SELF. A comprehensive simulation study was used to evaluate the performance of various estimators. The Bayes estimates with gamma informative priors and the Gibbs sampling technique had lower average absolute biases and mean squared errors than the ML and Bayes estimates with non-informative priors. When there is some prior knowledge, we recommend Bayes estimators.

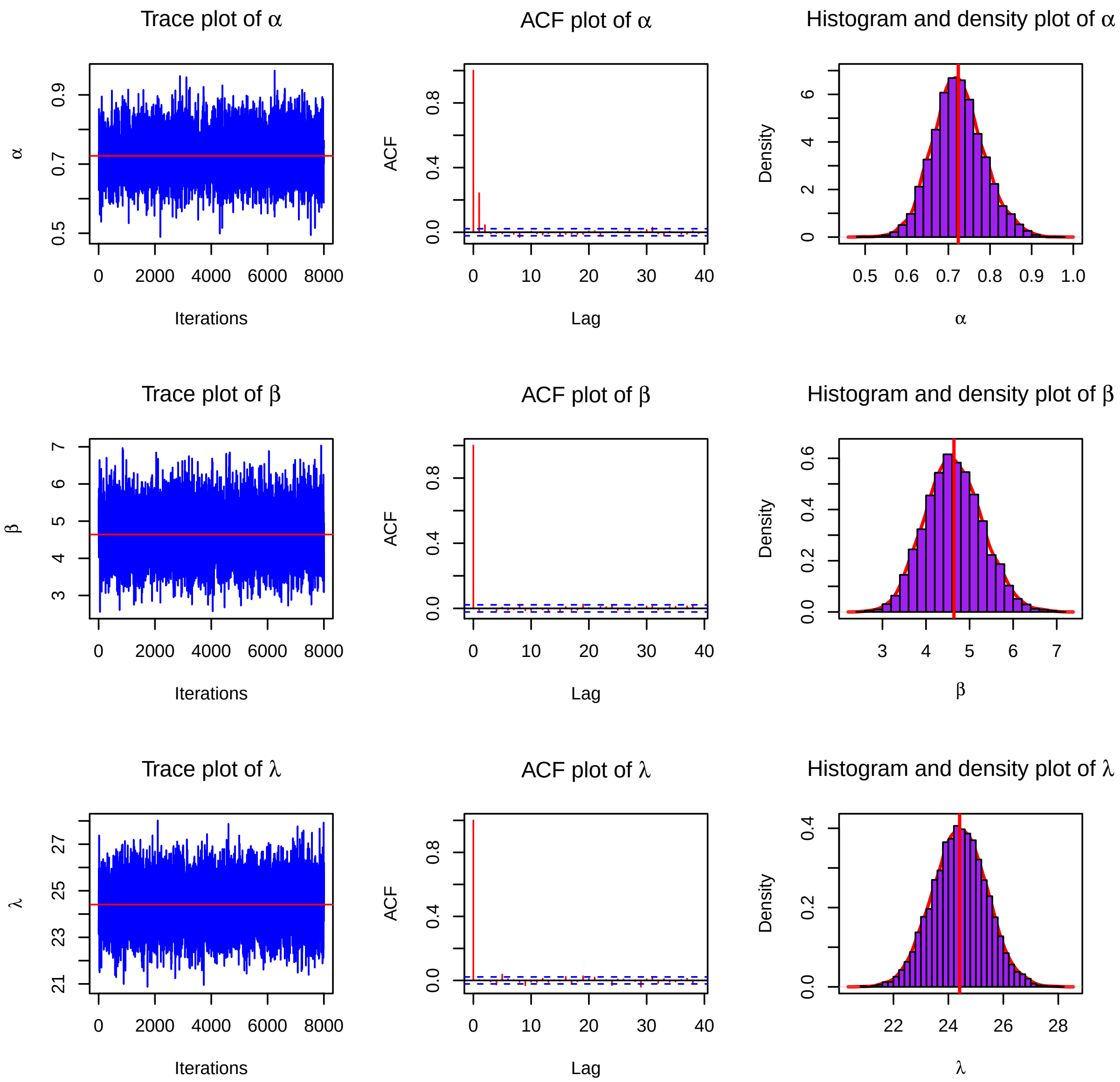

FIGURE 3.3: MCMC diagnostic plots of the parameters

Chapter 4

Classical and Bayesian Estimation of Stress-Strength Reliability for Inverse Pareto Lifetime Model using Progressively Censored Data*

4.1 Introduction

In this chapter, we deal with a problem from reliability theory and we estimate the stress-strength reliability (SSR) for IP lifetime model using progressively censored data from both classical and Bayesian approaches. The IP lifetime model already has been discussed in Chapter 2 under randomly censored data.

In life testing experiments the incomplete information commonly arises because of time limits and other restrictions on data collection or study. The incomplete information in life testing experiments is termed as censoring and it arises when components are removed or destroyed from the experiment before the final termination point. Therefore, censored samples are frequently available around us as a result and we use censored samples rather than the complete sample in life testing experiments. Several censoring schemes have been utilized in the literature to demonstrate the various motive belonging to the life testing experiments. Type-I and Type-II

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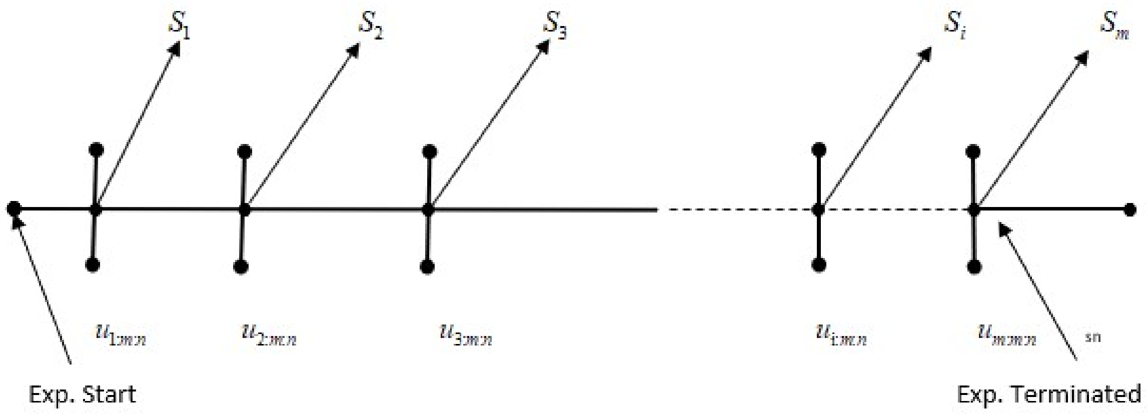


FIGURE 4.1: The schematic diagram of progressive censoring scheme.

censoring schemes are the two most common censoring schemes in the literature. These censoring schemes are used to save money and time by prefixing time or number of failures. These censoring schemes become ineffective when items are removed at intermittent stages from an experiment. To overcome such type difficulty, Cohen (1963) introduced a censoring scheme in the literature, known as progressive censoring scheme. It is one of the popular censoring scheme which supplies the adaptability of removals of the experimental units throughout the experiments. After that, many scholars have studied this censoring scheme for various lifetime models under different scenarios. Two excellent monographs on the progressive censoring scheme are given by Balakrishnan and Aggarwala (2000) and Balakrishnan and Cramer (2014), respectively. According to Hofmann et al. (2005) the progressive censoring schemes significantly improve upon the Type-II censoring scheme in many real-life situations. More details and applications of the progressive censoring scheme can be found in the following latest articles carried out by various scholars like: Kohansal and Rezakhah (2019), Aslam et al. (2020), Goel and Singh (2020), Abu-Moussa et al. (2021), Ghanbari et al. (2021), Asgharzadeh and Fallah (2021), Bedbur and Mies (2021), Wu and Gui (2021), Wu and Chang (2021), Hashem and Alyami (2021), and reference cited therein.

Mathematically, the progressive censoring can be articulated as follows; Let n test units are put on the life test, and only $m(m \leq n)$ failures are obtained. Suppose $U_{1:m:n}, U_{2:m:n}, \ldots, U_{m:m:n}$ be the obtained ordered lifetimes and m be the prefixed number of failures with prefixed censoring scheme $\underset{\sim}{S} = (S_1, S_2, \ldots, S_m)$. When the ith unit fails $(i = 1, 2, \ldots, m-1)$, S_i live units are randomly withdrawn from the experiments. Finally, the remaining $S_m = n - m - \sum_{i=1}^{m-1} S_i$ live units are withdrawn when the mth unit fails. The schematic diagram of the progressive censoring scheme is given in Figure 4.1.

Let $u_1, u_2, \ldots, u_m$ be a progressively Type-II censored sample with prefixed censoring scheme

$\underset{\sim}{S} = (S_1, S_2, \ldots, S_m)$ from a population with pdf $g_U(.)$ and cdf $G_U(.)$, the likelihood function is defined as, see, (Balakrishnan and Aggarwala, 2000)

$$L(u_{1:m:n}, u_{2:m:n}, \ldots, u_{m:m:n}) = K \prod_{i=1}^{m} g_U(u_{i:m:n}) \{1 - G_U(u_{i:m:n})\}^{S_i},$$
$$0 < u_{1:m:n} < u_{2:m:n} < \cdots < u_{m:m:n} < \infty \tag{4.1}$$

where, $K = n(n - S_1 - 1)(n - S_1 - S_2 - 2) \ldots (n - S_1 - S_2 - \cdots - S_{m-1} - m + 1)$.

Remarks: There are two particular cases of progressive Type II censoring scheme: (i) It becomes Type II censoring scheme when $S_i = 0;\ \forall\ i = 1, 2, \ldots m-1$ and $S_m = n - m$, and (ii) It becomes complete sample case when $S_i = 0;\ \forall\ i = 1, 2, \ldots, m$.

In reliability and life testing theory, the stress-strength reliability (SSR) model contains two independent random variables, one as a strength variable, say U and another as a stress variable, say V, the quantity $R = P(V < U)$ is known as SSR. Birnbaum (1956) studied the SSR model in connection with the classical Mann-Whitney statistic. The SSR system is applicable in many real-life problems. Johnstone (1983) showed an anti-tank sabot round being shot at a Soviet T-62 tank as an example of SSR in military applications. The Bayesian method was used to calculate the chances of a particular bullet penetrating its intended target. Another application of SSR was presented by Johnson (1988) in rocket engines. The maximal chamber pressure generated by the ignition of a solid propellant was denoted by V and the strength of the rocket chamber was denoted by U, so that SSR becomes the probability of successful firing of an engine. An excellent monograph on the several SSR models with their applications are given by Kotz et al. (2003). Some recent studies on SSR for different lifetime models based on complete samples are as follows: The Weibull lifetime model is discussed by Jia et al. (2017). Jovanović (2017) studied geometric-exponential lifetime model. The IP lifetime model is studied by Guo and Gui (2018). The generalized inverse Lindley lifetime model is discussed by Sharma (2018), Scaria et al. (2021) studied generalized Pareto lifetime model, the inverse Chen lifetime model is discussed by Agiwal (2021) and the references cited therein. Also, some recent studies on SSR for different lifetime models in case of progressive censoring are carried out by many scholars like: Maxwell lifetime model studied by Chaudhary and Tomer (2018). Yadav et al. (2018) studied IW lifetime. Two parameter Rayleigh lifetime model is discussed by Kohansal and Rezakhah (2019). Goel and Singh (2020) studied modified Weibull lifetime model. Abu-Moussa et al. (2021) discussed Rayleigh lifetime model, and references cited therein.

For a clear view of the study, the rest of the chapter is designed as follows: Section 4.2, deals with the model description. The maximum likelihood estimator (MLE) and asymptotic confidence interval (ACI) of SSR are presented in Section 4.3. The Bayes estimator and highest posterior density (HPD) credible interval of SSR has appeared in Section 4.4. The numerical computations are performed Section 4.5 to compare the ML and Bayes estimators of SSR, numerically. In Section 4.6, two different pairs of real data sets are analyzed to illustrate the proposed methodology. Finally, the concluding remarks are provided in Section 4.7.

4.2 The Model

The pdf and corresponding cdf of IPD with parameter θ, respectively, are given by

$$g_U(u;\theta) = \frac{\theta u^{\theta-1}}{(1+u)^{\theta+1}} \quad ; \theta > 0,\ u > 0, \tag{4.2}$$

$$G_U(u;\theta) = \left(\frac{u}{1+u}\right)^{\theta} \quad ; \theta > 0,\ u > 0, \tag{4.3}$$

Let U and V be independent random variables following IPD(θ_1) and IPD(θ_2), respectively, then the SSR is defined as

$$\begin{aligned} R = P(V < U) &= \int_0^\infty G_V(u) g_U du \\ &= \int_0^\infty \left(\frac{u}{1+u}\right)^{\theta_2} \frac{\theta_1 u^{\theta_1-1}}{(1+u)^{\theta_1+1}} du \\ &= \int_0^\infty \frac{\theta_1 u^{\theta_1+\theta_2-1}}{(1+u)^{\theta_1+\theta_2+1}} du \\ &= \frac{\theta_1}{\theta_1+\theta_2} = \delta(\theta_1,\theta_2) \quad \text{say.} \end{aligned} \tag{4.4}$$

4.3 Maximum Likelihood Estimation

The ML estimates of the unknown parameters θ_1 and θ_2 are developed in this section to get the ML estimate of SSR R. Let $u_{i:m_1:n_1}; i = 1,2,\ldots,m_1$, be the progressively Type II censored sample from IP(θ_1) with presumed censoring scheme $\underset{\sim}{S} = (S_1, S_2, \ldots, S_{m_1})$ and similarly let $v_{j:m_2:n_2};\ j = 1,2,\ldots,m_2$ be independent progressively Type-II censored sample from IP(θ_2) with presumed censoring scheme $\underset{\sim}{T} = (T_1, T_2, \ldots, T_{m_2})$, then using equations (4.2), (4.3) and

(6.7), the likelihood function is given by

$$\begin{aligned} L(\theta_1,\theta_2;\underset{\sim}{u},\underset{\sim}{v}) &= K_1K_2\prod_{i=1}^{m_1} g_U(u_i)[1-G_U(u_i)]^{S_i} \times \prod_{j=1}^{m_2} g_V(v_j)[1-G_V(v_j)]^{T_j} \\ &= K_1K_2\theta_1^{m_1}\theta_2^{m_2}\prod_{i=1}^{m}\frac{u_i^{\theta_1-1}}{(1+u_i)^{\theta_1+1}}\left[1-\left(\frac{u_i}{1+u_i}\right)^{\theta_1}\right]^{S_i} \\ &\quad \times \prod_{j=1}^{m_2}\frac{v_j^{\theta_2-1}}{(1+v_j)^{\theta_2+1}}\left[1-\left(\frac{v_j}{1+v_j}\right)^{\theta_2}\right]^{T_j} \end{aligned} \tag{4.5}$$

where,

$$K_1 = n_1(n_1-S_1-1)(n_1-S_1-S_2-2)\dots(n_1-S_1-S_2-\dots-S_{m_1-1}-m_1+1)$$

and

$$K_2 = n_2(n_2-T_1-1)(n_2-T_1-T_2-2)\dots(n_2-T_1-T_2-\dots-T_{m_2-1}-m_2+1)$$

. The corresponding log-likelihood function is obtained as

$$\begin{aligned} l(\theta_1,\theta_2) &= C+m_1\ln\theta_1+\theta_1\sum_{i=1}^{m_1}\ln\left(\frac{u_i}{1+u_i}\right)+\sum_{i=1}^{m_1}S_i\ln\left[1-\left(\frac{u_i}{1+u_i}\right)^{\theta_1}\right] \\ &\quad +m_2\ln\theta_2+\theta_2\sum_{j=1}^{m_2}\ln\left(\frac{v_i}{1+v_i}\right)+\sum_{j=1}^{m_2}T_j\ln\left[1-\left(\frac{v_j}{1+v_j}\right)^{\theta_2}\right], \end{aligned} \tag{4.6}$$

where, $C=\ln K_1+\ln K_2-\sum_{i=1}^{m_1}(\ln u_i+\ln(1+u_i))-\sum_{j=1}^{m_2}(\ln v_j+\ln(1+v_j))$. The following normal equations are obtained by differentiating the log-likelihood function w.r.t. θ_1 and θ_2, respectively:

$$\frac{\partial l(\theta_1,\theta_2)}{\partial\theta_1} = \frac{m_1}{\theta_1}+\sum_{i=1}^{m_1}\ln\left(\frac{u_i}{1+u_i}\right)-\sum_{i=1}^{m_1}S_i\frac{\left(\frac{u_i}{1+u_i}\right)^{\theta_1}\ln\left(\frac{u_i}{1+u_i}\right)}{\left[1-\left(\frac{u_i}{1+u_i}\right)^{\theta_1}\right]} = 0. \tag{4.7}$$

$$\text{and,}\quad \frac{\partial l(\theta_1,\theta_2)}{\partial\theta_2} = \frac{m_2}{\theta_2}+\sum_{j=1}^{m_2}\ln\left(\frac{v_j}{1+v_j}\right)-\sum_{j=1}^{m_2}T_j\frac{\left(\frac{v_j}{1+v_j}\right)^{\theta_2}\ln\left(\frac{v_j}{1+v_j}\right)}{\left[1-\left(\frac{v_j}{1+v_j}\right)^{\theta_2}\right]} = 0. \tag{4.8}$$

The ML estimates of θ_1 and θ_2, say $\hat{\theta}_1$ and $\hat{\theta}_2$ are the solutions of normal equations (4.7) and (4.8), respectively. Here, the closed form solutions are not available for equations (4.7) and

(4.8). A appropriate iterative technique can be utilised to get numerical solutions to these non-linear equations. A number of functions, such as *nlm*, *optim*, *maxLik*, and others, are available in the statistical software R to compute MLEs. Once the ML estimates of unknown parameters are computed, the ML estimate of SSR parameter R, say $\hat{R}$ is derived using invariance property of MLEs and is given by

$$\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2}. \tag{4.9}$$

4.3.1 Asymptotic Confidence Interval

Here, the ACI of SSR R is constructed using delta method as it is difficult to obtain exact distribution of $\hat{R}$. Let $\hat{\phi} = (\hat{\theta}_1, \hat{\theta}_2)$ be the ML estimates of unknown parameters $\phi = (\theta_1, \theta_2)$. The asymptotic variance of $\hat{R}$ using delta method, see, Krishnamoorthy and Lin (2010), is given by

$$Var(\hat{R}) = [b_C' I^{-1}(\phi) b_C],$$

where, $I(\phi) = -E \begin{bmatrix} \frac{\partial^2 l(\theta_1,\theta_2)}{\partial \theta_1^2} & \frac{\partial^2 l(\theta_1,\theta_2)}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 l(\theta_1,\theta_2)}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 l(\theta_1,\theta_2)}{\partial \theta_2^2} \end{bmatrix}$ is the Fisher information matrix and $b_C = \left(\frac{\partial R}{\partial \theta_1}, \frac{\partial R}{\partial \theta_2}\right)'$.

The observed Fisher information can be utilized as a consistent estimator of the Fisher information under modest regularity conditions. As a result, the observed variance of $\hat{R}$ is equal to

$$\hat{V}ar(\hat{R}) \simeq [b_C' I^{-1}(\phi) b_C]_{\phi = \hat{\phi}}.$$

The elements of partial derivatives in the Fisher information matrix $I(\phi)$ are given by

$$\frac{\partial^2 l(\theta_1, \theta_2)}{\partial \theta_1^2} = -\frac{m_1}{\theta_1^2} - \sum_{i=1}^{m_1} S_i \frac{\left\{\ln\left(\frac{u_i}{1+u_i}\right)\right\}^2 \left(\frac{u_i}{1+u_i}\right)^{\theta_1}}{\left[1 - \left(\frac{u_i}{1+u_i}\right)^{\theta_1}\right]^2},$$

$$\frac{\partial^2 l(\theta_1, \theta_2)}{\partial \theta_2^2} = -\frac{m_2}{\theta_2^2} - \sum_{j=1}^{m_2} T_j \frac{\left\{\ln\left(\frac{v_j}{1+v_j}\right)\right\}^2 \left(\frac{v_j}{1+v_j}\right)^{\theta_2}}{\left\{1 - \left(\frac{v_j}{1+v_j}\right)^{\theta_1}\right\}^2},$$

$$\frac{\partial^2 l(\theta_1, \theta_2)}{\partial \theta_1 \partial \theta_2} = \frac{\partial^2 l(\theta_1, \theta_2)}{\partial \theta_2 \partial \theta_1} = 0,$$

and the elements of b_C are given by

$$\frac{\partial R}{\partial \theta_1} = \frac{\theta_2}{(\theta_1+\theta_2)^2}, \quad \frac{\partial R}{\partial \theta_2} = -\frac{\theta_1}{(\theta_1+\theta_2)^2}.$$

Thus $\frac{\hat{R}-R}{\sqrt{\hat{V}ar(\hat{R})}} \sim N(0,1)$. Therefore, the $100(1-\xi)\%$ ACI of R is given by $\hat{R} \pm z_{\xi/2}\sqrt{\hat{V}ar(\hat{R})}$, where $z_{\xi/2}$ is the upper $(\xi/2)^{th}$ quantile of $N(0,1)$. Also, the coverage probability (CP) for R is given by

$$CP_R = \left[\left| \frac{\hat{R}-R}{\sqrt{\hat{V}ar(\hat{R})}} \right| \leq z_{\xi/2} \right].$$

4.4 Bayesian Estimation

In this part, we use the importance sampling (IS) approach to get the Bayes estimator of SSR R under the generalised entropy loss function (GELF) using non-informative and gamma informative priors.

4.4.1 Loss function

A suitable loss function must be specified in Bayesian estimation. The SELF is the most often used loss function in the literature. When over and under estimations of equal magnitude have the same effects, the SELF is appropriate. When the real loss is not symmetric in terms of over and under estimates, asymmetric loss functions are employed to illustrate the consequences of various inaccuracies. For this, a general-purpose loss function, such as GELF, can be employed. The GELF was proposed in the literature by Calabria and Pulcini (1996). This loss function is an extension of the entropy loss function and is defined by

$$L(\alpha,\hat{\alpha}) \propto \left[\left(\frac{\hat{\alpha}}{\alpha}\right)^q - q\ln\left(\frac{\hat{\alpha}}{\alpha}\right) - 1 \right]; \;\; q \neq 0,$$

where, $\hat{\alpha}$ is the decision rule which estimate α. When $q>0$, a positive error has more implications than a negative error, and when $q<0$, a negative error has greater effects. The Bayes estimator under GELF is calculated as follows:

$$\hat{\alpha} = E\left[\alpha^{-q}|\text{data}\right]^{-1/q}. \tag{4.10}$$

Remark: The Bayes estimator in equation (4.10) is reduced to a Bayes estimator under precautionary loss function (PLF), SELF, and entropy loss function (ELF) for $q = -2, \ -1$, and 1, respectively.

4.4.2 Prior and Posterior Distributions

We suppose the unknown parameters θ_1 and θ_2 are a-priori independent and have the following gamma distributions with their corresponding pdfs:

$$\eta_1(\theta_1) \propto \theta_1^{a_1-1}\exp(-b_1\theta_1); \;\; \theta_1 > 0, a_1, b_1 > 0,$$

$$\text{and} \quad \eta_2(\theta_2) \propto \theta_2^{a_2-1}\exp(-b_2\theta_2); \;\; \theta_2 > 0, a_2, b_2 > 0,$$

where, a_i, b_i; $i = 1,2$ are the hyper-parameters so chosen to reflect prior information about the parameters θ_1 and θ_2, respectively. As a result, the joint prior distribution of θ_1 and θ_2 can be expressed as

$$\eta(\theta_1,\theta_2) \propto \theta_1^{a_1-1}\theta_2^{a_2-1}\exp\{-(b_1\theta_1+b_2\theta_2)\}. \tag{4.11}$$

The selection of independent gamma priors is not unreasonable. The gamma distribution family is highly flexible, and it includes a variety of distributions. It is also worth noting that non-informative priors are special instances of independent gamma priors. Several researchers have utilised gamma priors in various contexts, including Guo and Gui (2018) Kumar (2018), Krishna et al. (2019), and many more. The posterior distribution of θ_1 and θ_2 is now obtained by incorporating joint prior distribution (4.11) to the likelihood function (4.5),

$$\begin{aligned}
\pi(\theta_1,\theta_2|\underset{\sim}{u},\underset{\sim}{v}) &= \frac{L(\theta_1,\theta_2;\text{data})\eta(\theta_1,\theta_2)}{\int_0^\infty\int_0^\infty L(\theta_1,\theta_2;\underset{\sim}{u},\underset{\sim}{v})g(\theta_1,\theta_2)d\theta_1 d\theta_2} \\
\Rightarrow \pi(\theta_1,\theta_2|\underset{\sim}{u},\underset{\sim}{v}) &\propto \theta_1^{m_1+a_1-1}\theta_2^{m_2+a_2-1}\exp\left\{-\theta_1\left[b_1-\sum_{i=1}^{m_1}\ln\left(\frac{u_i}{1+u_i}\right)\right]\right\} \\
&\quad \times\exp\left\{-\theta_2\left[b_2-\sum_{j=1}^{m_2}\ln\left(\frac{v_j}{1+v_j}\right)\right]\right\}\prod_{i=1}^{m_1}\left[1-\left(\frac{u_i}{1+u_i}\right)^{\theta_1}\right]^{S_i} \\
&\quad \times\prod_{j=1}^{m_2}\left[1-\left(\frac{v_j}{1+v_j}\right)^{\theta_2}\right]^{T_j}.
\end{aligned} \tag{4.12}$$

From the posterior distribution given in equation (4.12), we observe that the Bayes estimator for SSR R cannot obtain in closed form. Therefore, an approximation method, importance sampling technique is used to derive Bayes estimate of R.

4.4.3 Importance Sampling Technique

Here, the IS technique is used to construct the Bayes estimator and HPD credible interval of SSR R. The posterior distribution of θ_1 and θ_2 given in equation (4.12) can be rewritten as

$$\begin{aligned}\pi(\theta_1,\theta_2|\underset{\sim}{u},\underset{\sim}{v}) \propto\ & \theta_1^{m_1+a_1-1}\exp\left\{-\theta_1\left[b_1-\sum_{i=1}^{m_1}\ln\left(\frac{u_i}{1+u_i}\right)\right]\right\}\\ &\times\theta_2^{m_2+a_2-1}\exp\left\{-\theta_2\left[b_2-\sum_{j=1}^{m_2}\ln\left(\frac{v_j}{1+v_j}\right)\right]\right\}\\ &\times\exp\left\{\sum_{i=1}^{m_1}S_i\ln\left[1-\left(\frac{u_i}{1+u_i}\right)^{\theta_1}\right]+\sum_{j=1}^{m_2}T_j\left[1-\left(\frac{v_j}{1+v_j}\right)^{\theta_2}\right]\right\}\end{aligned}$$

$$\pi(\theta_1,\theta_2|\underset{\sim}{u},\underset{\sim}{v}) \propto f_{GA}(\theta_1;m_1+a_1,B_1)\,f_{GA}(\theta_2;m_2+a_2,B_2)\,W(\theta_1,\theta_2)=\pi_1(\theta_1,\theta_2|\text{data})\ \ (\text{say}),$$

where, $B_1=\left[b_1-\sum_{i=1}^{m_1}\ln\left(\frac{u_i}{1+u_i}\right)\right]$, $B_2=\left[b_2-\sum_{j=1}^{m_2}\ln\left(\frac{v_j}{1+v_j}\right)\right]$,

$$W(\theta_1,\theta_2)=\exp\left\{\sum_{i=1}^{m_1}S_i\ln\left[1-\left(\frac{u_i}{1+u_i}\right)^{\theta_1}\right]+\sum_{j=1}^{m_2}T_j\left[1-\left(\frac{v_j}{1+v_j}\right)^{\theta_2}\right]\right\}$$

and $f_{GA}(.;a,b)$ is a gamma distribution having shape and scale parameters a and b, respectively. Now the posterior expectation of $\phi(\theta_1,\theta_2)$ is given by

$$E[\phi(\theta_1,\theta_2)|\underset{\sim}{u},\underset{\sim}{v}]=\frac{\int_0^\infty\int_0^\infty\phi(\theta_1,\theta_2)\pi_1(\theta_1,\theta_2|\underset{\sim}{u},\underset{\sim}{v})d\theta_1d\theta_2}{\int_0^\infty\int_0^\infty\pi_1(\theta_1,\theta_2|\underset{\sim}{u},\underset{\sim}{v})d\theta_1d\theta_2} \tag{4.13}$$

The posterior mean $E[\phi(\theta_1,\theta_2)|\underset{\sim}{u},\underset{\sim}{v}]$ given in equation (4.13) is the ratio of two integrals and the closed form solution of this mean is not available. The IS approach is utilised to provide an approximate solution, and the following steps are taken into account for computation:

Step 1: Generate $\theta_1^{(1)}$ from $f_{GA}(\theta_1;m_1+a_1,B_1)$.
Step 2: Generate $\theta_2^{(1)}$ from $f_{GA}(\theta_2;m_2+a_2,B_2)$.
Step 3: Generate $\delta^{(1)}=\frac{\theta_1^{(1)}}{\theta_1^{(1)}+\theta_2^{(1)}}$ using equation (4.4).
Step 4: Repeat the above steps 1-3, M times to obtain the importance sample $(\delta^{(1)},\ldots,\delta^{(M)})$.

Now, using the IS technique under GELF, the approximate Bayes estimator of SSR is given by

$$\hat{R}_B = \left[\frac{\sum_{j=1}^{M} \left\{ \delta(\theta_1^{(j)}, \theta_2^{(j)}) \right\}^{-q} W(\theta_1^{(j)}, \theta_2^{(j)})}{\sum_{j=1}^{M} W(\theta_1^{(j)}, \theta_2^{(j)})} \right]^{-1/q}. \tag{4.14}$$

4.4.4 HPD Credible Interval

Using the produced importance sample, the HPD credible interval of SSR R can be constructed. Let $\delta_{(1)} < \delta_{(2)} < \cdots < \delta_{(M)}$ be the ordered values of $\delta^{(1)},\ \delta^{(2)}, \ldots, \delta^{(M)}$. Now, using the algorithm proposed by Chen and Shao (1999), the $100(1-\xi)\%$, where, $0 < \xi < 1$, HPD credible interval of SSR is given by $\left(\delta_{(j)},\ \ \delta_{(j+[(1-\xi)M])}\right)$, where j is chosen such that

$$\delta_{(j+[(1-\xi)M])} - \delta_{(j)} = \min_{1 \le i \le \xi M} \left(\delta_{(i+[(1-\xi)M])} - \delta_{(j)}\right), j = 1, 2, \ldots, M,$$

where, $[x]$ is the integral part of x.

