Chapter 3

Statistical Inference in Inverse Weibull Lifetime Model using Randomly Censored Data [*]

3.1 Introduction

The main objective of this chapter is to develop classical and Bayesian inferences about the associated model parameters and reliability characteristics of the inverse Weibull (IW) lifetime model using randomly censored data. The concept of random censoring has previously been thoroughly explored in Chapter 2.

The Weibull lifetime model is the most popular and widely used lifetime model in reliability and life testing experiments due to its flexible probability density and failure rate functions. The Weibull lifetime model can have an increasing, decreasing, or constant failure rate depending upon the values of its shape parameter. However, given lifetime data with a non-monotone failure rate pattern, the Weibull lifetime model does not provide a good parametric fit. This motivates authors to investigate other, more realistic lifetime models. The failure rate function of the IW lifetime model is either unimodal or decreasing depending on the shape parameter. There are a variety of real-life instances where data shows a non-monotone, unimodal failure rate, such as cancer patients' remission times, wind speed data, rainfall data, and so on. As a result, if an empirical investigation indicates that the underlying distribution's failure rate function has a unimodal form, the IW lifetime model may be utilised to examine such data sets.

[*]Part of this chapter has been published in the form of a research paper with the following details: Kumar, K., and Kumar, I. (2019). Estimation of inverse Weibull distribution based on randomly censored data. Statistica, 79(1): 47-74.

Recently, the IW lifetime model was studied by several researchers in different disciplines, for example: Kundu and Howlader (2010) studied Bayesian inferences and prediction of the IW lifetime model for type II censored data, Sultan et al. (2014) discussed Bayesian and ML estimation methods of the IW lifetime model parameters under progressive type II censoring, Akgül et al. (2016) used IW lifetime model for the wind speed data, Akgül and Şenoğlu (2018) compared different estimation methods for rainfall data fitted on IW lifetime model, Krishna et al. (2019) studied stress-strength reliability of IW lifetime model under progressive first failure censoring, Basheer et al. (2021) studied IW lifetime model for E-Bayesian and Hierarchical estimation procedures, the multi-component stress-strength for IW lifetime model is discussed by Jana and Bera (2022), the IW lifetime model under Type-I hybrid censoring is discussed by Kazemi and Azizpoor (2021) and reference cited therein.

The rest of the chapter is structured as follows: In Section 3.2, the IW lifetime model based on a randomly censored sample is discussed. We derive ML estimates of the parameters and reliability characteristics in Section 3.3. Based on the expected Fisher information (EFI) matrix, ACIs with corresponding CPs of the unknown parameters are also computed. The expected test time (ETT) of the experiment is discussed in Section 3.4, which is based on randomly censored data from IW lifetime model. In Section 3.5, Bayes estimators of the parameters and reliability characteristics under SELF with gamma informative and non-informative priors using TK approximation and MCMC techniques are obtained. The HPD credible intervals for the parameters based on MCMC techniques are also developed. Section 3.6 deals with an MC simulation study to compare the performance of the estimators developed in this chapter. In Section 3.7 findings are illustrated by a randomly censored real data set. Finally, a concluding remark is appeared in Section 3.8.

3.2 The Model

The pdf and corresponding cdf of IW lifetime model, respectively, are given by

$$f(x;\alpha,\beta) = \alpha\beta x^{-(\alpha+1)} e^{-\beta x^{-\alpha}}\ ;\ \ \alpha > 0, \beta > 0,\ x > 0, \tag{3.1}$$

$$F(x;\alpha,\beta) = e^{-\beta x^{-\alpha}}\ ;\ \alpha > 0, \beta > 0,\ x > 0. \tag{3.2}$$

The survival (or reliability) and failure rate (or hazard) functions, respectively, are given by

$$S(x;\alpha,\beta) = 1 - e^{-\beta x^{-\alpha}}\ ;\ \ \alpha > 0, \beta > 0,\ x > 0. \tag{3.3}$$

and

$$h(x;\alpha,\beta) = \frac{\alpha\beta x^{-(\alpha+1)}}{e^{\beta x^{-\alpha}} - 1} \;;\;\; \alpha > 0, \beta > 0,\; x > 0, \tag{3.4}$$

where, α and β are the shape and scale parameters, respectively. Figure 3.1 shows a visualization of the failure rate function of IW lifetime model for various values of the shape parameter α and scaling parameter β. Assume that the lifetime X and censoring time T, respectively, fol-

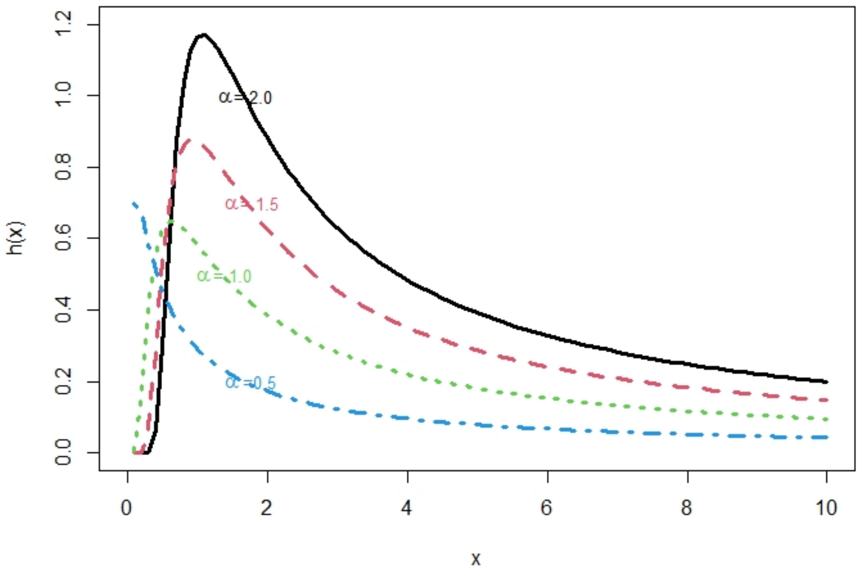

FIGURE 3.1: The plot of the failure rate function IW lifetime model with β=1.

low IW(α,β) and IW(α,λ). Then by using equation (2.6), the joint pdf of randomly censored variables (Y,D) is given by

$$f_{Y,D}(y,d,\alpha,\beta,\lambda) = \alpha\beta^{d}\lambda^{1-d}y^{-(\alpha+1)}e^{-y^{\alpha(\beta d+\lambda(1-d))}}(1-e^{-\lambda y^{-\alpha}})^{d}(1-e^{-\beta y^{-\alpha}})^{1-d} \tag{3.5}$$

and the probability of failure is obtained as

$$p = \text{P[An item fails]} = P[d=1] = P[X \leq T] = \int_0^\infty f_T(t)F_X(t)dt = \frac{\lambda}{\beta+\lambda}. \tag{3.6}$$

3.3 Maximum Likelihood Estimation

Let $(\mathbf{y},\mathbf{d}) = ((y_1,d_1),(y_2,d_2),\ldots,(y_n,d_n))$ be a randomly censored sample from model in (3.5). The likelihood function is given by

$$L(\alpha,\beta,\lambda|\mathbf{y},\mathbf{d}) = \alpha^n\beta^m\lambda^{(n-m)}\prod_{i=1}^{n} y_i^{-(\alpha+1)} e^{-\left(\beta\sum_{i=1}^{n} d_i y_i^{-\alpha}+\lambda\sum_{i=1}^{n}(1-d_i)y_i^{-\alpha}\right)} \prod_{i=1}^{n}\left(1-e^{-\lambda y_i^{-\alpha}}\right)^{d_i}\prod_{i=1}^{n}\left(1-e^{-\beta y_i^{-\alpha}}\right)^{(1-d_i)}, \tag{3.7}$$

where, $m = \sum_{i=1}^{n} d_i$ denotes the number of failures.
Thus, the log-likelihood function becomes

$$l(\alpha,\beta,\lambda) = n\ln\alpha + mln\beta + (n-m)\ln\lambda - (\alpha+1)S - \beta\sum_{i=1}^{n} d_i y_i^{-\alpha} - \lambda\sum_{i=1}^{n}(1-d_i)y_i^{-\alpha} + \sum_{i=1}^{n} d_i\ln\left(1-e^{-\lambda y_i^{-\alpha}}\right) + \sum_{i=1}^{n}(1-d_i)\ln\left(1-e^{-\beta y_i^{-\alpha}}\right), \tag{3.8}$$

where, $S = \sum_{i=1}^{n}\ln y_i$ denotes the log total time on test.
The corresponding normal equations are obtained as

$$\frac{\partial l(\alpha,\beta,\lambda)}{d\alpha} = \frac{m}{\beta} - S + \beta\sum_{i=1}^{n} d_i y_i^{-\alpha}\ln y_i + \lambda\sum_{i=1}^{n}(1-d_i)y_i^{-\alpha}\ln y_i - \sum_{i=1}^{n}\frac{d_i\lambda e^{-\lambda y_i^{-\alpha}} y_i^{-\alpha}\ln y_i}{\left(1-e^{-\lambda y_i^{-\alpha}}\right)} - \sum_{i=1}^{n}\frac{(1-d_i)\beta e^{-\beta y_i^{-\alpha}}\ln y_i}{\left(1-e^{-\beta y_i^{-\alpha}}\right)} = 0 \tag{3.9}$$

$$\frac{\partial l(\alpha,\beta,\lambda)}{d\beta} = \frac{m}{\beta} - \sum_{i=1}^{n} d_i y_i^{-\alpha} + \sum_{i=1}^{n}\frac{(1-d_i)e^{-\beta y_i^{-\alpha}} y_i^{-\alpha}}{\left(1-e^{-\beta y_i^{-\alpha}}\right)} = 0 \tag{3.10}$$

$$\frac{l(\alpha,\beta,\lambda)}{d\beta} = \frac{n-m}{\lambda} - \sum_{i=1}^{n}(1-d_i)y_i^{-\alpha} + \sum_{i=1}^{n}\frac{d_i e^{-\lambda y_i^{-\alpha}} y_i^{-\alpha}}{(1-e^{-\lambda y_i^{-\alpha}})} = 0 \tag{3.11}$$

