Chapter 4

Classical and Bayesian Estimation of Stress-Strength Reliability for Inverse Pareto Lifetime Model using Progressively Censored Data*

4.1 Introduction

In this chapter, we deal with a problem from reliability theory and we estimate the stress-strength reliability (SSR) for IP lifetime model using progressively censored data from both classical and Bayesian approaches. The IP lifetime model already has been discussed in Chapter 2 under randomly censored data.

In life testing experiments the incomplete information commonly arises because of time limits and other restrictions on data collection or study. The incomplete information in life testing experiments is termed as censoring and it arises when components are removed or destroyed from the experiment before the final termination point. Therefore, censored samples are frequently available around us as a result and we use censored samples rather than the complete sample in life testing experiments. Several censoring schemes have been utilized in the literature to demonstrate the various motive belonging to the life testing experiments. Type-I and Type-II

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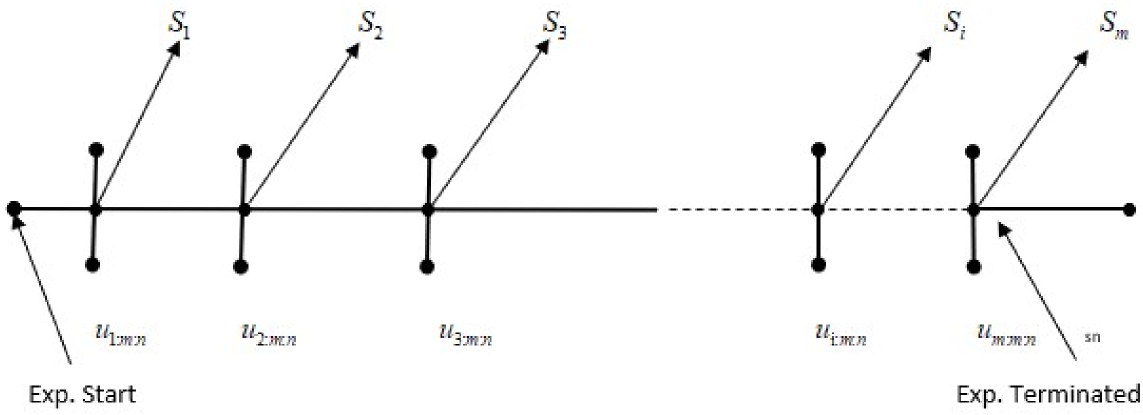


FIGURE 4.1: The schematic diagram of progressive censoring scheme.

censoring schemes are the two most common censoring schemes in the literature. These censoring schemes are used to save money and time by prefixing time or number of failures. These censoring schemes become ineffective when items are removed at intermittent stages from an experiment. To overcome such type difficulty, Cohen (1963) introduced a censoring scheme in the literature, known as progressive censoring scheme. It is one of the popular censoring scheme which supplies the adaptability of removals of the experimental units throughout the experiments. After that, many scholars have studied this censoring scheme for various lifetime models under different scenarios. Two excellent monographs on the progressive censoring scheme are given by Balakrishnan and Aggarwala (2000) and Balakrishnan and Cramer (2014), respectively. According to Hofmann et al. (2005) the progressive censoring schemes significantly improve upon the Type-II censoring scheme in many real-life situations. More details and applications of the progressive censoring scheme can be found in the following latest articles carried out by various scholars like: Kohansal and Rezakhah (2019), Aslam et al. (2020), Goel and Singh (2020), Abu-Moussa et al. (2021), Ghanbari et al. (2021), Asgharzadeh and Fallah (2021), Bedbur and Mies (2021), Wu and Gui (2021), Wu and Chang (2021), Hashem and Alyami (2021), and reference cited therein.

Mathematically, the progressive censoring can be articulated as follows; Let n test units are put on the life test, and only $m(m \leq n)$ failures are obtained. Suppose $U_{1:m:n}, U_{2:m:n}, \ldots, U_{m:m:n}$ be the obtained ordered lifetimes and m be the prefixed number of failures with prefixed censoring scheme $\underset{\sim}{S} = (S_1, S_2, \ldots, S_m)$. When the ith unit fails $(i = 1, 2, \ldots, m-1)$, S_i live units are randomly withdrawn from the experiments. Finally, the remaining $S_m = n - m - \sum\limits_{i=1}^{m-1} S_i$ live units are withdrawn when the mth unit fails. The schematic diagram of the progressive censoring scheme is given in Figure 4.1.

Let $u_1, u_2, \ldots, u_m$ be a progressively Type-II censored sample with prefixed censoring scheme

$\underset{\sim}{S} = (S_1, S_2, \ldots, S_m)$ from a population with pdf $g_U(.)$ and cdf $G_U(.)$, the likelihood function is defined as, see, (Balakrishnan and Aggarwala, 2000)

$$L(u_{1:m:n}, u_{2:m:n}, \ldots, u_{m:m:n}) = K \prod_{i=1}^{m} g_U(u_{i:m:n}) \{1 - G_U(u_{i:m:n})\}^{S_i}, \quad 0 < u_{1:m:n} < u_{2:m:n} < \cdots < u_{m:m:n} < \infty \tag{4.1}$$

where, $K = n(n - S_1 - 1)(n - S_1 - S_2 - 2) \ldots (n - S_1 - S_2 - \cdots - S_{m-1} - m + 1)$.