4.5 Numerical Computations

A Monte Carlo simulation study is provided in this section to assess the efficacy of the estimation methods developed in this chapter. The mean squared errors (MSEs) and average estimates (AEs) of the ML and Bayes estimators of RSS R are calculated. The Bayes estimate of SSR is computed in case of non-informative prior (Prior A) and informative gamma prior (Prior B) under GELF. Also, the average length (ALs) of 95% ACI and HPD credible intervals with their corresponding coverage probabilities (CP) of SSR R are obtained. For computation purpose, two independent progressively Type II censored samples $\underset{\sim}{u}$ and $\underset{\sim}{v}$ of sample sizes n_1 and n_2, effective sample sizes m_1 and m_2 are produced from IP(θ_1) and IP(θ_2) with prefixed censoring schemes S_i; $i = 1, 2, \ldots, m_1$ and T_j; $j = 1, 2, \ldots, m_2$, respectively, using the algorithm provided by Balakrishnan and Sandhu (1995). The several combinations of sample sizes (n_1, n_2), effective sample sizes (m_1, m_2), and prefixed censoring schemes $(\underset{\sim}{S}, \underset{\sim}{T})$ are considered. For simulation purpose, we assign $n = n_1 = n_2$, $m = m_1 = m_2$ and $CS = (\underset{\sim}{S} = \underset{\sim}{T})$, and these combinations are reported in Table 4.1. In Table 4.1, schemes [4], [8] and [12] are the cases for complete sample data.

We consider two sets of true values for the parameters $(\theta_1,\ \theta_2) = (1.5,\ 0.5)$ and $(\theta_1,\ \theta_2) = (0.5,\ 1.5)$ so that SSR R becomes, $R = 0.75$ and $R = 0.25$, respectively. In Bayesian computations, for informative priors the choices of hyper-parameters are chosen such that the prior means are exactly equal to true values of the parameters. For informative priors $\{(a_1, b_1) = (3,2),\ (a_2, b_2) = (2,4)\}$ and $\{(a_1, b_1) = (2,4),\ (a_2, b_2) = (3,2)\}$ are considered for above considered two sets of true values of the parameters, respectively. In case of non-informative priors, we consider $a_i = b_i = 0.0001;\ i = 1,2$. Also, we consider three different choices of $q = -2,\ -1,\ 1$ for GELF. We take $M = 10,000$ for importance sampling technique and consider 20% of M as burn-in-period. The entire process is repeated $1,000$ times. All computations in this article are done with the statistical software R (see R Core Team (2021)). All the simulated results are presented in Tables 4.2, 4.3, 4.4 and 4.5. From these simulation Tables, following conclusion are made:

In view of Tables 4.2 and 4.4, this experiment has brought up some interesting observations. In almost all cases, the output of ML and Bayes estimates of SSR in terms of MSEs are very adequate even for small sample sizes. MSEs are found to decrease as n and m increase. It confirms the consistent behavior of estimators of SSR. Also, the performance of Bayes estimators with Prior B is better than ML estimator even with Prior A in terms of MSEs, as Bayes estimators with Prior B includes prior information about the parameters.

In view of Tables 4.3 and 4.5 show that the average lengths of ACIs and HPD credible intervals are shrinking with increase in number of failures. According to Table 4.3 asymptotic intervals has smaller average length and than HPD credible intervals with Prior A and Prior B both. The coverage probability for HPD credible with Prior A and Prior B attains their prescribed confidence coefficient in almost all cases but ACI does not. Also, from Table 4.5 as the true value of SSR increases, the coverage probability for ACI estimator attain their prescribed confidence coefficient.

TABLE 4.1: Progressive censoring schemes used in simulation study.

n	m	CS	Schemes	n	m	CS	Schemes
20	15	[1]	(5*1,0*14)	30	24	[7]	(0*23,6*1)
	15	[2]	(1*2,0*5,1,0*5,1*2)		24	[8]	(0*30)
	15	[3]	(0*14,5*1)	40	35	[9]	(5*1,0*34)
	20	[4]	(0*20)		35	[10]	(1*1,0*8,1*1,0*6,1*1,0*8,1*1,0*8,1*1)
30	24	[5]	(6*1,0*23)		35	[11]	(0*34,5*1)
	24	[6]	(2*1,0*10,2*1,0*11,2*1)		40	[12]	(0*40)

TABLE 4.2: The AE and MSEs of ML and Bayes estimates of SSR, when $R = 0.75$.

			MLE		Bayes											
					Prior A						Prior B					
					q=-1		q=1		q=-2		q=-1		q=1		q=-2	
n	m	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
20	15	[1]	0.7437	0.0047	0.7387	0.0047	0.7326	0.0055	0.7420	0.0044	0.7377	0.0038	0.7365	0.0039	0.7415	0.0037
20	15	[2]	0.7399	0.0048	0.7355	0.0048	0.7334	0.0053	0.7452	0.0039	0.7397	0.0035	0.7377	0.0036	0.7397	0.0034
20	15	[3]	0.7425	0.0043	0.7382	0.0043	0.7333	0.0047	0.7450	0.0040	0.7390	0.0034	0.7398	0.0034	0.7416	0.0031
20	20	[4]	0.7439	0.0040	0.7395	0.0040	0.7343	0.0043	0.7439	0.0034	0.7410	0.0029	0.7414	0.0026	0.7448	0.0027
30	24	[5]	0.7473	0.0029	0.7439	0.0029	0.7386	0.0034	0.7435	0.0030	0.7436	0.0023	0.7400	0.0027	0.7432	0.0025
30	24	[6]	0.7435	0.0025	0.7403	0.0026	0.7420	0.0028	0.7451	0.0027	0.7452	0.0023	0.7408	0.0025	0.7449	0.0022
30	24	[7]	0.7482	0.0024	0.7452	0.0024	0.7420	0.0028	0.7451	0.0027	0.7445	0.0024	0.7443	0.0022	0.7437	0.0022
30	30	[8]	0.7473	0.0024	0.7444	0.0024	0.7390	0.0025	0.7456	0.0024	0.7418	0.0021	0.7425	0.0021	0.7465	0.0021
40	35	[9]	0.7477	0.0020	0.7452	0.0020	0.7452	0.0020	0.7449	0.0020	0.7455	0.0018	0.7423	0.0020	0.7462	0.0019
40	35	[10]	0.7449	0.0021	0.7426	0.0021	0.7436	0.0021	0.7443	0.0019	0.7473	0.0016	0.7411	0.0018	0.7463	0.0016
40	35	[11]	0.7495	0.0018	0.7473	0.0018	0.7372	0.0023	0.7446	0.0018	0.7449	0.0016	0.7430	0.0017	0.7470	0.0016
40	40	[12]	0.7475	0.0017	0.7452	0.0017	0.7424	0.0018	0.7452	0.0019	0.7443	0.0016	0.7423	0.0017	0.7450	0.0015

TABLE 4.3: AL and CPs of 95% asymptotic confidence/HPD credible intervals of SSR, when $R = 0.75$.

			ACI		HPD											
					Prior A						Prior B					
					q=-1		q=1		q=-2		q=-1		q=1		q=-2	
n	m	CS	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
20	15	[1]	0.5612	0.993	0.3000	0.973	0.2996	0.967	0.3000	0.968	0.2804	0.985	0.2779	0.981	0.2796	0.970
20	15	[2]	0.5283	0.987	0.3028	0.971	0.3000	0.977	0.2985	0.978	0.2792	0.978	0.2775	0.988	0.2807	0.981
20	15	[3]	0.5173	0.994	0.3019	0.985	0.3014	0.977	0.2985	0.985	0.2793	0.979	0.2759	0.982	0.2793	0.986
20	20	[4]	0.5177	0.993	0.2615	0.968	0.2616	0.963	0.2609	0.972	0.2474	0.978	0.2450	0.988	0.2466	0.977
30	24	[5]	0.4552	0.994	0.2382	0.968	0.2390	0.970	0.2395	0.973	0.2279	0.977	0.2278	0.974	0.2287	0.978
30	24	[6]	0.4314	0.996	0.2412	0.982	0.2379	0.974	0.2389	0.976	0.2268	0.991	0.2274	0.987	0.2281	0.984
30	24	[7]	0.4254	0.999	0.2383	0.981	0.2380	0.974	0.2391	0.976	0.2270	0.980	0.2256	0.980	0.2283	0.981
30	30	[8]	0.4243	0.996	0.2140	0.973	0.2151	0.969	0.2141	0.971	0.2074	0.980	0.2055	0.977	0.2057	0.967
40	35	[9]	0.3845	0.997	0.1987	0.977	0.1971	0.964	0.1992	0.979	0.1920	0.976	0.1922	0.985	0.1919	0.968
40	35	[10]	0.3696	0.994	0.1999	0.972	0.1980	0.976	0.1999	0.980	0.1911	0.983	0.1929	0.976	0.1923	0.980
40	35	[11]	0.3688	0.999	0.1977	0.976	0.2013	0.970	0.1998	0.978	0.1923	0.984	0.1920	0.982	0.1917	0.978
40	40	[12]	0.3670	0.996	0.1861	0.986	0.1864	0.977	0.1866	0.969	0.1812	0.981	0.1811	0.980	0.1815	0.981

TABLE 4.4: The AE and MSEs of ML and Bayes estimates of SSR, when $R = 0.25$.

					Bayes											
					Prior A						Prior B					
			MLE		q=-1		q=1		q=-2		q=-1		q=1		q=-2	
n	m	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
20	15	[1]	0.2570	0.0048	0.2620	0.0048	0.2492	0.0047	0.2681	0.0052	0.2598	0.0035	0.2499	0.0036	0.2687	0.0042
20	15	[2]	0.2604	0.0049	0.2649	0.0049	0.2454	0.0040	0.2669	0.0044	0.2593	0.0034	0.2456	0.0032	0.2673	0.0037
20	15	[3]	0.2566	0.0046	0.2609	0.0046	0.2475	0.0042	0.2625	0.0044	0.2589	0.0033	0.2487	0.0033	0.2657	0.0035
20	20	[4]	0.2558	0.0034	0.2602	0.0034	0.2450	0.0033	0.2689	0.0040	0.2602	0.0031	0.2441	0.0026	0.2659	0.0031
30	24	[5]	0.2554	0.0030	0.2588	0.0030	0.2476	0.0032	0.2639	0.0033	0.2560	0.0024	0.2498	0.0024	0.2611	0.0027
30	24	[6]	0.2534	0.0028	0.2565	0.0028	0.2455	0.0026	0.2628	0.0030	0.2545	0.0019	0.2445	0.0022	0.2593	0.0027
30	24	[7]	0.2534	0.0028	0.2563	0.0028	0.2450	0.0025	0.2606	0.0027	0.2548	0.0021	0.2462	0.0021	0.2593	0.0022
30	30	[8]	0.2544	0.0025	0.2574	0.0025	0.2479	0.0023	0.2607	0.0026	0.2597	0.0021	0.2449	0.0018	0.2590	0.0020
40	35	[9]	0.2515	0.0020	0.2540	0.0020	0.2473	0.0019	0.2608	0.0023	0.2565	0.0019	0.2461	0.0018	0.2580	0.0020
40	35	[10]	0.2548	0.0020	0.2571	0.0020	0.2458	0.0019	0.2590	0.0019	0.2555	0.0017	0.2489	0.0016	0.2569	0.0017
40	35	[11]	0.2522	0.0018	0.2545	0.0018	0.2491	0.0018	0.2582	0.0019	0.2553	0.0017	0.2479	0.0017	0.2562	0.0017
40	40	[12]	0.2534	0.0017	0.2557	0.0017	0.2473	0.0016	0.2569	0.0017	0.2558	0.0017	0.2465	0.0015	0.2580	0.0017

TABLE 4.5: AL and CPs of 95% asymptotic confidence/ HPD credible intervals of SSR, when $R = 0.25$.

					HPD											
					Prior A						Prior B					
			ACI		q=-1		q=1		q=-2		q=-1		q=1		q=-2	
n	m	CS	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
20	15	[1]	0.1903	0.927	0.3007	0.971	0.3024	0.962	0.2988	0.965	0.2790	0.982	0.2814	0.975	0.2802	0.976
20	15	[2]	0.1827	0.937	0.3028	0.973	0.3003	0.977	0.3003	0.975	0.2785	0.987	0.2778	0.985	0.2797	0.983
20	15	[3]	0.1760	0.932	0.3004	0.976	0.3012	0.979	0.2971	0.972	0.2781	0.983	0.2792	0.977	0.2786	0.986
20	20	[4]	0.1756	0.965	0.2621	0.979	0.2612	0.979	0.2629	0.971	0.2481	0.977	0.2462	0.979	0.2481	0.979
30	24	[5]	0.1536	0.936	0.2397	0.975	0.2391	0.962	0.2394	0.956	0.2276	0.979	0.2294	0.982	0.2276	0.977
30	24	[6]	0.1459	0.927	0.2390	0.979	0.2381	0.972	0.2398	0.973	0.2270	0.986	0.2261	0.981	0.2265	0.971
30	24	[7]	0.1424	0.935	0.2389	0.974	0.2379	0.982	0.2388	0.972	0.2268	0.985	0.2266	0.983	0.2269	0.989
30	30	[8]	0.1429	0.946	0.2149	0.967	0.2149	0.974	0.2140	0.961	0.2084	0.983	0.2051	0.985	0.2058	0.984
40	35	[9]	0.1284	0.926	0.1982	0.975	0.1987	0.971	0.1993	0.969	0.1928	0.967	0.1912	0.974	0.1918	0.976
40	35	[10]	0.1254	0.935	0.1997	0.980	0.1977	0.976	0.1990	0.981	0.1925	0.985	0.1925	0.981	0.1915	0.976
40	35	[11]	0.1230	0.959	0.1986	0.981	0.1993	0.971	0.1988	0.982	0.1922	0.982	0.1919	0.978	0.1911	0.979
40	40	[12]	0.1235	0.952	0.1865	0.980	0.1859	0.981	0.1855	0.972	0.1812	0.976	0.1799	0.977	0.1808	0.971

4.6 Real Data Analysis

The applicability of the considered model and methodology presented in this chapter is addressed in this section. We examine two distinct pairs of real data sets for this purpose.

4.6.1 Real Data Set I

This pair of real data sets are taken from Bain and Englehardt (1991). These data are the failure times (in hours) of the air conditioning system of two different aeroplanes. The failure times of the air conditioning system of two aeroplanes, respectively, are as follows:

Plane 720 (U): 1.2, 2.1, 2.6, 2.7, 2.9, 2.9, 4.8, 5.7, 5.9, 7.0, 7.4, 15.3, 32.6, 38.6, 50.2

Plane 7911 (V): 3.3, 4.7, 5.5, 5.6, 10.4, 17.6, 18.2, 22.0, 23.9, 24.6, 32.0.

Guo and Gui (2018) studied these data sets for SSR for IP lifetime model in a complete sample case. They showed that these data sets good fit the IP lifetime model. Before further proceeding, we perform Kolmogorov-Smirnov (KS) and Anderson-Darling (AD) goodness of fit tests to check whether the given data sets follow IPD or not using ML estimation. The ML estimates of the unknown parameters, KS, and AD test statistics with their corresponding p-values for these data sets are reported in Table 4.6. From Table 4.6, it is clear that p-values are greater than 0.05, corresponding to both KS and AD goodness of fit tests for these data sets. Therefore, we can assume that these data sets follow the IP lifetime model at a 5% level of significance. Now, using the four distinct progressive censoring techniques, the following four progressively

TABLE 4.6: Fitting of real data set I for IP lifetime model.

		KS Test		AD Test	
Data Set I	MLE	KS	p-value	AD	p-value
Plane 720 (U)	4.7844	0.1880	0.6638	0.4547	0.7907
Plane 7911 (V)	9.6022	0.2558	0.3996	0.7421	0.5212

censored samples are generated from the above complete sample data sets:

Scheme 1 : $(n_1 = 15,\ m_1 = 10)$, $S_1 = [5*1, 0*9]$, and $(n_2 = 11,\ m_2 = 8)$, $T_1 = [3*1, 0*7]$.

$$U : 1.2, 4.8, 5.7, 5.9, 7.0, 7.4, 15.3, 32.6, 38.6, 50.2$$

$$V : 3.3, 10.4, 17.6, 18.2, 22.0, 23.9, 24.6, 32.0$$

Scheme 2 : $(n_1 = 15,\ m_1 = 10)$, $S_2 = [1*1, 0*1, 1*1, 1*0, 1*1, 0*2, 1*1]$, and $(n_2 = 11,\ m_2 = 8)$ $T_2 = [1*1, 0*3, 1*1, 0*2, 1*1]$.

$$U : 1.2, 2.6, 2.7, 2.9, 4.8, 5.9, 7.0, 15.3, 32.6, 38.6$$

$$V : 3.3, 5.5, 5.6, 10.4, 17.6, 22.0, 23.9, 24.6$$

Scheme 3 : $(n_1 = 15,\ m_1 = 10)$, $S_3 = [0*9, 5*1]$, and $(n_2 = 11,\ m_2 = 8)$, $T_3 = [0*7, 3*1]$.

$$U : 1.2, 2.1, 2.6, 2.7, 2.9, 2.9, 4.8, 5.7, 5.9, 7.0$$

$$V : 3.3, 4.7, 5.5, 5.6, 10.4, 17.6, 18.2, 22.0$$

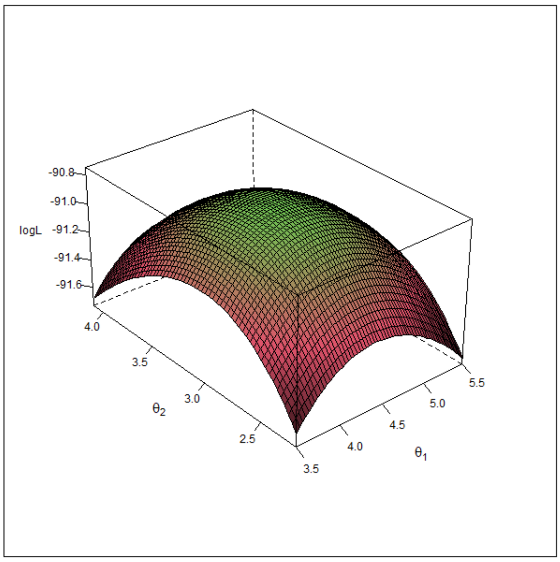
Scheme 4 : $(n_1 = 15,\ m_1 = 15)$, $S_4 = [0*15]$, and $(n_2 = 11,\ m_2 = 11)$, $T_4 = [0*11]$.

$$U : 1.2, 2.1, 2.6, 2.7, 2.9, 2.9, 4.8, 5.7, 5.9, 7.0, 7.4, 15.3, 32.6, 38.6, 50.2$$

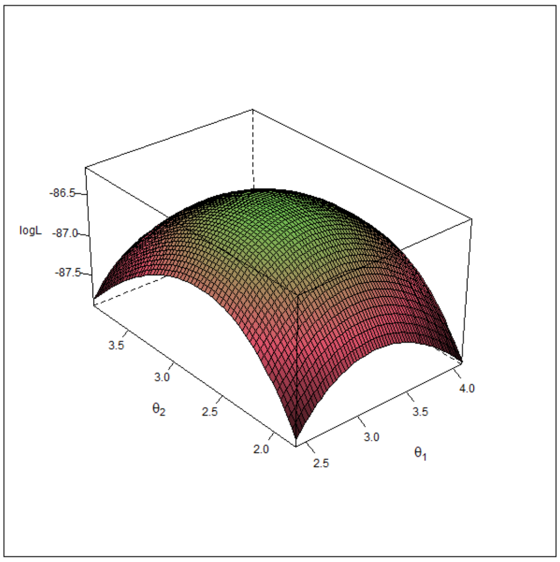
$$V : 3.3, 4.7, 5.5, 5.6, 10.4, 17.6, 18.2, 22.0, 23.9, 24.6, 32.0$$

Furthermore, for the applicability of considered methodology, we analyzed data set I under consideration of the proposed study. The ML, Bayes estimates, and 95% of asymptotic confidence/HPD credible intervals of SSR R are obtained. We further confirm the presence and uniqueness of the MLEs by plotting the log-likelihood function of the parameters θ_1 and θ_2 for four distinct progressively censored samples. These plots for four different censoring schemes are given in Figure 4.2. These plots show that the likelihood surfaces have curvature in both θ_1 and θ_2 directions, indicating that the MLEs $\hat{\theta}_1$ and $\hat{\theta}_2$ exist and are unique.

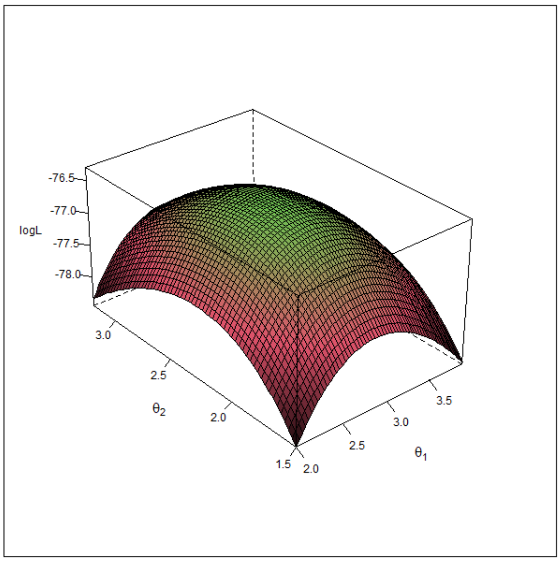
The Bayes estimates of SSR are computed using the importance sampling approach under GELF in the situation of non-informative priors because we do not have prior information. For the importance sampling approach, $M = 10,000$ samples are generated, with the burn-in period accounting for 20% of M. For GELF, we look at three distinct $q = -1, 1, -2$ values. Figure 4.3 shows the trace plots and histograms with posterior density plots based on importance samples for all four progressively censored data sets in Bayesian computations. From Figure 4.3, we observe that the trace plots represent fine mixing of the chains and converge to their stationary distributions. Also, histograms with corresponding density plots are almost symmetrical about their means in all cases. This shows good performance of the importance sampling technique and therefore, we can conclude that the Bayes estimates are good. In the case of real data set I estimation results are reported in Tables 4.7 for all four censoring schemes.



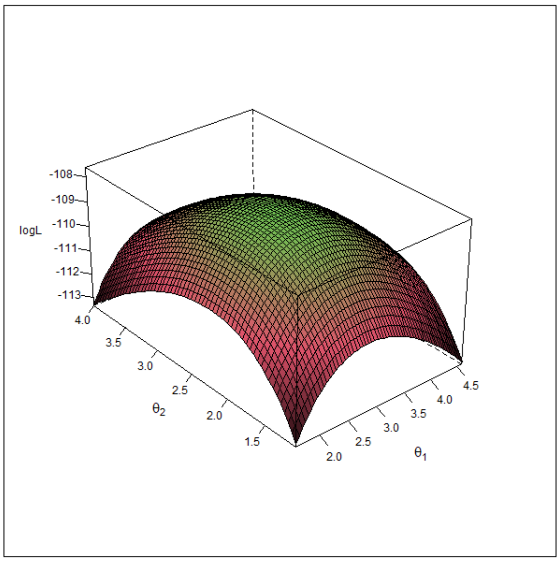
For censoring scheme (S_1, T_1).



For censoring scheme (S_2, T_2).



For censoring scheme (S_3, T_3).



For censoring scheme (S_4, T_4).

FIGURE 4.2: Plots of log-likelihood function of θ_1 and θ_2 for different censoring schemes in case of real data set I.

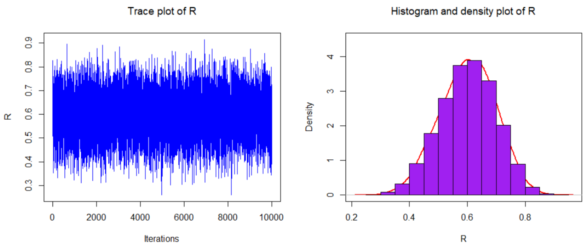
4.6.2 Real Data Set II

Here, we consider breakdown times (in minutes) of an insulating fluid between electrodes at different voltages 34 kV and 36 kV, respectively. These data sets are reported in (Nelson, 1982, p. 105). The breakdown times at two different electrodes, respectively, are as follows:

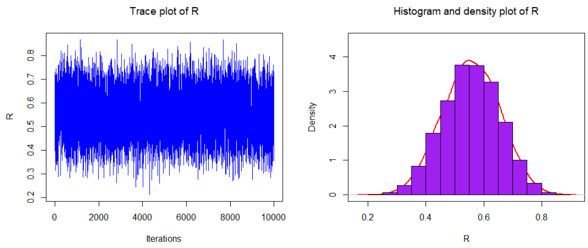
34 kV (U): 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06,

TABLE 4.7: The ML, Bayes estimates and 95% ACIs and HPD credible intervals of SSR in case of real data set I.

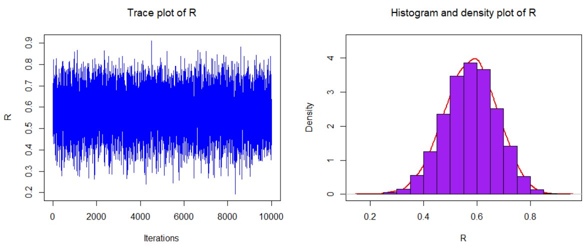
	ACI		q=-1		q=1		q=-2	
Schemes	$\hat{R}$	CI	$\hat{R}_B$	HPD	$\hat{R}_B$	HPD	$\hat{R}_B$	HPD
Scheme 1	0.3379	(0.1785,0.4973)	0.3470	(0.1381,0.5962)	0.3185	(0.1387,0.5980)	0.3600	(0.1310,0.5944)
Scheme 2	0.3451	(0.1946,0.4956)	0.3532	(0.1329,0.5913)	0.3287	(0.1369,0.5947)	0.3650	(0.1330,0.5904)
Scheme 3	0.3204	(0.1813,0.4596)	0.3295	(0.1180,0.5609)	0.3059	(0.1218,0.5629)	0.3409	(0.1194,0.5606)
Scheme 4	0.3326	(0.1899,0.4752)	0.3405	(0.1640,0.5500)	0.3177	(0.1578,0.5439)	0.3523	(0.1599,0.5461)



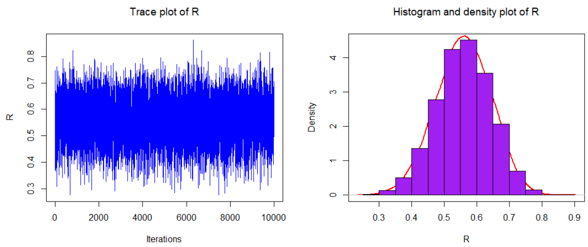
For censoring scheme (S_1, T_1).



For censoring scheme (S_2, T_2).



For censoring scheme (S_3, T_3).



For censoring scheme (S_4, T_4).

FIGURE 4.3: Trace plots and histogram with density plots of R for different censoring schemes in case of real data set I.

31.75, 32.52, 33.91, 36.71, 72.89.

36 kV (V): 0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.90, 3.67, 3.99, 5.35, 13.77, 25.50.

A similar procedure is followed in this sub-section as discussed in the case of real data sets I for fitting the real data sets. We perform KS and AD goodness of fit tests to check whether the given data sets follow the IP lifetime model or not. The ML estimates of the unknown parameters, KS, and AD test statistics with their corresponding p-values for these data sets are reported in Table 4.8. From Table 4.8, it is clear that these data sets follow the IP lifetime model at a 5% level of significance. Now, four progressively censored samples are generated from the above complete sample data sets based on following censoring schemes:

Scheme 1 : $(n_1 = 19,\ m_1 = 15),\ S_1 = [4*1, 0*14],$ and $(n_2 = 15,\ m_2 = 10),\ \ T_1 = [5*1, 0*$

TABLE 4.8: Fitting of real data set II for IP lifetime model.

Data Set	MLE	KS Test KS	KS Test p-value	AD Test AD	AD Test p-value
34 kV (U)	2.8327	0.2267	0.2433	1.5718	0.1605
36 kV (V)	2.2371	0.1937	0.5623	0.5227	0.7209

9].

$$U : 0.19, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89$$

$$V : 0.35, 2.07, 2.58, 2.71, 2.90, 3.67, 3.99, 5.35, 13.77, 25.50$$

Scheme 2 : $(n_1 = 19,\ m_1 = 15),\ \ S_2 = [1*1, 0*3, 1*1, 0*4, 1*1, 0*4, 1*1]$, and
$(n_2 = 15,\ m_2 = 10)\quad T_2 = [1*1, 0*1, 1*1, 0*1, 1*1, 0*1, 1*1, 0*2, 1*1]$.

$$U : 0.19, 0.96, 1.31, 2.78, 3.16, 4.67, 4.85, 6.50, 7.35, 8.01, 12.06, 31.75, 32.52, 33.91, 36.71$$

$$V : 0.35, 0.96, 0.99, 1.97, 2.07, 2.71, 2.90, 3.99, 5.35, 13.77$$

Scheme 3 : $(n_1 = 19,\ m_1 = 15),\ S_3 = [0*14, 4*1]$, and $(n_2 = 15,\ m_2 = 10),\quad T_3 = [0*9, 5*1]$.

$$U : 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75$$

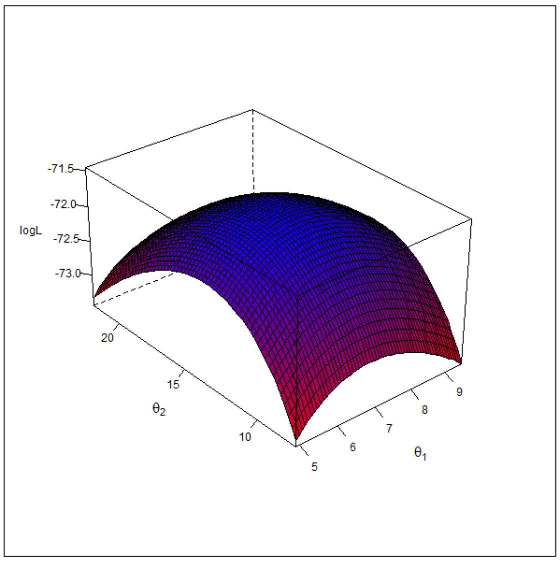
$$V : 0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.90$$

Scheme 4 : $(n_1 = 19,\ m_1 = 19),\ S_4 = [0*19]$, and $(n_2 = 15,\ m_2 = 15),\quad T_4 = [0*15]$.