The ML estimates of α, β and λ say $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$, respectively, are the solutions of the normal equations (3.9), (3.10) and (3.11). For the solution of the system of these normal equations, some suitable iterative procedure like the Newton-Raphson method can be used. It is important to note that in this study, we used a Broyden-Fletcher-Goldfarb-Shanno (BFGS) quasi-Newton method for computation. Once we get the desired ML estimates, using the invariance property of ML estimators, see, Zehna (1966), the ML estimates of the survival and failure rate functions, respectively, are obtained as

$$\hat{S}(t) = 1 - e^{-\hat{\beta} t^{-\hat{\alpha}}} \; ; \; t > 0 \quad \text{and} \quad \hat{h}(t) = \frac{\hat{\alpha}\hat{\beta} t^{-(\hat{\alpha}+1)}}{e^{\hat{\beta} x^{-\hat{\alpha}}} - 1} \; ; \; t > 0.$$

3.3.1 Expected Fisher Information Matrix

Here, we compute the Fisher information matrix for the construction of ACIs of the unknown parameters. Zheng and Gastwirth (2001) suggested the EFI matrix for randomly censored data using the failure rate functions. The Fisher information about parameters, say θ=(α, β, λ) contained in a randomly censored sample $(\mathbf{y}, \mathbf{d})$ of size n from the model in (3.5) is given by

$$I^{Y,D}(\theta) = n \times \begin{bmatrix} I_{11}(\theta) & I_{12}(\theta) & I_{13}(\theta) \\ & I_{22}(\theta) & I_{23}(\theta) \\ & & I_{33}(\theta) \end{bmatrix}$$

where,

$$\begin{aligned} I_{11}(\theta) &= \int_0^\infty \left(\frac{\partial}{\partial \alpha} \ln h_X(x) \right)^2 f_X(x) \bar{F}_T(x) dx + \int_0^\infty \left(\frac{\partial}{\partial \alpha} \ln h_T(x) \right)^2 f_T(x) \bar{F}_X(x) dx \\ &= \alpha\beta \int_0^\infty \left(\frac{1}{\alpha} - \ln x - \frac{\beta x^{-\alpha} \ln x}{1 - e^{-\beta x^{-\alpha}}} \right)^2 x^{-(\alpha+1)} e^{-\beta x^{-\alpha}} (1 - e^{-\lambda x^{-\alpha}}) dx \\ &+ \alpha\lambda \int_0^\infty \left(\frac{1}{\alpha} - \ln x - \frac{\lambda x^{-\alpha} \ln x}{1 - e^{-\lambda x^{-\alpha}}} \right)^2 x^{-(\alpha+1)} e^{-\lambda x^{-\alpha}} (1 - e^{-\beta x^{-\alpha}}) dx, \end{aligned}$$

$$\begin{aligned} I_{12}(\theta) &= \int_0^\infty \left(\frac{\partial}{\partial \alpha} \ln h_X(x) \right) \left(\frac{\partial}{\partial \beta} \ln h_X(x) \right) f_X \bar{F}_T(x) dx \\ &+ \int_0^\infty \left(\frac{\partial}{\partial \alpha} \ln h_T(x) \right) \left(\frac{\partial}{\partial \beta} \ln h_T(x) \right) f_T(x) \bar{F}_X(x) dx \\ &= \alpha\beta \int_0^\infty \left(\frac{1}{\alpha} - \ln x + \frac{\beta x^{-\alpha} \ln x}{1 - e^{-\beta x^{-\alpha}}} \right) \left(\frac{1}{\beta} + \frac{x^{-\alpha}}{1 - e^{-\beta x^{-\alpha}}} \right) \\ &\times x^{-(\alpha+1)} e^{-\beta x^{-\alpha}} (1 - e^{-\lambda x^{-\alpha}}) dx, \end{aligned}$$

$$\begin{aligned}I_{13}(\theta) &= \int_0^\infty \left(\frac{\partial}{\partial\alpha}\ln h_X(x)\right)\left(\frac{\partial}{\partial\lambda}\ln h_X(x)\right) f_X(x)\bar{F}_T(x)dx\\ &+\int_0^\infty \left(\frac{\partial}{\partial\alpha}\ln h_T(x)\right)\left(\frac{\partial}{\partial\lambda}\ln h_T(x)\right) f_T\bar{F}_X(x)dx\\ &=\alpha\lambda\int_0^\infty\left(\frac{1}{\alpha}-\ln x+\frac{\lambda x^{-\alpha}\ln x}{1-e^{-\lambda x^{-\alpha}}}\right)\left(\frac{1}{\lambda}-\frac{x^{-\alpha}}{1-e^{-\lambda x^{-\alpha}}}\right)\\ &\times x^{-(\alpha+1)}e^{-\lambda x^{-\alpha}}\left(1-e^{-\beta x^{-\alpha}}\right)dx,\end{aligned}$$

$$\begin{aligned}I_{22}(\theta) &= \int_0^\infty \left(\frac{\partial}{\partial\beta}\ln h_X(x)\right)^2 f_X(x)\bar{F}_T(x)dx+\int_0^\infty \left(\frac{\partial}{\partial\beta}\ln h_T(x)\right)^2 f_T(x)\bar{F}_X(x)dx\\ &=\alpha\beta\int_0^\infty\left(\frac{1}{\beta}-\frac{x^{-\alpha}}{1-e^{-\beta x^{-\alpha}}}\right)^2 x^{-(\alpha+1)}e^{-\beta x^{-\alpha}}(1-e^{-\lambda x^{-\alpha}})dx,\end{aligned}$$

$$\begin{aligned}I_{23}(\theta) &= \int_0^\infty \left(\frac{\partial}{\partial\beta}\ln h_X(x)\right)\left(\frac{\partial}{\partial\lambda}\ln h_X(x)\right) f_X(x)\bar{F}_T(x)dx\\ &+\int_0^\infty \left(\frac{\partial}{\partial\beta}\ln h_T(x)\right)\left(\frac{\partial}{\partial\lambda}\ln h_T(x)\right) f_T(x)\bar{F}_X(x)dx=0,\end{aligned}$$

$$\begin{aligned}I_{33}(\theta) &= \int_0^\infty \left(\frac{\partial}{\partial\lambda}\ln h_X(x)\right)^2 f_X(x)\bar{F}_T(x)dx+\int_0^\infty \left(\frac{\partial}{\partial\lambda}\ln h_T(x)\right)^2 f_T(x)\bar{F}_X(x)dx\\ &=\alpha\lambda\int_0^\infty\left(\frac{1}{\lambda}-\frac{x^{-\alpha}}{1-e^{-\lambda x^{-\alpha}}}\right)^2 x^{-(\alpha+1)}e^{-\lambda x^{-\alpha}}(1-e^{-\beta x^{-\alpha}})dx.\end{aligned}$$

Here, h_X and h_T are the failure rate functions of IW(α,β) and IW(α,λ), respectively. The elements of the EFI matrix $I^{Y,D}(\theta)$ need to be compute numerically.

Under some mild regularity conditions, $\hat{\theta}=(\hat{\alpha},\hat{\beta},\hat{\lambda})$ follows approximately trivariate normal distribution with mean $(\alpha,\ \beta,\ \lambda)$ and covariance matrix $\left[I^{Y,D}(\theta)\right]^{-1}$. In practice, covariance matrix $\left[I^{Y,D}(\theta)\right]^{-1}$ is estimated by observed covariance matrix $\left[I^{Y,D}(\hat{\theta})\right]^{-1}$ to obtain the required asymptotic CIs, see, Lawless (2003). Therefore, two sided equal tail 100(1-ξ)% ACIs for the parameters α, β and λ are, respectively, given by

$$(\hat{\alpha}\pm z_{\xi/2}\sqrt{\hat{V}ar(\hat{\alpha})}),\ (\hat{\beta}\pm z_{\xi/2}\sqrt{\hat{V}ar(\hat{\beta})}) \text{ and } (\hat{\lambda}\pm z_{\xi/2}\sqrt{\hat{V}ar(\hat{\lambda})}).$$

Here, $\hat{V}ar(\hat{\alpha})$, $\hat{V}ar(\hat{\beta})$ and $\hat{V}ar(\hat{\lambda})$ are diagonal elements of the observed covariance matrix $\left[I^{Y,D}(\hat{\theta})\right]^{-1}$ and $z_{\xi/2}$ is the upper $(\xi/2)^{th}$ percentile of the standard normal distribution. Also.

the CPs for the parameters α, β and λ are, respectively, given by

$$CP_\alpha = \left[\left|\frac{\hat{\alpha}-\alpha}{\sqrt{\hat{V}ar(\hat{\alpha})}}\right| \leq z_{\xi/2}\right], \quad CP_\beta = \left[\left|\frac{\hat{\beta}-\beta}{\sqrt{\hat{V}ar(\hat{\beta})}}\right| \leq z_{\xi/2}\right],$$

and

$$CP_\lambda = \left[\left|\frac{\hat{\lambda}-\alpha}{\sqrt{\hat{V}ar(\hat{\lambda})}}\right| \leq z_{\xi/2}\right].$$

3.4 Expected Time on Test

The ETT of a randomly censored life testing experiment is discussed in this section. In real life testing situations, ETT is beneficial for estimating the quantity of objects to be tested, as well as the duration and cost of the life testing experiment. ETT requires the following result:

Theorem 3.1. *In randomly censored sampling plan, the expectation of the largest order statistic* $Z = max(Y_1, Y_2, \ldots, Y_n)$ *is given by*
$E[Z] = \int_0^\infty \left[1-(1-\bar{F}_X(z)\bar{F}_T(z))^n\right]dz.$

Proof. Since, $Y_i, i = 1, 2, \ldots, n$ are iid, the cdf of Z is given by

$$F_Z(z) = P[Z \leq z] = P[max(Y_1, Y_2, \ldots, Y_n) \leq z] = \{P[Y_i \leq z]\}^n \;; z > 0.$$

Note that

$$\begin{aligned} F_Y(z) &= P[Y_i \leq z] = P\big[min(X_i, T_i) \leq z\big] = 1 - P[min(X_i, T_i) > z] \\ &= 1 - P[X_i > z]P[T_i > z] = 1 - \bar{F}_X(z)\bar{F}_T(z). \end{aligned}$$

Therefore,

$$E[Z] = \int_0^\infty (1 - F_Z(z))dz = \int_0^\infty \left[1-(1-\bar{F}_X(z)\bar{F}_T(z))^n\right]dz.$$

□

Now, if the failure time X follows IW(α, β) and censoring time T follows IW(α, λ), the ETT for randomly censored experiment is given by

$$ETT = \int_0^\infty \left[1 - \{1-(1-e^{-\beta z^{-\alpha}})(1-e^{-\lambda z^{-\alpha}})\}^n\right]dz \tag{3.12}$$

ETT obtained in equation (3.12) can be obtained numerically for the given values of the parameters and the sample size n. Also, the observed time on the test (OBTT) is given by

TABLE 3.1: Expected time on test (ETT) and the observed time on test (OBTT).