Remarks: There are two particular cases of progressive Type II censoring scheme: (i) It becomes Type II censoring scheme when $S_i = 0$; $\forall\, i = 1, 2, \ldots m-1$ and $S_m = n - m$, and (ii) It becomes complete sample case when $S_i = 0$; $\forall\, i = 1, 2, \ldots, m$.

In reliability and life testing theory, the stress-strength reliability (SSR) model contains two independent random variables, one as a strength variable, say U and another as a stress variable, say V, the quantity $R = P(V < U)$ is known as SSR. Birnbaum (1956) studied the SSR model in connection with the classical Mann-Whitney statistic. The SSR system is applicable in many real-life problems. Johnstone (1983) showed an anti-tank sabot round being shot at a Soviet T-62 tank as an example of SSR in military applications. The Bayesian method was used to calculate the chances of a particular bullet penetrating its intended target. Another application of SSR was presented by Johnson (1988) in rocket engines. The maximal chamber pressure generated by the ignition of a solid propellant was denoted by V and the strength of the rocket chamber was denoted by U, so that SSR becomes the probability of successful firing of an engine. An excellent monograph on the several SSR models with their applications are given by Kotz et al. (2003). Some recent studies on SSR for different lifetime models based on complete samples are as follows: The Weibull lifetime model is discussed by Jia et al. (2017). Jovanović (2017) studied geometric-exponential lifetime model. The IP lifetime model is studied by Guo and Gui (2018). The generalized inverse Lindley lifetime model is discussed by Sharma (2018), Scaria et al. (2021) studied generalized Pareto lifetime model, the inverse Chen lifetime model is discussed by Agiwal (2021) and the references cited therein. Also, some recent studies on SSR for different lifetime models in case of progressive censoring are carried out by many scholars like: Maxwell lifetime model studied by Chaudhary and Tomer (2018). Yadav et al. (2018) studied IW lifetime. Two parameter Rayleigh lifetime model is discussed by Kohansal and Rezakhah (2019). Goel and Singh (2020) studied modified Weibull lifetime model. Abu-Moussa et al. (2021) discussed Rayleigh lifetime model, and references cited therein.

For a clear view of the study, the rest of the chapter is designed as follows: Section 4.2, deals with the model description. The maximum likelihood estimator (MLE) and asymptotic confidence interval (ACI) of SSR are presented in Section 4.3. The Bayes estimator and highest posterior density (HPD) credible interval of SSR has appeared in Section 4.4. The numerical computations are performed Section 4.5 to compare the ML and Bayes estimators of SSR, numerically. In Section 4.6, two different pairs of real data sets are analyzed to illustrate the proposed methodology. Finally, the concluding remarks are provided in Section 4.7.

4.2 The Model

The pdf and corresponding cdf of IPD with parameter θ, respectively, are given by

$$g_U(u;\theta) = \frac{\theta u^{\theta-1}}{(1+u)^{\theta+1}} \quad ;\ \theta > 0,\ u > 0, \tag{4.2}$$

$$G_U(u;\theta) = \left(\frac{u}{1+u}\right)^{\theta} \quad ;\ \theta > 0,\ u > 0, \tag{4.3}$$

Let U and V be independent random variables following IPD(θ_1) and IPD(θ_2), respectively, then the SSR is defined as

$$\begin{aligned} R = P(V<U) &= \int_0^\infty G_V(u) g_U du \\ &= \int_0^\infty \left(\frac{u}{1+u}\right)^{\theta_2} \frac{\theta_1 u^{\theta_1-1}}{(1+u)^{\theta_1+1}} du \\ &= \int_0^\infty \frac{\theta_1 u^{\theta_1+\theta_2-1}}{(1+u)^{\theta_1+\theta_2+1}} du \\ &= \frac{\theta_1}{\theta_1+\theta_2} = \delta(\theta_1,\theta_2) \quad \text{say.} \end{aligned} \tag{4.4}$$

4.3 Maximum Likelihood Estimation

The ML estimates of the unknown parameters θ_1 and θ_2 are developed in this section to get the ML estimate of SSR R. Let $u_{i:m_1:n_1}; i = 1,2,\ldots,m_1$, be the progressively Type II censored sample from IP(θ_1) with presumed censoring scheme $\underset{\sim}{S} = (S_1, S_2, \ldots, S_{m_1})$ and similarly let $v_{j:m_2:n_2};\ j = 1,2,\ldots,m_2$ be independent progressively Type-II censored sample from IP(θ_2) with presumed censoring scheme $\underset{\sim}{T} = (T_1, T_2, \ldots, T_{m_2})$, then using equations (4.2), (4.3) and

(6.7), the likelihood function is given by

$$
\begin{aligned}
L(\theta_1,\theta_2;\underset{\sim}{u},\underset{\sim}{v}) &= K_1K_2\prod_{i=1}^{m_1} g_U(u_i)[1-G_U(u_i)]^{S_i}\times\prod_{j=1}^{m_2} g_V(v_j)[1-G_V(v_j)]^{T_j}\\
&= K_1K_2\theta_1^{m_1}\theta_2^{m_2}\prod_{i=1}^{m}\frac{u_i^{\theta_1-1}}{(1+u_i)^{\theta_1+1}}\left[1-\left(\frac{u_i}{1+u_i}\right)^{\theta_1}\right]^{S_i}\\
&\quad\times\prod_{j=1}^{m_2}\frac{v_j^{\theta_2-1}}{(1+v_j)^{\theta_2+1}}\left[1-\left(\frac{v_j}{1+v_j}\right)^{\theta_2}\right]^{T_j}
\end{aligned}
\tag{4.5}
$$