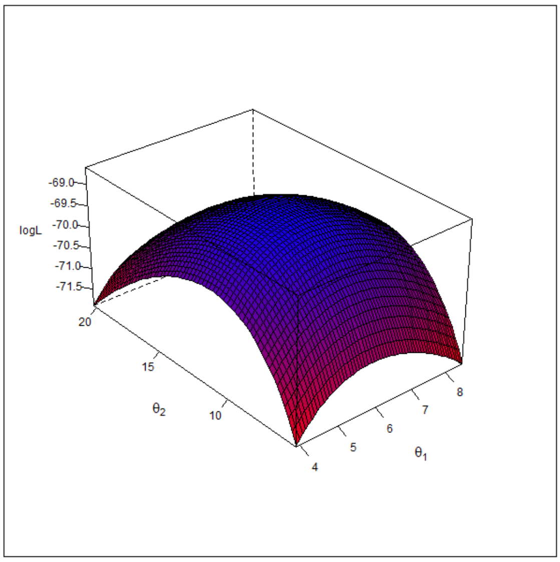
U : 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89

$$V : 0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.90, 3.67, 3.99, 5.35, 13.77, 25.50$$

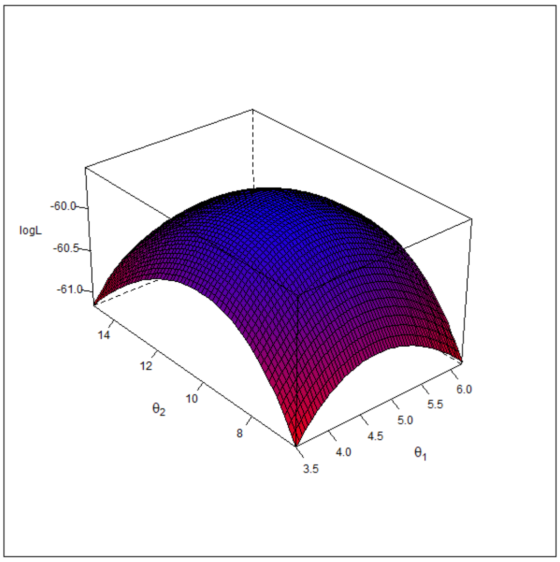
Similarly as we have discussed in case of real data set I, we analyze data set II for the applicability of considered methodology. The ML, Bayes estimates, and 95% of ACIs and HPD credible intervals of SSR R are obtained. To confirm the existence and uniqueness of the MLEs, we display the log-likelihood function of the parameters θ_1 and θ_2 for four distinct progressively censored samples. Figure 4.4 shows these graphs for four distinct censoring schemes. In the



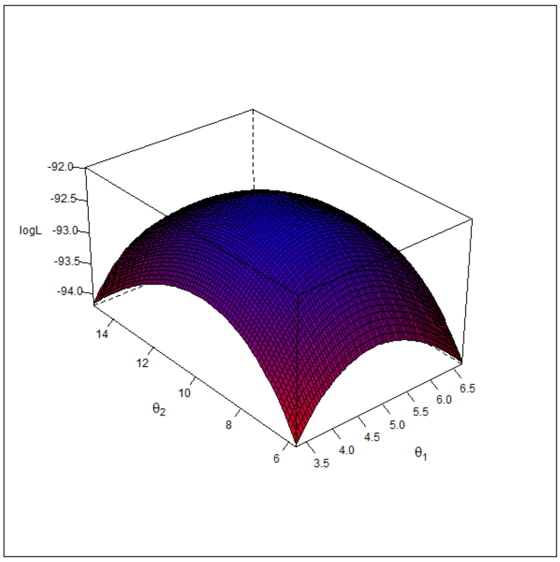
For censoring scheme (S_1, T_1).



For censoring scheme (S_2, T_2).



For censoring scheme (S_3, T_3).



For censoring scheme (S_4, T_4).

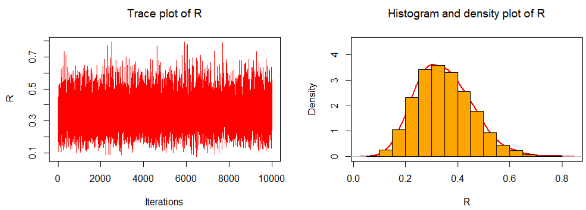
FIGURE 4.4: Plots of log-likelihood function of θ_1 and θ_2 for different censoring schemes in case of real data set II.

case of real data set II, these graphs demonstrate that the likelihood surfaces exhibit curvature in both θ_1 and θ_2 directions, suggesting that the ML estimates $\hat{\theta}_1$ and $\hat{\theta}_2$ exist and are unique.

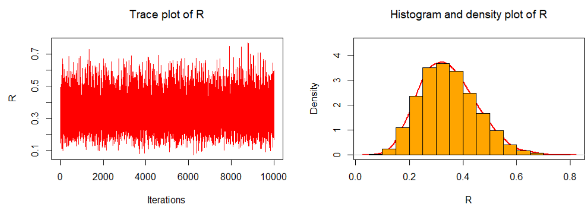
The Bayes estimate of SSR are obtained in case of non-informative priors as we do not have prior information, using the IS procedure under GELF. For the IS technique, $M = 10,000$ observations are generated and first 20% observations are considered as burn-in-period. Again here, we consider three different values of $q = -2, \ -1, \ 1$ for GELF. In Bayesian computations, the trace plots and histograms with posterior density plots based on importance samples are plotted for all four progressively censored data sets and are given in Figure 4.5. From this Figure we observe that the trace plots represent fine mixing of the chains and converge to their stationary distributions. Also, histograms with corresponding density plots are almost symmetrical about their means in all cases. This shows good performance of the IS technique and therefore, we can conclude that the Bayes estimates are good. In case of real data set II estimation results are reported in Tables 4.9 for all four pairs of progressively censored samples.

TABLE 4.9: The ML, Bayes estimates and 95% asymptotic confidence/HPD credible intervals of SSR in case of real data set II.

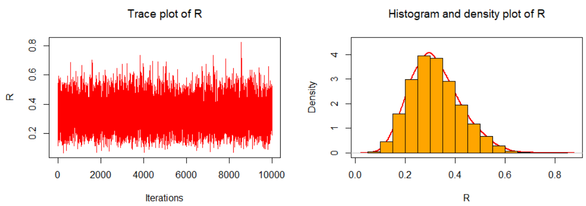
	MLE		q=-1		q=1		q=-2	
Schemes	$\hat{R}$	ACI	$\hat{R}_B$	HPD	$\hat{R}_B$	HPD	$\hat{R}_B$	HPD
Scheme 1	0.5923	(0.3667,0.8179)	0.5890	(0.3839,0.8050)	0.5737	(0.3863,0.8059)	0.5960	(0.3869,0.8069)
Scheme 2	0.5515	(0.3507,0.7523)	0.5775	(0.3832,0.8050)	0.5644	(0.3830,0.8021)	0.5830	(0.3849,0.8061)
Scheme 3	0.5205	(0.3314,0.7095)	0.5903	(0.3933,0.8113)	0.5775	(0.3967,0.8135)	0.5971	(0.3973,0.8138)
Scheme 4	0.5188	(0.3303,0.7074)	0.5922	(0.4073,0.7674)	0.5797	(0.4083,0.7696)	0.5971	(0.4029,0.7692)



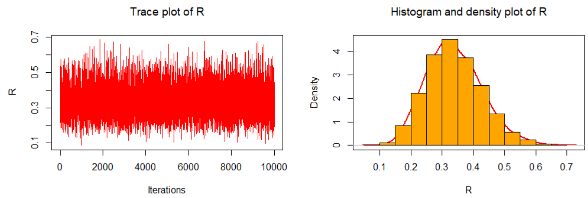
For censoring scheme (S_1, T_1).



For censoring scheme (S_2, T_2).



For censoring scheme (S_3, T_3).



For censoring scheme (S_4, T_4).

FIGURE 4.5: Trace plots and histogram with density plots of R for different censoring schemes in case of real data set II.

4.7 Concluding Remarks

In this chapter, we discussed the problem of estimation of SSR $R = P(V < U)$ for the IP lifetime model using progressively censored data. We derived ML estimate and 95% of asymptotic confidence interval with corresponding coverage probability of SSR. We computed Bayes estimates in case of both informative and non-informative priors under generalized entropy loss function using importance sampling technique. Also, 95% HPD credible interval of SSR was constructed. The performance of ML and Bayes estimators of SSR were examined by computational analysis using a Monte Carlo simulation. The computational results suggested that the Bayes estimator is more precise than the ML estimator and these can be used for all practical purposes when the prior information is available. Two pairs of real data sets were also discussed for practical applicability of considered methodology developed in this chapter. The methodology and estimation results studied in this article will be beneficial to reliability practitioners in real life situations. In this chapter, iterative and approximation methods were used for ML and Bayesian computations, respectively. In future work exact estimation procedures can be developed. Also, we can obtain optimum censoring plans to achieve the optimum accuracy of the estimators. More work is needed along with these directions as future scope.

Chapter 5

Classical and Bayesian Estimation in Inverse Pareto Lifetime Model using Progressively First Failure Censored Data

5.1 Introduction

The main objective of this chapter is to develop statistical inferences for the associated parameter and reliability characteristics of the IP lifetime model using the progressively first failure censored (PFFC) data from both a classical and Bayesian perspective.

Because of the severe competition in the market, product reliability is typically improving with the advancement of manufacturing technologies. Generally, in life-testing experiments, observing the failure time for all test units often takes a long time, resulting in a substantial increase in experimental time and cost. As a consequence of the time and cost constraints of the experiments, censoring is a regular phenomenon in reliability and life-testing experiments. Many researchers have investigated the Type-I censoring scheme, in which the life-testing experiment terminates when the experimental period exceeds the prescribed time, and the Type-II censoring scheme, in which the life-testing experiment terminates when the number of recorded failure units meets the intended aim. One of their limitations is that none of them allows control units to be removed during the experiment. It may be required to remove test units in some circumstances. For example, in certain exceptional instances, the unit failure is beyond the control of the experimenters and might be triggered by unforeseen laboratory equipment damage. It's also possible to remove test units from the experiment on purpose to free up laboratory equipment and supplies for other projects, as well as save time and money. Because of such limitations, Cohen (1963) introduced progressive censoring in the literature, which allows the adaptability to remove the test units before they fails from the ongoing experiment. For more details one may refer Balakrishnan and Aggarwala (2000) and Balakrishnan and Cramer (2014).

It may not always be able to fulfill the test's time and cost constraints. As a result, distinct censoring schemes have been introduced one after the other to boost the efficiency of testing procedures. When testing materials are inexpensive, we may conduct the test by putting m groups with k items within each group of n individual items. During this procedure, the first failures in each group are recorded, and the assessment will not be completed until all groups have experienced the first failure. Such a situation of the testing plan was proposed by Balasooriya (1995) called a first-failure censoring scheme.

Furthermore, Wu and Kuş (2009) suggested a novel censoring plan by combining progressive and first failure censoring schemes, known as the progressive first-failure censoring scheme (PFFCS) and data collected by using this scheme is termed as progressively first failure censored (PFFC) data. This censoring scheme bears some special cases to other censoring schemes, due to its compatible features with other censoring plans, this censoring scheme has gained a lot of coverage in literature under multiple scenarios. For example, the estimation of SSR for GIE lifetime model is studied by Krishna et al. (2017), Kayal et al. (2019) developed inferences on Chen lifetime model, Bi et al. (2020) studied bathtub shaped lifetime model for reliability estimation, the statistical inferences for inverse power Lomax lifetime model is discussed by Shi and Shi (2021), estimation of SSR for generalized Maxwell lifetime model is discussed by Saini et al. (2021a), the estimation of multicomponent SSR for Bur Type XII lifetime model is discussed by Saini et al. (2021b) etc.

Practically, the PFFCS is defined as follows: Assume that in a real-life testing experiment, n classes of individuals are being tested at the same time, each with k test units, and they are entirely independent to one another. During the experiment, when the first failure unit, say $X^{G}_{1:m:n:k}$ occurs, the group it belongs to, as well as any G_1 live groups from remaining live $n-1$ groups are randomly discarded from the experiment. Similarly, at the second failure unit, say $X^{G}_{2:m:n:k}$, the group it belongs to, as well as any G_2 live groups in the remaining live $n-2-G_1$ groups, are excluded from the experiment at random. This process is continued until the mth failed unit, say $X^{G}_{m:m:n:k}$ occurs, at that point all remaining G_m live groups are removed from the experiment. Here m and $\underset{\sim}{G} = (G_1, G_2, \ldots, G_m)$ are the prefixed number of failures and censoring schemes, respectively, in such a way that $n = m + \sum_{j=1}^{m} G_j$. Then $X^{G}_{1:m:n:k} < X^{G}_{2:m:n:k} < \cdots < X^{G}_{m:m:n:k}$ are recorded as PFFC ordered sample with prefixed censoring schemes $\underset{\sim}{G} = (G_1, G_2, \ldots, G_m)$. To further demonstrate this censoring scheme, Figure 5.1 depicts the PFFC sample generation procedure. It's worth noting that the PFFCS has the following special cases:

(a) It reduces to complete sample case, when $k = 1, n = m$ and $G_j = 0;\ j = 1,2,\ldots m$.

(b) It become conventional type-II censoring plan, if $k = 1$ and $G_j = (0,\ n-m);\ j = 1,2,\ldots m-1$.

(c) It becomes progressively-II censoring plan, when $k = 1$.

(d) It reduces to first-failure censoring plan, when $G_j = 0;\ j = 1,2,\ldots n$.

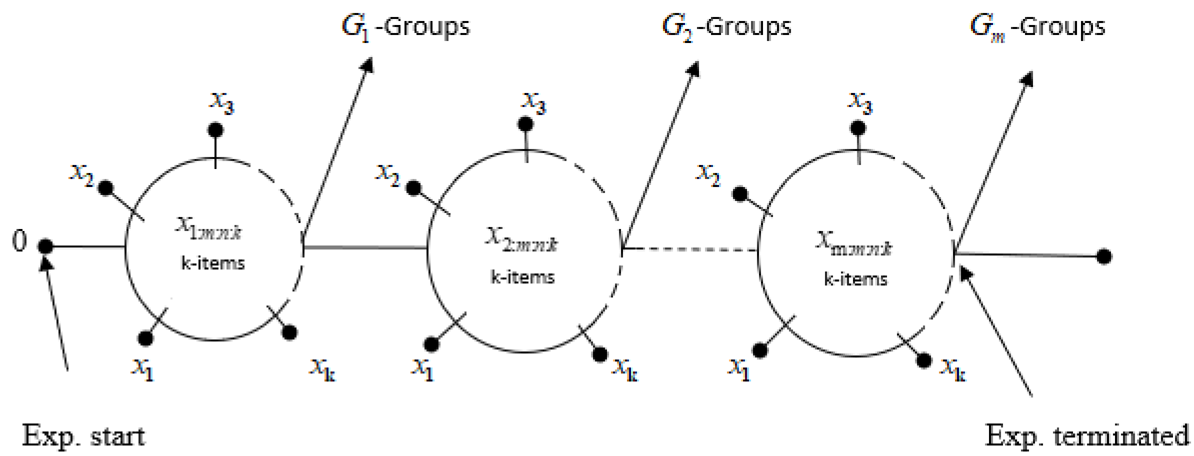


FIGURE 5.1: Schematic diagram of PFFCS.

Suppose the lifetimes of $n \times k$ test units are put on a life testing experiment following a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$, then the joint pdf of $X^G_{1:m:n:k}, X^G_{2:m:n:k}, \ldots, X^G_{m:m:n:k}$ is expressed as, see, (Wu and Kuş, 2009)

$$L\left(x^G_{1:\ m:\ n:\ k},\ x^G_{2:\ m:\ n:\ k},\ \ldots, x^G_{m:\ m:\ n:\ k}\right) = Ak^m \prod_{j=1}^{m} f_X(x^G_{j:\ m:\ n:\ k})\{1 - F_X(x^G_{j:\ m:\ n:\ k})\}^{k(G_j+1)-1},$$
$$0 < x^G_{j:\ m:\ n:\ k} < \infty; \quad \forall j = 1,2,\ldots m \tag{5.1}$$

where, $A = \ n(n-G_1-1)\ (n-G_1-G_2-2)\ \ldots\ (n-G_1-G_2-\ \ldots\ -G_{m-1}\ -m+1)$.

This chapter is organized as follows: The IP lifetime model based on PFFC data is discussed in Section 5.2. In Section 5.3 is devoted to derive MLEs of parameter and reliability characteristics. Also, derived asymptotic and bootstrap CIs for the associated model parameter. The Bayes estimators of parameter and reliability characteristics under SELF using three approximation techniques, namely TK approximation, importance sampling (IS) and M-H algorithm are discussed in Section 5.4. Also, we derived HPD credible interval for the associated model parameter. Section 5.5 presents the numerical computations using Monte Carlo simulations.

For demonstration purpose, a real data analysis is presented in Section 5.6. Finally, a concluding remarks are presented in Section 5.7.

5.2 The Model

The IP lifetime model has already been discussed in the following Chapters 2 and 4 under random censoring and progressive censoring schemes, respectively. Here, we also describe pdf, cdf and reliability characteristics of IP lifetime model for quick response under consideration of this chapter. The pdf and cdf of IP lifetime model with parameter θ, respectively, are given by

$$f_X(x;\theta) = \frac{\theta x^{\theta-1}}{(1+x)^{\theta+1}} \quad ; \theta > 0,\ x > 0, \tag{5.2}$$

and

$$F_X(x) = \left(\frac{x}{1+x}\right)^{\theta}; \quad \theta \geq 0,\ \ x > 0. \tag{5.3}$$

Also, the corresponding reliability (or survival) and hazard (pr failure rate) functions of IP lifetime model, respectively, are given by

$$R(x;\theta) = 1 - \left(\frac{x}{1+x}\right)^{\theta} \quad ; \theta > 0,\ x > 0, \tag{5.4}$$

and

$$h(x;\theta) = \frac{\theta x^{\theta-1}}{(1+x)^{\theta+1}\left[1-\left(\frac{x}{1+x}\right)^{\theta}\right]} \quad ; \theta > 0, x > 0. \tag{5.5}$$

As the moment of IP lifetime model does not in closed form, therefore, we consider median time to system failure (MdTSF) and given as

$$MdTSF = \frac{1}{2^{1/\theta}-1} \ ; \theta > 0. \tag{5.6}$$

5.3 Classical Estimation

In case of classical estimation method, the associated parameter and reliability characteristics are estimated by using ML estimation, asymptotic confidence and bootstrap confidence intervals methods.

5.3.1 Maximum Likelihood Estimation

This section is devoted to derive ML estimates of the associated model parameter θ and reliability characteristics $R(t)$, $h(t)$ and $MdTSF$, respectively. Also, obtain asymptotic and bootstrap CIs of θ. Let $x^{\underset{\sim}{G}}_{i:m:n:k}$; $\quad i=1,2,\ldots,m$, be the PFFC sample from IP lifetime model with pre-fixed number of failures m and censoring plan $\underset{\sim}{G}=(G_1,G_2,\ldots,G_m)$. For notation simplicity, hereafter we use $\underset{\sim}{x}=(x_1,x_2,\ldots,x_m)$ as PFFC sample. Then, using (6.6), (6.2) and (5.4), the likelihood function becomes

$$L(\underset{\sim}{x},\theta)=Ak^m\theta^m\prod_{i=1}^{m}\frac{x_i^{\theta-1}}{(1+x_i)^{\theta+1}}\left[1-\left(\frac{x_1}{1+x_i}\right)^{\theta}\right]^{k(G_i+1)-1} \tag{5.7}$$

where, $A=n(n-G_1-1)(n-G_1-G_2-2)\ldots(n-G_1-G_2-\cdots-G_{m-1}-m+1)$. The log-likelihood function is obtained as

$$l(\underset{\sim}{x},\theta)=C+m\ln\theta+\theta\sum_{i=1}^{m}\ln\left(\frac{x_i}{1+x_i}\right)+\sum_{i=1}^{m}[k(G_i+1)-1]\ln\left[1-\left(\frac{x_i}{1+x_i}\right)^{\theta}\right], \tag{5.8}$$

where, $C=\ln A+m\ln k-\sum\limits_{i=1}^{m}\ln[x_i(1+x_i)]$. The solution of the following normal equation of log-likelihood yields the ML estimate of θ,

$$\frac{\partial l(\underset{\sim}{x},\theta)}{\partial\theta}=\frac{m}{\theta}+\sum_{i=1}^{m}\ln\left(\frac{x_i}{1+x_i}\right)-\sum_{i=1}^{m}[k(G_i+1)-1]\frac{\left(\frac{x_i}{1+x_i}\right)^{\theta}\ln\left(\frac{x_i}{1+x_i}\right)}{\left[1-\left(\frac{x_i}{1+x_i}\right)^{\theta}\right]}=0. \tag{5.9}$$

Here, the ML estimate of θ is the solution of (5.9), and because the closed form solution for (5.9) is not accessible, a suitable numerical iterative technique can be employed to compute the ML estimate of θ numerically. Once we get the ML estimate of θ say $\hat{\theta}$, then using the invariance property of ML estimation, we can obtain the ML estimates of $R(t)$, $h(t)$, and $MdTSF$, respectively.

$$\hat{R}(t)=1-\left(\frac{t}{1+t}\right)^{\hat{\theta}}, \tag{5.10}$$

$$\hat{h}(t)=\frac{\hat{\theta}t^{\hat{\theta}-1}}{(1+t)^{\hat{\theta}+1}\left[1-\left(\frac{t}{1+t}\right)^{\hat{\theta}}\right]}. \tag{5.11}$$

$$\widehat{MdTSF}=\frac{1}{2^{1/\hat{\theta}}-1}. \tag{5.12}$$

Now, under mild regularity constraints, the MLE of θ is asymptotically normally distributed i.e. $\hat{\theta} \sim N(\theta, I^{-1}(\hat{\theta}))$, where $I(\hat{\theta})$ is the observed Fisher information,

$$I(\hat{\theta}) = E\left[-\frac{\partial^2 l(x,\theta)}{\partial \theta^2}\right]_{\theta=\hat{\theta}}, \tag{5.13}$$

with,

$$\frac{\partial^2 l(\underset{\sim}{x},\theta)}{\partial \theta^2} = -\frac{m}{\theta^2} - \sum_{i=1}^{m} [k(G_i+1)-1]\left\{\ln\left(\frac{x_i}{1+x_i}\right)\right\}^2 \frac{\left(\frac{x_i}{1+x_i}\right)^{\theta}}{\left[1-\left(\frac{x_i}{1+x_i}\right)^{\theta}\right]^2}$$

Suppose $\hat{Var}(\hat{\theta}) = I^{-1}(\hat{\theta})$ is the observed variance of $\hat{\theta}$, the asymptotic CI of θ can be obtained as

$$\hat{\theta} \pm z_{\xi/2}\sqrt{\hat{Var}(\hat{\theta})},$$

here, $z_{\xi/2}$ is the upper $(\xi/2)^{th}$ percentile of N(0,1). Also, the coverage probability (CP) for θ is given by

$$CP_{\theta} = \left[\left|\frac{\hat{\theta}-\theta}{\sqrt{\hat{Var}(\hat{\theta})}}\right| \leq z_{\xi/2}\right].$$

5.3.2 Bootstrap Confidence Intervals

In literature, Efron (1979) was the first who developed the bootstrap approach. This technique employs as a simple resampling technique that permits inferential statistics to be constructed, when samples are not sufficiently large or need heavy assumptions about the underlying distribution. Later on this concept has been applied in several applications. For more details one may refer Efron (1982), Hall (1988), Davison and Hinkley (1997). In the literature, several bootstrap techniques have been developed. In this chapter, we employ two bootstrap techniques as percentile bootstrap (boot-p) and Student's t bootstrap (boot-t) based on t-statistic to construct bootstrap CIs of the associated parameter θ of IP lifetime model. In order to compute two parametric bootstrap CIs of θ, the following steps are used as follows:

5.3.2.1 Percentile Bootstrap (boot-p) Confidence Interval

Step 1: Produce a PFFC sample $x = (x_1, x_2, \ldots, x_m)$ from the IP lifetime model with a prefixed censoring scheme $\underset{\sim}{G} = (G_1, G_2, \ldots, G_m)$ and an effective sample size of m, and then compute the ML estimate $\hat{\theta}$ of θ

Step 2: Produce an independent bootstrap PFFC sample, say $\underset{\sim}{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)$ using $\hat{\theta}$.

Then, obtain the bootstrap ML estimate, say $\hat{\theta}^*$ of θ based on the generated bootstrap sample $\underset{\sim}{x}^*$.

Step 3: Replicate the Step 2, B times to generate a sequence of bootstrap ML estimates $\hat{\theta}_i^*$; $\quad i = 1,2,\ldots,B$.

Step 4: Let $\hat{\theta}^*_{(1)} \leq \hat{\theta}^*_{(2)} \leq \cdots \leq \hat{\theta}^*_{(B)}$ denote the ordered values of $\hat{\theta}_i$ for $i = 1,2,\ldots,B$. The approximate $100(1-\alpha)\%$ boot-p CI of θ is given by $\left(\hat{\theta}^*_{[(\alpha/2)\times B]}, \hat{\theta}^*_{[(1-\alpha/2)\times B]}\right)$, where $[a]$ is the integral part of a.

5.3.2.2 Student's t Bootstrap (boot-t) Confidence Interval

Step 1 and Step 2 are same as in boot-p procedure.

Step 3: Obtain the boot-t statistic $\tau^* = \frac{\hat{\theta}^*-\hat{\theta}}{\sqrt{I^{-1}(\hat{\theta}^*)}}$ for $\hat{\theta}^*$.

Step 4: Replicate steps 2-3, B times to generate a sequence of boot-t statistics τ_i^*; $i = 1,2,\ldots,B$.

Step 5: Suppose $\tau^*_{(1)} \leq \tau^*_{(2)} \leq \cdots \leq \tau^*_{(B)}$ be the ordered values of τ_i^* for $i = 1,2,\ldots,B$.
Thus, the approximate $100(1-\alpha)\%$ boot-t CI of θ is given by

$$\left(\hat{\theta} - \tau^*_{[(1-\alpha/2)\times B]}\sqrt{I^{-1}(\hat{\theta}^*)}, \hat{\theta} - \tau^*_{[(\alpha/2)\times B]}\sqrt{I^{-1}(\hat{\theta}^*)}\right).$$

5.4 Bayesian Estimation

This part focuses on developing Bayesian estimate methods for unknown parameters and reliability characteristics of the IP lifetime model using PFFC data under SELF. Let us consider the prior belief of an unknown parameter θ is measured to follow a gamma distribution with hyper-parameters a and b, and the corresponding pdf of prior belief is termed as

$$p(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}; \;\; \theta > 0, \; a,b > 0.$$

Therefore, by incorporating prior belief in maximum likelihood function in (5.7), the posterior distribution become

$$\pi(\theta|\underline{x}) \propto \theta^{m+a-1}\exp\left\{-\theta\left(b-\sum_{i=1}^{m}\ln\left(\frac{x_i}{1+x_i}\right)\right)\right\}\exp\left\{[k(G_i+1)-1]\ln\left[1-\left(\frac{x_i}{1+x_i}\right)^{\theta}\right]\right\}. \tag{5.14}$$

The posterior mean under SELF is the Bayes estimator of every parametric function. Here, it can be seen that the posterior distribution does not belong to any well known family of distributions, so it is quite difficult to obtain the posterior mean. In addition, the ideal posterior means are ratios of two integrals that cannot be simplified in some expressions of the closed form. In order to solve these integrals, we proposed using the following approximation methods: The TK approximation, IS, and M-H algorithm techniques.