λ	n	$\alpha=2,\ \beta=0.5$			$\alpha=2,\ \beta=1$			$\alpha=2,\ \beta=2$		
		ETT	OBTT		ETT	OBTT		ETT	OBTT	
			AB	MSE		AB	MSE		AB	MSE
0.5	20	1.7601	0.0582	0.6400	2.0871	0.0833	0.5272	2.4592	0.0509	0.5224
	30	1.9620	0.0482	0.6377	2.3279	0.0573	0.6017	2.7482	0.0805	0.6903
	40	2.1175	0.0702	0.678	2.5132	0.0859	0.6609	2.9702	0.0889	0.4232
	50	2.2456	0.0912	0.8631	2.6659	0.0867	0.6270	3.1530	0.0751	0.6062
	60	2.3556	0.0718	0.9476	2.7969	0.0857	0.7214	3.3096	0.0775	0.7966
1	20	2.0871	0.0659	0.644	2.4891	0.0887	0.6800	2.9515	0.0673	0.6545
	30	2.3279	0.0891	0.619	2.7747	0.0753	0.6754	3.2921	0.0696	0.7035
	40	2.5132	0.0829	0.6797	2.9946	0.0764	0.6559	3.5542	0.0700	0.7217
	50	2.6659	0.0830	0.7519	3.1758	0.0961	0.7262	3.7702	0.0894	0.7541
	60	2.7969	0.0878	0.8634	3.3313	0.0801	0.7952	3.9554	0.0994	0.7428
2	20	2.4592	0.0191	0.6023	2.9515	0.0869	0.6880	3.5202	0.0864	0.5600
	30	2.7482	0.0935	0.7368	3.2921	0.0779	0.6381	3.9240	0.0965	0.5507
	40	2.9702	0.0819	0.7145	3.5542	0.0817	0.7594	4.2350	0.0904	0.7119
	50	3.1530	0.0836	0.7161	3.7702	0.0983	0.7037	4.4913	0.0824	0.7524
	60	3.3096	0.0680	0.7428	3.9554	0.0941	0.7269	4.7111	0.0836	0.7905

$OBTT = max(y_1, y_2, \ldots, y_n)$. We compute, the average absolute bias (AB) and mean squared error (MSE) for OBTT based on 1,000 randomly censored simulated samples from the model in (3.5). The values of ETT and AB, MSE for OBTT under randomly censored IW lifetime model for different values of the parameters and sample size n are reported in table 3.1. Table 3.1 shows that the OBTT estimates the ETT quite closely and efficiently.

3.5 Bayesian Estimation

For Bayesian estimation, we use the piece-wise independent gamma priors described below for the parameters α, β, and λ:

$$g_1(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)}\alpha^{a_1-1}e^{-b_1\alpha}\ ; \alpha, a_1, b_1 > 0,$$
$$g_2(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)}\beta^{a_2-1}e^{-b_2\beta}\ ; \beta, a_2, b_2 > 0,$$
$$g_3(\lambda) = \frac{b_3^{a_3}}{\Gamma(a_3)}\lambda^{a_3-1}e^{-b_3\lambda}\ ; \lambda, a_3, b_3 > 0, \text{ respectively.}$$

Thus, the joint prior distribution of α, β and λ can be written as

$$g(\alpha,\beta,\lambda) \propto \alpha^{a_1-1}\beta^{a_2-1}\lambda^{a_3-1}e^{-(b_1\alpha+b_2\beta+b_3\lambda)} \tag{3.13}$$

The piece-wise independent gamma priors assumption is a reasonable one. These priors have been utilised by many authors on the shape and scale parameters of IW lifetime model, see, for example Singh et al. (2013), Krishna et al. (2019). It is also noted that the non-informative priors are the special cases of independent gamma priors when hyper-parameters $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0$ in (3.13).

Based on the observed randomly censored data $(\mathbf{y}, \mathbf{d})$, likelihood function in (3.7) and joint prior distribution of (α, β, λ) in (3.13), the joint posterior distribution of α, β and λ is given by

$$\pi(\alpha,\beta,\lambda|\mathbf{y},\mathbf{d}) = \frac{L(\mathbf{y},\mathbf{d}|\alpha,\beta,\lambda)g(\alpha,\beta,\lambda)}{\int_0^\infty\int_0^\infty\int_0^\infty L(\mathbf{y},\mathbf{d}|\alpha,\beta,\lambda)g(\alpha,\beta,\lambda)d\alpha d\beta d\lambda}$$

$$\pi(\alpha,\beta,\lambda|\mathbf{y},\mathbf{d}) \propto \alpha^{n+a_1-1}e^{-\alpha\left(b_1+\sum_{i=1}^{n}\ln y_i\right)}\beta^{m+a_2-1}e^{-\beta\left(b_2+\sum_{i=1}^{n}d_i y_i^{-\alpha}\right)}\lambda^{n-m+a_3-1} e^{-\lambda\left(b_2+\sum_{i=1}^{n}(1-d_i)y_i^{-\alpha}\right)}\prod_{i=1}^{n}(1-e^{-\lambda y_i^{-\alpha}})^{d_i}\prod_{i=1}^{n}(1-e^{-\beta y_i^{-\alpha}})^{1-d_i} \tag{3.14}$$

Thus, the Bayes estimator of any function of α, β and λ, say, $\phi(\alpha,\beta,\lambda)$ under SELF is the posterior expectation of $\phi(\alpha,\beta,\lambda)$ and is given by

$$E[\phi(\alpha,\beta,\lambda)|\mathbf{y},\mathbf{d}] = \frac{\int_0^\infty\int_0^\infty\int_0^\infty \phi(\alpha,\beta,\lambda)L(\mathbf{y},\mathbf{d}|\alpha,\beta,\lambda)g(\alpha,\beta,\lambda)d\alpha d\beta d\lambda}{\int_0^\infty\int_0^\infty\int_0^\infty L(\mathbf{y},\mathbf{d}|\alpha,\beta,\lambda)g(\alpha,\beta,\lambda)d\alpha d\beta d\lambda} \tag{3.15}$$

The Bayes estimator is in the form of a ratio of two integrals for which there is no closed form solution, as shown in the accompanying equation (3.15). As a result, the following integral ratio can be solved numerically. To construct the Bayes estimators, we employ the TK approximation approach given by Tierney and Kadane (1986), as well as MCMC techniques such as the Gibbs sampling methods followed by the M-H algorithm.

3.5.1 TK Approximation Method

According to TK approximation method, the approximate Bayes estimator of $\phi(\alpha,\beta,\lambda)$ under SELF is given by

$$\hat{\phi}_{TK} = E[\phi(\alpha,\beta,\lambda)|\mathbf{y},\mathbf{d}] = \frac{\int_0^\infty\int_0^\infty\int_0^\infty e^{n\delta_\phi^*(\alpha,\beta,\lambda)} d\alpha d\beta d\lambda}{\int_0^\infty\int_0^\infty\int_0^\infty e^{n\delta(\alpha,\beta,\lambda)} d\alpha d\beta d\lambda} \tag{3.16}$$

where, $\delta(\alpha,\beta,\lambda) = \frac{1}{n}[l(\alpha,\beta,\lambda)+\rho(\alpha,\beta,\lambda)]$ and $\delta^*(\alpha,\beta,\lambda) = \left[\delta(\alpha,\beta,\lambda)+\frac{1}{n}\ln\phi(\alpha,\beta,\lambda)\right]$, here, $l(\alpha,\beta,\lambda)$ is the log-likelihood function and $\rho(\alpha,\beta,\lambda) = \ln g(\alpha,\beta,\lambda)$.
The expression (3.16) is approximated by the TK method as

$$\hat{\phi}(\alpha,\beta,\lambda) = \sqrt{\frac{|\Sigma^*|}{|\Sigma|}} e^{n[\delta_\phi^*(\hat{\alpha}_{\delta^*},\hat{\beta}_{\delta^*},\hat{\lambda}_{\delta^*})-\delta(\hat{\alpha}_{\delta},\hat{\beta}_{\delta},\hat{\lambda}_{\delta})]}, \tag{3.17}$$

where, $|\Sigma^*|$ and $|\Sigma|$ are the determinants of inverse of negative hessian of $\delta^*(\alpha,\beta,\lambda)$ and $\delta(\alpha,\beta,\lambda)$ at $(\hat{\alpha}_{\delta^*},\hat{\beta}_{\delta^*},\hat{\lambda}_{\delta^*})$ and $(\hat{\alpha}_{\delta},\hat{\beta}_{\delta},\hat{\lambda}_{\delta})$, respectively. Also, $(\hat{\alpha}_{\delta^*},\hat{\beta}_{\delta^*},\hat{\lambda}_{\delta^*})$ and $(\hat{\alpha}_{\delta},\hat{\beta}_{\delta},\hat{\lambda}_{\delta})$ maximize $\delta^*(\alpha,\beta,\lambda)$ and $\delta(\alpha,\beta,\lambda)$, respectively. Next, we observe that

$$\begin{aligned}\delta(\alpha,\beta,\lambda) = \frac{1}{n}\Bigg[&(n+a_1-1)\ln\alpha+(m+a_2-1)\ln\beta+(n-m+a_3-1)\ln\lambda-(\alpha+1)S\\ &-\beta\Big(b_2+\sum_{i=1}^n d_i y_i^{-\alpha}\Big)-\lambda\Big(b_3+\sum_{i=1}^n(1-d_i)y_i^{-\alpha}\Big)-b_1\alpha\\ &+\sum_{i=1}^n d_i\ln\Big(1-e^{-\lambda y_i^{-\alpha}}\Big)+\sum_{i=1}^n(1-d_i)\ln\Big(1-e^{-\beta y_i^{-\alpha}}\Big)\Bigg]\end{aligned}$$