where,

$$K_1 = n_1(n_1-S_1-1)(n_1-S_1-S_2-2)\dots(n_1-S_1-S_2-\cdots-S_{m_1-1}-m_1+1)$$

and

$$K_2 = n_2(n_2-T_1-1)(n_2-T_1-T_2-2)\dots(n_2-T_1-T_2-\cdots-T_{m_2-1}-m_2+1)$$

. The corresponding log-likelihood function is obtained as

$$
\begin{aligned}
l(\theta_1,\theta_2) &= C+m_1\ln\theta_1+\theta_1\sum_{i=1}^{m_1}\ln\left(\frac{u_i}{1+u_i}\right)+\sum_{i=1}^{m_1}S_i\ln\left[1-\left(\frac{u_i}{1+u_i}\right)^{\theta_1}\right]\\
&\quad+m_2\ln\theta_2+\theta_2\sum_{j=1}^{m_2}\ln\left(\frac{v_i}{1+v_i}\right)+\sum_{j=1}^{m_2}T_j\ln\left[1-\left(\frac{v_j}{1+v_j}\right)^{\theta_2}\right],
\end{aligned}
\tag{4.6}
$$

where, $C=\ln K_1+\ln K_2-\sum_{i=1}^{m_1}(\ln u_i+\ln(1+u_i))-\sum_{j=1}^{m_2}(\ln v_j+\ln(1+v_j))$. The following normal equations are obtained by differentiating the log-likelihood function w.r.t. θ_1 and θ_2, respectively:

$$\frac{\partial l(\theta_1,\theta_2)}{\partial\theta_1} = \frac{m_1}{\theta_1}+\sum_{i=1}^{m_1}\ln\left(\frac{u_i}{1+u_i}\right)-\sum_{i=1}^{m_1}S_i\frac{\left(\frac{u_i}{1+u_i}\right)^{\theta_1}\ln\left(\frac{u_i}{1+u_i}\right)}{\left[1-\left(\frac{u_i}{1+u_i}\right)^{\theta_1}\right]}=0. \tag{4.7}$$

$$\text{and,}\quad \frac{\partial l(\theta_1,\theta_2)}{\partial\theta_2} = \frac{m_2}{\theta_2}+\sum_{j=1}^{m_2}\ln\left(\frac{v_j}{1+v_j}\right)-\sum_{j=1}^{m_2}T_j\frac{\left(\frac{v_j}{1+v_j}\right)^{\theta_2}\ln\left(\frac{v_j}{1+v_j}\right)}{\left[1-\left(\frac{v_j}{1+v_j}\right)^{\theta_2}\right]}=0. \tag{4.8}$$

The ML estimates of θ_1 and θ_2, say $\hat{\theta}_1$ and $\hat{\theta}_2$ are the solutions of normal equations (4.7) and (4.8), respectively. Here, the closed form solutions are not available for equations (4.7) and

(4.8). A appropriate iterative technique can be utilised to get numerical solutions to these non-linear equations. A number of functions, such as *nlm*, *optim*, *maxLik*, and others, are available in the statistical software R to compute MLEs. Once the ML estimates of unknown parameters are computed, the ML estimate of SSR parameter R, say $\hat{R}$ is derived using invariance property of MLEs and is given by

$$\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2}. \tag{4.9}$$

4.3.1 Asymptotic Confidence Interval

Here, the ACI of SSR R is constructed using delta method as it is difficult to obtain exact distribution of $\hat{R}$. Let $\hat{\phi} = (\hat{\theta}_1, \hat{\theta}_2)$ be the ML estimates of unknown parameters $\phi = (\theta_1, \theta_2)$. The asymptotic variance of $\hat{R}$ using delta method, see, Krishnamoorthy and Lin (2010), is given by

$$Var(\hat{R}) = [b_C' I^{-1}(\phi) b_C],$$

where, $I(\phi) = -E \begin{bmatrix} \frac{\partial^2 l(\theta_1,\theta_2)}{\partial \theta_1^2} & \frac{\partial^2 l(\theta_1,\theta_2)}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 l(\theta_1,\theta_2)}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 l(\theta_1,\theta_2)}{\partial \theta_2^2} \end{bmatrix}$ is the Fisher information matrix and $b_C = \left(\frac{\partial R}{\partial \theta_1}, \frac{\partial R}{\partial \theta_2}\right)'$.

The observed Fisher information can be utilized as a consistent estimator of the Fisher information under modest regularity conditions. As a result, the observed variance of $\hat{R}$ is equal to

$$\hat{V}ar(\hat{R}) \simeq [b_C' I^{-1}(\phi) b_C]_{\phi=\hat{\phi}}.$$

The elements of partial derivatives in the Fisher information matrix $I(\phi)$ are given by

$$\frac{\partial^2 l(\theta_1,\theta_2)}{\partial \theta_1^2} = -\frac{m_1}{\theta_1^2} - \sum_{i=1}^{m_1} S_i \frac{\left\{\ln\left(\frac{u_i}{1+u_i}\right)\right\}^2 \left(\frac{u_i}{1+u_i}\right)^{\theta_1}}{\left[1 - \left(\frac{u_i}{1+u_i}\right)^{\theta_1}\right]^2},$$

$$\frac{\partial^2 l(\theta_1,\theta_2)}{\partial \theta_2^2} = -\frac{m_2}{\theta_2^2} - \sum_{j=1}^{m_2} T_j \frac{\left\{\ln\left(\frac{v_j}{1+v_j}\right)\right\}^2 \left(\frac{v_j}{1+v_j}\right)^{\theta_2}}{\left\{1 - \left(\frac{v_j}{1+v_j}\right)^{\theta_1}\right\}^2},$$

$$\frac{\partial^2 l(\theta_1,\theta_2)}{\partial \theta_1 \partial \theta_2} = \frac{\partial^2 l(\theta_1,\theta_2)}{\partial \theta_2 \partial \theta_1} = 0,$$

and the elements of b_C are given by

$$\frac{\partial R}{\partial \theta_1} = \frac{\theta_2}{(\theta_1+\theta_2)^2}, \quad \frac{\partial R}{\partial \theta_2} = -\frac{\theta_1}{(\theta_1+\theta_2)^2}.$$