5.4.1 TK Approximation

Here, the TK approximation procedure is used to compute the point Bayes estimates of the parameter and reliability characteristics. For the parametric function $\phi(\theta)$, the posterior mean is given as follows:

$$J(x) = \frac{\int_0^\infty \phi(\theta)\exp\{L(\underset{\sim}{x},\theta)+\rho(\phi)\}d\theta}{\int_0^\infty \exp\{L(\underset{\sim}{x},\theta)+\rho(\phi)\}d\theta}, \tag{5.15}$$

where, $L(\underset{\sim}{x},\theta)$ is log-likelihood function and $\rho(\theta) = \ln p(\theta)$. Using TK approximation, we can write $J(x)$ as an explicit form, we have $\delta(\theta) = \frac{L(\underset{\sim}{x},\theta)+\rho(\theta)}{mk}$ and $\delta^*(\theta) = \delta(\theta) + \frac{\ln\phi(\theta)}{mk}$, and assume that $\hat{\phi}_\delta(\theta)$ and $\hat{\phi}_{\delta^*}(\theta)$ maximizes the functions $\delta(\theta)$ and $\delta^*(\theta)$, respectively. Then according to the TK approximation method $J(x)$ can be described as

$$J(x) = \left(\frac{\det(\Delta^*_\phi)}{\det(\Delta_\phi)}\right)^{\frac{1}{2}} \exp\left[mk\left\{\delta^*_\phi(\hat{\theta}_{\delta^*}) - \delta(\hat{\theta})\right\}\right]. \tag{5.16}$$

Here, we need to compute $\det(\Delta^*_\phi)$ and $\det(\Delta_\phi)$ which is the determinants of negative inverse Hessian of $\delta^*(\theta)$ and $\delta_\phi(\theta)$. By incorporating prior distribution to the log-likelihood function, the Bayes estimator of θ using the TK approximation is computed, and $\delta(\theta)$ is given as

$$\delta(\theta) = \frac{1}{mk}\left[(m+a-1)\ln\theta - \theta\left\{b - \sum_{i=1}^{m}\ln\left(\frac{x_i}{1+x_1}\right)\right\} + \sum_{i=1}^{m}[k(G_i+1)-1]\ln\left\{1-\left(\frac{x_i}{1+x_i}\right)^\theta\right\}\right]$$

Therefore, $\hat{\theta}_\delta$ is computed by solving the following non-linear equation

$$\frac{\partial\delta(\theta)}{\partial\theta} = \frac{m+a-1}{\theta} - b + \sum^{m}\ln\left(\frac{x_i}{1+x_i}\right) - \sum_{i=1}^{m}[k(G_i+1)-1]\frac{\left(\frac{x_i}{1+x_i}\right)^\theta \ln\left(\frac{x_i}{1+x_i}\right)}{\left[1-\left(\frac{x_i}{1+x_i}\right)^\theta\right]} = 0.$$

Also,

$$\frac{\partial^2\delta(\theta)}{\partial\theta^2}=\frac{1}{mk}\left\{-\frac{m+a-1}{\theta^2}-\sum_{j=1}^{m_2}[k(G_i+1)-1]\frac{\left\{\ln\left(\frac{x_i}{1+x_i}\right)\right\}^2\left(\frac{x_i}{1+x_i}\right)^{\theta}}{\left[1-\left(\frac{x_i}{1+x_i}\right)^{\theta}\right]^2}\right\},$$

Hence,

$$\det(\Delta_\phi)=\left|\frac{\partial^2\delta(\theta)}{\partial\theta^2}\right|_{\theta=\hat{\theta}}^{-1}. \tag{5.17}$$

Since, $\delta^*(\theta)$ is the function of $\phi(\theta)$, the Bayes estimator of $\phi(\theta)$ is computed by considering

$$\delta^*(\theta)=\delta(\theta)+\frac{\ln\phi(\theta)}{mk}. \tag{5.18}$$

Now, for $\phi(\theta)=\theta$, then $\hat{\theta}^*$ is computed by solving the following equation

$$\frac{\partial\delta^*(\theta)}{\partial\theta}=\frac{\partial\delta(\theta)}{\partial\theta}+\frac{1}{mk}\frac{1}{\theta}=0. \tag{5.19}$$

Also, using the derivative

$$\frac{\partial^2\delta^*(\theta)}{\partial\theta^2}=\frac{\partial^2\delta(\theta)}{\partial\theta^2}-\frac{1}{mk}\frac{1}{\theta^2}, \tag{5.20}$$

we get $\det(\Delta_\theta^*)$ as

$$\det(\Delta_\theta^*)=\left|\frac{\partial^2\delta^*(\theta)}{\partial\theta^2}\right|_{\theta=\hat{\theta}^*}^{-1}.$$

Thus, the Bayes estimator of θ is finally obtained by

$$\hat{\theta}_{TK}=\left(\frac{\det(\Delta_\theta^*)}{\det(\Delta_\theta)}\right)^{\frac{1}{2}}\exp\left[mk\left\{\delta_\theta^*(\hat{\theta}_{\delta^*})-\delta(\hat{\theta}_\delta)\right\}\right].$$

Similarly, the Bayes estimators of reliability characteristics $R(t), h(t)$, and $MdTSF$, respectively are given as follows

$$\hat{R}_{TK}(t)=\left(\frac{\det(\Delta_{R(t)}^*)}{\det(\Delta_{R(t)})}\right)^{\frac{1}{2}}\exp\left[mk\left\{\delta_{R(t)}^*(\hat{R}_{\delta^*(t)})-\delta(\hat{R}_\delta(t))\right\}\right],$$

$$\hat{h}_{TK}(t)=\left(\frac{\det(\Delta_{h(t)}^*)}{\det(\Delta_{h(t)})}\right)^{\frac{1}{2}}\exp\left[mk\left\{\delta_{h(t)}^*(\hat{h}_{\delta^*(t)})-\delta(\hat{h}_\delta(t))\right\}\right]$$

and

$$\widehat{MdTSF}_{TK}(t) = \left(\frac{\det(\Delta^*_{MdTSF})}{\det(\Delta_{MdTSF})}\right)^{\frac{1}{2}} \exp\left[mk\left\{\delta^*_{MdTSF}(\widehat{MdTSF}_{\delta^*}) - \delta(\widehat{MdTSF}_{\delta})\right\}\right].$$

5.4.2 Importance Sampling Technique

The importance sampling (IS) approach is used to find the Bayes estimator of the parameter and reliability characteristics under SELF. The posterior distribution described in (5.14) can be rewritten as

$$\pi(\theta|\underset{\sim}{x}) \propto f_{GA}(\theta; m+a, S)\, U(\theta) \tag{5.21}$$

where, $S = \left[b - \sum_{i=1}^{m} \ln\left(\frac{x_i}{1+x_i}\right)\right]$ and $U(\theta) = \exp\left\{\sum_{i=1}^{m}[k(G_i+1)-1]\ln\left[1-\left(\frac{x_i}{1+x_i}\right)^{\theta}\right]\right\}$ and $f_{GA}(.;p,q)$ is a gamma density with shape p and scale q parameters, respectively. Under SELF, the Bayes estimator of $\phi(\theta)$, a function of θ, is now given by

$$\hat{\phi}_{IS}(\theta) = E[\phi(\theta)|\underset{\sim}{x}] = \frac{\int_0^{\infty} \phi(\theta)\pi(\theta|\underset{\sim}{x})d\theta}{\int_0^{\infty} \pi(\theta|\underset{\sim}{x})d\theta}. \tag{5.22}$$

Therefore, we do not need to compute the normalizing constant to approximate $\hat{\phi}IS(\theta)$ given in (5.22) using the IS techniques. The steps below are used for programming purposes:

Step 1: Produce $\theta^{(1)}$ from $f_{GA}(\theta; m+a, b+S)$.

Step 2: To obtain importance sample, repeat the above Step 1, M times, $(\theta^{(1)}), (\theta^{(2)}), \ldots, (\theta^{(M)})$.

Now we can obtain the approximate Bayes estimates of the function of parameter $\phi(\theta)$ as follows:

$$\hat{\phi}_{IS}(\theta) = \frac{\sum_{j=1}^{M} \phi(\theta^{(j)})U(\theta^{(j)})}{\sum_{j=1}^{M} U(\theta^{(j)})}. \tag{5.23}$$

Hence, the Bayes estimates of parameter and reliability characteristics under SELF using IS method are, respectively given by

$$\hat{\theta}_{IS} = \frac{\sum_{j=1}^{M} \theta^{(j)} U(\theta^{(j)})}{\sum_{j=1}^{M} U(\theta^{(j)})}, \quad \hat{R}_{IS}(t) = \frac{\sum_{j=1}^{M} \left(1 - \left(\frac{1}{1+t}\right)^{\theta^{(j)}}\right) U(\theta^{(j)})}{\sum_{j=1}^{M} U(\theta^{(j)})}; \quad t > 0,$$

$$\hat{h}_{IS}(t) = \frac{\sum_{j=1}^{M} \frac{\theta^{(j)} t^{\theta^{(j)}-1}}{(1+t)^{\theta^{(j)}+1}\left(1-\left(\frac{t}{1+t}\right)^{\theta^{(j)}}\right)} U(\theta^{(j)})}{\sum_{j=1}^{M} U(\theta^{(j)})}; \quad t > 0, \quad \widehat{MdTSF} = \frac{\sum_{j=1}^{M} \frac{1}{2^{\frac{1}{\theta^{(j)}}}-1} U(\theta^{(j)})}{\sum_{j=1}^{M} U(\theta^{(j)})}.$$

5.4.3 Metropolis-Hastings Algorithm

Here, we consider one of the popular MCMC technique as M-H algorithm to compute the Bayes estimates of parameter and reliability characteristics. We take candidate point from a normal distribution to draw samples from the posterior distribution of $\theta|\underset{\sim}{x}$ from (5.14). For programming or computation purposes, the following steps are carried out:

Step 1: Consider initial guess value of θ say $\theta^{(0)}$.

Step 2: From the proposal density $\eta(\theta^{(j)}|\theta^{(j-1)})$, generate a candidate point $\theta_c^{(j)}$.

Step 3: Generate u using a uniform distribution $U(0,1)$.

Step 4: Obtain $A\left(\theta_c^{(j)}|\theta^{(j-1)}\right) = \min\left\{\frac{\pi\left(\theta_c^{(j)}|\underset{\sim}{x}\right)\eta\left(\theta^{(j-1)}|\theta_c^{(j)}\right)}{\pi\left(\theta^{(j-1)}|\underset{\sim}{x}\right)\eta\left(\theta_c^{(j)}|\theta^{(j-1)}\right)}, 1\right\}$.

Step 5: If $u \leq A$ set $\theta^{(j)} = \theta_c^{(j)}$ with acceptance rate A otherwise $\theta^{(j)} = \theta^{(j-1)}$.

Step 6: In order to compute the parameter sequence of θ, repeat steps 1-5, for $j = 1, 2, \ldots, M$, say $\left\{\theta^{(1)}, \theta^{(2)}, \theta^{(3)}, \ldots, \theta^{(M)}\right\}$.

Using the $(M - M_0)$ observations, where M_0 is the burn-in period, we get an estimate. Hence, the approximate Bayes estimate using M-H algorithm procedure under SELF is given by

$$\hat{\phi}_{MH}(\theta) = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \phi(\theta^{(j)}).$$

Thus, using the M-H algorithm, the Bayes estimates of the parameter θ and the reliability characteristics $R(t), h(t)$, and $MdTSF$ are computed under SELF as follows:

$$\hat{\theta}_{MH} = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\theta^{(j)}, \ \hat{R}_{MH}(t) = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\left[1-\left(\frac{t}{1+t}\right)^{\theta^{(j)}}\right],$$

$$\hat{h}_{MH}(t) = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\frac{\theta^{(j)}t^{\theta^{(j)}-1}}{(1+t)^{\theta^{(j)}+1}\left[1-\left(\frac{t}{1+t}\right)^{\theta^{(j)}}\right]}, \quad \widehat{MdTSF} = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\frac{1}{(2^{1/\theta^{(j)}}-1)}.$$

5.4.4 HPD Credible Interval

In this subsection, the HPD credible interval of θ can be obtained using generated MCMC sample. Suppose $\theta_{(1)} < \theta_{(2)} < \cdots < \theta_{(M)}$ denotes the ordered values of $\theta^{(1)}, \ \theta^{(2)}, \ldots, \theta^{(M)}$. Thus, $100(1-\xi)\%$, where, $0 < \xi < 1$, HPD credible interval of θ is given by $\left(\theta_{(j)}, \ \theta_{(j+[(1-\xi)M])}\right)$, where j is chosen such that

$$\theta_{(j+[(1-\xi)M])} - \theta_{(j)} = \min_{1\le i\le \xi M}\left(\theta_{(i+[(1-\xi)M])} - \theta_{(i)}\right), j = 1,2,\ldots,M,$$

where, $[x]$ is the integer part of x.

5.5 Numerical Computations

To analyze the impact of the different estimators produced in this chapter, extensive numerical computations are done in this section. The estimators are compared with their corresponding average estimates (AE) and mean squared errors (MSE). For computations, first of all we generated PFFC samples for different combinations of (k,n,m) with prefixed censoring plans $\underset{\sim}{G}$ and distinct values of a model parameter θ. To generate PFFC samples, we use the algorithm suggested by Balakrishnan and Sandhu (1995) with some modifications in such a way that, the PFFC sample $x_1, x_2, \ldots, x_m$ can be viewed as a progressively censored sample from a population with cdf $(1-(1-F(x))^k)$, see, Wu and Kuş (2009). To see the behaviour of estimation methods, the following parameters are taken as follows: number of items within each group $k = 3,5$, number of groups $n = 20,30$ and prefixed number of failures $m = (80,100)\%$ of n with prefixed censoring plans $\underset{\sim}{G}$, respectively. Also, two sets of parameter values are taken as $\theta = 0.5$ and $\theta = 1.5$, respectively. For each n, four different failure plans are adopted, and out of these, three are common for each n. The three different common failure plans are as follows:

Plan 1: $[(k,n,m),(G_1 = n-m, G_i = 0, \quad \forall \quad i = 2,3,\ldots m)]$, in this case $(n-m)$ groups are discarded from the test at the first failure only,

Plan 2: $[(k,n,m),(G_i = 0, \quad \forall i = 1,2,\ldots,m-1, G_m = n-m)]$, in this case $(n-m)$ groups are removed at mth failure, and

Plan 3: $[(k,n=m), G_i = 0, \quad \forall i = 1,2,\ldots,m]$ this is the case of first failure censored sample.

TABLE 5.1: Several combinations of progressive censoring plans

(n,m)	CS	Plans	(n,m)	CS	Plans
(20,16)	1	(4,0*15)	(30,24)	5	(6,0*23)
	1	(1,0*4,1,0*4,1,0*4,1)		6	(2,0*11,2,0*10,2)
	3	(0*15,4)		7	(0*23,6)
(20,20)	4	(0*20)	(20,30)	8	(0*30)

The simplified notations are used for different combinations of censoring plans as shown in the Table 6.1. In addition, $t = 0.80$ (in time units) is taken as mission time to compute the reliability characteristics. The ML estimate of parameter and reliability characteristics are computed in the case of a non-Bayesian estimation process. The interval estimates of the associated model parameter θ are also computed using asymptotic and bootstrap (boot-p & boot-t) CIs, as well as their respective coverage probabilities.

Furthermore, employing an informative gamma prior, Bayes estimates of parameter and reliability characteristics are derived under SELF (Prior 1). Its related hyper-parameters (a,b) for Prior 1 are set so that the prior mean is $\theta = a/b$, i.e. $\theta = a/b$. Therefore, chosen $(a,b) = (3,2)$ and $(a,b) = (1.2,2.4)$ for $\theta = 1.5$ and 0.5, respectively. For non-informative prior (Prior 0), hyper-parameters are taken as $(a,b) \to (0,0)$. To obtain Bayes estimates, the TK approximation, IS, and M-H algorithms are utilized. $M = 10,000$ samples are generated for the IS and M-H algorithms, of which $M_0 = 2000$ is considered as the burn-in period. Also, obtained 95% HPD credible interval for the parameter θ, as well as the coverage probability.

The simulations are carried out with $N = 1000$ replications. Then, the AEs with corresponding MSEs of different estimates are computed. Suppose $\hat{\phi}_j$ is the estimate of ϕ for the jth sample, then $\mathrm{AE} = \frac{1}{N}\sum_{j=1}^{N}\hat{\phi}_j$, $\mathrm{MSE} = \frac{1}{N}\sum_{j=1}^{N}(\hat{\phi}_j - \phi)^2$. Also, the average lengths (AL) with corresponding coverage probabilities (CP) of 95% ACI, bootstrap (boot-p & boot-t) CI, and HPD credible intervals of parameter θ are computed. All the simulated results are summarizes in the following Tables 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, 5.9, 5.10, 5.11.

TABLE 5.2: AE and MSEs of ML and Bayes estimates of θ, when $\theta = 1.5$.

		MLE		TK				IS				MH			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	1	1.5627	0.0669	1.5705	0.0686	1.5603	0.0567	1.5604	0.0679	1.5533	0.0559	1.5703	0.0685	1.5599	0.0566
	2	1.5632	0.0642	1.5692	0.0655	1.5608	0.0534	1.5361	0.0708	1.5394	0.0557	1.5690	0.0655	1.5605	0.0534
	3	1.5690	0.0609	1.5739	0.0620	1.5499	0.0516	1.5069	0.0652	1.5053	0.0521	1.5738	0.0620	1.5499	0.0518
	4	1.5182	0.0471	1.5225	0.0475	1.5273	0.0457	1.5000	0.0516	1.5099	0.0465	1.5222	0.0475	1.5272	0.0457
	5	1.5450	0.0431	1.5500	0.0438	1.5359	0.0359	1.5022	0.0470	1.4964	0.0384	1.5501	0.0438	1.5357	0.0359
	6	1.5353	0.0384	1.5392	0.0388	1.5328	0.0346	1.4368	0.0463	1.4518	0.0405	1.5388	0.0388	1.5328	0.0345
	7	1.5292	0.0333	1.5323	0.0336	1.5445	0.0371	1.3721	0.0502	1.4054	0.0441	1.5322	0.0337	1.5444	0.0371
	8	1.5185	0.0322	1.5213	0.0324	1.5175	0.0317	1.4447	0.0368	1.4539	0.0361	1.5214	0.0324	1.5175	0.0317
5	1	1.5429	0.0462	1.5506	0.0473	1.5477	0.0422	1.4338	0.0542	1.4540	0.0456	1.5507	0.0475	1.5475	0.0422
	2	1.5369	0.0377	1.5434	0.0385	1.5471	0.0346	1.3512	0.0560	1.3999	0.0436	1.5431	0.0385	1.5475	0.0347
	3	1.5289	0.0347	1.5345	0.0353	1.5326	0.0327	1.2977	0.0730	1.3489	0.0562	1.5347	0.0353	1.5329	0.0328
	4	1.5157	0.0325	1.5206	0.0329	1.5160	0.0292	1.3527	0.0570	1.3787	0.0459	1.5205	0.0330	1.5160	0.0292
	5	1.5320	0.0291	1.5370	0.0296	1.5222	0.0250	1.2919	0.0691	1.3180	0.0572	1.5368	0.0296	1.5224	0.0250
	6	1.5268	0.0256	1.5311	0.0260	1.5277	0.0240	1.2088	0.1056	1.2498	0.0828	1.5309	0.0260	1.5280	0.0240
	7	1.5360	0.0254	1.5398	0.0258	1.5241	0.0223	1.1688	0.1292	1.1907	0.1122	1.5397	0.0258	1.5241	0.0223
	8	1.5070	0.0192	1.5102	0.0193	1.5101	0.0193	1.2180	0.0962	1.2455	0.0809	1.5101	0.0193	1.5100	0.0193

TABLE 5.3: AL and CPs of 95% ACI, Bootstrap CIs and HPD intervals of θ, when $\theta = 1.5$.

				Bootstrap				HPD			
		ACI		boot-p		boot-t		Prior 0		Prior 1	
k	CS	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
3	1	0.9357	0.948	1.0218	0.927	0.9531	0.949	1.0556	0.969	1.0127	0.979
	2	0.8930	0.940	0.9746	0.924	0.9108	0.938	1.0090	0.964	0.9726	0.975
	3	0.8750	0.941	0.9552	0.923	0.8937	0.945	0.9890	0.966	0.9431	0.970
	4	0.8414	0.952	0.8662	0.947	0.8428	0.948	0.9498	0.980	0.9238	0.968
	5	0.7600	0.942	0.8129	0.938	0.7733	0.947	0.8576	0.967	0.8300	0.979
	6	0.7192	0.946	0.7668	0.938	0.7316	0.948	0.8113	0.967	0.7912	0.968
	7	0.6961	0.952	0.7412	0.953	0.7075	0.957	0.7859	0.979	0.7762	0.965
	8	0.6873	0.944	0.6988	0.949	0.6872	0.945	0.7759	0.972	0.7580	0.965
5	1	0.7675	0.937	0.8199	0.925	0.7833	0.939	0.8693	0.964	0.8461	0.965
	2	0.7296	0.954	0.7766	0.940	0.7441	0.957	0.8266	0.977	0.8114	0.981
	3	0.7077	0.955	0.7523	0.940	0.7214	0.957	0.7995	0.976	0.7829	0.978
	4	0.6940	0.949	0.7067	0.954	0.6978	0.950	0.7849	0.974	0.7654	0.978
	5	0.6257	0.940	0.6569	0.927	0.6356	0.944	0.7071	0.967	0.6877	0.972
	6	0.5942	0.951	0.6228	0.942	0.6043	0.947	0.6704	0.970	0.6611	0.972
	7	0.5803	0.943	0.6076	0.936	0.5896	0.945	0.6554	0.960	0.6393	0.977
	8	0.5635	0.960	0.5699	0.967	0.5657	0.964	0.6358	0.979	0.6270	0.976

TABLE 5.4: AE and MSEs of ML and Bayes estimates of $R(t)$, when $t = 0.80$ and $R(t) = 0.7037$.

		MLE		TK				IS				MH			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	1	0.7127	0.0032	0.7094	0.0031	0.7082	0.0027	0.7076	0.0031	0.7070	0.0027	0.7093	0.0031	0.7082	0.0026
	2	0.7131	0.0031	0.7097	0.0030	0.7091	0.0025	0.7027	0.0035	0.7047	0.0027	0.7097	0.0030	0.7090	0.0025
	3	0.7148	0.0029	0.7114	0.0028	0.7068	0.0024	0.6969	0.0034	0.6974	0.0028	0.7114	0.0028	0.7068	0.0024
	4	0.7037	0.0025	0.7003	0.0024	0.7019	0.0024	0.6956	0.0028	0.6984	0.0025	0.7003	0.0024	0.7019	0.0024
	5	0.7105	0.0022	0.7082	0.0021	0.7057	0.0018	0.6979	0.0025	0.6971	0.0022	0.7083	0.0021	0.7056	0.0018
	6	0.7086	0.0019	0.7064	0.0019	0.7054	0.0017	0.6829	0.0030	0.6869	0.0025	0.7063	0.0019	0.7054	0.0017
	7	0.7076	0.0017	0.7054	0.0017	0.7081	0.0018	0.6671	0.0036	0.6757	0.0030	0.7053	0.0017	0.7081	0.0018
	8	0.7051	0.0017	0.7029	0.0017	0.7021	0.0017	0.6856	0.0023	0.6877	0.0023	0.7029	0.0017	0.7021	0.0017
5	1	0.7098	0.0023	0.7080	0.0023	0.7079	0.0021	0.6813	0.0035	0.6869	0.0028	0.7080	0.0023	0.7079	0.0021
	2	0.7090	0.0020	0.7073	0.0019	0.7087	0.0017	0.6614	0.0042	0.6744	0.0030	0.7072	0.0019	0.7088	0.0017
	3	0.7074	0.0018	0.7056	0.0018	0.7055	0.0017	0.6469	0.0057	0.6610	0.0042	0.7057	0.0018	0.7056	0.0017
	4	0.7044	0.0018	0.7026	0.0017	0.7019	0.0016	0.6617	0.0042	0.6691	0.0033	0.7025	0.0017	0.7019	0.0016
	5	0.7087	0.0015	0.7074	0.0015	0.7044	0.0013	0.6461	0.0054	0.6536	0.0043	0.7074	0.0015	0.7044	0.0013
	6	0.7077	0.0014	0.7066	0.0013	0.7060	0.0013	0.6221	0.0085	0.6345	0.0065	0.7065	0.0013	0.7060	0.0013
	7	0.7100	0.0013	0.7088	0.0013	0.7054	0.0012	0.6099	0.0107	0.6171	0.0090	0.7088	0.0013	0.7054	0.0012
	8	0.7035	0.0011	0.7023	0.0011	0.7023	0.0011	0.6254	0.0076	0.6337	0.0063	0.7023	0.0011	0.7023	0.0011

TABLE 5.5: AE and MSEs of ML and Bayes estimates of $h(t)$, when $t = 0.80$ and $h(t) = 0.4386$.

		MLE		TK				IS				MH			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	1	0.4279	0.0027	0.4287	0.0027	0.4302	0.0022	0.4304	0.0027	0.4314	0.0023	0.4286	0.0027	0.4302	0.0022
	2	0.4277	0.0026	0.4286	0.0026	0.4298	0.0021	0.4351	0.0029	0.4339	0.0023	0.4286	0.0026	0.4298	0.0021
	3	0.4263	0.0025	0.4274	0.0024	0.4319	0.0021	0.4407	0.0027	0.4407	0.0022	0.4273	0.0024	0.4319	0.0021
	4	0.4366	0.0020	0.4376	0.0020	0.4364	0.0019	0.4420	0.0022	0.4398	0.0020	0.4376	0.0020	0.4364	0.0019
	5	0.4307	0.0018	0.4313	0.0018	0.4339	0.0015	0.4408	0.0020	0.4418	0.0017	0.4312	0.0018	0.4339	0.0015
	6	0.4326	0.0016	0.4332	0.0016	0.4343	0.0014	0.4543	0.0022	0.4510	0.0019	0.4332	0.0016	0.4343	0.0014
	7	0.4337	0.0014	0.4344	0.0014	0.4319	0.0015	0.4682	0.0025	0.4609	0.0022	0.4344	0.0014	0.4319	0.0015
	8	0.4359	0.0014	0.4366	0.0014	0.4374	0.0014	0.4523	0.0017	0.4504	0.0017	0.4366	0.0014	0.4373	0.0014
5	1	0.4313	0.0019	0.4313	0.0019	0.4317	0.0017	0.4553	0.0025	0.4508	0.0021	0.4313	0.0019	0.4316	0.0017
	2	0.4323	0.0016	0.4324	0.0016	0.4313	0.0014	0.4728	0.0028	0.4620	0.0021	0.4324	0.0016	0.4312	0.0014
	3	0.4338	0.0015	0.4340	0.0015	0.4343	0.0014	0.4849	0.0038	0.4733	0.0028	0.4339	0.0015	0.4342	0.0014
	4	0.4366	0.0014	0.4369	0.0014	0.4376	0.0013	0.4725	0.0029	0.4666	0.0023	0.4368	0.0014	0.4376	0.0013
	5	0.4329	0.0013	0.4330	0.0012	0.4359	0.0011	0.4859	0.0036	0.4798	0.0029	0.4330	0.0012	0.4358	0.0011
	6	0.4339	0.0011	0.4340	0.0011	0.4346	0.0010	0.5051	0.0056	0.4954	0.0043	0.4340	0.0011	0.4345	0.0010
	7	0.4319	0.0011	0.4321	0.0011	0.4352	0.0010	0.5146	0.0069	0.5092	0.0059	0.4321	0.0011	0.4352	0.0010
	8	0.4379	0.0009	0.4381	0.0008	0.4381	0.0008	0.5027	0.0050	0.4962	0.0042	0.4381	0.0008	0.4381	0.0008

TABLE 5.6: AE and MSEs of ML and Bayes estimates of $MdTSF$, when $MdTSF = 1.7024$.

		MLE		TK				IS				MH			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	1	1.7923	0.1350	1.8042	0.1386	1.7895	0.1144	1.7897	0.1370	1.7795	0.1126	1.8039	0.1383	1.7890	0.1143
	2	1.7930	0.1295	1.8022	0.1322	1.7901	0.1079	1.7552	0.1427	1.7596	0.1123	1.8020	0.1323	1.7897	0.1079
	3	1.8011	0.1229	1.8088	0.1252	1.7746	0.1042	1.7137	0.1313	1.7112	0.1049	1.8086	0.1253	1.7747	0.1044
	4	1.7290	0.0948	1.7357	0.0957	1.7426	0.0920	1.7039	0.1038	1.7179	0.0936	1.7353	0.0956	1.7424	0.0919
	5	1.7668	0.0868	1.7745	0.0884	1.7543	0.0724	1.7066	0.0946	1.6983	0.0772	1.7746	0.0883	1.7541	0.0724
	6	1.7531	0.0773	1.7591	0.0783	1.7499	0.0696	1.6140	0.0926	1.6350	0.0811	1.7586	0.0783	1.7500	0.0696
	7	1.7443	0.0670	1.7493	0.0677	1.7665	0.0748	1.5222	0.1000	1.5693	0.0880	1.7491	0.0678	1.7664	0.0747
	8	1.7292	0.0648	1.7337	0.0652	1.7283	0.0638	1.6250	0.0736	1.6380	0.0722	1.7338	0.0652	1.7282	0.0638
5	1	1.7640	0.0930	1.7754	0.0955	1.7712	0.0851	1.6097	0.1084	1.6383	0.0914	1.7755	0.0959	1.7710	0.0851
	2	1.7553	0.0759	1.7650	0.0776	1.7702	0.0697	1.4928	0.1114	1.5615	0.0870	1.7647	0.0776	1.7707	0.0699
	3	1.7439	0.0698	1.7523	0.0711	1.7497	0.0659	1.4173	0.1450	1.4896	0.1118	1.7526	0.0712	1.7500	0.0660
	4	1.7252	0.0654	1.7326	0.0662	1.7261	0.0586	1.4950	0.1134	1.5316	0.0915	1.7325	0.0663	1.7260	0.0587
	5	1.7482	0.0586	1.7558	0.0597	1.7347	0.0502	1.4090	0.1373	1.4458	0.1137	1.7555	0.0596	1.7350	0.0503
	6	1.7409	0.0515	1.7472	0.0524	1.7424	0.0484	1.2921	0.2093	1.3497	0.1644	1.7470	0.0524	1.7428	0.0484
	7	1.7539	0.0511	1.7595	0.0519	1.7373	0.0450	1.2360	0.2559	1.2666	0.2224	1.7594	0.0520	1.7373	0.0449
	8	1.7126	0.0385	1.7175	0.0388	1.7173	0.0389	1.3049	0.1910	1.3435	0.1607	1.7174	0.0389	1.7172	0.0389

TABLE 5.7: AE and MSEs of ML and Bayes estimates of θ, when $\theta = 0.5$.