Then, $(\hat{\alpha}_{\delta},\hat{\beta}_{\delta},\hat{\lambda}_{\delta})$ are computed by solving the following non-linear equations

$$\begin{aligned}\frac{\partial\delta}{\partial\alpha} &= \frac{n+a_1-1}{\alpha}-S+\beta\sum_{i=1}^n d_i y_i^{-\alpha}\ln y_i+\lambda\sum_{i=1}^n(1-d_i)y_i^{-\alpha}\ln y_i-b_1\\ &\quad -\lambda\sum_{i=1}^n\frac{d_i e^{-\lambda y^{-\alpha}} y_i^{-\alpha}\ln y_i}{\Big(1-e^{-\lambda y_i^{-\alpha}}\Big)}-\beta\sum_{i=1}^n\frac{(1-d_i)e^{-\beta y_i^{-\alpha}} y_i^{-\alpha}\ln y_i}{\Big(1-e^{-\beta y_i^{-\alpha}}\Big)}=0\\ \frac{\partial\delta}{\partial\beta} &= \frac{m+a_2-1}{\beta}-\Big(b_2+\sum_{i=1}^n d_i y_i^{-\alpha}\Big)+\sum_{i=1}^n\frac{(1-d_i)e^{-\beta y_i^{-\alpha}} y_i^{-\alpha}}{\Big(1-e^{-\beta y_i^{-\alpha}}\Big)}=0\\ \frac{\partial\delta}{\partial\lambda} &= \frac{n-m+a_3-1}{\lambda}-\Big(b_3+\sum_{i=1}^n(1-d_i)y_i^{-\alpha}\Big)+\sum_{i=1}^n\frac{d_i e^{-\lambda y_i^{-\alpha}} y_i^{-\alpha}}{\Big(1-e^{-\lambda y_i^{-\alpha}}\Big)}=0\end{aligned}$$

Now, obtain $|\Sigma|$ from

$$\Sigma^{-1} = \frac{1}{n}\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix},$$

where,

$$\begin{aligned}
\delta_{11} = -\frac{\partial^2 \delta}{\partial \alpha^2} &= \frac{n+a_1-1}{\alpha^2} + \beta \sum_{i=1}^{n} d_i y_i^{-\alpha} (\ln y_i)^2 + \lambda \sum_{i=1}^{n} (1-d_i) y_i^{-\alpha} (\ln y_i)^2 \\
&+ \lambda \sum_{i=1}^{n} \frac{d_i y_i^{-\alpha} (\ln y_i)^2 e^{-\lambda y_i^{-\alpha}} (e^{-\lambda y_i^{-\alpha}} + \lambda y_i^{-\alpha} - 1)}{\left(1-e^{-\lambda y_i^{-\alpha}}\right)^2} \\
&+ \beta \sum_{i=1}^{n} \frac{(1-d_i) y_i^{-\alpha} (\ln y_i)^2 e^{-\beta y_i^{-\alpha}} (e^{-\beta y_i^{-\alpha}} + \beta y_i^{-\alpha} - 1)}{\left(1-e^{-\beta y_i^{-\alpha}}\right)^2} \\
\delta_{12} = \delta_{21} = -\frac{\partial^2 \delta}{\partial \alpha \partial \beta} &= -\sum_{i=1}^{n} d_i y_i^{-\alpha} \ln y_i - \sum_{i=1}^{n} \frac{(1-d_i) y_i^{-\alpha} \ln y_i e^{-\beta y_i^{-\alpha}} (e^{-\beta y_i^{-\alpha}} + \beta y_i^{-\alpha} - 1)}{\left(1-e^{-\beta y_i^{-\alpha}}\right)^2} \\
\delta_{13} = \delta_{31} = -\frac{\partial^2 \delta}{\partial \alpha \partial \lambda} &= -\sum_{i=1}^{n} (1-d_i) y_i^{-\alpha} \ln y_i - \sum_{i=1}^{n} \frac{d_i y_i^{-\alpha} \ln y_i e^{-\lambda y_i^{-\alpha}} (e^{-\lambda y_i^{-\alpha}} + \lambda y_i^{-\alpha} - 1)}{\left(1-e^{-\lambda y_i^{-\alpha}}\right)^2} \\
\delta_{22} = -\frac{\partial^2 \delta}{\partial \beta^2} &= \frac{m+a_2-1}{\beta^2} + \sum_{i=1}^{n} \frac{(1-d_i) y_i^{-2\alpha} e^{-\beta y_i^{-\alpha}}}{(1-e^{-\beta y_i^{-\alpha}})^2}, \quad \delta_{23} = \delta_{32} = -\frac{\partial^2 \delta}{\partial \beta \partial \lambda} = 0, \\
\delta_{33} = \frac{\partial^2 \delta}{\partial \lambda^2} &= \frac{n-m+a_3-1}{\lambda^2} + \sum_{i=1}^{n} \frac{d_i y_i^{-2\alpha} e^{-\lambda y_i^{-\alpha}}}{\left(1-e^{-\lambda y_i^{-\alpha}}\right)^2}
\end{aligned}$$

In order to compute the Bayes estimator of α we take $\phi(\alpha,\beta,\lambda) = \alpha$ and accordingly function $\delta^*(\alpha,\beta,\lambda)$ becomes

$$\delta^*_\alpha(\alpha,\beta,\lambda) = \delta(\alpha,\beta,\lambda) + \frac{1}{n}\ln\alpha$$

and then $(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*}, \hat{\lambda}_{\delta^*})$ are obtained as solution of the following non-linear equations

$$\frac{\partial \delta^*_\alpha}{\partial \alpha} = \frac{\partial \delta}{\partial \alpha} + \frac{1}{\alpha} = 0, \quad \frac{\partial \delta^*_\alpha}{\partial \beta} = \frac{\partial \delta}{\partial \beta} = 0, \quad \frac{\partial \delta^*_\alpha}{\partial \lambda} = \frac{\partial \delta}{\partial \lambda} = 0$$

and obtain $|\Sigma^*|$ from

$$\Sigma^{*\,-1}_\alpha = \frac{1}{n}\begin{bmatrix} \delta^*_{11} & \delta^*_{12} & \delta^*_{13} \\ \delta^*_{21} & \delta^*_{22} & \delta^*_{23} \\ \delta^*_{31} & \delta^*_{32} & \delta^*_{33} \end{bmatrix},$$

where,

$$\delta_{11}^* = -\frac{\partial^2 \delta_\alpha^*}{\partial \alpha^2} = -\frac{\partial^2 \delta}{\partial \alpha^2} + \frac{1}{\alpha^2},\ \delta_{12}^* = \delta_{12},\ \delta_{13}^* = \delta_{13},\ \delta_{21}^* = \delta_{21},\ \delta_{22}^* = \delta_{22},\ \delta_{23}^* = \delta_{23},\ \delta_{31}^* = \delta_{31},$$
$$\delta_{32}^* = \delta_{32},\ \delta_{33}^* = \delta_{33}.$$

Thus, the approximate Bayes estimator of α under SELF is given by

$$\hat{\alpha}_{TK} = \sqrt{\frac{|\Sigma_\alpha^*|}{|\Sigma|}} e^{n[\delta_\alpha^*(\hat{\alpha}_{\delta^*},\hat{\beta}_{\delta^*},\hat{\lambda}_{\delta^*}) - \delta(\hat{\alpha}_\delta,\hat{\beta}_\delta,\hat{\lambda}_\delta)]}$$

Similarly, we can derive the approximate Bayes estimator of β and λ as

$$\hat{\beta}_{TK} = \sqrt{\frac{|\Sigma_\beta^*|}{|\Sigma|}} e^{n[\delta_\beta^*(\hat{\alpha}_{\delta^*},\hat{\beta}_{\delta^*},\hat{\lambda}_{\delta^*}) - \delta(\hat{\alpha}_\delta,\hat{\beta}_\delta,\hat{\lambda}_\delta)]}$$

$$\hat{\lambda}_{TK} = \sqrt{\frac{|\Sigma_\lambda^*|}{|\Sigma|}} e^{n[\delta_\lambda^*(\hat{\alpha}_{\delta^*},\hat{\beta}_{\delta^*},\hat{\lambda}_{\delta^*}) - \delta(\hat{\alpha}_\delta,\hat{\beta}_\delta,\hat{\lambda}_\delta)]},$$

respectively. Next, We compute the Bayes estimator of survival function S(t).
In this case, $\phi(\alpha,\beta,\lambda) = 1 - e^{-\beta t^{-\alpha}}$, then

$$\delta_\lambda^*(\alpha,\beta,\lambda) = \delta(\alpha,\beta,\lambda) + \frac{1}{n}\ln\left(1 - e^{-\beta t^{-\alpha}}\right).$$

Now compute $(\hat{\alpha}_{\delta^*},\hat{\beta}_{\delta^*},\hat{\lambda}_{\delta^*})$ by solving the following non-linear equations:

$$\frac{\partial \delta_{S(t)}^*}{\partial \alpha} = \frac{\partial \delta}{\partial \alpha} - \frac{\beta e^{-\beta t^{-\alpha}} t^{-\alpha} \ln t}{(1 - e^{-\beta t^{-\alpha}})} = 0,\ \frac{\partial \delta_{S(t)}^*}{\partial \beta} = \frac{\partial \delta}{\partial \beta} + \frac{e^{-\beta t^{-\alpha}} t^{-\alpha}}{(1 - e^{-\beta t^{-\alpha}})} = 0,\ \frac{\partial \delta_{S(t)}^*}{\partial \lambda} = \frac{\partial \delta}{\partial \lambda} = 0$$

Now we find $\Sigma_{S(t)}^*$ from

$${\Sigma_{S(t)}^*}^{-1} = \frac{1}{n}\begin{bmatrix} \delta_{S(t)11}^* & \delta_{S(t)12}^* & \delta_{S(t)13}^* \\ \delta_{S(t)21}^* & \delta_{S(t)22}^* & \delta_{S(t)23}^* \\ \delta_{S(t)31}^* & \delta_{S(t)32}^* & \delta_{S(t)33}^* \end{bmatrix},$$

where,

$$\delta^*_{S(t)11} = -\frac{\partial^2 \delta^*_{S(t)}}{\partial \alpha^2} = -\frac{\partial^2 \delta}{\partial \alpha^2} - \frac{\beta e^{-\beta t^{-\alpha}} t^{-\alpha} (\ln t)^2 (e^{-\beta t^{-\alpha}} + \beta t^{-\alpha} - 1)}{(1 - e^{-\beta t^{-\alpha}})^2},$$

$$\delta^*_{S(t)12} = \delta^*_{S(t)21} = -\frac{\partial^2 \delta^*_{S(t)}}{\partial \alpha \partial \beta} = -\frac{\partial^2 \delta}{\partial \alpha \partial \beta} - \frac{e^{-\beta t^{-\alpha}} t^{-\alpha} \ln t (e^{-\beta t^{-\alpha}} + \beta t^{-\alpha} - 1)}{(1 - e^{-\beta t^{-\alpha}})^2},$$

$$\delta^*_{S(t)13} = -\frac{\partial^2 \delta^*_{S(t)}}{\partial \alpha \partial \lambda} = -\frac{\partial^2 \delta}{\partial \alpha \partial \lambda}, \; \delta^*_{S(t)22} = -\frac{\partial^2 \delta^*_{S(t)}}{\partial \beta^2} = -\frac{\partial^2 \delta}{\partial \beta^2} + \frac{t^{-2\alpha} e^{-\beta t^{-\alpha}}}{(1 - e^{-\beta t^{-\alpha}})^2},$$

$$\delta^*_{S(t)23} = \delta_{23}, \delta^*_{S(t)31} = \delta_{31}, \; \delta^*_{S(t)33} = \delta_{33} \qquad .$$