Thus $\frac{\hat{R}-R}{\sqrt{\hat{V}ar(\hat{R})}} \sim N(0,1)$. Therefore, the $100(1-\xi)\%$ ACI of R is given by $\hat{R} \pm z_{\xi/2}\sqrt{\hat{V}ar(\hat{R})}$, where $z_{\xi/2}$ is the upper $(\xi/2)^{th}$ quantile of $N(0,1)$. Also, the coverage probability (CP) for R is given by

$$CP_R = \left[\left|\frac{\hat{R}-R}{\sqrt{\hat{V}ar(\hat{R})}}\right| \leq z_{\xi/2}\right].$$

4.4 Bayesian Estimation

In this part, we use the importance sampling (IS) approach to get the Bayes estimator of SSR R under the generalised entropy loss function (GELF) using non-informative and gamma informative priors.

4.4.1 Loss function

A suitable loss function must be specified in Bayesian estimation. The SELF is the most often used loss function in the literature. When over and under estimations of equal magnitude have the same effects, the SELF is appropriate. When the real loss is not symmetric in terms of over and under estimates, asymmetric loss functions are employed to illustrate the consequences of various inaccuracies. For this, a general-purpose loss function, such as GELF, can be employed. The GELF was proposed in the literature by Calabria and Pulcini (1996). This loss function is an extension of the entropy loss function and is defined by

$$L(\alpha,\hat{\alpha}) \propto \left[\left(\frac{\hat{\alpha}}{\alpha}\right)^q - q\ln\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right]; \;\; q \neq 0,$$

where, $\hat{\alpha}$ is the decision rule which estimate α. When $q > 0$, a positive error has more implications than a negative error, and when $q < 0$, a negative error has greater effects. The Bayes estimator under GELF is calculated as follows:

$$\hat{\alpha} = E\left[\alpha^{-q}|\text{data}\right]^{-1/q}. \tag{4.10}$$

Remark: The Bayes estimator in equation (4.10) is reduced to a Bayes estimator under precautionary loss function (PLF), SELF, and entropy loss function (ELF) for $q = -2,\ -1$, and 1, respectively.

4.4.2 Prior and Posterior Distributions

We suppose the unknown parameters θ_1 and θ_2 are a-priori independent and have the following gamma distributions with their corresponding pdfs:

$$\eta_1(\theta_1) \propto \theta_1^{a_1-1}\exp(-b_1\theta_1);\ \ \theta_1 > 0, a_1, b_1 > 0,$$

$$\text{and}\quad \eta_2(\theta_2) \propto \theta_2^{a_2-1}\exp(-b_2\theta_2);\ \ \theta_2 > 0, a_2, b_2 > 0,$$

where, $a_i, b_i;\ i = 1,2$ are the hyper-parameters so chosen to reflect prior information about the parameters θ_1 and θ_2, respectively. As a result, the joint prior distribution of θ_1 and θ_2 can be expressed as

$$\eta(\theta_1,\theta_2) \propto \theta_1^{a_1-1}\theta_2^{a_2-1}\exp\{-(b_1\theta_1 + b_2\theta_2)\}. \tag{4.11}$$

The selection of independent gamma priors is not unreasonable. The gamma distribution family is highly flexible, and it includes a variety of distributions. It is also worth noting that non-informative priors are special instances of independent gamma priors. Several researchers have utilised gamma priors in various contexts, including Guo and Gui (2018) Kumar (2018), Krishna et al. (2019), and many more. The posterior distribution of θ_1 and θ_2 is now obtained by incorporating joint prior distribution (4.11) to the likelihood function (4.5),

$$\pi(\theta_1,\theta_2|\underset{\sim}{u},\underset{\sim}{v}) = \frac{L(\theta_1,\theta_2;\text{data})\eta(\theta_1,\theta_2)}{\int_0^\infty\int_0^\infty L(\theta_1,\theta_2;\underset{\sim}{u},\underset{\sim}{v})g(\theta_1,\theta_2)d\theta_1 d\theta_2}$$

$$\Rightarrow \pi(\theta_1,\theta_2|\underset{\sim}{u},\underset{\sim}{v}) \propto \theta_1^{m_1+a_1-1}\theta_2^{m_2+a_2-1}\exp\left\{-\theta_1\left[b_1 - \sum_{i=1}^{m_1}\ln\left(\frac{u_i}{1+u_i}\right)\right]\right\}$$
$$\times\exp\left\{-\theta_2\left[b_2 - \sum_{j=1}^{m_2}\ln\left(\frac{v_j}{1+v_j}\right)\right]\right\}\prod_{i=1}^{m_1}\left[1-\left(\frac{u_i}{1+u_i}\right)^{\theta_1}\right]^{S_i}$$
$$\times\prod_{j=1}^{m_2}\left[1-\left(\frac{v_j}{1+v_j}\right)^{\theta_2}\right]^{T_j}. \tag{4.12}$$

From the posterior distribution given in equation (4.12), we observe that the Bayes estimator for SSR R cannot obtain in closed form. Therefore, an approximation method, importance sampling technique is used to derive Bayes estimate of R.