		MLE		TK				IS				MH			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	1	0.5211	0.0082	0.5237	0.0084	0.5211	0.0066	0.5201	0.0088	0.5183	0.0068	0.5236	0.0084	0.5211	0.0066
	2	0.5188	0.0070	0.5208	0.0071	0.5218	0.0061	0.5099	0.0076	0.5114	0.0063	0.5206	0.0071	0.5218	0.0061
	3	0.5172	0.0065	0.5188	0.0066	0.5205	0.0058	0.4975	0.0067	0.5019	0.0059	0.5188	0.0066	0.5206	0.0058
	4	0.5105	0.0058	0.5119	0.0059	0.5116	0.0052	0.5048	0.0061	0.5056	0.0056	0.5118	0.0059	0.5116	0.0052
	5	0.5161	0.0048	0.5178	0.0049	0.5192	0.0050	0.5014	0.0053	0.5062	0.0051	0.5177	0.0049	0.5192	0.0050
	6	0.5143	0.0044	0.5156	0.0045	0.5155	0.0042	0.4805	0.0050	0.4841	0.0045	0.5155	0.0045	0.5154	0.0042
	7	0.5110	0.0038	0.5120	0.0038	0.5122	0.0036	0.4572	0.0055	0.4628	0.0051	0.5120	0.0038	0.5121	0.0036
	8	0.5038	0.0034	0.5047	0.0034	0.5064	0.0035	0.4800	0.0042	0.4819	0.0037	0.5046	0.0034	0.5063	0.0035
5	1	0.5130	0.0050	0.5155	0.0051	0.5173	0.0051	0.4747	0.0060	0.4855	0.0054	0.5156	0.0052	0.5173	0.0051
	2	0.5163	0.0045	0.5185	0.0046	0.5137	0.0041	0.4574	0.0064	0.4587	0.0058	0.5187	0.0046	0.5136	0.0041
	3	0.5137	0.0042	0.5156	0.0043	0.5117	0.0041	0.4379	0.0079	0.4436	0.0070	0.5156	0.0043	0.5116	0.0041
	4	0.5088	0.0038	0.5105	0.0039	0.5073	0.0035	0.4527	0.0062	0.4577	0.0055	0.5104	0.0039	0.5073	0.0035
	5	0.5095	0.0031	0.5111	0.0031	0.5122	0.0031	0.4312	0.0077	0.4375	0.0066	0.5110	0.0031	0.5122	0.0031
	6	0.5112	0.0031	0.5126	0.0031	0.5087	0.0028	0.4057	0.0114	0.4093	0.0105	0.5125	0.0031	0.5088	0.0028
	7	0.5084	0.0025	0.5097	0.0026	0.5076	0.0025	0.3854	0.0151	0.3910	0.0138	0.5096	0.0026	0.5076	0.0025
	8	0.5011	0.0023	0.5022	0.0023	0.5035	0.0022	0.4046	0.0111	0.4098	0.0101	0.5022	0.0023	0.5036	0.0022

TABLE 5.8: AL and CPs of 95% ACI, Bootstrap CIs, and HPD intervals of θ, when $\theta = 0.5$.

		Bootstrap						HPD			
		ACI		boot-p		boot-t		Prior 0		Prior 1	
k	CS	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
3	1	0.3120	0.931	0.3410	0.914	0.3179	0.931	0.3506	0.952	0.3433	0.969
	2	0.2963	0.943	0.3236	0.928	0.3024	0.945	0.3331	0.961	0.3282	0.976
	3	0.2884	0.932	0.3143	0.924	0.2940	0.935	0.3245	0.967	0.3206	0.973
	4	0.2830	0.943	0.2910	0.939	0.2837	0.944	0.3180	0.973	0.3129	0.972
	5	0.2540	0.946	0.2714	0.933	0.2585	0.947	0.2850	0.965	0.2826	0.960
	6	0.2410	0.940	0.2569	0.931	0.2451	0.946	0.2702	0.955	0.2673	0.974
	7	0.2326	0.954	0.2481	0.945	0.2368	0.962	0.2608	0.974	0.2582	0.977
	8	0.2280	0.945	0.2320	0.943	0.2280	0.945	0.2554	0.963	0.2542	0.976
5	1	0.2551	0.943	0.2727	0.927	0.2607	0.944	0.2871	0.962	0.2845	0.960
	2	0.2451	0.951	0.2605	0.939	0.2497	0.954	0.2759	0.970	0.2700	0.973
	3	0.2378	0.948	0.2529	0.934	0.2423	0.950	0.2673	0.967	0.2624	0.962
	4	0.2331	0.942	0.2364	0.938	0.2336	0.939	0.2621	0.969	0.2574	0.975
	5	0.2081	0.947	0.2184	0.944	0.2115	0.951	0.2338	0.967	0.2316	0.967
	6	0.1989	0.944	0.2081	0.936	0.2018	0.944	0.2227	0.967	0.2194	0.963
	7	0.1921	0.956	0.2008	0.951	0.1948	0.961	0.2158	0.969	0.2130	0.971
	8	0.1874	0.953	0.1893	0.953	0.1879	0.950	0.2097	0.969	0.2089	0.968

TABLE 5.9: AE and MSEs of ML and Bayes estimates of $R(t)$, when $t = 0.80$ and $R(t) = 0.3333$.

		MLE		TK				IS				MH			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	1	0.3430	0.0022	0.3430	0.0022	0.3420	0.0017	0.3412	0.0023	0.3406	0.0018	0.3429	0.0022	0.3420	0.0017
	2	0.3420	0.0019	0.3418	0.0019	0.3426	0.0016	0.3362	0.0020	0.3373	0.0017	0.3417	0.0019	0.3426	0.0016
	3	0.3412	0.0018	0.3409	0.0018	0.3420	0.0015	0.3299	0.0019	0.3325	0.0016	0.3409	0.0018	0.3421	0.0016
	4	0.3378	0.0016	0.3374	0.0016	0.3374	0.0014	0.3338	0.0017	0.3343	0.0016	0.3373	0.0016	0.3374	0.0014
	5	0.3410	0.0013	0.3410	0.0013	0.3418	0.0013	0.3324	0.0015	0.3350	0.0014	0.3410	0.0013	0.3418	0.0013
	6	0.3401	0.0012	0.3400	0.0012	0.3400	0.0011	0.3214	0.0015	0.3235	0.0013	0.3400	0.0012	0.3399	0.0011
	7	0.3384	0.0011	0.3382	0.0011	0.3384	0.0010	0.3088	0.0017	0.3119	0.0016	0.3382	0.0011	0.3384	0.0010
	8	0.3346	0.0010	0.3344	0.0010	0.3353	0.0010	0.3213	0.0013	0.3224	0.0011	0.3343	0.0010	0.3353	0.0010
5	1	0.3393	0.0014	0.3397	0.0014	0.3407	0.0014	0.3181	0.0018	0.3240	0.0016	0.3398	0.0014	0.3407	0.0014
	2	0.3412	0.0012	0.3415	0.0012	0.3390	0.0011	0.3087	0.0020	0.3096	0.0018	0.3416	0.0012	0.3390	0.0011
	3	0.3399	0.0012	0.3401	0.0012	0.3380	0.0011	0.2979	0.0025	0.3012	0.0022	0.3400	0.0012	0.3380	0.0011
	4	0.3373	0.0011	0.3374	0.0011	0.3358	0.0010	0.3062	0.0020	0.3091	0.0017	0.3374	0.0011	0.3357	0.0010
	5	0.3378	0.0009	0.3381	0.0009	0.3387	0.0008	0.2943	0.0025	0.2980	0.0021	0.3380	0.0009	0.3387	0.0008
	6	0.3387	0.0009	0.3389	0.0009	0.3369	0.0008	0.2798	0.0037	0.2819	0.0034	0.3389	0.0009	0.3369	0.0008
	7	0.3373	0.0007	0.3375	0.0007	0.3364	0.0007	0.2679	0.0049	0.2712	0.0045	0.3374	0.0007	0.3364	0.0007
	8	0.3335	0.0006	0.3335	0.0006	0.3343	0.0006	0.2792	0.0036	0.2822	0.0032	0.3335	0.0006	0.3343	0.0006

TABLE 5.10: AE and MSEs of ML and Bayes estimates of $h(t)$, when $t = 0.80$ and $h(t) = 0.6944$.

		MLE		TK				IS				MH			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	1	0.6885	0.0007	0.6881	0.0007	0.6887	0.0006	0.6891	0.0008	0.6895	0.0006	0.6880	0.0007	0.6887	0.0006
	2	0.6891	0.0006	0.6888	0.0006	0.6885	0.0005	0.6920	0.0007	0.6915	0.0006	0.6888	0.0006	0.6884	0.0005
	3	0.6896	0.0006	0.6894	0.0006	0.6888	0.0005	0.6956	0.0006	0.6943	0.0005	0.6894	0.0006	0.6888	0.0005
	4	0.6916	0.0005	0.6914	0.0005	0.6915	0.0005	0.6935	0.0005	0.6932	0.0005	0.6914	0.0005	0.6914	0.0005
	5	0.6898	0.0004	0.6895	0.0004	0.6891	0.0004	0.6944	0.0005	0.6929	0.0005	0.6895	0.0004	0.6891	0.0004
	6	0.6903	0.0004	0.6901	0.0004	0.6902	0.0004	0.7006	0.0005	0.6995	0.0004	0.6902	0.0004	0.6902	0.0004
	7	0.6913	0.0003	0.6912	0.0003	0.6911	0.0003	0.7076	0.0005	0.7059	0.0005	0.6912	0.0003	0.6911	0.0003
	8	0.6935	0.0003	0.6934	0.0003	0.6928	0.0003	0.7007	0.0004	0.7001	0.0003	0.6934	0.0003	0.6929	0.0003
5	1	0.6908	0.0004	0.6902	0.0005	0.6897	0.0004	0.7024	0.0005	0.6991	0.0005	0.6902	0.0005	0.6897	0.0004
	2	0.6897	0.0004	0.6893	0.0004	0.6907	0.0004	0.7076	0.0006	0.7072	0.0005	0.6892	0.0004	0.6907	0.0004
	3	0.6905	0.0004	0.6901	0.0004	0.6913	0.0004	0.7135	0.0007	0.7117	0.0006	0.6901	0.0004	0.6913	0.0004
	4	0.6920	0.0003	0.6917	0.0003	0.6926	0.0003	0.7090	0.0006	0.7075	0.0005	0.6917	0.0003	0.6926	0.0003
	5	0.6917	0.0003	0.6914	0.0003	0.6910	0.0003	0.7155	0.0007	0.7135	0.0006	0.6914	0.0003	0.6910	0.0003
	6	0.6912	0.0003	0.6909	0.0003	0.6921	0.0002	0.7233	0.0011	0.7222	0.0010	0.6909	0.0003	0.6921	0.0002
	7	0.6920	0.0002	0.6918	0.0002	0.6924	0.0002	0.7296	0.0014	0.7278	0.0013	0.6918	0.0002	0.6924	0.0002
	8	0.6942	0.0002	0.6940	0.0002	0.6936	0.0002	0.7236	0.0010	0.7220	0.0009	0.6940	0.0002	0.6936	0.0002

TABLE 5.11: AE and MSEs of ML and Bayes estimates of $MdTSF$, when $MdTSF = 0.3333$.

k	CS	MLE		TK				IS				MH			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
		AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	1	0.3621	0.0130	0.3675	0.0135	0.3636	0.0105	0.3630	0.0140	0.3602	0.0107	0.3674	0.0135	0.3637	0.0105
	2	0.3588	0.0111	0.3633	0.0114	0.3642	0.0097	0.3496	0.0119	0.3511	0.0098	0.3631	0.0114	0.3642	0.0097
	3	0.3568	0.0102	0.3607	0.0105	0.3624	0.0093	0.3340	0.0103	0.3390	0.0091	0.3607	0.0105	0.3625	0.0093
	4	0.3483	0.0091	0.3519	0.0092	0.3513	0.0081	0.3430	0.0095	0.3437	0.0087	0.3519	0.0093	0.3513	0.0082
	5	0.3548	0.0076	0.3584	0.0078	0.3602	0.0078	0.3378	0.0082	0.3438	0.0079	0.3584	0.0078	0.3602	0.0079
	6	0.3525	0.0069	0.3555	0.0071	0.3552	0.0067	0.3118	0.0075	0.3162	0.0067	0.3554	0.0071	0.3550	0.0067
	7	0.3482	0.0059	0.3508	0.0060	0.3508	0.0056	0.2834	0.0078	0.2900	0.0072	0.3507	0.0060	0.3508	0.0056
	8	0.3393	0.0052	0.3417	0.0053	0.3438	0.0054	0.3111	0.0061	0.3132	0.0055	0.3415	0.0053	0.3437	0.0054
5	1	0.3511	0.0079	0.3558	0.0081	0.3579	0.0080	0.3052	0.0087	0.3182	0.0081	0.3559	0.0081	0.3579	0.0080
	2	0.3550	0.0071	0.3591	0.0074	0.3530	0.0064	0.2840	0.0091	0.2855	0.0083	0.3593	0.0074	0.3528	0.0064
	3	0.3517	0.0065	0.3554	0.0067	0.3505	0.0064	0.2608	0.0109	0.2674	0.0097	0.3553	0.0067	0.3504	0.0064
	4	0.3456	0.0059	0.3489	0.0060	0.3449	0.0055	0.2782	0.0087	0.2840	0.0077	0.3488	0.0060	0.3448	0.0055
	5	0.3461	0.0048	0.3492	0.0049	0.3505	0.0048	0.2524	0.0107	0.2596	0.0093	0.3491	0.0049	0.3505	0.0048
	6	0.3482	0.0047	0.3509	0.0049	0.3460	0.0043	0.2229	0.0155	0.2268	0.0143	0.3508	0.0049	0.3461	0.0043
	7	0.3447	0.0039	0.3471	0.0040	0.3444	0.0038	0.1998	0.0202	0.2061	0.0187	0.3470	0.0040	0.3445	0.0038
	8	0.3356	0.0034	0.3378	0.0035	0.3394	0.0034	0.2213	0.0151	0.2271	0.0138	0.3377	0.0035	0.3395	0.0034

From these findings, the following interpretations are drawn: the MSEs of ML and Bayes estimates of parameter and reliability characteristics decrease as n increases in almost all cases. Also, it is seen that Bayes estimates have smaller MSEs than ML estimates in almost all cases. Also, the Bayes estimates using Prior 1 performed quite better than Prior 0, as it includes prior information. It is also observed that the MSEs are decreasing with an increasing number of individuals within each group. The ALs of asymptotic, bootstrap (boot-p, boot-t) and HPD narrow down with an increase in n in almost all cases. In the case of HPD, ALs are more narrow as compared to the asymptotic and bootstrap confidence intervals. In almost all cases, the CPs of ML and Bayes estimates of θ achieve the desired confidence coefficient.

5.6 Real Data Analysis

In this section, we analyzed a real data set as an example to illustrate the situation of life testing experiments for IP lifetime model with PFFC data. Here, we take head and neck cancer data from Efron (1988). These data are survival times (in days) treated with combined radiotherapy and chemotherapy of 45 patients suffering from head and neck cancer disease and given as follows:

12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46,58.36, 63.47, 68.46,78.26,74.47,81,43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209,249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776.

Recently, Sharma et al. (2015) and Sharma (2018) studied and fitted head neck cancer data to the inverse Lindley (IL) and generalized inverse Lindley (GIL) lifetime models, respectively. To begin, we assess the failure rate function of the data set using the scaled total time on test (TTT) transform. The scaled TTT is calculated as follows:

$$\psi(r/n) = \left[\sum_{j=1}^{r} t_{(i)} + (n-r)t_r\right] \Bigg/ \left(\sum_{j=1}^{r} t_{(i)}\right),$$

where, $t_{(i)}, \quad i = 1, 2, \ldots, n$ represent the ith order statistic and $r = 1, 2, \ldots, n$. If the plot $(r/n, \psi(r/n))$ is convex (concave), the failure rate function has a decreasing (increasing) shape. If it start concave and then become convex (begins convex and then becomes concave), the failure rate function is upside down bathtub shaped (bathtub shaped), respectively, for more details about scale TTT, see, Mudholkar et al. (1996). The scaled TTT plot of head-neck cancer data set is given in Figure 6.1. This Figure suggests that the head-neck cancer data set follow upside down bathtub shaped failure rate function. This empirical behaviour of failure rate function is quite similar to the considered IPD model. Further, check whether the considered real data

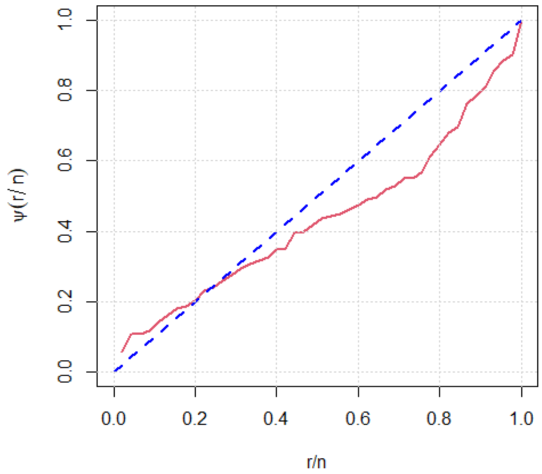


FIGURE 5.2: TTT plot for plots for head-neck cancer disease data.

set is good-fit to the IP lifetime model or not using some goodness-of-fit tests. Kolmogrov-Smirnov (KS) and Anderson-Darling (AD) goodness-of-fit tests were employed in this study, and test statistics and p-values were obtained. The ML estimates of associated parameters are also used to generate two information theoretic criteria based on the log-likelihood function, namely AIC and BIC. To assess the goodness-of-fit, test statistics and p-values are used. Then, based on the considered real data set, compare the fitting of the IP lifetime model with the IL and GIL lifetime models. The best lifetime model has the lowest AIC, BIC, $-\ln L$, KS, and AD test statistics and the greatest p-value for the KS and AD tests. The fittings of IP lifetime model and competitive models are reported in table 5.12. From Table 5.12, it is noticed that IP, IL and

TABLE 5.12: Summery of fitted models for head-neck cancer disease data.

					AD Test		KS Test	
Models	MLE	AIC	BIC	-ln$L$	statistic	p-value	statistic	p-value
IP	$\hat{\theta} = 76.4848$	570.9288	572.7354	284.4644	0.4095	0.8386	0.0783	0.9255
IL	$\hat{\theta} = 76.3539$	571.0640	572.8707	284.5320	0.4190	0.8290	0.0818	0.9002
GIL	$\hat{\alpha} = 1.0248$	573.0149	576.6282	284.5074	0.4153	0.8326	0.0890	0.8376
	$\hat{\beta} = 83.9189$							

GIL all models are good-fitted to the consider data set. Among all the fitted models IP lifetime model outperform as it has lower AIC, BIC, $-\ln L$, KS and AD statistics with high p-value. So choice of IP lifetime model is quite reasonable for this data set.

Furthermore, the first failure censored sample was collected by randomly arranging the considered complete data-set into $n = 15$ groups with $k = 3$ sample points inside each group. The observation with '+' sings are first failure observations in the respective groups as shown in

TABLE 5.13: First failure censored head neck cancer disease data.

Group / Items	1	2	3	4	5	6	7	8
(i)	74.47	155.00	63.47+	23.56+	173.00	47.38	12.20+	195.00
(ii)	43.00+	130.00+	194.00	119.00	58.36+	41.35	68.46	725.00
(iii)	140.00	159.00	519.00	432.00	84.00	37.00+	110.00	23.74+
Group / Items	**9**	**10**	**11**	**12**	**13**	**14**	**15**	
(i)	55.46+	339.00	133.00	209.00	94.00+	633.00	469.00	
(ii)	1776.00	817.00	281.00	112.00+	319.00	146.00	92.00	
(iii)	78.26	179.00+	25.87+	127.00	249.00	31.98+	81.00+	

Table 5.13. Consequently, the ordered first failure censored sample is given by

$$12.20, 23.56, 23.74, 25.87, 31.98, 37, 43, 55.46, 58.36, 63.47, 81, 94, 112, 130, 179.$$

Now, applying four different progressive censoring plans on the above first failure censored sample with prefixed number failure $m = 10$. The four different censoring plans and their corresponding PFFC samples are as follows:

Scheme 1: $k = 3, n = 15, m = 10, \underset{\sim}{G} = (5, 0*9)$,
$\underset{\sim}{x}$ = 12.20, 43, 55.46, 58.36, 63.47, 81, 94, 112, 130, 179.

Scheme 2: $k = 3, n = 15, m = 10, \underset{\sim}{G} = (1, 0*2, 1, 0*2, 2, 0*2, 1)$,
$\underset{\sim}{x}$ = 12.20, 23.74, 25.87, 31.98, 43, 55.46, 58.36, 94, 112, 130.

Scheme 3: $k = 3, n = 15, m = 10, \underset{\sim}{G} = (0*9, 5)$,
$\underset{\sim}{x}$ = 12.20, 23.56, 23.74, 25.87, 31.98, 37, 43, 55.46, 58.36, 63.47.

Scheme 4: $k = 3, n = m = 15, \underset{\sim}{G} = (0*15)$,
$\underset{\sim}{x}$ = 12.20, 23.56, 23.74, 25.87, 31.98, 37, 43, 55.46, 58.36, 63.47, 81, 94, 112, 130, 179.

The ML and Bayes estimators of parameter and reliability characteristics under consideration of different censoring plans are obtained and reported in Table 5.14. The reliability characteristics $R(t)$ and $h(t)$ are computed at mission time t as median of the considered data. The Bayes estimates of parameter and reliability characteristics are obtained using non-informative prior as information about considered data are unavailable. For M-H algorithm and importance sampling, $M = 10,000$ Markov chains are generated and $M_0 = 2500$ are taken as burn-in-period. The 95% ACI, boot-p, boot-t CIs and HPD credible intervals are computed and tabulated in Table 5.15. For bootstrap confidence intervals each PFFC samples are replicated by $B = 1000$

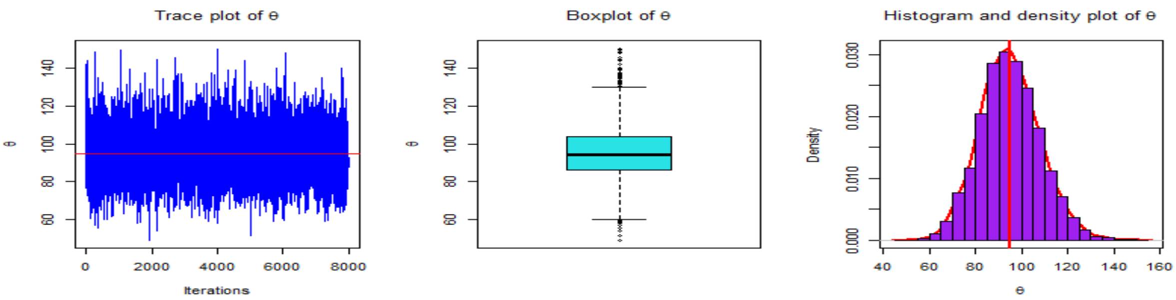
times. Also, Figure 6.3 shows the diagnostic plots of Markov chains for all censoring schemes under consideration of real data set, which verifies the convergence of stationary distributions for generation of Markov chain from posterior. The trace plot shows a random scatter about the mean and shows fine mixture of the parameter chains. The boxplots and histograms of generated samples shows the posterior distribution are almost symmetric i.e. posterior mean can be the best estimate in almost all censoring schemes under consideration of real data set.

TABLE 5.14: ML and Bayes estimates of parameter and reliability characteristics under consideration of head-neck cancer disease data for $k = 3, n = 15, m = 10$.

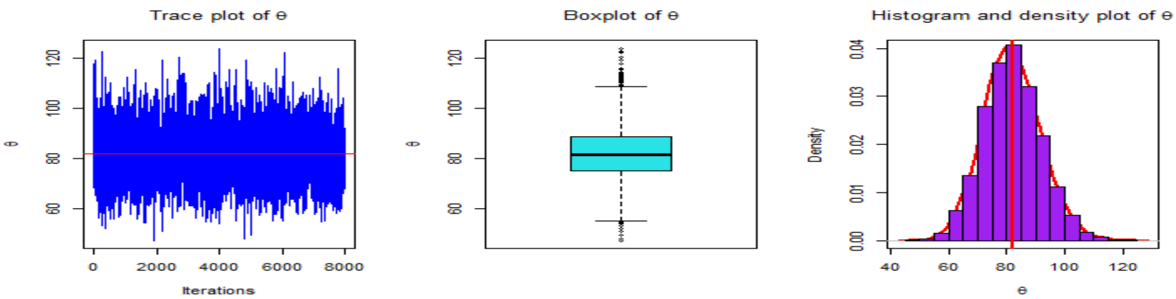
Schemes Parameters	Scheme 1	Scheme 2	Scheme 3	Scheme 4
$\hat{\theta}$	90.9509	83.8542	74.7172	74.9301
$\hat{\theta}_{\text{TK}}$	91.3190	84.2256	75.0075	75.1549
$\hat{\theta}_{\text{IS}}$	87.4737	81.3659	72.6638	72.9190
$\hat{\theta}_{\text{MH}}$	91.5311	83.3144	73.0104	94.0702
$\hat{R}(t)$	0.7174	0.8404	0.8479	0.7909
$\hat{R}_{\text{TK}}(t)$	0.7186	0.8339	0.8417	0.7851
$\hat{R}_{\text{IS}}(t)$	0.7112	0.8325	0.8367	0.8421
$\hat{R}_{\text{MH}}(t)$	0.7034	0.8315	0.8398	0.7819
$\hat{h}(t)$	0.0069	0.0076	0.0085	0.0086
$\hat{h}_{\text{TK}}(t)$	0.0070	0.0077	0.0086	0.0087
$\hat{h}_{\text{IS}}(t)$	0.0069	0.0078	0.0088	0.0070
$\widehat{h}_{\text{MH}}(t)$	0.0071	0.0079	0.0088	0.0089
$\widehat{MdTSF}$	130.7150	120.4767	107.2949	107.6020
$\widehat{MdTSF}_{\text{TK}}$	131.2270	121.0074	107.7120	107.9234
$\widehat{MdTSF}_{\text{IS}}$	131.5521	119.6980	104.8326	135.2153
$\widehat{MdTSF}_{\text{MH}}$	125.6985	116.8870	104.3325	104.7007

TABLE 5.15: The 95% asymptotic, boot-p, boot-t confidence and HPD credible intervals of parameter θ under consideration of head-neck cancer disease data.

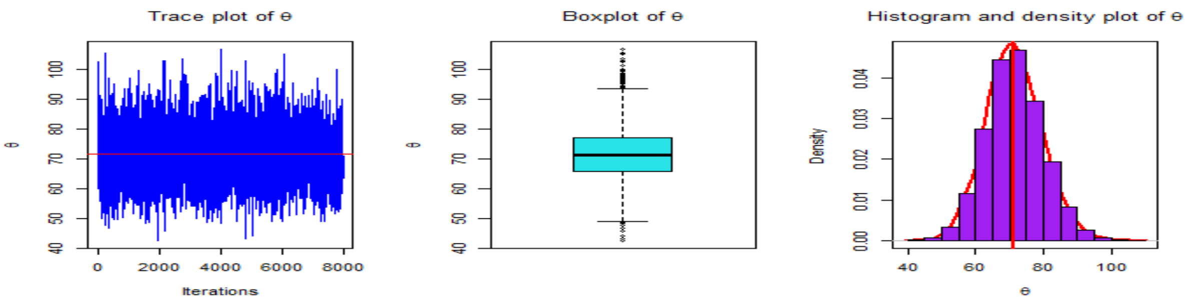
Schemes Parameters	Scheme 1	Scheme 2	Scheme 3	Scheme 4
$\hat{\theta}_{\text{ACI}}$	(56.14, 125.75)	(55.56, 112.14)	(50.49, 98.93)	(50.94, 98.91)
$\hat{\theta}_{\text{boot-p}}$	(66.13, 162.29)	(54.61, 129.40)	(45.54, 91.92)	(52.83, 111.10)
$\hat{\theta}_{\text{boot-t}}$	(44.66, 127.29)	(53.94, 129.27)	(60.82, 119.90)	(50.61, 106.11)
$\hat{\theta}_{\text{HPD}}$	(45.02, 78.92)	(78.92, 83.85)	(70.21, 75.15)	(70.46, 75.35)



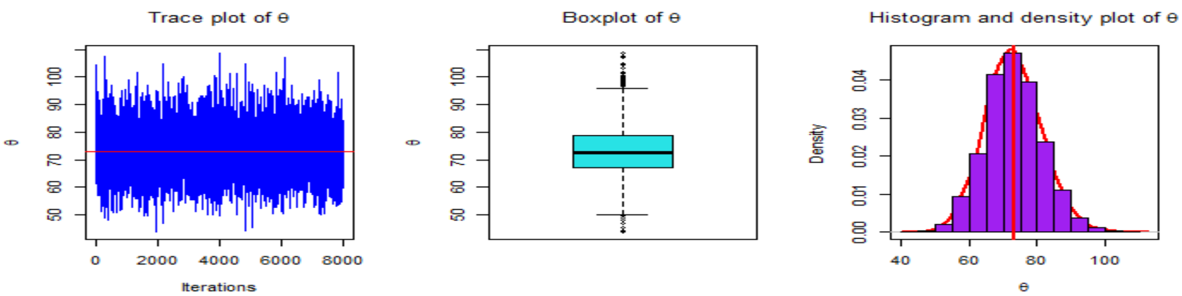
(A) Scheme 1.



(B) Scheme 2.



(C) Scheme 3.



(D) Scheme 4.

FIGURE 5.3: MCMC diagnostic plots for different censoring schemes under consideration of head-neck cancer disease data.

5.7 Concluding Remarks

In this chapter, some inference procedures about the parameter and reliability characteristics for IP lifetime under PFFC data were developed. The ML and Bayes estimates of unknown parameter and reliability characteristics were computed. For Bayesian estimation, TK approximation, importance sampling, and the M-H algorithm using non-informative and gamma informative priors under SELF were considered. Based on the asymptotic normality of ML estimates and bootstrap methods, the 95% asymptotic, boot-p, and boot-t CIs of the parameter were constructed. Also, the HPD credible interval of a parameter based on MCMC samples was computed. An extensive numerical computation was performed to determine the potentiality of different estimators developed in this chapter. A real data set was studied to determine the feasibility of the considered IP lifetime model. From the simulated results, it is observed that Bayesian estimation using MCMC followed by the M-H algorithm outperforms. Therefore, we recommend the use of this consider methodology for all practical purpose in classical as well as Bayesian point of views.