Thus, the Bayes estimator of $S(t)$ is given by

$$\hat{S}(t)_{TK} = \sqrt{\frac{|\Sigma^*_{S(t)}|}{|\Sigma|}} e^{n[\delta^*_{S(t)}(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*}, \hat{\lambda}_{\delta^*}) - \delta(\hat{\alpha}_\delta, \hat{\beta}_\delta, \hat{\lambda}_\delta)]}$$

Similarly, the Bayes estimator of failure rate function is given by

$$\hat{h}(t)_{TK} = \sqrt{\frac{|\Sigma^*_{h(t)}|}{|\Sigma|}} e^{n[\delta^*_{h(t)}(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*}, \hat{\lambda}_{\delta^*}) - \delta(\hat{\alpha}_\delta, \hat{\beta}_\delta, \hat{\lambda}_\delta)]},$$

3.5.2 Gibbs Sampling Method

To do sample-based inference, we consider adopting one of the MCMC methods as Gibbs sampling technique to produce a random sample from the joint posterior distribution. For detailed study about MCMC techniques and their applications, one may refer Robert and Casella (2004) and Gelman et al. (2013). The Gibbs sampling approach uses the full conditional posterior densities of α, $beta$, and $lambda$, respectively

$$\pi_1(\beta \mid \alpha, \mathbf{y}, \mathbf{d}) = \beta^{m+a_2-1} e^{-\beta\left(b_2 + \sum_{i=1}^{n} d_i y_i^{-\alpha}\right)} \prod_{i=1}^{n} \left(1 - e^{-\beta y_i^{-\alpha}}\right)^{(1-d_i)}, \tag{3.18}$$

$$\pi_2(\lambda \mid \alpha, \mathbf{y}, \mathbf{d}) = \lambda^{n-m+a_3-1} e^{-\lambda\left(b_3 + \sum_{i=1}^{n} (1-d_i) y_i^{-\alpha}\right)} \prod_{i=1}^{n} \left(1 - e^{-\beta y_i^{-\alpha}}\right)^{d_i}, \tag{3.19}$$

$$\begin{aligned} \pi_3(\alpha \mid \beta, \lambda, \mathbf{y}, \mathbf{d}) = \alpha^{n+a_1-1} e^{-\alpha\left(b_1 + \sum_{i=1}^{n} \ln y_i\right)} e^{-\left(\beta \sum_{i=1}^{n} d_i y_i^{-\alpha} + \lambda \sum_{i=1}^{n} (1-d_i) y_i^{-\alpha}\right)} \\ \prod_{i=1}^{n} (1 - e^{-\lambda y_i^{-\alpha}})^{d_i} \prod_{i=1}^{n} (1 - e^{-\beta y_i^{-\alpha}})^{(1-d_i)}. \end{aligned} \tag{3.20}$$

To produce samples from the full conditional posterior densities (3.18), (3.19) and (3.20), we utilise the following algorithm:

Gibbs Sampler Algorithm

Step 1: Start with initial guess of α, β and λ say $\alpha^{(0)}$, $\beta^{(0)}$ and $\lambda^{(0)}$.
Step 2: Set $j = 1$.
Step 3: Generate $\beta^{(j)}$ from $\pi_1(\beta \mid \alpha^{(j-1)}, \mathbf{y}, \mathbf{d})$ in (3.18) using M-H algorithm with normal proposal density.
Step 4: Generate $\lambda^{(j)}$ from $\pi_2(\lambda \mid \alpha^{(j-1)}, \mathbf{y}, \mathbf{d})$ in (3.19) using M-H algorithm with normal proposal density.
Step 5: Generate $\alpha^{(j)}$ from $\pi_3(\alpha \mid \beta^{(j-1)}, \lambda^{(j-1)}, \mathbf{y}, \mathbf{d})$ in (3.20) using M-H algorithm with normal proposal density.
Step 6: Set $j = j+1$ and repeat steps 3-5 for all $j = 1, 2, \ldots, M$ to obtain MCMC samples

$$\left(\alpha^{(1)}, \beta^{(1)}, \lambda^{(1)}\right), \left(\alpha^{(2)}, \beta^{(2)}, \lambda^{(2)}\right), \ldots, \left(\alpha^{(M)}, \beta^{(M)}, \lambda^{(M)}\right).$$

Now, the approximate Bayes estimator of $\phi(\alpha, \beta, \lambda)$, can be obtained as

$$\hat{\phi}_{GS}(\alpha, \beta, \lambda) = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \phi\left(\alpha^{(j)}, \beta^{(j)}, \lambda^{(j)}\right), \tag{3.21}$$

where, M_0 is the burn-in period i.e. a number of iterations in Markov chain before the stationary distribution is achieved. Thus, taking, $\phi(\alpha, \beta, \lambda) = \alpha, \beta$ and λ, the Bayes estimators of the parameters α, β and λ under SELF, respectively, are given by

$$\hat{\alpha}_{GS} = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \alpha^{(j)}, \quad \hat{\beta}_{GS} = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \beta^{(j)}, \text{ and}$$

$$\hat{\lambda}_{GS} = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \lambda^{(j)}.$$

Also, the Bayes estimators of the survival and failure rate functions, respectively, are given by

$$\hat{S}(t)_{GS} = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \left(1 - e^{-\beta^{(j)} t^{-\alpha^{(j)}}}\right) \; ; \, t > 0, \text{ and}$$

$$\hat{h}(t)_{GS} = \frac{1}{M-M_0} \sum_{j=M_0+1}^{M} \frac{\alpha^{(j)}\beta^{(j)}t^{-(\alpha^{(j)}+1)}}{e^{\beta^{(j)}t^{-\alpha^{(j)}}}-1} \; ; \; t > 0.$$

3.5.3 HPD Credible Intervals

Using the produced MCMC samples, we now compute the HPD credible interval of the unknown parameters. Let $\alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(M-M_0)}$ be the ordered values of $\alpha^{(M_0+1)}, \alpha^{(M_0+2)}, \ldots, \alpha^{(M)}$. Then 100(1-$\xi$)% of HPD credible intervals of the parameter α, is given by

$$\left(\alpha_{(j)},\ \alpha_{(j+[(1-\xi)(M-M_0)])}\right),\ \ 0 < \xi < 1$$

where j is chosen such that

$$\alpha_{(j+[(1-\xi)(M-M_0)])} - \alpha_{(j)} = \min_{1\le i\xi \le (M-M_0)} \left(\alpha_{(i+[(1-\xi)(M-M_0)])} - \alpha_{(i)}\right);\ j = 1,2,\ldots,(M-M_0),$$

here, $[x]$ is the largest integer less than or equal to x, see, Chen and Shao (1999).

Similarly, 100(1-ξ)% HPD credible intervals for β and λ can be constructed.

3.6 Numerical Computations

In this section, we run a simulation study to compare the proposed estimators developed in the preceding sections. All of the computations were done with the statistical software R, see, R Core Team (2021). In this simulation study, we use five distinct sample sizes n= 20, 30, 40, 50, and 60. In all situations, the true value of $\lambda = 1.0$ is used, as well as two distinct values of β=0.5, 1.5, and two different values of α=0.5, 2. Under SELF, non-informative as well as gamma informative priors are considered for Bayesian computation. In case of informative priors following values of hyper-parameters $(a_1,b_1,a_2,b_2,a_3,b_3)$ are taken so that prior means are exactly equal to the true values of the parameters: $(2,4,2,4,2,2), (2,4,3,2,2,2), (4,2,2,4,2,2)$ and $(4,2,3,2,2,2)$.

For each case, the ML and Bayes estimates of the unknown parameters, survival and failure rate functions are computed. The mission time $t = 0.80$ is taken for survival and failure rate functions. The TK approximation and Gibbs sampling methods are used to compute the Bayes estimators of the parameters and reliability characteristics. The 95% asymptotic CIs based on expected Fisher information matrix and HPD credible intervals based on Gibbs sampling method are constructed. The integrals associated with expected Fisher information matrix are solved using the *integrate* function of software R. We take, $M = 10,000$ with burn-in period $M_0 = 2,000$ for Gibbs sampling method. The whole process was simulated 1,000 times and the average absolute biases (AB) with the corresponding mean squared errors (MSE) are computed for different estimators. Also, the average length (AL) and the coverage probabilities (CP) of 95% asymptotic confidence and HPD credible intervals are calculated. The results of the simulation study are reported in following Tables 3.2 , 3.6, 3.3, 3.7, 3.4, 3.8, 3.5, 3.9.

In simulation tables, the short notations TK stands for Tierney-Kadane method, GS stand for Gibbs sampling method, P1 for non-informative prior and P2 for gamma informative prior. From these results the following conclusions are made:

(i) The AB and MSEs of the ML and Bayes estimators of the parameters and reliability characteristics decrease as the sample size increases in all situations.

(ii) The AB and MSE decrease as the failure time parameter β increases.

(iii) In terms of both AB and MSEs, Bayes estimates outperform ML estimates as they include prior knowledge. In terms of both AB and MSEs, the Gibbs sampling approach outperforms the TK approximation method.

(iv) As the sample size n increases, the ALs of all intervals shrinks. Also, the ALs of HPD credible intervals are less than the ALs of ACIs.

(v) The CPs achieve their required confidence levels quite satisfactorily in classical estimation. However, when the true value of parameter $\alpha = 0.5$ is used in the Bayesian estimate approach, CPs only reach their nominal level. In most situations, as the true value of α and the sample size n increase, they don't hit their nominal levels.

TABLE 3.2: The ML and Bayes estimates of parameters and reliability characteristics, when α=0.5, β=0.5, λ=1, t=0.8, $S(t)$= 0.4282, $h(t)$= 0.4665.