4.4.3 Importance Sampling Technique

Here, the IS technique is used to construct the Bayes estimator and HPD credible interval of SSR R. The posterior distribution of θ_1 and θ_2 given in equation (4.12) can be rewritten as

$$\pi(\theta_1,\theta_2|\underset{\sim}{u},\underset{\sim}{v}) \propto \theta_1^{m_1+a_1-1}\exp\left\{-\theta_1\left[b_1-\sum_{i=1}^{m_1}\ln\left(\frac{u_i}{1+u_i}\right)\right]\right\}$$
$$\times\,\theta_2^{m_2+a_2-1}\exp\left\{-\theta_2\left[b_2-\sum_{j=1}^{m_2}\ln\left(\frac{v_j}{1+v_j}\right)\right]\right\}$$
$$\times\exp\left\{\sum_{i=1}^{m_1}S_i\ln\left[1-\left(\frac{u_i}{1+u_i}\right)^{\theta_1}\right]+\sum_{j=1}^{m_2}T_j\left[1-\left(\frac{v_j}{1+v_j}\right)^{\theta_2}\right]\right\}$$

$$\pi(\theta_1,\theta_2|\underset{\sim}{u},\underset{\sim}{v}) \propto f_{GA}(\theta_1;m_1+a_1,B_1)\,f_{GA}(\theta_2;m_2+a_2,B_2)\,W(\theta_1,\theta_2)=\pi_1(\theta_1,\theta_2|\text{data})\ \ (\text{say}),$$

where, $B_1=\left[b_1-\sum\limits_{i=1}^{m_1}\ln\left(\frac{u_i}{1+u_i}\right)\right]$, $B_2=\left[b_2-\sum\limits_{j=1}^{m_2}\ln\left(\frac{v_j}{1+v_j}\right)\right]$,

$$W(\theta_1,\theta_2)=\exp\left\{\sum_{i=1}^{m_1}S_i\ln\left[1-\left(\frac{u_i}{1+u_i}\right)^{\theta_1}\right]+\sum_{j=1}^{m_2}T_j\left[1-\left(\frac{v_j}{1+v_j}\right)^{\theta_2}\right]\right\}$$

and $f_{GA}(.;a,b)$ is a gamma distribution having shape and scale parameters a and b, respectively. Now the posterior expectation of $\phi(\theta_1,\theta_2)$ is given by

$$E[\phi(\theta_1,\theta_2)|\underset{\sim}{u},\underset{\sim}{v}]=\frac{\int_0^\infty\int_0^\infty\phi(\theta_1,\theta_2)\pi_1(\theta_1,\theta_2|\underset{\sim}{u},\underset{\sim}{v})d\theta_1d\theta_2}{\int_0^\infty\int_0^\infty\pi_1(\theta_1,\theta_2|\underset{\sim}{u},\underset{\sim}{v})d\theta_1d\theta_2} \tag{4.13}$$

The posterior mean $E[\phi(\theta_1,\theta_2)|\underset{\sim}{u},\underset{\sim}{v}]$ given in equation (4.13) is the ratio of two integrals and the closed form solution of this mean is not available. The IS approach is utilised to provide an approximate solution, and the following steps are taken into account for computation:

Step 1: Generate $\theta_1^{(1)}$ from $f_{GA}(\theta_1;m_1+a_1,B_1)$.
Step 2: Generate $\theta_2^{(1)}$ from $f_{GA}(\theta_2;m_2+a_2,B_2)$.
Step 3: Generate $\delta^{(1)}=\frac{\theta_1^{(1)}}{\theta_1^{(1)}+\theta_2^{(1)}}$ using equation (4.4).
Step 4: Repeat the above steps 1-3, M times to obtain the importance sample $(\delta^{(1)},\ldots,\delta^{(M)})$.

Now, using the IS technique under GELF, the approximate Bayes estimator of SSR is given by

$$\hat{R}_B = \left[\frac{\sum_{j=1}^{M} \left\{ \delta(\theta_1^{(j)}, \theta_2^{(j)}) \right\}^{-q} W(\theta_1^{(j)}, \theta_2^{(j)})}{\sum_{j=1}^{M} W(\theta_1^{(j)}, \theta_2^{(j)})} \right]^{-1/q}. \tag{4.14}$$

4.4.4 HPD Credible Interval

Using the produced importance sample, the HPD credible interval of SSR R can be constructed. Let $\delta_{(1)} < \delta_{(2)} < \cdots < \delta_{(M)}$ be the ordered values of $\delta^{(1)},\ \delta^{(2)}, \ldots, \delta^{(M)}$. Now, using the algorithm proposed by Chen and Shao (1999), the $100(1-\xi)\%$, where, $0 < \xi < 1$, HPD credible interval of SSR is given by $\left(\delta_{(j)},\ \delta_{(j+[(1-\xi)M])}\right)$, where j is chosen such that

$$\delta_{(j+[(1-\xi)M])} - \delta_{(j)} = \min_{1 \le i \le \xi M} \left(\delta_{(i+[(1-\xi)M])} - \delta_{(i)}\right), j = 1, 2, \ldots, M,$$

where, $[x]$ is the integral part of x.