Chapter 6

Statistical Inference of Shannon's Entropy from Maxwell Lifetime Model using Progressively First Failure Censored Data

6.1 Introduction

In this chapter, we consider a problem from information theory and proposed to developed statistical inference of Shannon's entropy for the Maxwell lifetime model based on PFFC data. The construction and application of the PFFCS have already been explored in depth in Chapter 5. Information theory provides a simple approach for measuring the uncertainty and reciprocal information of random variables as entropy measures. The applications of entropy are described in a variety of fields, including computer science, molecular biology, hydrology, meteorology, and others. For example, in the study of trends in gene sequences, molecular biologists use the principle of Shannon's entropy. For more details, one may refer Cover (1999), which contains an excellent monograph on the information theory and implications of the concept of entropy in various disciplines. Shannon's entropy is the most widely used entropy in statistical and information theory, and it was introduced by Shannon (1948). Let X be a random variable with pdf $f(.)$, the Shannon's entropy of X is expressed as follow

$$H(f) = E\left[-\ln\ f(X)\right] = -\int_{-\infty}^{\infty} f(x) \ln\ \left(f(x)\right)\ dx. \tag{6.1}$$

Recently, some attempts have been made by many scholars in parametric statistical inferences to measure the entropy for different lifetime models based on complete as well as censored data. For example, entropy for several shifted exponential populations is studied by Kayal and

Kumar (2013), Cho et al. (2014) discussed entropy for Rayleigh lifetime model under doubly generalized Type-II hybrid censoring scheme, based on generalized progressive hybrid censoring Liu and Gui (2019) estimated entropy, Du et al. (2018) established a statistical inference of information entropy for the log-logistic lifetime model based on progressively Type-I interval censored data, Yu et al. (2019) studied Shannon's entropy for the inverse Weibull lifetime model using PFFC data, Rajesh and Sunoj (2021) discussed Shannon's entropy based on length-bias and Type-I censoring, Hassan and Zaky (2021) developed Bayes estimate of entropy for the Lomax lifetime model based on record data, Shakhatreh et al. (2021) discussed differential entropy for the Weibull lifetime model in the case of objective Bayesian, and references were cited therein. The main objective of this chapter is to develop classical and Bayesian inferences for the associated parameter and Shannon's entropy of the Maxwell lifetime model using PFFC data. The Maxwell-Boltzmann distribution was first proposed by James Clerk Maxwell and Ludwig Boltzmann in late 1800 as a distribution of velocities in a gas at a given temperature, for more details, see, Bekker and Roux (2005). The Maxwell-Boltzmann distribution, popularly known as the Maxwell (MW) lifetime model, is widely used in the fields of chemistry and physics for a variety of purposes. Many basic properties of gases, such as pressure and diffusion, are explained by the MW lifetime model. The MW lifetime model has recently gained popularity as a well-known lifetime model in the literature. This lifetime model has been widely investigated by various researchers for modelling several lifetime data scenarios. For example, Krishna and Malik (2009) studied the MW lifetime model under Type-II censored data, Krishna and Malik (2012) discussed the MW lifetime model under progressive censoring, Krishna et al. (2015) discussed the MW lifetime model under randomly censored data, Tomer and Panwar (2015) established estimation procedures for the MW lifetime model using Type-I progressive hybrid censoring, and Bayesian analysis for MW lifetime model is discussed by Panwar and Tomer (2019), etc. The remainder of this chapter is as follows: Section 6.2 deals with the model description. Classical estimation methods such as ML, ACIs, and bootstrap CIs methods are developed in Section 6.3. Section 6.4, devoted to Bayesian estimation methods using TK approximation and MCMC techniques. Extensive numerical simulations are done in Section 6.5 to demonstrate the influence of numerous estimators created in this chapter. The application of the considered methodology is examined by a real data analysis in Section 6.6. Finally, in Section 6.7, there are some concluding remarks.

6.2 The Model

Let X be random variable following MW lifetime model with parameter λ i.e. $X \sim \mathrm{MW}(\lambda)$, the pdf, cdf and failure rate (or hazard) function, respectively, are given by

$$f(x;\lambda) = \frac{4}{\sqrt{\pi}}\frac{1}{\lambda^{3/2}}x^2 e^{-\frac{x^2}{\lambda}}; \quad 0 < x < \infty, \lambda > 0, \tag{6.2}$$

$$F(x;\lambda) = \Gamma\left(\frac{x^2}{\lambda}, \frac{3}{2}\right); \quad 0 \le x < \infty, \lambda > 0, \tag{6.3}$$

$$\text{and,} \quad h(x) = \frac{4}{\sqrt{\pi}\lambda^{3/2}}\frac{x^2 e^{-x^2\lambda}}{[1-\Gamma(x^2/\lambda, 3/2)]}; \quad x > 0,\ \lambda > 0. \tag{6.4}$$

where, $\Gamma(t,b) = \frac{1}{\Gamma(b)}\int\limits_0^\infty e^{-x}x^{b-1}dx$ is the incomplete gamma ratio. The failure rate of the MW is increasing, see Krishna and Malik (2012). Now using equations (6.2) and (6.1), the Shannon's entropy is given by

$$\begin{aligned}
H(f) = E\left[-\ln f(x)\right] &= -\int\limits_0^\infty f(x)\ln f(x)dx \\
&= -\int\limits_0^\infty f(x)\left\{\ln 4 - \frac{1}{2}\ln\pi - \frac{3}{2}\ln\lambda + 2\ln x - \frac{x^2}{\lambda}\right\}dx \\
&= -A\int\limits_0^\infty f(x)dx - 2\int\limits_0^\infty \ln x\, f(x)dx + \frac{1}{\lambda}\int\limits_0^\infty x^2 f(x)dx \\
&= -A - \frac{8}{\sqrt{\pi}}\frac{1}{\lambda^{3/2}}\int\limits_0^\infty x^2\ln x e^{-x^2/\lambda}dx + \frac{4}{\sqrt{\pi}}\frac{1}{\lambda^{5/2}}\int\limits_0^\infty x^4 e^{-x^2/\lambda}dx
\end{aligned}$$

$$H(f) = \frac{1}{2}\ln\lambda + \gamma + \frac{1}{2}\ln\pi - \frac{1}{2} \simeq H(\lambda) \quad \text{(say)}, \tag{6.5}$$

where, $A = \ln 4 - \frac{1}{2}\ln\pi - \frac{3}{2}\ln\lambda$, $\int\limits_0^\infty f(x)dx = 1$, and γ is a Euler–Mascheroni constant.

6.3 Classical Estimation

In this part, we use the expectation-maximization (EM) approach to create ML estimates of the related parameter λ and entropy $H(\lambda)$. Based on ML estimates we constructed ACIs of λ and

$H(\lambda)$. Also, we construct the bootstrap CIs for λ and $H(\lambda)$.

6.3.1 Maximum Likelihood Estimation

Let $x_{j:m:n:k};\quad j=1,2,\ldots m$ be the PFFC sample drown from MW(λ), with presumed censoring plans $\underset{\sim}{G}$ and the number of groups n, each group having k individuals with effective sample size m. Then, the likelihood function using equations (6.2), (6.3) and (5.1) is given by

$$L(\underset{\sim}{x};\lambda) = Ak^m\left(\frac{4}{\sqrt{\pi}}\right)^m \lambda^{-\frac{3m}{2}} \exp\left\{-\frac{1}{\lambda}\sum_{j=1}^{m} x_j^2\right\}\prod_{j=1}^{m} x_j^2\left[1-\Gamma\left(\frac{x_j^2}{\lambda},\frac{3}{2}\right)\right]^{k(G_j+1)-1}, \tag{6.6}$$

where, $A = n(n-G_1-1)(n-G_1-G_2-2)\ldots(n-G_1-G_2-\cdots-G_{m-1}-m+1)$.

To begin, assume that the observed and censored data are represented by $\underset{\sim}{X} = (x_{1:m:n:k}, x_{2:m:n:k},\ldots,x_{m:m:n:k})$, and $\underset{\sim}{Z} = \left(z_{11},\ldots,z_{1[k(G_1+1)-1]},\ldots,z_{m1},\ldots,z_{m[k(G_m+1)-1]}\right)$, respectively. The combined forms of complete sample is given by $\underset{\sim}{Y} = (\underset{\sim}{X},\underset{\sim}{Z})$. After ignoring the additive constant, we have log-likelihood function

$$L_c(\underset{\sim}{Y};\lambda) = -\frac{3nk}{2}\ln\lambda - \frac{1}{\lambda}\sum_{i=1}^{m} x_i^2 - \frac{1}{\lambda}\sum_{j=1}^{k(G_i+1)-1} Z_{ij}^2. \tag{6.7}$$

We must compute the pseudo log-likelihood function for the E-step. It can be calculated from complete sample by replacing any Z_{ij} function, such as $\eta(Z_{ij})$, with $E\left[\eta(Z_{ij}|Z_{ij}>x_i)\right]$. As a result, the pseudo log-likelihood function is given as follows

$$L_c(\underset{\sim}{Y};\lambda) = -\frac{3nk}{2}\ln\,\lambda - \frac{1}{\lambda}\sum_{i=1}^{m} x_i^2 - \frac{1}{\lambda}\sum_{j=1}^{k(G_i+1)-1} E\left[Z_{ik}^2|Z_{ij>w_i}\right]. \tag{6.8}$$

For given $X_i = x$, the conditional distribution of Z_{ij}, follows a truncated MW lifetime model with left truncation at x_i. That is,

$$f(Z_{ij}|x_i,\lambda) = \frac{f(Z_{ij},\lambda)}{1-F(x_i,\lambda)};\quad Z_{ij}>x_i\;\; i=\;1,2,\;\ldots\;,m\;\;\text{and}\;\; j=\;1,2,\;\ldots\;,[k(G_i+1)-1]. \tag{6.9}$$

The conditional expectation in (6.9) can be defined in the following way:

$$A(c,\lambda)\;=\;E\left[\,Z_{ij}^2|Z_{ij}>c\right] = \frac{3\lambda}{2[1-F(x_i,\lambda)]}\left[1-\Gamma\left(\frac{c^2}{\lambda},\frac{3}{2}\right)\right]. \tag{6.10}$$

The M-step now implies trying to maximize of the pseudo log-likelihood function, with the appropriate value of (6.8) being replaced. If the rth stage estimate of λ is $\lambda^{(r)}$, the $(r+1)^{th}$ stage estimate $\lambda^{(r+1)}$ can be estimated by maximizing the following equation

$$L_c^*(\underset{\sim}{W};\lambda) = -\frac{3nk}{2}\,\ln\,\lambda - \frac{1}{\lambda}\sum_{i=1}^{m}[k(G_i+1)-1]\,A(x_i,\lambda^{(r)}). \tag{6.11}$$

The following equations are used to compute $\lambda^{(r+1)}$:

$$h(\lambda) = \lambda, \tag{6.12}$$

$$\text{where,}\quad h(\lambda) = \frac{2\left[\sum_{i=1}^{m} x_j + \sum_{i=1}^{m}[k(G_i+1)-1]A(x_i,\lambda^{(r)})\right]}{3nk}. \tag{6.13}$$

Then, in the next iteration, $\lambda^{(r+1)}$ is utilized as the new true value of λ. The ML estimate of λ is computed by replacing the E-step and M-step until convergence, given the starting value $\lambda^{(0)}$ of unknown parameter. Using the invariance features of ML estimates, the ML estimate of entropy $H(\lambda)$ is computed by simply plugin $\hat{\lambda}$ in (6.5. Thus the ML estimate of entropy is given by

$$\hat{H}(\hat{\lambda}) = \frac{1}{2}\ln\hat{\lambda} + \gamma + \frac{1}{2}\ln\pi - \frac{1}{2}.$$

6.3.2 Asymptotic Confidence Interval

The EM approach can be used to calculate the asymptotic variance-covariance (VC) matrix for ML estimates, see Louis (1982). For this purpose, the following concepts are used as follows:

$$\text{Observe information} \;=\; \text{Complete information} \;-\; \text{Missing infomation} \tag{6.14}$$

Now, let us define $\underset{\sim}{X}$ be observed data, $\underset{\sim}{Y}$ be complete data and I_X be the corresponding observed information, I_Y be the corresponding complete information, and $I_{Y|X}(\lambda)$ be the missing information. Then, the equation (6.14) can be expressed as follows:

$$I_X(\lambda) = I_Y(\lambda) - I_{Y|X}(\lambda). \tag{6.15}$$

The complete information I_Y is given by

$$I_Y(\lambda) = -E\left[\frac{\partial^2 L_c(Y;\lambda)}{\partial\lambda^2}\right] = \frac{3nk}{2\lambda^2}. \tag{6.16}$$

The Fisher information in one observation is supplied by x_i, which is censored at the moment of the ith failure.

$$I^i_{Y|X}(\lambda) = -E_{Z_i|x_i}\left[\frac{\partial^2 \ln f(Z_{ij}|x_i,\lambda)}{\partial \lambda^2}\right] = -\frac{3}{2}\frac{1}{\lambda} + \psi'(\lambda) + \frac{3}{\lambda^2[1-F(x_i,\lambda)]}\left[1-\Gamma\left(\frac{x_i^2}{\lambda},\frac{5}{2}\right)\right].$$

Therefore, the expected information for the conditional distribution of $Y|X$ (i.e. the missing information) is

$$I_{Y|X}(\lambda) = \sum_{i=1}^{m}[k(G_i+1)-1]I_{Y|X}(\lambda). \tag{6.17}$$

We now obtain the observed information matrix $I_X(\lambda)$ by substituting equations (6.16) and (6.17) in equation (6.14). Thus, $I_X^{-1}(\lambda) = \left[I_Y(\lambda) - I_{Y|X}(\lambda)\right]^{-1}$ can be used to derive the VC matrix of parameter λ. Thus, an approximate $(1-\alpha)100\%$ CIs for λ is obtained as $\hat{\lambda} \pm z_{\alpha/2}\sqrt{Var(\hat{\lambda})}$, here, $z_{\alpha/2}$ is the upper $(\alpha/2)^{th}$ percentile of N(0,1). Also, the coverage probability (CP) for λ is given by

$$CP_\lambda = \left[\left|\frac{\hat{\lambda}-\lambda}{\sqrt{\hat{V}ar(\hat{\lambda})}}\right| \leq z_{\alpha/2}\right].$$

The delta approach is now used to construct the ACI of entropy $H(\lambda)$. Assuming $\hat{\lambda}$ is the ML estimate of unknown parameter λ, the asymptotic variance of $\hat{H}$ using the delta technique is given by (Krishnamoorthy and Lin, 2010).

$$Var(H) = [b_C' I_X^{-1}(\lambda) b_C],$$

here, $b_C = \frac{\partial H(\lambda)}{\partial \lambda} = \frac{1}{2\lambda}$.

Under moderate regularity criteria, the OFI matrix is used as a consistent estimator of the Fisher information. As a result, the observed variance of $\hat{H}$ is equal to

$$\hat{V}ar(\hat{H}) \simeq [b_C' I_X^{-1}(\lambda) b_C]_{\lambda=\hat{\lambda}}.$$

Thus $\frac{\hat{H}-H}{\sqrt{\hat{V}ar(\hat{H})}} \sim N(0,1)$. Therefore, $(1-\alpha)100\%$ ACI of H is given by $\hat{H} \pm z_{\alpha/2}\sqrt{\hat{V}ar(\hat{H})}$. Also, the coverage probability (CP) of H is given by

$$CP_H = \left[\left|\frac{\hat{H}-H}{\sqrt{\hat{V}ar(\hat{H})}}\right| \leq z_{\alpha/2}\right].$$

6.3.3 Bootstrap Confidence Intervals

Using the similar concept as in (5.3.2), we construct the boot-p and boot-t CIs of associated parameter λ and entropy $H(\lambda)$ of MW lifetime model. Let $X_1, X_2, \ldots, X_m$ be a PFFC samples of effective sample size m drawn from MW(λ). Then the bootstrap procedures and algorithms will be same as we have already been discussed in subsection (5.3.2).

6.3.3.1 Boot-p Confidence Intervals

Let $\hat{\lambda}_{(j)}$ and $\hat{H}_{(j)}$; $j = 1, 2, \ldots, B$ denotes the ordered values of boot-p samples $\hat{\lambda}_j$ and $\hat{H}_j$, respectively. Thus, $(1-\alpha)100\%$ boot-p CIs of λ and H, respectively, are given by

$$\left(\hat{\lambda}^*_{[(\alpha/2)B]}, \hat{\lambda}^*_{[(1-\alpha/2)B]}\right) \text{ and } \left(\hat{H}^*_{[(\alpha/2)B]}, \hat{H}^*_{[(1-\alpha/2)B]}\right),$$

where, $[a]$ is the integral part of a.

6.3.3.2 Boot-t Confidence Intervals

Let $\left(\tau^*_{1(1)} \leq \tau_{1(2)} \leq \cdots \leq \tau_{1(B)}\right)$ and $\left(\tau^*_{1(1)} \leq \tau_{1(2)} \leq \cdots \leq \tau_{1(B)}\right)$ denotes the ordered values of boot-t samples $\tau^*_{i(j)}$ for $j = 1, 2, \ldots, B, i = 1, 2$, respectively. Thus, $(1-\alpha)100\%$ boot-t CIs for λ and $H(\lambda)$, respectively, are given by

$$\left(\hat{\lambda} - \tau_{1[(1-\alpha/2)B]}\sqrt{I_X^{-1}(\hat{\lambda})}, \hat{\lambda} - \tau_{1[(\alpha/2)B]}\sqrt{I_X^{-1}(\hat{\lambda})}\right),$$

$$\left(\hat{H} - \tau_{2[(1-\alpha/2)B]}\sqrt{I_X^{-1}(\hat{H})}, \hat{H} - \tau_{2[(\alpha/2)B]}\sqrt{I_X^{-1}(\hat{H})}\right).$$

6.4 Bayesian Estimation

In this section, we derive Bayes estimators and HPD credible intervals for the parameter λ and entropy $H(\lambda)$ using the LINEX loss function. The Bayesian approach to reliability inference necessitates the inclusion of experimental data, as well as prior belief in the parameters and technical knowledge of failure mechanisms, in the inferential methods. As a result, Bayesian methods are frequently used to small sample data, which is especially useful in the case of costly life testing studies. The inverted gamma distribution is a common natural conjugate prior density for the parameter λ of the MW lifetime model in Bayesian estimation, see (Bekker and Roux, 2005), (Chaudhary et al., 2017). Therefore, we consider the prior distribution of unknown parameter λ assumes to follow an inverted gamma distribution with the following pdf:

$$g(\lambda) \propto \frac{1}{\lambda^{a+1}} \exp(-b/\lambda); \quad \lambda > 0, a, b > 0, \tag{6.18}$$

a and b are hyper-parameters, respectively. The joint posterior distribution of λ is provided by employing the likelihood function in (6.6) and the prior distribution in (6.18).

$$\pi(\lambda|\underset{\sim}{X}) = \eta \frac{1}{\lambda^{\frac{3m}{2}+a+1}} exp\left[-\frac{1}{\lambda}\left(\sum_{i=1}^{m} x_i^2 + b\right)\right] \prod_{i=1}^{m}\left[1-\Gamma\left(\frac{x_i^2}{\lambda}, \frac{3}{2}\right)\right]^{k(G_i+1)-1}, \tag{6.19}$$

where, η^{-1} is the normalizing constant and is given by

$$\eta^{-1} = \int_0^\infty \frac{1}{\lambda^{\frac{3m}{2}+a+1}} exp\left[-\frac{1}{\lambda}\left(\sum_{i=1}^{m} x_i^2 + b\right)\right] \prod_{i=1}^{m}\left[1-\Gamma\left(\frac{x_i^2}{\lambda}, \frac{3}{2}\right)\right]^{k(G_i+1)-1} d\lambda.$$

We have already defined the LINEX loss function in Chapter 2. For convenience, we again define the LINEX loss function as follows:

$$L(\Delta) = e^{c\Delta} - c\Delta - 1; \quad c \neq 0, \quad \Delta = \hat{\lambda} - \lambda, \tag{6.20}$$

where c is the LINEX loss parameter. Under this loss function, the Bayes estimator of any function of the parameter λ, say $\phi(\lambda)$, is given by

$$E[\phi(\lambda)] = -\frac{1}{c} \ln \left[\frac{\int_0^\infty e^{-c\phi(\lambda)} \pi(\lambda|\underset{\sim}{X}) d\lambda}{\int_0^\infty \pi(\lambda|\underset{\sim}{X}) d\lambda}\right]. \tag{6.21}$$

As seen in the preceding equation (6.21), the Bayes estimators are in the form of a ratio of two integrals for which there is no closed form solution. It is feasible to get a numerical solution for the aforementioned integral ratio. We propose utilizing two approximation approaches to solve the aforementioned ratio of integrals: the TK and MCMC methods.

6.4.1 TK Approximation Method

According to TK approximation's method proposed by Tierney and Kadane (1986), the approximation of the posterior mean of $\phi(\lambda)$ is given by

$$E[\phi(\lambda)|\underset{\sim}{X}] = \frac{\int_0^\infty e^{n\delta_\phi^*(\lambda)}d\lambda}{\int_0^\infty e^{n\delta(\lambda)}d\lambda} \simeq \left(\frac{|\Sigma_\phi^*|}{|\Sigma|}\right)^{\frac{1}{2}} e^{n[\delta_\phi^*(\hat{\lambda}_\phi^*)-\delta(\hat{\lambda}_\phi)]} \tag{6.22}$$

where, $\delta(\lambda) = \frac{1}{n}[l(\lambda)+\rho(\lambda)]$, and $\delta^*(\lambda) = \delta(\lambda)+\frac{1}{n}\ln\phi(\lambda)$, here, $l(\lambda)$ is the log-likelihood function and $\rho(\lambda) = \ln g(\lambda)$. Also, $|\Sigma_\phi^*|$ and $|\Sigma|$ are the determinants of inverse of the negative hessian of $\delta^*(\lambda)$ and $\delta(\lambda)$ at $\hat{\lambda}_{\delta^*}$ and $\hat{\lambda}_\delta$, respectively. Also, $\hat{\lambda}_\delta$ and $\hat{\lambda}_{\delta^*}$ maximize $\delta(\lambda)$ and $\delta^*(\lambda)$, respectively. Next, we observe that

$$\delta(\lambda) = \frac{1}{n}\Bigg[-\left(\frac{3m}{2}+a+1\right)\ln\lambda - \frac{1}{\lambda}\left(\sum_{i=1}^m x_i^2+b\right)+2\sum_{i=1}^m \ln x_i + \sum_{i=1}^m [k(G_i+1)-1]\ln\left(1-\Gamma\left(\frac{x_i^2}{\lambda},\frac{3}{2}\right)\right)\Bigg]$$

Then, by solving the following non-linear equation, $\hat{\lambda}_\delta$ is computed:

$$\frac{\partial\delta(\lambda)}{\partial\lambda} = \frac{1}{n}\left[-\left(\frac{3m}{2}+a+1\right)\frac{1}{\lambda}+\frac{1}{\lambda^2}\left(\sum_{i=1}^m x_i^2+b\right)+\sum_{i=1}^m[k(G_i+1)-1]\psi(\lambda)\right],$$

where

$$\psi(\lambda) = \frac{\partial\ln\left[1-\Gamma\left(\frac{x_i^2}{\lambda},\frac{3}{2}\right)\right]}{\partial\lambda} = -\frac{3}{2\lambda\left[1-\Gamma\left(\frac{x_i^2}{\lambda},\frac{3}{2}\right)\right]}\left\{\frac{3}{2}\Gamma\left(\frac{x_i^2}{\lambda},\frac{5}{2}\right)-\Gamma\left(\frac{x_i^2}{\lambda},\frac{3}{2}\right)\right\}.$$

Now, obtain $|\Sigma|$ from $\Sigma^{-1} = \frac{1}{n}\left(-\frac{\partial^2\delta(\lambda)}{\partial\lambda^2}\right)$,

$$\Sigma^{-1} = -\frac{1}{n}\left[\left(\frac{3m}{2}+a+1\right)\frac{1}{\lambda^2}-\frac{2}{\lambda^3}\left(\sum_{i=1}^m x_i^2+b\right)+\sum_{i=1}^m[k(G_i+1)-1]\psi'(\lambda)\right],$$

where,

$$\psi'(\lambda) = \frac{\partial\psi(\lambda)}{\partial\lambda} = -\frac{3}{2\lambda^2}\left\{\frac{\left[\left\{1-\Gamma\left(x_i^2/\lambda,3/2\right)\right\}Q_1+Q_2\right]}{\left\{1-\Gamma\left(x_i/\lambda,3/2\right)\right\}}\right\},$$

where,

$$Q_1 = \frac{5}{2}\left[\Gamma\left(x_i^2/\lambda, 7/2\right) - 2\Gamma\left(x_i^2/\lambda, 5/2\right) + \Gamma\left(x_i^2/\lambda, 3/2\right)\right]$$

and

$$Q_2 = \frac{3}{2}\left[\Gamma\left(x_i^2/\lambda, 5/2\right) - \Gamma\left(x_i^2/\lambda, 3/2\right)\right].$$

We use $\phi(\lambda) = e^{-c\lambda}$ to compute the Bayes estimator of λ under the LINEX loss function, and the function $\delta * (\lambda)$ becomes

$$\delta^*(\lambda) = \delta(\lambda) - \frac{c\lambda}{n}.$$

Then, by solving the following non-linear equation, $\hat{\lambda}^*_{\delta^*}$ is obtained:

$$\frac{\partial \delta^*(\lambda)}{\partial \lambda} = \frac{\partial \delta(\lambda)}{\partial \lambda} - \frac{c}{n} = 0, \quad \text{and obtain} \quad |\Sigma^*| \ \text{ from } \ \Sigma^{*-1}_{\lambda} = -\frac{1}{n}\left(\frac{\partial^2 \delta^*(\lambda)}{\partial \lambda^2}\right).$$

Under the LINEX loss function, the approximate Bayes estimator of λ is given by

$$\hat{\lambda}_{TK} = -\frac{1}{c}\ln\left[\left(\frac{|\Sigma^*_{\lambda}|}{|\Sigma|}\right)^{\frac{1}{2}} \exp\left\{n\left[\delta^*_{\lambda}(\hat{\lambda}^*_{\delta}) - \delta(\hat{\lambda}_{\delta})\right]\right\}\right].$$

Similarly, the Bayes estimator of entropy $H(\lambda)$ is given by

$$\hat{H}_{TK} = -\frac{1}{c}\ln\left[\left(\frac{|\Sigma^*_{H}|}{|\Sigma|}\right)^{\frac{1}{2}} \exp\left\{n\left[\delta^*_{H}(\hat{H}^*_{\delta}) - \delta(\hat{H}_{\delta})\right]\right\}\right].$$

6.4.2 MCMC Method

Here, the Bayes estimates of parameter and entropy are computed using the MCMC technique followed by M-H algorithm. We create a candidate points from a normal distribution to generate a sequence of sample from the posterior distribution of λ using data $\underset{\sim}{X}$ in (6.19). For the purpose of computation, the following steps are used:

Step 1: Begin with guess value $\lambda^{(0)}$ for λ.

Step 2: Create a candidate point $\lambda_c^{(j)}$ from the proposal density $\eta(\lambda^{(j)}|\lambda^{(j-1)})$.

Step 3: Create u from uniform $(0,1)$.

Step 4: Compute $\alpha(\lambda_c^{(j)}|\lambda^{(j-1)}) = \min\left\{\frac{\pi\left(\lambda_c^{(j)}|\underset{\sim}{w}\right)\eta\left(\lambda^{(j-1)}|\lambda_c^{(j)}\right)}{\pi\left(\lambda^{(j-1)}|\underset{\sim}{w}\right)\eta\left(\lambda_c^{(j)}|\lambda^{(j-1)}\right)}, 1\right\}$.

Step 5: If $u \leq \alpha$ set $\lambda^{(j)} = \lambda_c^{(j)}$ with acceptance probability α otherwise $\lambda^{(j)} = \lambda^{(j-1)}$.

Step 6: Compute $H^{(j)} = H(\lambda^{(j)})$ using (6.1).

Step 7: For $j = 1,2,\ldots,M$, repeat steps 2-6 to get the sequence of the parameter λ as $(\lambda_1,\lambda_2,\ldots,\lambda_M)$ and the entropy H as $(H_1,H_2,\ldots,H_M)$, respectively.

To get an independent sample from the stationary distribution of the Markov chain, which is typically the posterior distribution, we discard first $\lambda_j's$ and $H_j's$, where, $M_0; j = 1,2,\ldots,M_0$ is the burn-in-period. Therefore, the Bayes estimators of the parameter λ and entropy $H(\lambda)$ under the LINEX loss function, respectively, are given by

$$\hat{\lambda}_{MH} = -\frac{1}{c}\ln\left[\frac{1}{M-M_0}\sum_{j=M_0+1}^{M} e^{-c\lambda_j}\right],$$

$$\hat{H}_{MH} = -\frac{1}{c}\ln\left[\frac{1}{M-M_0}\sum_{j=M_0+1}^{M} e^{-cH(\lambda_j)}\right].$$

6.4.3 HPD Credible Interval Estimation

Using the generated MCMC samples based on the M-H algorithm, we now construct the HPD credible intervals for the parameter λ and entropy $H(\lambda)$. Let $\lambda_{(1)} < \lambda_{(2)} < \cdots < \lambda_{(M)}$ represent the ordered values of $\lambda_1,\lambda_2,\ldots,\lambda_M$. Then $(1-\alpha)100\%,\quad 0<\alpha<1$, HPD credible interval for λ is given by $\left(\lambda_{(j)},\lambda_{(j+[(1-\alpha)M])}\right)$, where j is chosen such that

$$\lambda_{j+[(1-\alpha)M]} - \lambda_{(j)} = \min_{1\le i\le \alpha M}\left(\lambda_{(i+[(1-\alpha)M])} - \lambda_{(i)}\right); \quad j = 1,2,\ldots,M.$$

Similarly, HPD credible interval for $H(\lambda)$ is computed as $\left(H_{(j)},H_{(j+[(1-\alpha)M])}\right)$, where j is chosen such that

$$H_{j+[(1-\alpha)M]} - H_{(j)} = \min_{1\le i\le \alpha M}\left(H_{(i+[(1-\alpha)M])} - H_{(i)}\right); \quad j = 1,2,\ldots,M.$$

6.5 Numerical Computations

In this section, we perform an extensive numerical computation in terms of an MC simulation to understand the impact of distinct estimators created in the previous sections. The impact of different estimators are evaluated by their average estimates (AE) and mean squared errors (MSE). The AE and MSEs of ML and Bayes estimators of the parameter and entropy are used in the MC simulation to determine the influence of different estimators established in the previous sections. The Bayes estimators of parameter and entropy are computed in the case of non-informative prior (Prior 0) and informative inverted gamma prior (Prior 1) under the LINEX loss function. Also, we obtain average lengths (AL) of 95% ACIs, bootstrap CIs, and HPD credible intervals with their corresponding coverage probabilities (CP). To obtain bootstrap CIs of the parameter λ and entropy $H(\lambda)$, we generate $B = 1000$ bootstrap samples for the prescribed sample under consideration.