		$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$		$\hat{S}(t)$		$\hat{h}(t)$	
n	Methods	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
	MLE	0.0856	0.0132	0.0037	0.0274	0.2759	0.3763	0.0813	0.0101	0.1097	0.0213
	TK P1	0.0844	0.0128	0.1246	0.0254	0.2721	0.1604	0.0755	0.0089	0.1078	0.0212
20	TK P2	0.0703	0.0086	0.1060	0.0178	0.2101	0.0805	0.0646	0.0064	0.0871	0.0120
	GS P1	0.0545	0.0044	0.1027	0.0183	0.2322	0.1088	0.0617	0.0061	0.0626	0.0061
	GS P2	0.0492	0.0037	0.0905	0.0136	0.1823	0.0593	0.055	0.0047	0.0536	0.0043
	MLE	0.0597	0.0060	0.0989	0.0154	0.2037	0.0764	0.0609	0.0058	0.0758	0.0096
	TK P1	0.0631	0.0067	0.1011	0.0169	0.2056	0.0755	0.0616	0.0061	0.0822	0.0114
30	TK P2	0.0529	0.0046	0.0854	0.0116	0.1788	0.0566	0.0520	0.0042	0.0654	0.0070
	GS P1	0.0476	0.0034	0.0829	0.0124	0.1812	0.0588	0.0496	0.0041	0.0543	0.0044
	GS P2	0.0454	0.0030	0.0754	0.0096	0.1596	0.0457	0.0450	0.0032	0.0497	0.0036
	MLE	0.0513	0.0045	0.0869	0.0119	0.1756	0.0534	0.0538	0.0045	0.0667	0.0075
	TK P1	0.0520	0.0048	0.0849	0.0118	0.1741	0.0527	0.0043	0.0428	0.0664	0.0080
40	TK P2	0.0465	0.0036	0.0773	0.0095	0.1591	0.0432	0.0478	0.0036	0.0595	0.0059
	GS P1	0.0482	0.0032	0.0727	0.0091	0.1561	0.0434	0.0428	0.0030	0.0543	0.0041
	GS P2	0.0457	0.0029	0.0671	0.0076	0.1432	0.0362	0.0402	0.0026	0.0495	0.0035
	MLE	0.0460	0.0036	0.0789	0.0096	0.1493	0.0370	0.0485	0.0036	0.0060	0.0551
	TK P1	0.0481	0.0039	0.0794	0.0098	0.1531	0.0390	0.0484	0.0036	0.0626	0.0063
50	TK P2	0.0424	0.0030	0.0719	0.0080	0.1382	0.0317	0.0439	0.0030	0.0050	0.0502
	GS P1	0.0479	0.0031	0.0668	0.0077	0.1376	0.0330	0.0392	0.0025	0.0523	0.0039
	GS P2	0.0456	0.0028	0.0629	0.0065	0.1255	0.0273	0.0370	0.0022	0.0035	0.2159
	MLE	0.0427	0.0029	0.0714	0.0081	0.1385	0.0321	0.0437	0.0030	0.0546	0.0048
	TK P1	0.0402	0.0026	0.0664	0.0070	0.1302	0.0283	0.0402	0.0025	0.0508	0.0041
60	TK P2	0.0396	0.0026	0.0698	0.0077	0.1308	0.0283	0.0421	0.0027	0.0513	0.0043
	GS P1	0.0459	0.0029	0.0632	0.0065	0.1188	0.0246	0.0367	0.0021	0.0514	0.0036
	GS P2	0.0483	0.0031	0.0634	0.0067	0.1199	0.0248	0.0366	0.0021	0.0539	0.0039

TABLE 3.3: ML and Bayes estimates of parameters and reliability characteristics, when α=0.5, β=1.5, λ=1, t=0.8, $S(t)$= 0.8131, $h(t)$= 0.2410.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$		$\hat{S}(t)$		$\hat{h}(t)$	
		AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
20	MLE	0.0819	0.0121	0.3834	0.4124	0.2143	0.0775	0.0649	0.0065	0.0750	0.0095
	TK P1	0.0817	0.0120	0.3941	0.3528	0.2256	0.0928	0.0645	0.0063	0.0728	0.0090
	TK P2	0.0715	0.0089	0.2898	0.1526	0.1847	0.0563	0.0532	0.0044	0.0607	0.0062
	GS P1	0.0760	0.0109	0.3318	0.2129	0.2004	0.0729	0.0615	0.0057	0.0622	0.0063
	GS P2	0.0659	0.0078	0.2558	0.1110	0.1651	0.0451	0.0507	0.0039	0.0520	0.0044
30	MLE	0.0620	0.0066	0.2756	0.1413	0.1711	0.0508	0.0521	0.0043	0.0561	0.0053
	TK P1	0.0618	0.0068	0.2843	0.1541	0.1680	0.0479	0.0520	0.0042	0.0564	0.0054
	TK P2	0.0574	0.0056	0.2381	0.0993	0.1553	0.0411	0.0456	0.0033	0.0490	0.0041
	GS P1	0.0577	0.0060	0.2527	0.1155	0.1523	0.0398	0.0492	0.0037	0.0482	0.0038
	GS P2	0.0538	0.0052	0.2157	0.0788	0.1411	0.0344	0.0435	0.0030	0.0422	0.0030
40	MLE	0.0532	0.0048	0.2301	0.0948	0.1396	0.0318	0.0440	0.0030	0.0465	0.0035
	TK P1	0.0532	0.0049	0.2296	0.0918	0.1491	0.0374	0.0445	0.003	0.0481	0.0038
	TK P2	0.0503	0.0043	0.2069	0.0738	0.1300	0.0275	0.0394	0.0024	0.0419	0.0029
	GS P1	0.0498	0.0042	0.2094	0.0740	0.1359	0.0322	0.0420	0.0027	0.0410	0.0027
	GS P2	0.0477	0.0039	0.1882	0.0596	0.1186	0.0233	0.0375	0.0022	0.0354	0.0020
50	MLE	0.0464	0.0036	0.2072	0.0746	0.1244	0.0258	0.0405	0.0026	0.0430	0.0030
	TK P1	0.0453	0.0034	0.2056	0.0705	0.1309	0.0278	0.0398	0.0024	0.0421	0.0029
	TK P2	0.0443	0.0033	0.1909	0.0621	0.1174	0.0230	0.0373	0.0022	0.0398	0.0026
	GS P1	0.0423	0.0030	0.1885	0.0584	0.1196	0.0244	0.0376	0.0021	0.0357	0.0021
	GS P2	0.0426	0.0031	0.1763	0.0523	0.1104	0.0206	0.0354	0.0019	0.0340	0.0019
60	MLE	0.0432	0.0031	0.1772	0.0520	0.1185	0.0226	0.0354	0.0020	0.0370	0.0021
	TK P1	0.0405	0.0027	0.1874	0.0577	0.1135	0.0209	0.0363	0.0021	0.0381	0.0023
	TK P2	0.0417	0.0028	0.1657	0.0452	0.1130	0.0205	0.0332	0.0017	0.0347	0.0019
	GS P1	0.0378	0.0024	0.1739	0.0493	0.1044	0.0183	0.0346	0.0019	0.0324	0.0016
	GS P2	0.0388	0.0025	0.1538	0.0389	0.1042	0.0178	0.0314	0.0015	0.0292	0.0013

TABLE 3.4: The ML and Bayes estimates of parameters and reliability characteristics, when α=2, β=0.5, λ=1, t=0.8, $S(t)$= 0.5422 , $h(t)$= 1.6493.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$		$\hat{S}(t)$		$\hat{h}(t)$	
		AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
	MLE	0.3528	0.2323	0.1287	0.0273	0.2571	0.1577	0.0791	0.0101	0.4018	0.3022
	TK P1	0.3438	0.2205	0.1265	0.0268	0.2642	0.2276	0.0763	0.0093	0.3984	0.2968
20	TK P2	0.2699	0.1255	0.0967	0.0149	0.2165	0.0806	0.0618	0.0060	0.2953	0.1534
	GS P1	0.2906	0.1184	0.1826	0.0514	0.2578	0.1249	0.0879	0.0115	0.3748	0.1815
	GS P2	0.2492	0.086	0.1448	0.0309	0.2149	0.0807	0.0731	0.0078	0.3102	0.1249
	MLE	0.2529	0.1106	0.0997	0.0164	0.2059	0.0768	0.0611	0.0061	0.2933	0.1521
	TK P1	0.2489	0.1067	0.0986	0.0162	0.2081	0.0797	0.0596	0.0059	0.2920	0.1507
30	TK P2	0.2103	0.0734	0.0869	0.0122	0.1788	0.0579	0.0554	0.0048	0.2396	0.0968
	GS P1	0.2946	0.1127	0.1866	0.0476	0.2455	0.1027	0.0875	0.0104	0.3945	0.1876
	GS P2	0.2686	0.0932	0.1661	0.0370	0.2178	0.0783	0.0798	0.0088	0.3611	0.1532
	MLE	0.2164	0.0801	0.0904	0.0123	0.1784	0.0535	0.0558	0.0048	0.2562	0.1081
	TK P1	0.2136	0.0778	0.0898	0.0122	0.1796	0.0549	0.0548	0.0046	0.2551	0.1073
40	TK P2	0.1858	0.0571	0.0763	0.0091	0.1545	0.0388	0.0480	0.0035	0.2106	0.0739
	GS P1	0.3059	0.1149	0.1898	0.0465	0.2436	0.0927	0.0885	0.0103	0.4102	0.1931
	GS P2	0.2871	0.1011	0.1718	0.0373	0.2183	0.0711	0.0808	0.0085	0.3831	0.1664
	MLE	0.1839	0.0592	0.0766	0.0092	0.1531	0.0402	0.0477	0.0035	0.2164	0.0811
	TK P1	0.1822	0.0578	0.0760	0.0091	0.1533	0.0410	0.0470	0.0035	0.2159	0.0807
50	TK P2	0.1625	0.0465	0.0699	0.0077	0.1368	0.0307	0.0436	0.0030	0.1892	0.0637
	GS P1	0.3188	0.1185	0.1927	0.0452	0.2367	0.0828	0.0892	0.0099	0.4249	0.1992
	GS P2	0.2999	0.1052	0.1821	0.0398	0.2148	0.0665	0.0855	0.0090	0.4044	0.1793
	MLE	0.1677	0.0450	0.0719	0.0080	0.1349	0.0289	0.0441	0.0030	0.2006	0.0642
	TK P1	0.1662	0.0441	0.0715	0.0080	0.1349	0.0292	0.0436	0.0029	0.2001	0.0639
60	TK P2	0.1458	0.0364	0.0621	0.0059	0.1270	0.0273	0.0386	0.0023	0.1715	0.0487
	GS P1	0.3255	0.1210	0.1933	0.0450	0.2272	0.0721	0.0891	0.0097	0.4310	0.2030
	GS P2	0.3104	0.1098	0.1824	0.0385	0.2277	0.0706	0.0853	0.0087	0.4125	0.1831