4.5 Numerical Computations

A Monte Carlo simulation study is provided in this section to assess the efficacy of the estimation methods developed in this chapter. The mean squared errors (MSEs) and average estimates (AEs) of the ML and Bayes estimators of RSS R are calculated. The Bayes estimate of SSR is computed in case of non-informative prior (Prior A) and informative gamma prior (Prior B) under GELF. Also, the average length (ALs) of 95% ACI and HPD credible intervals with their corresponding coverage probabilities (CP) of SSR R are obtained. For computation purpose, two independent progressively Type II censored samples $\underset{\sim}{u}$ and $\underset{\sim}{v}$ of sample sizes n_1 and n_2, effective sample sizes m_1 and m_2 are produced from IP(θ_1) and IP(θ_2) with prefixed censoring schemes S_i; $i = 1, 2, \ldots, m_1$ and T_j; $j = 1, 2, \ldots, m_2$, respectively, using the algorithm provided by Balakrishnan and Sandhu (1995). The several combinations of sample sizes (n_1, n_2), effective sample sizes (m_1, m_2), and prefixed censoring schemes $(\underset{\sim}{S}, \underset{\sim}{T})$ are considered. For simulation purpose, we assign $n = n_1 = n_2$, $m = m_1 = m_2$ and $CS = (\underset{\sim}{S} = \underset{\sim}{T})$, and these combinations are reported in Table 4.1. In Table 4.1, schemes [4], [8] and [12] are the cases for complete sample data.

We consider two sets of true values for the parameters $(\theta_1,\ \theta_2) = (1.5,\ 0.5)$ and $(\theta_1,\ \theta_2) = (0.5,\ 1.5)$ so that SSR R becomes, $R = 0.75$ and $R = 0.25$, respectively. In Bayesian computations, for informative priors the choices of hyper-parameters are chosen such that the prior means are exactly equal to true values of the parameters. For informative priors $\{(a_1, b_1) = (3,2),\ (a_2, b_2) = (2,4)\}$ and $\{(a_1, b_1) = (2,4),\ (a_2, b_2) = (3,2)\}$ are considered for above considered two sets of true values of the parameters, respectively. In case of non-informative priors, we consider $a_i = b_i = 0.0001;\ i = 1,2$. Also, we consider three different choices of $q = -2,\ -1,\ 1$ for GELF. We take $M = 10,000$ for importance sampling technique and consider 20% of M as burn-in-period. The entire process is repeated 1,000 times. All computations in this article are done with the statistical software R (see R Core Team (2021)). All the simulated results are presented in Tables 4.2, 4.3, 4.4 and 4.5. From these simulation Tables, following conclusion are made:

In view of Tables 4.2 and 4.4, this experiment has brought up some interesting observations. In almost all cases, the output of ML and Bayes estimates of SSR in terms of MSEs are very adequate even for small sample sizes. MSEs are found to decrease as n and m increase. It confirms the consistent behavior of estimators of SSR. Also, the performance of Bayes estimators with Prior B is better than ML estimator even with Prior A in terms of MSEs, as Bayes estimators with Prior B includes prior information about the parameters.

In view of Tables 4.3 and 4.5 show that the average lengths of ACIs and HPD credible intervals are shrinking with increase in number of failures. According to Table 4.3 asymptotic intervals has smaller average length and than HPD credible intervals with Prior A and Prior B both. The coverage probability for HPD credible with Prior A and Prior B attains their prescribed confidence coefficient in almost all cases but ACI does not. Also, from Table 4.5 as the true value of SSR increases, the coverage probability for ACI estimator attain their prescribed confidence coefficient.

TABLE 4.1: Progressive censoring schemes used in simulation study.

n	m	CS	Schemes	n	m	CS	Schemes
20	15	[1]	(5*1,0*14)	30	24	[7]	(0*23,6*1)
	15	[2]	(1*2,0*5,1,0*5,1*2)		24	[8]	(0*30)
	15	[3]	(0*14,5*1)	40	35	[9]	(5*1,0*34)
	20	[4]	(0*20)		35	[10]	(1*1,0*8,1*1,0*6,1*1,0*8,1*1,0*8,1*1)
30	24	[5]	(6*1,0*23)		35	[11]	(0*34,5*1)
	24	[6]	(2*1,0*10,2*1,0*11,2*1)		40	[12]	(0*40)

TABLE 4.2: The AE and MSEs of ML and Bayes estimates of SSR, when $R = 0.75$.