For the simulation purpose, the PFFC samples are generated with several combinations of $(k,n,m,\underset{\sim}{G})$ for distinct values of associated parameter λ from MW(λ). To generate PFFC samples, we use the algorithm proposed by Balakrishnan and Sandhu (1995), with the addition that the PFFC sample $x_1,x_2,...,x_m$ can be viewed as a progressively censored sample from a population with cdf $\left[1-(1-F(x))^k\right]$, see Wu and Kuş (2009). Here, we considered two sets of parameter values $\theta = 0.75$ and $\theta = 1.5$, for which the corresponding entropy becomes 0.5057 and 0.8523, respectively. We consider group sizes $k = 3,5$, number of groups $n = 20,50$ with effective sample sizes $m = 40,80\%$ of n. For each n, we adopt three different censoring plans $\underset{\sim}{G}$ and these plans are common for each n. The different common failure plans $\underset{\sim}{G}$ for each effective sample m are as follows:

Plan 1: If $[(k,n,m),(G_1 = n-m, G_j = 0, \quad \forall \quad j = 2,3,\ldots m)]$, in this case $(n-m)$ groups are discarded from the experiment at the first failure only,

Plan 2: If $[(k,n,m),(G_j = 0, \quad \forall j = 1,2,\ldots,m-1, G_m = n-m)]$, in this case $(n-m)$ groups are discarded at mth failure, and

Plan 3: If $[(k,n=m),G_j = 0, \quad \forall j = 1,2,\ldots,m]$ this is the case of first failure censored sample.

The simplified notations are used for different combinations of censoring plans ([CS]) which are summaries in the Table 6.1. Note: the notation used in censoring schemes like $(0*7)$ denotes $(0,0,0,0,0,0,0)$ and $(4*3)$ stands for $(4,4,4)$. For Bayesian calculations of parameter

TABLE 6.1: Several combinations of censoring plans.

n	m	[CS]	Schemes	n	m	[CS]	Schemes
20	8	[1]	(12*1,0*7)	50	20	[7]	(30*1,0*19)
		[2]	(4*3,0*5)			[8]	(5*6,0*14)
		[3]	(0*7,12*1)			[9]	(0*19,30*1)
20	16	[4]	(4*1,0*15)	50	40	[10]	(10*1,0*39)
		[5]	(2*2,0*14)			[11]	(5*2,0*38)
		[6]	(0*15,4*1)			[12]	(0*39,10*1)

and entropy, the hyper-parameters (a,b) are chosen in such a way that the prior mean is exactly equal to the true values of the parameter, i.e. $\lambda = \frac{a}{b}$. Here, we consider $(a,b) = (3,4)$ and $(3,2)$ for $\theta = 0.75$ and 1.5, respectively. In case of Prior 0, hyper-parameters are taken as $(a,b) = (0.0001,0.0001)$. In order to derive Bayes estimators under LINEX loss function, we consider two choice of loss function parameter as $c = (-0.5$ and $0.5)$. Two approximation techniques such as TK approximation and M-H algorithm are used for Bayesian computations. For M-H algorithm, we generate $M = 10,000$ samples and $M_0 = 20\%$ of M is considered as burn-in period. All the simulated results for several combinations of censoring plans are summarizes in the following Tables 6.2, 6.3, 6.4, 6.5, 6.6, 6.7. These findings lead to the following conclusions:

In view of Tables 6.2, 6.3, 6.5, 6.6, this experiment has brought up some interesting observations. In almost all cases, the ML and Bayes estimates output of parameter and entropy in terms of MSEs are very adequate even for small sample sizes. MSEs are found to decrease as n or m rise. It checks the consistent behavior of the estimators. Also, the performance of Bayes estimators with Prior 1 is better than ML estimators even with Prior 0 in terms of MSEs, as Bayes estimators with Prior 1 includes some prior information. Also, Bayes estimators computed using M–H algorithm outperform the TK approximation procedure.

With the reference of Tables 6.4 and 6.7, the average lengths (AL) of ACIs, boot-p, boot-t CIs, and HPD credible intervals are shrinking with an increase in the number of failures m. It is also observed that the HPD credible intervals with Prior 1 have the smallest ALs as compared to others. It is also fair to say that all four intervals have reasonable coverage probabilities. Also, it is seen that the ALs of boot-p confidence intervals outperform as it has smaller ALs to ACI and boot-t both.

TABLE 6.2: Average ML and Bayes estimates of parameter λ, when $\lambda = 0.75$.

		$\hat{\lambda}_{ML}$		$\hat{\lambda}_{TK}$								$\hat{\lambda}_{MH}$							
				$c=-0.5$				$c=0.5$				$c=-0.5$				$c=0.5$			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	[CS]	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	[1]	0.7545	0.0404	0.8146	0.0529	0.9171	0.0598	0.7880	0.0442	0.8900	0.0479	0.7704	0.0092	0.8151	0.0120	0.7616	0.0085	0.8065	0.0106
	[2]	0.7538	0.0395	0.8132	0.0514	0.9142	0.0582	0.7872	0.0432	0.8877	0.0468	0.7700	0.0090	0.8140	0.0117	0.7613	0.0083	0.8055	0.0104
	[3]	0.7519	0.0365	0.8086	0.0470	0.9044	0.0532	0.7848	0.0399	0.8799	0.0432	0.7685	0.0083	0.8102	0.0107	0.7605	0.0077	0.8023	0.0096
	[4]	0.7563	0.0205	0.7885	0.0241	0.8454	0.0272	0.7762	0.0219	0.8328	0.0240	0.7627	0.0050	0.7893	0.0061	0.7579	0.0048	0.7846	0.0056
	[5]	0.7563	0.0204	0.7884	0.0240	0.8452	0.0272	0.7762	0.0219	0.8327	0.0239	0.7627	0.0050	0.7892	0.0061	0.7578	0.0048	0.7845	0.0056
	[6]	0.7558	0.0197	0.7872	0.0231	0.8424	0.0261	0.7755	0.0211	0.8303	0.0230	0.7623	0.0048	0.7881	0.0058	0.7576	0.0046	0.7835	0.0054
	[7]	0.7553	0.0166	0.7813	0.0189	0.8276	0.0211	0.7719	0.0176	0.8179	0.0190	0.7605	0.0041	0.7825	0.0048	0.7567	0.0039	0.7787	0.0045
	[8]	0.7550	0.0162	0.7807	0.0184	0.8261	0.0205	0.7714	0.0171	0.8166	0.0185	0.7603	0.0040	0.7818	0.0047	0.7565	0.0038	0.7781	0.0044
	[9]	0.7539	0.0148	0.7783	0.0167	0.8204	0.0186	0.7698	0.0157	0.8117	0.0169	0.7595	0.0037	0.7795	0.0043	0.7560	0.0035	0.7760	0.0040
	[10]	0.7518	0.0088	0.7650	0.0094	0.7895	0.0100	0.7606	0.0091	0.7850	0.0094	0.7552	0.0022	0.7671	0.0024	0.7532	0.0022	0.7651	0.0023
	[11]	0.7518	0.0088	0.7650	0.0094	0.7895	0.0100	0.7606	0.0091	0.7849	0.0094	0.7552	0.0022	0.7671	0.0024	0.7532	0.0022	0.7651	0.0023
	[12]	0.7515	0.0085	0.7644	0.0090	0.7881	0.0096	0.7602	0.0087	0.7837	0.0091	0.7550	0.0021	0.7665	0.0023	0.7531	0.0021	0.7646	0.0022
5	[1]	0.7532	0.0384	0.8115	0.0498	0.9107	0.0563	0.7864	0.0420	0.8848	0.0455	0.7695	0.0087	0.8126	0.0113	0.7610	0.0081	0.8044	0.0101
	[2]	0.7526	0.0376	0.8104	0.0486	0.9083	0.0550	0.7857	0.0411	0.8829	0.0445	0.7691	0.0085	0.8117	0.0111	0.7608	0.0079	0.8035	0.0099
	[3]	0.7508	0.0351	0.8065	0.0449	0.9000	0.0509	0.7836	0.0384	0.8763	0.0415	0.7678	0.0079	0.8084	0.0103	0.7600	0.0074	0.8007	0.0092
	[4]	0.7555	0.0194	0.7867	0.0227	0.8412	0.0256	0.7751	0.0207	0.8293	0.0226	0.7621	0.0048	0.7876	0.0057	0.7575	0.0046	0.7831	0.0053
	[5]	0.7555	0.0194	0.7866	0.0226	0.8411	0.0256	0.7751	0.0207	0.8292	0.0226	0.7621	0.0048	0.7875	0.0057	0.7575	0.0046	0.7830	0.0053
	[6]	0.7550	0.0188	0.7856	0.0219	0.8386	0.0247	0.7744	0.0200	0.8271	0.0219	0.7617	0.0046	0.7865	0.0055	0.7573	0.0044	0.7821	0.0051
	[7]	0.7547	0.0157	0.7799	0.0179	0.8242	0.0199	0.7709	0.0167	0.8150	0.0180	0.7600	0.0039	0.7811	0.0046	0.7563	0.0038	0.7774	0.0043
	[8]	0.7544	0.0154	0.7794	0.0175	0.8230	0.0194	0.7706	0.0163	0.8139	0.0176	0.7599	0.0038	0.7805	0.0044	0.7562	0.0037	0.7769	0.0042
	[9]	0.7534	0.0143	0.7773	0.0161	0.8182	0.0178	0.7692	0.0151	0.8098	0.0162	0.7591	0.0035	0.7785	0.0041	0.7558	0.0034	0.7751	0.0039
	[10]	0.7514	0.0084	0.7643	0.0089	0.7876	0.0094	0.7601	0.0086	0.7833	0.0089	0.7549	0.0021	0.7663	0.0023	0.7530	0.0020	0.7644	0.0022
	[11]	0.7514	0.0084	0.7642	0.0089	0.7876	0.0094	0.7601	0.0086	0.7833	0.0089	0.7549	0.0021	0.7663	0.0023	0.7530	0.0020	0.7644	0.0022
	[12]	0.7511	0.0081	0.7638	0.0086	0.7863	0.0091	0.7597	0.0083	0.7822	0.0086	0.7547	0.0020	0.7657	0.0022	0.7529	0.0020	0.7639	0.0021

TABLE 6.3: Average ML and Bayes estimates of entropy H, when $\lambda = 0.75$ and $H = 0.5057$.

		$\hat{\lambda}_{ML}$		$\hat{\lambda}_{TK}$								$\hat{\lambda}_{MH}$							
				$c=-0.5$				$c=0.5$				$c=-0.5$				$c=0.5$			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	[CS]	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	[1]	0.4906	0.0189	0.5126	0.0187	0.5823	0.0151	0.5033	0.0187	0.5746	0.0139	0.5073	0.0033	0.5371	0.0037	0.5036	0.0033	0.5338	0.0035
	[2]	0.4905	0.0185	0.5125	0.0183	0.5811	0.0148	0.5034	0.0183	0.5735	0.0137	0.5073	0.0032	0.5365	0.0036	0.5036	0.0032	0.5333	0.0035
	[3]	0.4905	0.0172	0.5120	0.0170	0.5768	0.0138	0.5035	0.0170	0.5696	0.0129	0.5071	0.0030	0.5347	0.0034	0.5037	0.0030	0.5317	0.0032
	[4]	0.5009	0.0091	0.5126	0.0091	0.5507	0.0083	0.5079	0.0091	0.5465	0.0079	0.5074	0.0020	0.5252	0.0021	0.5053	0.0020	0.5232	0.0021
	[5]	0.5009	0.0091	0.5126	0.0091	0.5506	0.0083	0.5079	0.0091	0.5464	0.0079	0.5074	0.0020	0.5251	0.0021	0.5053	0.0020	0.5232	0.0020
	[6]	0.5009	0.0088	0.5124	0.0088	0.5493	0.0080	0.5079	0.0088	0.5453	0.0077	0.5074	0.0019	0.5246	0.0020	0.5054	0.0019	0.5227	0.0020
	[7]	0.5020	0.0074	0.5114	0.0074	0.5424	0.0068	0.5077	0.0074	0.5390	0.0066	0.5073	0.0016	0.5219	0.0017	0.5056	0.0016	0.5203	0.0017
	[8]	0.5019	0.0072	0.5113	0.0072	0.5417	0.0067	0.5077	0.0072	0.5384	0.0064	0.5072	0.0016	0.5215	0.0017	0.5056	0.0016	0.5200	0.0017
	[9]	0.5018	0.0066	0.5110	0.0066	0.5391	0.0062	0.5076	0.0066	0.5360	0.0060	0.5071	0.0015	0.5204	0.0016	0.5056	0.0015	0.5190	0.0015
	[10]	0.5030	0.0039	0.5078	0.0039	0.5242	0.0037	0.5060	0.0039	0.5225	0.0037	0.5063	0.0009	0.5142	0.0009	0.5054	0.0009	0.5134	0.0009
	[11]	0.5030	0.0039	0.5078	0.0039	0.5242	0.0037	0.5060	0.0039	0.5224	0.0037	0.5063	0.0009	0.5142	0.0009	0.5054	0.0009	0.5134	0.0009
	[12]	0.5030	0.0038	0.5077	0.0038	0.5235	0.0036	0.5060	0.0038	0.5218	0.0035	0.5062	0.0009	0.5139	0.0009	0.5054	0.0009	0.5131	0.0009
5	[1]	0.4905	0.0180	0.5123	0.0178	0.5795	0.0144	0.5034	0.0178	0.5721	0.0134	0.5072	0.0032	0.5359	0.0036	0.5036	0.0032	0.5327	0.0034
	[2]	0.4904	0.0177	0.5122	0.0175	0.5785	0.0142	0.5035	0.0175	0.5712	0.0132	0.5072	0.0031	0.5354	0.0035	0.5036	0.0031	0.5323	0.0033
	[3]	0.4903	0.0166	0.5118	0.0164	0.5749	0.0134	0.5036	0.0164	0.5679	0.0125	0.5070	0.0029	0.5339	0.0033	0.5037	0.0029	0.5309	0.0031
	[4]	0.5008	0.0087	0.5123	0.0087	0.5488	0.0079	0.5079	0.0086	0.5448	0.0076	0.5073	0.0019	0.5243	0.0020	0.5054	0.0019	0.5225	0.0020
	[5]	0.5008	0.0086	0.5123	0.0087	0.5488	0.0079	0.5079	0.0086	0.5447	0.0076	0.5073	0.0019	0.5243	0.0020	0.5054	0.0019	0.5225	0.0020
	[6]	0.5008	0.0084	0.5122	0.0084	0.5476	0.0077	0.5079	0.0084	0.5437	0.0074	0.5073	0.0018	0.5238	0.0020	0.5054	0.0018	0.5220	0.0019
	[7]	0.5019	0.0070	0.5112	0.0070	0.5409	0.0065	0.5077	0.0070	0.5376	0.0063	0.5072	0.0016	0.5212	0.0017	0.5056	0.0016	0.5197	0.0016
	[8]	0.5018	0.0069	0.5111	0.0069	0.5403	0.0064	0.5077	0.0069	0.5371	0.0062	0.5072	0.0015	0.5209	0.0016	0.5056	0.0015	0.5194	0.0016
	[9]	0.5016	0.0064	0.5108	0.0064	0.5381	0.0060	0.5076	0.0064	0.5351	0.0058	0.5070	0.0014	0.5199	0.0015	0.5056	0.0014	0.5185	0.0015
	[10]	0.5029	0.0038	0.5077	0.0038	0.5233	0.0035	0.5060	0.0037	0.5216	0.0035	0.5062	0.0009	0.5138	0.0009	0.5054	0.0009	0.5130	0.0009
	[11]	0.5029	0.0038	0.5077	0.0037	0.5233	0.0035	0.5060	0.0037	0.5216	0.0035	0.5062	0.0009	0.5138	0.0009	0.5054	0.0009	0.5130	0.0009
	[12]	0.5029	0.0036	0.5076	0.0036	0.5227	0.0034	0.5059	0.0036	0.5211	0.0034	0.5062	0.0009	0.5135	0.0009	0.5054	0.0009	0.5128	0.0009

TABLE 6.4: The 95% ACI, bootstrap CI and HPD credible intervals of parameter λ and entropy $H(\lambda)$, when $\lambda = 0.75$ and $H = 0.5057$.

		ACI				Bootstrap								HPD							
						λ				$H(\lambda)$				λ				$H(\lambda)$			
		λ		$H(\lambda)$		Boot-p		Boot-t		Boot-p		Boot-t		Prior 0		Prior 1		Prior 0		Prior 1	
k	[CS]	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AE	MSE	AL	CP	AL	CP
3	[1]	0.7932	0.912	0.5219	0.948	0.7799	0.927	0.9120	0.952	0.5256	0.927	0.5312	0.952	0.5075	0.9550	0.5050	0.9530	0.3363	0.9540	0.3158	0.947
	[2]	0.7836	0.912	0.5163	0.947	0.7706	0.927	0.9003	0.952	0.5207	0.927	0.5258	0.952	0.5022	0.9550	0.5004	0.9520	0.3329	0.9540	0.3133	0.948
	[3]	0.7530	0.914	0.4981	0.951	0.7403	0.925	0.8678	0.951	0.5055	0.925	0.5089	0.951	0.4855	0.9550	0.4854	0.9530	0.3220	0.9540	0.3051	0.948
	[4]	0.5667	0.935	0.3746	0.948	0.5617	0.945	0.5906	0.950	0.3721	0.945	0.3722	0.950	0.3789	0.9560	0.3767	0.9520	0.2514	0.9530	0.2413	0.948
	[5]	0.5662	0.935	0.3743	0.948	0.5612	0.945	0.5900	0.950	0.3719	0.945	0.3718	0.950	0.3786	0.9560	0.3764	0.9520	0.2512	0.9530	0.2411	0.948
	[6]	0.5562	0.936	0.3680	0.945	0.5513	0.945	0.5797	0.951	0.3657	0.946	0.3658	0.951	0.3725	0.9570	0.3707	0.9520	0.2472	0.9540	0.2377	0.947
	[7]	0.5050	0.943	0.3343	0.945	0.5240	0.955	0.5420	0.954	0.3458	0.955	0.3459	0.954	0.3410	0.9570	0.3394	0.9520	0.2265	0.9530	0.2189	0.949
	[8]	0.4988	0.943	0.3304	0.945	0.5177	0.956	0.5347	0.954	0.3419	0.956	0.3416	0.954	0.3371	0.9570	0.3358	0.9520	0.2239	0.9540	0.2167	0.949
	[9]	0.4772	0.942	0.3165	0.944	0.4938	0.954	0.5112	0.953	0.3273	0.954	0.3273	0.953	0.3234	0.9560	0.3228	0.9520	0.2149	0.9520	0.2089	0.949
	[10]	0.3566	0.938	0.2372	0.941	0.3708	0.950	0.3802	0.950	0.2472	0.950	0.2472	0.950	0.2459	0.9490	0.2452	0.9470	0.1637	0.9470	0.1607	0.943
	[11]	0.3565	0.938	0.2371	0.941	0.3707	0.950	0.3801	0.950	0.2471	0.950	0.2471	0.950	0.2458	0.9490	0.2451	0.9470	0.1636	0.9470	0.1606	0.943
	[12]	0.3498	0.939	0.2327	0.942	0.3637	0.950	0.3731	0.950	0.2427	0.950	0.2427	0.950	0.2413	0.9480	0.2407	0.9470	0.1607	0.9440	0.1579	0.945
5	[1]	0.7726	0.913	0.5098	0.948	0.7598	0.925	0.8892	0.952	0.5151	0.925	0.5201	0.952	0.4962	0.9560	0.4951	0.9520	0.3290	0.9540	0.3104	0.948
	[2]	0.7646	0.912	0.5052	0.947	0.7515	0.924	0.8799	0.953	0.5107	0.924	0.5158	0.953	0.4918	0.9560	0.4913	0.9520	0.3261	0.9540	0.3083	0.948
	[3]	0.7383	0.914	0.4895	0.950	0.7256	0.923	0.8526	0.951	0.4979	0.923	0.5011	0.951	0.4772	0.9550	0.4783	0.9530	0.3167	0.9540	0.3012	0.948
	[4]	0.5517	0.936	0.3651	0.946	0.5464	0.945	0.5748	0.951	0.3628	0.945	0.3629	0.951	0.3697	0.9570	0.3681	0.9520	0.2453	0.9530	0.2362	0.949
	[5]	0.5513	0.937	0.3649	0.946	0.5460	0.945	0.5743	0.951	0.3625	0.945	0.3626	0.951	0.3694	0.9570	0.3679	0.9520	0.2452	0.9530	0.2361	0.949
	[6]	0.5424	0.936	0.3592	0.945	0.5377	0.946	0.5660	0.951	0.3576	0.946	0.3577	0.951	0.3640	0.9570	0.3628	0.9520	0.2416	0.9530	0.2331	0.95
	[7]	0.4920	0.943	0.3260	0.946	0.5094	0.953	0.5271	0.953	0.3369	0.953	0.3370	0.953	0.3328	0.9570	0.3317	0.9520	0.2210	0.9540	0.2143	0.949
	[8]	0.4870	0.943	0.3228	0.946	0.5043	0.954	0.5214	0.953	0.3336	0.954	0.3336	0.953	0.3296	0.9570	0.3287	0.9520	0.2189	0.9540	0.2125	0.949
	[9]	0.4684	0.941	0.3109	0.944	0.4840	0.952	0.5017	0.954	0.3215	0.952	0.3214	0.954	0.3178	0.9560	0.3175	0.9520	0.2112	0.9530	0.2056	0.949
	[10]	0.3474	0.938	0.2312	0.942	0.3609	0.950	0.3702	0.950	0.2409	0.950	0.2409	0.950	0.2397	0.9480	0.2392	0.9470	0.1596	0.9460	0.1569	0.945
	[11]	0.3473	0.938	0.2311	0.942	0.3608	0.950	0.3701	0.950	0.2408	0.950	0.2408	0.950	0.2396	0.9480	0.2391	0.9470	0.1596	0.9460	0.1568	0.945
	[12]	0.3413	0.940	0.2272	0.942	0.3546	0.949	0.3638	0.950	0.2368	0.949	0.2369	0.950	0.2357	0.9490	0.2353	0.9470	0.1569	0.9460	0.1544	0.945

TABLE 6.5: Average ML and Bayes estimates of λ, when $\lambda = 1.5$.

		MLE		TK								MH							
				$c=-0.5$				$c=0.5$				$c=-0.5$				$c=0.5$			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	[CS]	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	[1]	1.5091	0.1618	1.6600	0.2363	1.4669	0.1355	1.5527	0.1645	1.3987	0.1216	1.5499	0.0387	1.4760	0.0209	1.5146	0.0328	1.4495	0.0211
	[2]	1.5077	0.1579	1.6563	0.2291	1.4676	0.1330	1.5518	0.1609	1.4006	0.1196	1.5488	0.0377	1.4764	0.0206	1.5143	0.0320	1.4503	0.0208
	[3]	1.5039	0.1460	1.6446	0.2075	1.4691	0.1247	1.5486	0.1494	1.4065	0.1130	1.5453	0.0346	1.4774	0.0196	1.5131	0.0298	1.4528	0.0198
	[4]	1.5126	0.0819	1.5900	0.1015	1.4936	0.0746	1.5410	0.0841	1.4549	0.0692	1.5302	0.0207	1.4884	0.0146	1.5110	0.0189	1.4720	0.0146
	[5]	1.5124	0.0817	1.5899	0.1013	1.4936	0.0745	1.5410	0.0840	1.4550	0.0691	1.5302	0.0206	1.4884	0.0146	1.5109	0.0188	1.4720	0.0146
	[6]	1.5116	0.0789	1.5869	0.0972	1.4939	0.0722	1.5399	0.0811	1.4565	0.0671	1.5292	0.0199	1.4888	0.0142	1.5106	0.0182	1.4728	0.0142
	[7]	1.5106	0.0663	1.5725	0.0788	1.4964	0.0616	1.5347	0.0680	1.4650	0.0578	1.5250	0.0167	1.4910	0.0126	1.5095	0.0155	1.4773	0.0126
	[8]	1.5099	0.0647	1.5710	0.0767	1.4966	0.0602	1.5341	0.0663	1.4659	0.0566	1.5245	0.0163	1.4912	0.0124	1.5093	0.0152	1.4778	0.0123
	[9]	1.5078	0.0593	1.5653	0.0695	1.4970	0.0556	1.5316	0.0608	1.4688	0.0525	1.5225	0.0149	1.4919	0.0115	1.5085	0.0139	1.4795	0.0115
	[10]	1.5035	0.0353	1.5346	0.0384	1.4972	0.0340	1.5168	0.0357	1.4810	0.0329	1.5124	0.0089	1.4947	0.0077	1.5044	0.0086	1.4872	0.0077
	[11]	1.5035	0.0352	1.5346	0.0383	1.4972	0.0340	1.5168	0.0357	1.4810	0.0329	1.5124	0.0089	1.4947	0.0077	1.5044	0.0086	1.4872	0.0077
	[12]	1.5030	0.0339	1.5332	0.0368	1.4972	0.0328	1.5161	0.0344	1.4816	0.0318	1.5119	0.0085	1.4949	0.0074	1.5042	0.0082	1.4876	0.0074
5	[1]	1.5064	0.1535	1.6521	0.2210	1.5937	0.1384	1.5506	0.1566	1.4027	0.1172	1.5475	0.0365	1.5231	0.0243	1.5138	0.0312	1.4512	0.0205
	[2]	1.5051	0.1503	1.6491	0.2154	1.5920	0.1359	1.5498	0.1536	1.4043	0.1155	1.5466	0.0357	1.5227	0.0239	1.5135	0.0306	1.4519	0.0202
	[3]	1.5015	0.1403	1.6391	0.1976	1.5864	0.1278	1.5470	0.1440	1.4093	0.1098	1.5435	0.0331	1.5213	0.0226	1.5125	0.0286	1.4541	0.0193
	[4]	1.5110	0.0776	1.5857	0.0954	1.5606	0.0749	1.5394	0.0798	1.4573	0.0662	1.5288	0.0195	1.5176	0.0156	1.5104	0.0179	1.4732	0.0140
	[5]	1.5109	0.0775	1.5856	0.0952	1.5605	0.0748	1.5393	0.0797	1.4573	0.0661	1.5287	0.0195	1.5176	0.0155	1.5104	0.0179	1.4732	0.0140
	[6]	1.5100	0.0750	1.5830	0.0918	1.5589	0.0726	1.5383	0.0772	1.4586	0.0644	1.5279	0.0188	1.5171	0.0151	1.5101	0.0173	1.4739	0.0137
	[7]	1.5093	0.0630	1.5691	0.0744	1.5502	0.0613	1.5333	0.0646	1.4668	0.0553	1.5238	0.0159	1.5151	0.0132	1.5090	0.0148	1.4783	0.0121
	[8]	1.5087	0.0616	1.5679	0.0727	1.5494	0.0602	1.5328	0.0633	1.4676	0.0543	1.5234	0.0155	1.5149	0.0129	1.5089	0.0145	1.4787	0.0119
	[9]	1.5067	0.0571	1.5631	0.0667	1.5461	0.0559	1.5307	0.0587	1.4701	0.0509	1.5217	0.0143	1.5139	0.0121	1.5082	0.0134	1.4801	0.0111
	[10]	1.5028	0.0335	1.5329	0.0363	1.5246	0.0330	1.5160	0.0339	1.4819	0.0314	1.5118	0.0084	1.5078	0.0076	1.5041	0.0081	1.4878	0.0073
	[11]	1.5028	0.0334	1.5328	0.0363	1.5246	0.0330	1.5160	0.0339	1.4819	0.0314	1.5117	0.0084	1.5078	0.0076	1.5041	0.0081	1.4878	0.0073
	[12]	1.5023	0.0324	1.5317	0.0350	1.5237	0.0319	1.5154	0.0328	1.4824	0.0304	1.5113	0.0081	1.5075	0.0074	1.5039	0.0079	1.4881	0.0071

TABLE 6.6: Average ML and Bayes estimates of entropy H, when $\lambda = 1.5$ and $H = 0.8523$.