TABLE 3.5: The ML and Bayes estimates of parameters and reliability characteristics, when α=2, β=1.5, λ=1, t=0.8, $S(t)$= 0.9040, $h(t)$= 0.6220.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$		$\hat{S}(t)$		$\hat{h}(t)$	
		AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
	MLE	0.3267	0.1896	0.3681	0.2911	0.2091	0.0725	0.0432	0.0028	0.2198	0.0790
	TK P1	0.3194	0.1812	0.3747	0.3139	0.2071	0.0902	0.0435	0.0030	0.2130	0.0756
20	TK P2	0.2702	0.1226	0.2853	0.1458	0.1809	0.0552	0.0390	0.0024	0.1809	0.0538
	GS P1	0.2805	0.1381	0.2451	0.1128	0.1793	0.0574	0.0341	0.0018	0.1482	0.0352
	GS P2	0.2530	0.1118	0.1933	0.0641	0.1538	0.0390	0.0304	0.0015	0.1244	0.0252
	MLE	0.2542	0.1134	0.2796	0.1319	0.1670	0.0451	0.0376	0.0022	0.1865	0.0551
	TK P1	0.2503	0.1095	0.2817	0.1354	0.1659	0.0446	0.0374	0.0022	0.1826	0.0537
30	TK P2	0.2338	0.0939	0.2263	0.0962	0.1581	0.0417	0.0306	0.0015	0.1460	0.0360
	GS P1	0.2266	0.0878	0.1999	0.0674	0.1546	0.0394	0.0287	0.0013	0.1265	0.0255
	GS P2	0.2168	0.0814	0.1588	0.0447	0.1471	0.0360	0.0238	0.0009	0.1015	0.0169
	MLE	0.2083	0.0718	0.2435	0.1039	0.1404	0.0313	0.0325	0.0016	0.1571	0.0390
	TK P1	0.2060	0.0700	0.2448	0.1059	0.1398	0.0311	0.0323	0.0016	0.1542	0.0378
40	TK P2	0.2033	0.0698	0.2025	0.0694	0.1366	0.0312	0.0285	0.0013	0.1313	0.0271
	GS P1	0.1886	0.0597	0.1763	0.0544	0.1478	0.0342	0.0252	0.0010	0.1079	0.0185
	GS P2	0.1911	0.0612	0.1480	0.0361	0.1399	0.0316	0.0224	0.0008	0.0927	0.0133
	MLE	0.1874	0.0584	0.2089	0.0766	0.1293	0.0274	0.0290	0.0013	0.1391	0.0319
	TK P1	0.1853	0.0569	0.2096	0.0776	0.1289	0.0272	0.0293	0.0014	0.1378	0.0314
50	TK P2	0.1748	0.0489	0.1983	0.0670	0.1178	0.0221	0.0260	0.0011	0.1192	0.0228
	GS P1	0.1684	0.0470	0.1554	0.0421	0.1469	0.0333	0.0225	0.0008	0.0978	0.0154
	GS P2	0.1619	0.0426	0.1827	0.0560	0.1283	0.0259	0.0207	0.0007	0.0853	0.0114
	MLE	0.1740	0.0479	0.1854	0.0585	0.1174	0.0224	0.0260	0.0010	0.1240	0.0242
	TK P1	0.1724	0.0469	0.1859	0.0590	0.1170	0.0223	0.0259	0.0011	0.1225	0.0238
60	TK P2	0.1595	0.0419	0.1715	0.0478	0.1097	0.0190	0.0242	0.0009	0.1144	0.0205
	GS P1	0.1549	0.0387	0.1429	0.0334	0.1400	0.0295	0.0204	0.0006	0.0885	0.0119
	GS P2	0.1419	0.0333	0.1310	0.0283	0.1338	0.0263	0.0191	0.0006	0.0821	0.0105

TABLE 3.6: The AL and CPs of 95% ACIs and HPD credible intervals of parameters when α=0.5, β=0.5, λ=1.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$	
		AL	CP	AL	CP	AL	CP
	ACI	0.3671	0.935	0.5843	0.920	1.2637	0.943
20	HPD P1	0.2281	0.930	0.5834	0.932	0.3498	0.935
	HPD P2	0.2153	0.941	0.3299	0.941	0.7012	0.945
	ACI	0.2851	0.930	0.4827	0.935	0.9586	0.938
30	HPD P1	0.1784	0.938	0.2902	0.941	0.6215	0.944
	HPD P2	0.1718	0.932	0.2814	0.946	0.5878	0.952
	ACI	0.2433	0.947	0.4135	0.930	0.8181	0.938
40	HPD P1	0.1511	0.941	0.2556	0.941	0.5355	0.940
	HPD P2	0.1471	0.946	0.2452	0.945	0.5157	0.941
	ACI	0.2159	0.940	0.3718	0.943	0.7231	0.940
50	HPD P1	0.1338	0.940	0.2296	0.947	0.4791	0.940
	HPD P2	0.1312	0.948	0.2231	0.948	0.464	0.9453
	ACI	0.1954	0.949	0.3426	0.936	0.6567	0.943
60	HPD P1	0.1204	0.948	0.2123	0.948	0.4344	0.942
	HPD P2	0.1189	0.949	0.2072	0.947	0.4255	0.952

TABLE 3.7: AL and CPs of 95% ACIs and HPD credible intervals of parameters when α=0.5, β=1.5, λ=1.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$	
		AL	CP	AL	CP	AL	CP
	ACI	0.3642	0.952	1.7017	0.953	0.9861	0.929
20	HPD P1	0.2497	0.939	1.1026	0.937	0.6654	0.929
	HPD P2	0.2395	0.976	0.9716	0.945	0.6184	0.931
	ACI	0.2840	0.954	1.2835	0.943	0.8124	0.942
30	HPD P1	0.1969	0.940	0.8683	0.941	0.5399	0.942
	HPD P2	0.1909	0.952	0.8055	0.957	0.5249	0.943
	ACI	0.2418	0.935	1.1032	0.961	0.6874	0.95
40	HPD P1	0.1684	0.939	0.7378	0.944	0.4715	0.943
	HPD P2	0.1633	0.947	0.7133	0.946	0.4521	0.955
	ACI	0.2152	0.949	0.9663	0.947	0.6188	0.951
50	HPD P1	0.1486	0.942	0.6615	0.942	0.4219	0.949
	HPD P2	0.1459	0.946	0.6368	0.955	0.4111	0.952
	ACI	0.1955	0.945	0.8734	0.965	0.5618	0.936
60	HPD P1	0.1339	0.944	0.6076	0.937	0.3819	0.945
	HPD P2	0.1328	0.948	0.582	0.955	0.3756	0.955

TABLE 3.8: The AL and CPs of 95% ACIs and HPD credible intervals of parameters when α=2, β=0.5, λ=1.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$	
		AL	CP	AL	CP	AL	CP
	ACI	1.4658	0.918	0.5896	0.929	1.2104	0.933
20	HPD P1	0.8695	0.938	0.3921	0.932	0.7789	0.939
	HPD P2	0.8107	0.941	0.3579	0.935	0.7164	0.942
	ACI	1.1445	0.948	0.4825	0.928	0.9524	0.926
30	HPD P1	0.6727	0.946	0.3210	0.935	0.6383	0.936
	HPD P2	0.6475	0.952	0.3050	0.941	0.6041	0.940
	ACI	0.9762	0.939	0.4165	0.934	0.8203	0.943
40	HPD P1	0.5659	0.940	0.2717	0.936	0.5553	0.944
	HPD P2	0.5487	0.943	0.2606	0.943	0.5298	0.948
	ACI	0.8637	0.941	0.3722	0.936	0.7243	0.937
50	HPD P1	0.4904	0.943	0.2339	0.938	0.4926	0.941
	HPD P2	0.4820	0.944	0.2290	0.941	0.4724	0.952
	ACI	0.7808	0.949	0.3400	0.935	0.6517	0.947
60	HPD P1	0.4355	0.952	0.2011	0.940	0.4427	0.947
	HPD P2	0.4307	0.951	0.1989	0.943	0.4372	0.951

TABLE 3.9: The AL and CPs of 95% ACIs and HPD credible intervals of parameters when α=2, β=1.5, λ=1.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$	
		AL	CP	AL	CP	AL	CP
	ACI	1.4535	0.948	1.6405	0.955	1.0017	0.931
20	HPD P1	0.9936	0.946	1.0499	0.940	0.6852	0.936
	HPD P2	0.9293	0.949	0.9477	0.945	0.6452	0.945
	ACI	1.1502	0.957	1.2786	0.945	0.7987	0.937
30	HPD P1	0.7845	0.955	0.8503	0.929	0.5575	0.940
	HPD P2	0.7476	0.952	0.7969	0.958	0.5387	0.942
	ACI	0.9687	0.954	1.1014	0.948	0.6938	0.958
40	HPD P1	0.6611	0.945	0.7416	0.923	0.4875	0.944
	HPD P2	0.6452	0.948	0.7010	0.959	0.4722	0.948
	ACI	0.8667	0.946	0.9628	0.931	0.6200	0.943
50	HPD P1	0.5901	0.945	0.6541	0.912	0.4373	0.943
	HPD P2	0.5740	0.946	0.6338	0.949	0.4216	0.952
	ACI	0.7814	0.945	0.8769	0.934	0.5626	0.938
60	HPD P1	0.5319	0.945	0.5994	0.938	0.3976	0.940
	HPD P2	0.5196	0.951	0.5781	0.947	0.3889	0.945

3.7 Real Data Analysis

With the help of real data, we demonstrate the estimation techniques developed in this chapter. Here, we consider a secondary data set having remission times (in weeks) of a group of 30 leukemia patients who received similar treatments. This data set is reported in Lawless (2003) and observations with + sign are censored times.

1, 1, 2, 4, 4, 6, 6, 6, 7, 8, 9, 9, 10, 12, 13, 14, 18, 19, 24, 26, 29, 31+, 42, 45+, 50+, 57, 60, 71+, 85+, 91.

Before going further, we fit the randomly censored IW lifetime model and compared its fitting with some well-known lifetime models like generalized inverted exponential (GIE), gamma, and Weibull lifetime models for this data set. We calculate ML estimates of the associated unknown parameters together with some useful measure of goodness-of-fit tests, namely, the negative log-likelihood function $-\ln L$, the AIC defined by $AIC = 2 \times k - 2 \times \ln L$, proposed by Akaike (1974) and BIC defined by $BIC = k \times \ln(n) - 2 \times \ln L$, proposed by Schwarz (1978), where k is the number of parameters in the model, n is the number of observations in the given data set, L is the maximized value of the likelihood function for the estimated model and Kolmogorov-Smirnov (KS) statistic with its p-values. The lowest $-\ln L$, AIC, BIC, and KS statistics, as well as the highest p-value, indicate the optimal lifetime model. Table 3.10 contains the results of the ML estimates as well as goodness-of-fit test measures. These findings

TABLE 3.10: Summary fit of the real data set of remission times (in weeks) of 30 leukemia patients.