					Bayes											
					Prior A						Prior B					
			MLE		q=-1		q=1		q=-2		q=-1		q=1		q=-2	
n	m	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
20	15	[1]	0.7437	0.0047	0.7387	0.0047	0.7326	0.0055	0.7420	0.0044	0.7377	0.0038	0.7365	0.0039	0.7415	0.0037
20	15	[2]	0.7399	0.0048	0.7355	0.0048	0.7334	0.0053	0.7452	0.0039	0.7397	0.0035	0.7377	0.0036	0.7397	0.0034
20	15	[3]	0.7425	0.0043	0.7382	0.0043	0.7333	0.0047	0.7450	0.0040	0.7390	0.0034	0.7398	0.0034	0.7416	0.0031
20	20	[4]	0.7439	0.0040	0.7395	0.0040	0.7343	0.0043	0.7439	0.0034	0.7410	0.0029	0.7414	0.0026	0.7448	0.0027
30	24	[5]	0.7473	0.0029	0.7439	0.0029	0.7386	0.0034	0.7435	0.0030	0.7436	0.0023	0.7400	0.0027	0.7432	0.0025
30	24	[6]	0.7435	0.0025	0.7403	0.0026	0.7420	0.0028	0.7451	0.0027	0.7452	0.0023	0.7408	0.0025	0.7449	0.0022
30	24	[7]	0.7482	0.0024	0.7452	0.0024	0.7420	0.0028	0.7451	0.0027	0.7445	0.0024	0.7443	0.0022	0.7437	0.0022
30	30	[8]	0.7473	0.0024	0.7444	0.0024	0.7390	0.0025	0.7456	0.0024	0.7418	0.0021	0.7425	0.0021	0.7465	0.0021
40	35	[9]	0.7477	0.0020	0.7452	0.0020	0.7452	0.0020	0.7449	0.0020	0.7455	0.0018	0.7423	0.0020	0.7462	0.0019
40	35	[10]	0.7449	0.0021	0.7426	0.0021	0.7436	0.0021	0.7443	0.0019	0.7473	0.0016	0.7411	0.0018	0.7463	0.0016
40	35	[11]	0.7495	0.0018	0.7473	0.0018	0.7372	0.0023	0.7446	0.0018	0.7449	0.0016	0.7430	0.0017	0.7470	0.0016
40	40	[12]	0.7475	0.0017	0.7452	0.0017	0.7424	0.0018	0.7452	0.0019	0.7443	0.0016	0.7423	0.0017	0.7450	0.0015

TABLE 4.3: AL and CPs of 95% asymptotic confidence/HPD credible intervals of SSR, when $R = 0.75$.

					HPD											
					Prior A						Prior B					
			ACI		q=-1		q=1		q=-2		q=-1		q=1		q=-2	
n	m	CS	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
20	15	[1]	0.5612	0.993	0.3000	0.973	0.2996	0.967	0.3000	0.968	0.2804	0.985	0.2779	0.981	0.2796	0.970
20	15	[2]	0.5283	0.987	0.3028	0.971	0.3000	0.977	0.2985	0.978	0.2792	0.978	0.2775	0.988	0.2807	0.981
20	15	[3]	0.5173	0.994	0.3019	0.985	0.3014	0.977	0.2985	0.985	0.2793	0.979	0.2759	0.982	0.2793	0.986
20	20	[4]	0.5177	0.993	0.2615	0.968	0.2616	0.963	0.2609	0.972	0.2474	0.978	0.2450	0.988	0.2466	0.977
30	24	[5]	0.4552	0.994	0.2382	0.968	0.2390	0.970	0.2395	0.973	0.2279	0.977	0.2278	0.974	0.2287	0.978
30	24	[6]	0.4314	0.996	0.2412	0.982	0.2379	0.974	0.2389	0.976	0.2268	0.991	0.2274	0.987	0.2281	0.984
30	24	[7]	0.4254	0.999	0.2383	0.981	0.2380	0.974	0.2391	0.976	0.2270	0.980	0.2256	0.980	0.2283	0.981
30	30	[8]	0.4243	0.996	0.2140	0.973	0.2151	0.969	0.2141	0.971	0.2074	0.980	0.2055	0.977	0.2057	0.967
40	35	[9]	0.3845	0.997	0.1987	0.977	0.1971	0.964	0.1992	0.979	0.1920	0.976	0.1922	0.985	0.1919	0.968
40	35	[10]	0.3696	0.994	0.1999	0.972	0.1980	0.976	0.1999	0.980	0.1911	0.983	0.1929	0.976	0.1923	0.980
40	35	[11]	0.3688	0.999	0.1977	0.976	0.2013	0.970	0.1998	0.978	0.1923	0.984	0.1920	0.982	0.1917	0.978
40	40	[12]	0.3670	0.996	0.1861	0.986	0.1864	0.977	0.1866	0.969	0.1812	0.981	0.1811	0.980	0.1815	0.981

TABLE 4.4: The AE and MSEs of ML and Bayes estimates of SSR, when $R = 0.25$.

					Bayes											
					Prior A						Prior B					
			MLE		q=-1		q=1		q=-2		q=-1		q=1		q=-2	
n	m	CS	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE	AE	MSE
20	15	[1]	0.2570	0.0048	0.2620	0.0048	0.2492	0.0047	0.2681	0.0052	0.2598	0.0035	0.2499	0.0036	0.2687	0.0042
20	15	[2]	0.2604	0.0049	0.2649	0.0049	0.2454	0.0040	0.2669	0.0044	0.2593	0.0034	0.2456	0.0032	0.2673	0.0037
20	15	[3]	0.2566	0.0046	0.2609	0.0046	0.2475	0.0042	0.2625	0.0044	0.2589	0.0033	0.2487	0.0033	0.2657	0.0035
20	20	[4]	0.2558	0.0034	0.2602	0.0034	0.2450	0.0033	0.2689	0.0040	0.2602	0.0031	0.2441	0.0026	0.2659	0.0031
30	24	[5]	0.2554	0.0030	0.2588	0.0030	0.2476	0.0032	0.2639	0.0033	0.2560	0.0024	0.2498	0.0024	0.2611	0.0027
30	24	[6]	0.2534	0.0028	0.2565	0.0028	0.2455	0.0026	0.2628	0.0030	0.2545	0.0019	0.2445	0.0022	0.2593	0.0027
30	24	[7]	0.2534	0.0028	0.2563	0.0028	0.2450	0.0025	0.2606	0.0027	0.2548	0.0021	0.2462	0.0021	0.2593	0.0022
30	30	[8]	0.2544	0.0025	0.2574	0.0025	0.2479	0.0023	0.2607	0.0026	0.2597	0.0021	0.2449	0.0018	0.2590	0.0020
40	35	[9]	0.2515	0.0020	0.2540	0.0020	0.2473	0.0019	0.2608	0.0023	0.2565	0.0019	0.2461	0.0018	0.2580	0.0020
40	35	[10]	0.2548	0.0020	0.2571	0.0020	0.2458	0.0019	0.2590	0.0019	0.2555	0.0017	0.2489	0.0016	0.2569	0.0017
40	35	[11]	0.2522	0.0018	0.2545	0.0018	0.2491	0.0018	0.2582	0.0019	0.2553	0.0017	0.2479	0.0017	0.2562	0.0017
40	40	[12]	0.2534	0.0017	0.2557	0.0017	0.2473	0.0016	0.2569	0.0017	0.2558	0.0017	0.2465	0.0015	0.2580	0.0017