		MLE		TK								MH							
				$c=-0.5$				$c=0.5$				$c=-0.5$				$c=0.5$			
				Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1		Prior 0		Prior 1	
k	[CS]	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
3	[1]	0.8372	0.0189	0.8592	0.0187	0.8066	0.0174	0.8499	0.0187	0.7991	0.0181	0.8539	0.0033	0.8329	0.0024	0.8501	0.0033	0.8298	0.0025
	[2]	0.8371	0.0185	0.8590	0.0183	0.8074	0.0171	0.8499	0.0183	0.8000	0.0178	0.8538	0.0032	0.8333	0.0024	0.8502	0.0032	0.8302	0.0025
	[3]	0.8371	0.0172	0.8586	0.0170	0.8101	0.0160	0.8501	0.0170	0.8031	0.0166	0.8537	0.0030	0.8342	0.0023	0.8502	0.0030	0.8314	0.0024
	[4]	0.8475	0.0091	0.8592	0.0091	0.8305	0.0087	0.8545	0.0091	0.8263	0.0089	0.8540	0.0020	0.8413	0.0016	0.8519	0.0020	0.8394	0.0017
	[5]	0.8475	0.0091	0.8592	0.0091	0.8305	0.0087	0.8545	0.0091	0.8263	0.0089	0.8540	0.0020	0.8413	0.0016	0.8519	0.0020	0.8394	0.0017
	[6]	0.8475	0.0088	0.8590	0.0088	0.8312	0.0084	0.8545	0.0088	0.8272	0.0086	0.8540	0.0019	0.8416	0.0016	0.8519	0.0019	0.8398	0.0016
	[7]	0.8485	0.0074	0.8580	0.0074	0.8348	0.0071	0.8543	0.0074	0.8314	0.0073	0.8538	0.0016	0.8433	0.0014	0.8521	0.0016	0.8418	0.0014
	[8]	0.8485	0.0072	0.8579	0.0072	0.8353	0.0070	0.8543	0.0072	0.8320	0.0071	0.8538	0.0016	0.8435	0.0014	0.8521	0.0016	0.8420	0.0014
	[9]	0.8483	0.0066	0.8575	0.0066	0.8366	0.0064	0.8542	0.0066	0.8336	0.0065	0.8537	0.0015	0.8442	0.0013	0.8522	0.0015	0.8428	0.0013
	[10]	0.8496	0.0039	0.8544	0.0039	0.8425	0.0039	0.8525	0.0039	0.8407	0.0039	0.8529	0.0009	0.8472	0.0009	0.8520	0.0009	0.8463	0.0009
	[11]	0.8496	0.0039	0.8544	0.0039	0.8425	0.0039	0.8525	0.0039	0.8407	0.0039	0.8529	0.0009	0.8472	0.0009	0.8520	0.0009	0.8463	0.0009
	[12]	0.8495	0.0038	0.8543	0.0038	0.8428	0.0037	0.8525	0.0038	0.8411	0.0038	0.8528	0.0009	0.8474	0.0008	0.8520	0.0009	0.8465	0.0008
5	[1]	0.8371	0.0180	0.8589	0.0178	0.8505	0.0124	0.8500	0.0178	0.8011	0.0174	0.8538	0.0032	0.8483	0.0023	0.8502	0.0032	0.8306	0.0025
	[2]	0.8370	0.0177	0.8588	0.0175	0.8504	0.0122	0.8500	0.0175	0.8020	0.0171	0.8537	0.0031	0.8484	0.0023	0.8502	0.0031	0.8310	0.0024
	[3]	0.8369	0.0166	0.8583	0.0164	0.8503	0.0117	0.8502	0.0164	0.8046	0.0161	0.8536	0.0029	0.8485	0.0022	0.8502	0.0029	0.8319	0.0023
	[4]	0.8474	0.0087	0.8589	0.0087	0.8536	0.0072	0.8545	0.0086	0.8276	0.0085	0.8539	0.0019	0.8511	0.0016	0.8519	0.0019	0.8400	0.0016
	[5]	0.8474	0.0086	0.8589	0.0087	0.8536	0.0072	0.8545	0.0086	0.8276	0.0085	0.8539	0.0019	0.8511	0.0016	0.8519	0.0019	0.8400	0.0016
	[6]	0.8474	0.0084	0.8587	0.0084	0.8536	0.0070	0.8545	0.0084	0.8283	0.0082	0.8539	0.0018	0.8511	0.0015	0.8520	0.0018	0.8403	0.0016
	[7]	0.8485	0.0070	0.8578	0.0070	0.8535	0.0061	0.8543	0.0070	0.8325	0.0069	0.8538	0.0016	0.8515	0.0013	0.8522	0.0016	0.8422	0.0014
	[8]	0.8484	0.0069	0.8577	0.0069	0.8535	0.0060	0.8543	0.0069	0.8329	0.0068	0.8537	0.0015	0.8515	0.0013	0.8522	0.0015	0.8424	0.0014
	[9]	0.8482	0.0064	0.8574	0.0064	0.8534	0.0056	0.8542	0.0064	0.8342	0.0063	0.8536	0.0014	0.8515	0.0012	0.8521	0.0014	0.8431	0.0013
	[10]	0.8495	0.0038	0.8543	0.0038	0.8521	0.0035	0.8525	0.0037	0.8413	0.0037	0.8528	0.0009	0.8517	0.0008	0.8520	0.0009	0.8466	0.0008
	[11]	0.8495	0.0038	0.8543	0.0037	0.8521	0.0035	0.8525	0.0037	0.8413	0.0037	0.8528	0.0009	0.8517	0.0008	0.8520	0.0009	0.8466	0.0008
	[12]	0.8495	0.0036	0.8542	0.0036	0.8520	0.0034	0.8525	0.0036	0.8416	0.0036	0.8528	0.0009	0.8517	0.0008	0.8520	0.0009	0.8468	0.0008

TABLE 6.7: The 95% ACI, bootstrap CI and HPD credible intervals of parameter λ and entropy H, when $\lambda = 1.5$ and $H = 0.8523$.

		ACI				Bootstrap								HPD							
						θ				$H(\theta)$				θ				$H(\theta)$			
		θ		$H(\theta)$		Boot-p		Boot-t		Boot-p		Boot-t		Prior 0		Prior 1		Prior 0		Prior 1	
k	[CS]	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AE	MSE	AL	CP	AL	CP
3	[1]	1.5864	0.912	0.5257	0.948	1.5597	0.927	1.8239	0.952	0.5324	0.927	0.5335	0.952	1.0150	0.9550	0.8815	0.9600	0.3363	0.9540	0.3055	0.9590
	[2]	1.5672	0.912	0.5198	0.947	1.5411	0.927	1.8007	0.952	0.5271	0.927	0.5279	0.952	1.0044	0.9550	0.8740	0.9600	0.3329	0.9540	0.3027	0.9590
	[3]	1.5059	0.914	0.5007	0.951	1.4806	0.925	1.7356	0.951	0.5105	0.925	0.5103	0.951	0.9710	0.9550	0.8506	0.9600	0.3220	0.9540	0.2941	0.9590
	[4]	1.1333	0.935	0.3746	0.948	1.1233	0.945	1.1811	0.950	0.3721	0.945	0.3721	0.950	0.7579	0.9560	0.7004	0.9580	0.2514	0.9530	0.2387	0.9580
	[5]	1.1323	0.935	0.3743	0.948	1.1224	0.945	1.1799	0.950	0.3719	0.945	0.3718	0.950	0.7573	0.9560	0.7000	0.9580	0.2512	0.9530	0.2385	0.9580
	[6]	1.1125	0.936	0.3680	0.945	1.1025	0.946	1.1595	0.951	0.3657	0.946	0.3658	0.951	0.7451	0.9570	0.6900	0.9580	0.2472	0.9540	0.2350	0.9580
	[7]	1.0100	0.943	0.3343	0.945	1.0479	0.954	1.0839	0.954	0.3458	0.955	0.3459	0.954	0.6820	0.9570	0.6394	0.9580	0.2265	0.9530	0.2170	0.9570
	[8]	0.9977	0.943	0.3304	0.944	1.0355	0.956	1.0695	0.954	0.3419	0.956	0.3416	0.954	0.6741	0.9570	0.6327	0.9580	0.2239	0.9540	0.2146	0.9580
	[9]	0.9544	0.941	0.3165	0.944	0.9876	0.954	1.0225	0.953	0.3273	0.954	0.3273	0.953	0.6469	0.9560	0.6094	0.9580	0.2149	0.9520	0.2065	0.9570
	[10]	0.7133	0.938	0.2372	0.941	0.7416	0.950	0.7605	0.950	0.2472	0.950	0.2472	0.950	0.4918	0.9490	0.4758	0.9480	0.1637	0.9470	0.1602	0.9470
	[11]	0.7130	0.938	0.2371	0.941	0.7413	0.950	0.7601	0.950	0.2471	0.950	0.2471	0.950	0.4916	0.9490	0.4756	0.9480	0.1636	0.9470	0.1601	0.9470
	[12]	0.6995	0.938	0.2327	0.942	0.7275	0.949	0.7461	0.950	0.2427	0.950	0.2427	0.950	0.4826	0.9480	0.4674	0.9470	0.1607	0.9440	0.1573	0.9460
5	[1]	1.5452	0.913	0.5129	0.948	1.5195	0.925	1.7784	0.952	0.5210	0.925	0.5219	0.952	0.9924	0.9560	0.8658	0.9600	0.3290	0.9540	0.3050	0.9570
	[2]	1.5293	0.912	0.5081	0.947	1.5031	0.925	1.7597	0.953	0.5162	0.925	0.5175	0.953	0.9836	0.9560	0.8594	0.9600	0.3261	0.9540	0.3027	0.9570
	[3]	1.4766	0.914	0.4917	0.950	1.4514	0.923	1.7052	0.951	0.5024	0.923	0.5023	0.951	0.9544	0.9550	0.8385	0.9600	0.3167	0.9540	0.2950	0.9570
	[4]	1.1034	0.936	0.3651	0.946	1.0927	0.945	1.1496	0.951	0.3628	0.945	0.3629	0.951	0.7394	0.9570	0.6853	0.9580	0.2453	0.9530	0.2345	0.9570
	[5]	1.1026	0.937	0.3649	0.946	1.0919	0.945	1.1488	0.951	0.3626	0.945	0.3627	0.951	0.7389	0.9570	0.6849	0.9580	0.2452	0.9530	0.2344	0.9570
	[6]	1.0849	0.936	0.3592	0.945	1.0753	0.946	1.1320	0.951	0.3576	0.946	0.3577	0.951	0.7280	0.9570	0.6759	0.9580	0.2416	0.9530	0.2312	0.9570
	[7]	0.9840	0.944	0.3260	0.946	1.0189	0.953	1.0542	0.953	0.3369	0.953	0.3370	0.953	0.6656	0.9570	0.6254	0.9580	0.2210	0.9540	0.2130	0.9570
	[8]	0.9739	0.943	0.3228	0.946	1.0086	0.954	1.0427	0.953	0.3336	0.954	0.3336	0.953	0.6592	0.9570	0.6199	0.9590	0.2189	0.9540	0.2111	0.9570
	[9]	0.9368	0.941	0.3109	0.944	0.9681	0.952	1.0034	0.954	0.3215	0.953	0.3214	0.954	0.6355	0.9560	0.5995	0.9580	0.2112	0.9530	0.2041	0.9560
	[10]	0.6948	0.938	0.2312	0.942	0.7218	0.950	0.7404	0.950	0.2409	0.950	0.2409	0.950	0.4794	0.9480	0.4644	0.9480	0.1596	0.9450	0.1565	0.9480
	[11]	0.6945	0.938	0.2311	0.942	0.7216	0.950	0.7401	0.950	0.2408	0.950	0.2408	0.950	0.4793	0.9480	0.4643	0.9480	0.1596	0.9460	0.1564	0.9480
	[12]	0.6826	0.940	0.2272	0.942	0.7091	0.949	0.7275	0.950	0.2368	0.949	0.2368	0.950	0.4713	0.9490	0.4569	0.9470	0.1569	0.9460	0.1539	0.9480

6.6 Real Data Application

A real data analysis is done in this portion to demonstrate the feasibility of the considered MW lifetime model and methodology used in this chapter. Here, we consider the tensile strength (in GPa) of 100 observations of carbon fibers, which are as follows:

3.70, 3.11, 4.42, 3.28, 3.75, 2.96 ,3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27 ,2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79 ,1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22 ,5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83 ,1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22 ,2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

Originally this set was reported by Nichols and Padgett (2006) and further studied by Mohammed et al. (2017) and Xie and Gui (2020), respectively. To begin, we use the scaled TTT transform to examine the behaviour of the failure rate function of the considered data set. The scaled TTT is calculated as follows:

$$\psi(r/n) = \left[\sum_{j=1}^{r} t_{(i)} + (n-r)t_r\right] \Big/ \left(\sum_{j=1}^{r} t_{(i)}\right),$$

where, $t_{(i)}, \quad i = 1, 2, \ldots, n$ denotes the ith order statistic and $r = 1, 2, \ldots, n$. If the plot $(r/n, \psi(r/n))$ is convex (concave), the failure rate function has a decreasing (increasing) shape. For more details one may refer Mudholkar et al. (1996). The scaled TTT plot of the considered data set is displayed in Figure 6.1. From Figure 6.1, it is clear that the considered data set follows increasing failure rate function. This empirical behavior of failure rate function is quite similar to the considered MW lifetime model.

Further, we check whether the considered data set is well fitted to the MW lifetime model or not using some goodness-of-fit tests. Here, we consider Kolmogrov-Smirnov (KS) and Anderson-Darling (AD) tests statistics along with their corresponding p-values. The KS and AD statistics with their corresponding p-values (in parenthesis) are 0.0884 (0.4145) and 0.7977 (0.4824), respectively. According to the obtained p-values, the considered model is fitted quite well for the considered real data set. Also, to see the feasibility of fitting graphically for the considered real data set, we plot empirical & fitted cdfs and probability-probability (P-P) plots of considered MW lifetime model and displayed in Figure 6.2. Figure 6.2, also suggests the considered model is well fitted with the considered real data set.

Moreover, to illustrate the methodologies adopted throughout the study, we can consider the PFFC data. After dividing the above-mentioned complete data set of sample size 100 into

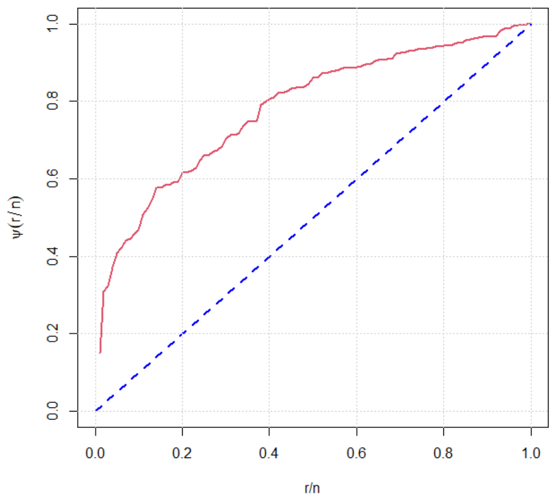


FIGURE 6.1: TTT plots under consideration of real data set.

$n = 25$ groups, each with $k = 4$ items, the FFC sample has been collected. The grouped data and the corresponding FFC samples are reported in Table 6.8. The items with "+" within each group indicate the first failure. Next, we generate six PFFC samples using 6 different combinations of censoring plans for the obtained first failure censored data in Table 6.8, with m=(40 & 80) % of n. The several censoring plans and the corresponding PFFC samples are presented in Table 6.9. We construct the ML and Bayes estimates of λ and $H(\lambda)$ for all censoring plans. We utilise non-informative priors to construct Bayes estimators since we don't have prior knowledge. Under the LINEX loss function, the Bayes estimators are calculated using TK approximation and MCMC techniques at two distinct values of the loss parameter c= (-0.5 & 0.5). For parameter λ and entropy $H(\lambda)$, we also build 95% ACI, boot-p, boot-t CIs, and HPD credible intervals. ML and Bayes estimates of the parameter and entropy are shown in Table 6.10. Also, various interval estimates of parameter and entropy are reported in Table 6.11 and 6.12, respectively. We use graphical diagnostic tools such as the trace map, boxplot, and histogram with Gaussian density plots to confirm the convergence of their stationary distributions, as illustrated in Figure 6.3. A random dispersion around the mean (shown by a thick red line) and a fine variety of parameter sequences can be seen in the trace plot. As shown by the boxplots and histograms of created samples, the posterior distribution is almost symmetric, meaning that the mean may be chosen as the best estimate of the parameter.

TABLE 6.8: Grouped real data set (Observation with "+" indicates the first failure (FF) in the group).

Groups → Items↓	1	2	3	4	5	6	7	8	9	10	11	12	13
I	3.27	2.05	3.33	1.87	2.03	3.68	2.87	2.67+	1.84	1.73	2.82	2.77	2.73
II	2.97	1.61	4.38	2.97	2.12	2.68	1.59+	2.96	0.39+	3.19	2.41+	2.17	2.88
III	3.11	3.11	1.69	1.57+	0.85+	4.90	1.89	3.09	2.17	1.57+	3.60	3.51	3.75
IV	2.03+	1.25+	1.18+	1.59	1.84	2.38+	2.43	4.20	2.35	2.93	3.22	1.08+	1.69+
FF Obs.	2.03	1.25	1.18	1.57	0.85	2.38	1.59	2.67	0.39	1.57	2.41	1.08	1.69
Ordered FF obs.	0.39	0.81	0.85	0.98	1.08	1.12	1.18	1.22	1.25	1.36	1.41	1.47	1.57
Groups→ Items↓	14	15	16	17	18	19	20	21	22	23	24	25	
I	5.56	2.81	2.55	4.70	1.36+	3.22	1.61	3.15	4.91	3.31	2.76	0.98+	
II	1.41+	3.39	2.17	2.59	2.83	1.71+	2.85	4.42	1.17	1.92	5.08	3.39	
III	2.48	3.68	3.56	3.19	2.74	3.65	1.47+	2.00+	1.12+	1.80+	3.28	2.50	
IV	2.95	0.81+	1.22+	2.56+	2.53	3.15	2.79	2.81	3.70	2.55	2.48+	3.31	
FF Obs.	1.41	0.81	1.22	2.56	1.36	1.71	1.47	2.00	1.12	1.80	2.48	0.98	
Ordered FF obs.	1.57	1.59	1.69	1.71	1.80	2.00	2.03	2.38	2.41	2.48	2.56	2.67	

TABLE 6.9: Censoring schemes and progressively first failure censored samples corresponding to considered real data set.

k	[CS]	Schemes	Progressively first failure censored samples
4	[1]	(15,0*9)	0.39 ,1.80 ,1.84 ,2.03 ,2.12 ,2.17, 2.48, 2.50, 2.73 ,2.77
	[2]	(5*3,0*7)	0.39 ,1.18, 1.57, 2.03, 2.12, 2.17, 2.48, 2.50, 2.73, 2.77
	[3]	(0*9,15)	0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.18, 1.22, 1.25, 1.36
4	[4]	(5,0*19)	0.39, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.59, 1.61, 1.69 ,1.80, 1.84, 2.03, 2.12, 2.17, 2.48, 2.50, 2.73, 2.77
	[5]	(2,3,0*18)	0.39, 0.98, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.59, 1.61, 1.69, 1.80, 1.84, 2.03, 2.12, 2.17, 2.48, 2.50, 2.73, 2.77
	[6]	(0*19,5)	0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.59, 1.61, 1.69, 1.80, 1.84, 2.03, 2.12

TABLE 6.10: ML and Bayes estimates of λ and $H(\lambda)$ under consideration of real data set for $k=4, n=25$, m=(40 & 80)% of n and c=(-0.5 & 0.5).

				TK Bayes				M-H Bayes			
		MLE		$c=-0.5$		$c=0.5$		$c=-0.5$		$c=0.5$	
k	[CS]	$\hat{\lambda}$	$\hat{H}$	$\hat{\lambda}$	H	$\hat{\lambda}$	$\hat{H}$	$\hat{\lambda}$	$\hat{H}$	$\hat{\lambda}$	$\hat{\lambda}$
4	[1]	9.2897	1.7640	13.4153	1.7800	8.7078	1.7714	9.6163	1.7467	8.5906	1.7436
	[2]	10.6695	1.8333	11.5620	1.8493	9.9031	1.8411	11.1367	1.8167	9.8262	1.8137
	[3]	5.6674	1.5169	6.5958	1.5330	5.5463	1.5304	5.7261	1.5016	5.3784	1.4988
4	[4]	6.6806	1.5992	7.2764	1.6083	6.5572	1.6046	6.7445	1.5903	6.4575	1.5887
	[5]	6.7637	1.6054	7.3707	1.6145	6.6362	1.6107	6.8301	1.5965	6.5371	1.5949
	[6]	5.7635	1.5254	6.2046	1.5344	5.6944	1.5308	5.8011	1.5169	5.5937	1.5153

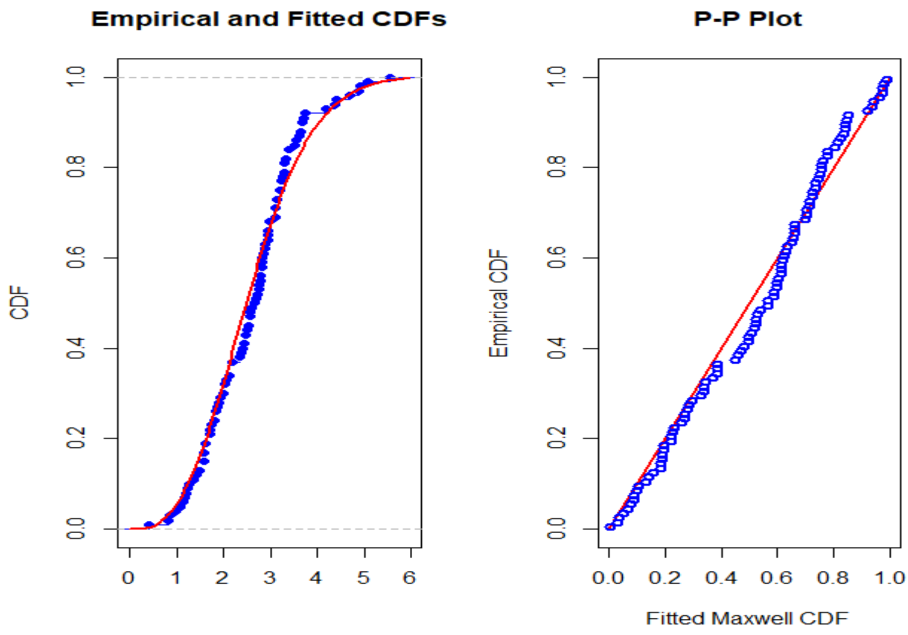


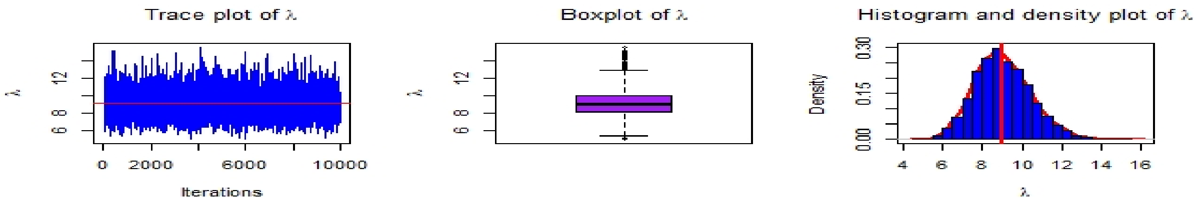
FIGURE 6.2: Empirical and fitted Maxwell distribution plots for real data.

TABLE 6.11: The 95% ACI, boot-p, boot-t confidence/HPD credible intervals of parameter λ, under consideration of real data set for $k = 4, n = 25$ m=(40 & 80)% of n and c=(-0.5 & 0.5).

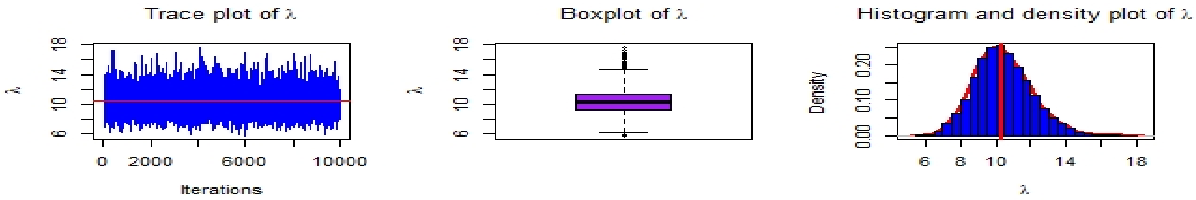
k	[CS]	ACI	Bootstrap		HPD credible intercals	
			boot-p	boot-t	c=-0.5	c=0.5
4	[1]	(4.973, 13.606)	(6.636, 12.366)	(6.213, 11.944)	(6.357, 11.872)	(6.357, 11.872)
	[2]	(5.804, 15.535)	(7.816, 13.602)	(7.736, 13.523)	(7.365, 13.587)	(7.365, 13.587)
	[3]	(3.157, 8.178)	(3.880, 8.365)	(2.970, 7.454)	(3.967, 7.187)	(3.967, 7.187)
	[4]	(4.478, 8.883)	(5.006, 8.771)	(4.590, 8.355)	(5.125, 8.058)	(5.125,8.058)
	[5]	(4.538, 8.989)	(5.044, 8.824)	(4.703, 8.483)	(5.192, 8.156)	(5.192, 8.156)
	[6]	(3.893, 7.634)	(4.064, 7.962)	(3.565, 7.463)	(4.443, 6.937)	(4.443, 6.938)

6.7 Concluding Remarks

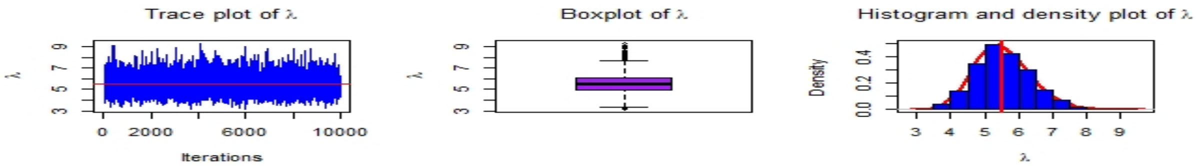
In this chapter, we developed some inferences of associated parameters and entropy of MW lifetime model based on PFFC data from both classical and Bayesian points of view. For classical estimation procedures, we applied the EM algorithm to compute ML estimates. Also, we obtained asymptotic and two bootstraps (boot-p & boot-t) confidence intervals. Further, in the case of Bayesian estimation procedures, we applied two approximation techniques such as TK approximation and M-H algorithm to approximate the Bayes estimators under LINEX loss function. In addition, we use the M-H algorithm to compute the HPD credible intervals. Moreover, a numerical computation is performed employing a Monte Carlo simulation and real data applications to know the performance of different estimators and the potentiality of



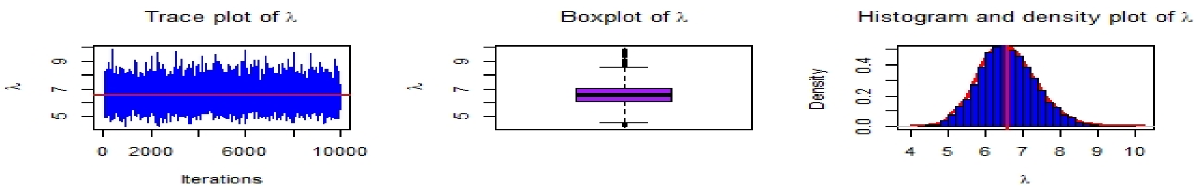
(A) Scheme 1.



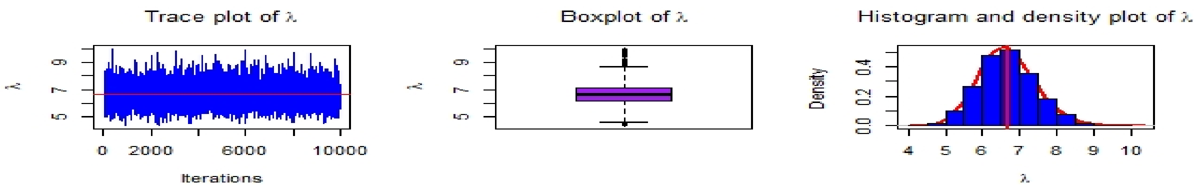
(B) Scheme 2.



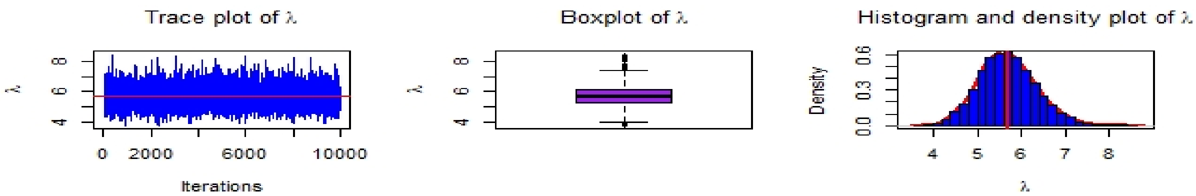
(C) Scheme 3.



(D) Scheme 4.



(E) Scheme 5.



(F) Scheme 6.

FIGURE 6.3: MCMC diagnostic plots for different censoring schemes under consideration of real data set.

TABLE 6.12: The 95% ACI, boot-p, boot-t confidence/HPD credible intervals of entropy H, under consideration of real data set for $k = 4, n = 25$ m=(40 & 80)% of n and c=(-0.5 & 0.5).

k	[CS]	ACI	Bootstrap		HPD credible intercals	
			boot-p	boot-t	c=-0.5	c=0.5
4	[1]	(1.532, 1.996)	(1.596, 1.907)	(1.621, 1.932)	(1.591, 1.900)	(1.591, 1.900)
	[2]	(1.605, 2.061)	(1.678, 1.955)	(1.712, 1.989)	(1.664, 1.968)	(1.664, 1.968)
	[3]	(1.295, 1.738)	(1.327, 1.712)	(1.322, 1.706)	(1.353, 1.648)	(1.353, 1.648)
	[4]	(1.434, 1.764)	(1.455, 1.735)	(1.463, 1.743)	(1.472, 1.698)	(1.472, 1.698)
	[5]	(1.441, 1.770)	(1.459, 1.738)	(1.472, 1.752)	(1.478, 1.704)	(1.478, 1.704)
	[6]	(1.363, 1.688)	(1.351, 1.687)	(1.364, 1.700)	(1.400, 1.623)	(1.400, 1.623)

considered methodologies and models. Finally, based on the observed data we recommend the use of Bayesian estimation of the parameter and entropy based on the MCMC method for the MW lifetime model when the prior information about the parameter is available. If prior information is not available then the ML estimates may be are preferred.

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