Model	MLE	-lnL	AIC	BIC	KS Test	
					Statistic	p-value
$X \sim$ IW(α,β) $T \sim$ IW(α,λ)	$\hat{\alpha}$=0.7774 $\hat{\beta}$=4.9231 $\hat{\lambda}$=33.3523	137.7351	281.4701	285.6737	0.1373	0.624
$X \sim$ GIE(α,β) $T \sim$ GIE(α,λ)	$\hat{\alpha}$=0.6619 $\hat{\beta}$= 4.7952 $\hat{\lambda}$=63.1718	138.2794	282.5587	286.7623	0.1599	0.4272
$X \sim$ Weibull(α,β) $T \sim$ Weibull(α,λ)	$\hat{\alpha}$=0.9714 $\hat{\beta}$=0.0365 $\hat{\lambda}$=0.0073	140.4595	286.9191	291.1227	0.1556	0.4621
$X \sim$ gamma(α,β) $T \sim$ gamma(α,λ)	$\hat{\alpha}$=1.0441 $\hat{\beta}$=0.0346 $\hat{\lambda}$=0.0072	140.4587	286.9175	291.1211	0.1710	0.3444

demonstrate that for the data set under consideration, a randomly censored IW lifetime model is the appropriate choice. For the fitting of randomly censored data via the graphs, we also consider the Kaplan-Meier (KM) product limit estimator. The KM product-limit estimator for survival function was proposed by Kaplan and Meier (1958) and is given by

$$\hat{S}(t) = \prod_{y_i \le t} \left(1 - \frac{1}{n_i}\right)^{d_i},$$

where, n_i is the number of items survived at time y_i and $d_i = 1$ if item failed, 0 otherwise. Figure 3.2 shows the graphs of the KM estimator and the estimated survival functions of the considered models. The survival function estimate for the IW lifetime model is fairly close to that provided by the KM estimator, as seen in 3.2. As a result, the KM estimator recommends using the IW lifetime model to describe this data set. We provide the results of the estimation

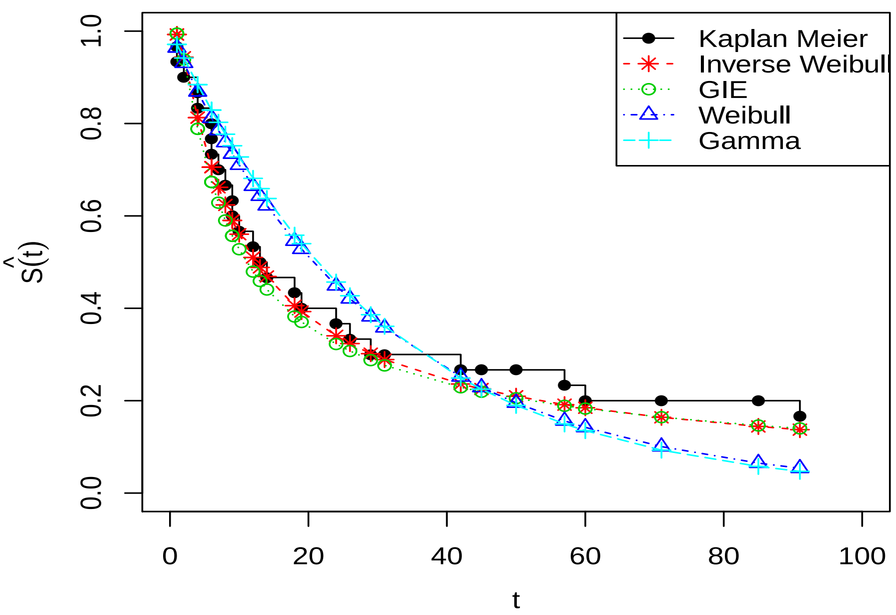

FIGURE 3.2: The plot of estimates of survival functions of considered models

techniques examined in this article based on the above real data set in Table 3.11. The unknown parameters and reliability characteristics are estimated using ML and Bayes methods. We use the median of the data as the mission time ($t = 13.5$) for reliability characteristics. The Bayes estimates are derived using non-informative priors under SELF since we don't have any prior information about the parameters. The TK approximation and Gibbs sampling procedures are used to get the Bayes estimates of parameters. We create a Markov chain using $M = 1,00,000$ for the Gibbs sampling technique using the MH algorithm. The first $M_0 = 20,000$ observations are discarded as burn-in observations, and every $10th$ observation is used as the iid observation of produced MCMC samples of α, β, and λ. We also use graphical diagnostic tools like trace, autocorrelation function (ACF), and histogram with Gaussian kernel density plots to assess the

TABLE 3.11: The ML and Bayes estimates of the parameters and reliability characteristics corresponding to the real data set of remission times (in weeks) of 30 leukemia patients.

Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{S}(t)$	$\hat{h}(t)$
MLE	0.7774	4.9231	33.3524	0.4784	0.0409
	(0.5768, 0.9781)	(2.7112, 7.1349)	(2.3046, 64.4001)		
TK	0.7759	4.9254	36.4505	0.4760	0.0410
MCMC	0.7238	4.6407	24.4085	0.5048	0.0370
	(0.6146, 0.8573)	(3.3702, 5.8947)	(22.4474, 26.2966)		

convergence of their stationary distributions. Figure 3.3 shows the trace, ACF, and histogram with Gaussian kernel plots for the parameters. For all parameters, the trace plots demonstrate a random dispersion around the mean value (represented by a solid line) and fine chain mixing. Chains exhibit very low autocorrelations, as shown by ACF plots. The marginal distributions of the parameters are remarkably symmetrical, as seen by the histogram plots of the produced MCMC samples, implying that the mean is the best estimate for the parameters. These graphs are, in fact, indicative of rapid MCMC convergence.

3.8 Concluding Remarks

The IW lifetime model is a useful lifetime model for representing the failure rate functions with unimodal behaviour. The classical and Bayesian estimation procedures for the parameters and reliability characteristics of IWD under the random censoring model were discussed in this chapter. The MLEs for the unknown parameters as well as the reliability characteristics were calculated. Based on expected Fisher information, asymptotic confidence intervals for the parameters are also calculated. ETT was computed for a randomly censored experiment. TK approximation and Gibbs sampling methods were used to approximate Bayes estimators of the parameters and reliability characteristics under SELF. A comprehensive simulation study was used to evaluate the performance of various estimators. The Bayes estimates with gamma informative priors and the Gibbs sampling technique had lower average absolute biases and mean squared errors than the ML and Bayes estimates with non-informative priors. When there is some prior knowledge, we recommend Bayes estimators.

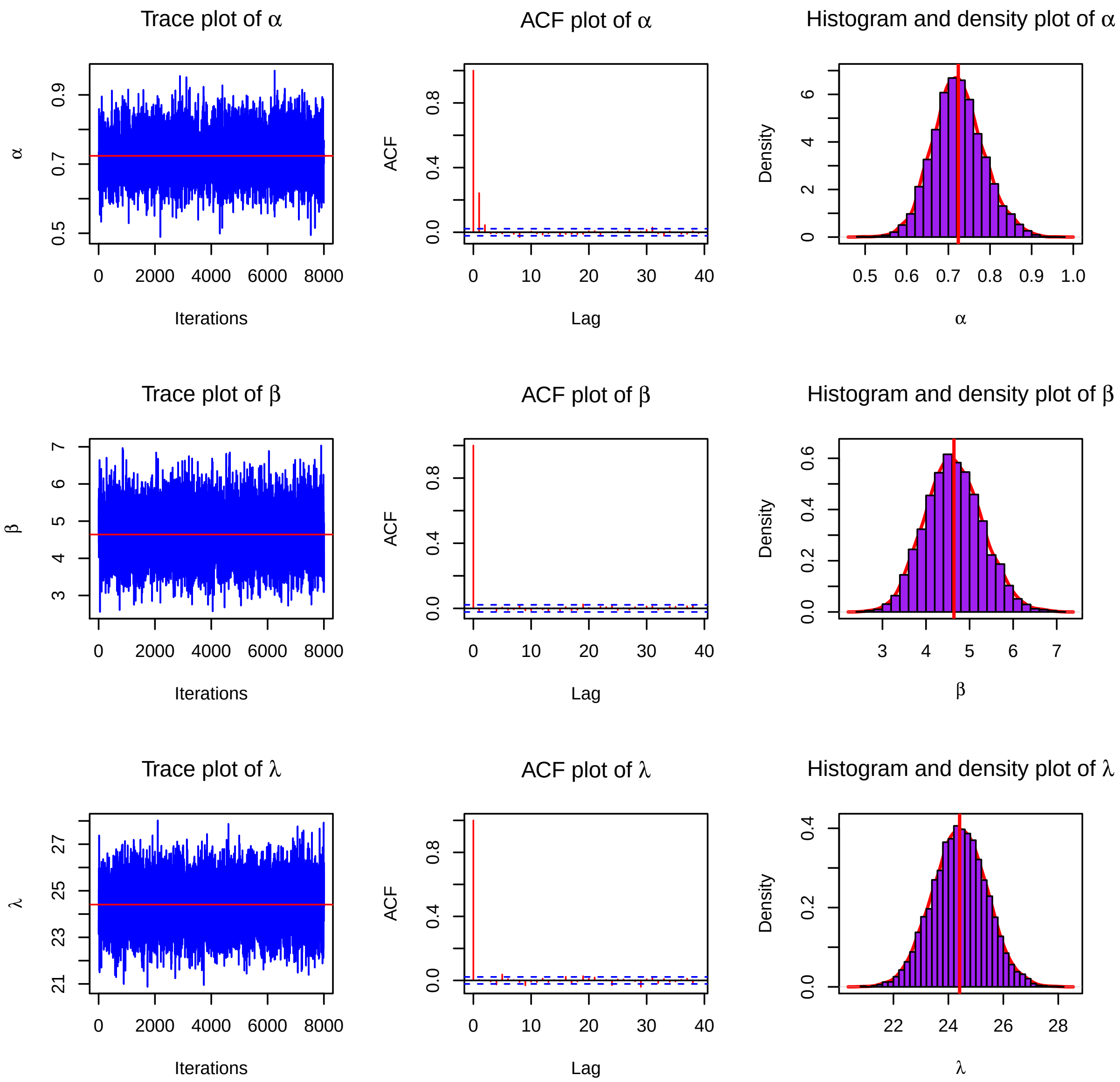

FIGURE 3.3: MCMC diagnostic plots of the parameters