TABLE 4.5: AL and CPs of 95% asymptotic confidence/ HPD credible intervals of SSR, when $R = 0.25$.

					HPD											
					Prior A						Prior B					
			ACI		q=-1		q=1		q=-2		q=-1		q=1		q=-2	
n	m	CS	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
20	15	[1]	0.1903	0.927	0.3007	0.971	0.3024	0.962	0.2988	0.965	0.2790	0.982	0.2814	0.975	0.2802	0.976
20	15	[2]	0.1827	0.937	0.3028	0.973	0.3003	0.977	0.3003	0.975	0.2785	0.987	0.2778	0.985	0.2797	0.983
20	15	[3]	0.1760	0.932	0.3004	0.976	0.3012	0.979	0.2971	0.972	0.2781	0.983	0.2792	0.977	0.2786	0.986
20	20	[4]	0.1756	0.965	0.2621	0.979	0.2612	0.979	0.2629	0.971	0.2481	0.977	0.2462	0.979	0.2481	0.979
30	24	[5]	0.1536	0.936	0.2397	0.975	0.2391	0.962	0.2394	0.956	0.2276	0.979	0.2294	0.982	0.2276	0.977
30	24	[6]	0.1459	0.927	0.2390	0.979	0.2381	0.972	0.2398	0.973	0.2270	0.986	0.2261	0.981	0.2265	0.971
30	24	[7]	0.1424	0.935	0.2389	0.974	0.2379	0.982	0.2388	0.972	0.2268	0.985	0.2266	0.983	0.2269	0.989
30	30	[8]	0.1429	0.946	0.2149	0.967	0.2149	0.974	0.2140	0.961	0.2084	0.983	0.2051	0.985	0.2058	0.984
40	35	[9]	0.1284	0.926	0.1982	0.975	0.1987	0.971	0.1993	0.969	0.1928	0.967	0.1912	0.974	0.1918	0.976
40	35	[10]	0.1254	0.935	0.1997	0.980	0.1977	0.976	0.1990	0.981	0.1925	0.985	0.1925	0.981	0.1915	0.976
40	35	[11]	0.1230	0.959	0.1986	0.981	0.1993	0.971	0.1988	0.982	0.1922	0.982	0.1919	0.978	0.1911	0.979
40	40	[12]	0.1235	0.952	0.1865	0.980	0.1859	0.981	0.1855	0.972	0.1812	0.976	0.1799	0.977	0.1808	0.971

4.6 Real Data Analysis

The applicability of the considered model and methodology presented in this chapter is addressed in this section. We examine two distinct pairs of real data sets for this purpose.

4.6.1 Real Data Set I

This pair of real data sets are taken from Bain and Englehardt (1991). These data are the failure times (in hours) of the air conditioning system of two different aeroplanes. The failure times of the air conditioning system of two aeroplanes, respectively, are as follows:

Plane 720 (U): 1.2, 2.1, 2.6, 2.7, 2.9, 2.9, 4.8, 5.7, 5.9, 7.0, 7.4, 15.3, 32.6, 38.6, 50.2

Plane 7911 (V): 3.3, 4.7, 5.5, 5.6, 10.4, 17.6, 18.2, 22.0, 23.9, 24.6, 32.0.

Guo and Gui (2018) studied these data sets for SSR for IP lifetime model in a complete sample case. They showed that these data sets good fit the IP lifetime model. Before further proceeding, we perform Kolmogorov-Smirnov (KS) and Anderson-Darling (AD) goodness of fit tests to check whether the given data sets follow IPD or not using ML estimation. The ML estimates of the unknown parameters, KS, and AD test statistics with their corresponding p-values for these data sets are reported in Table 4.6. From Table 4.6, it is clear that p-values are greater than 0.05, corresponding to both KS and AD goodness of fit tests for these data sets. Therefore, we can assume that these data sets follow the IP lifetime model at a 5% level of significance. Now, using the four distinct progressive censoring techniques, the following four progressively

TABLE 4.6: Fitting of real data set I for IP lifetime model.

Data Set I	MLE	KS Test		AD Test	
		KS	p-value	AD	p-value
Plane 720 (U)	4.7844	0.1880	0.6638	0.4547	0.7907
Plane 7911 (V)	9.6022	0.2558	0.3996	0.7421	0.5212

censored samples are generated from the above complete sample data sets:

Scheme 1 : $(n_1 = 15,\ m_1 = 10)$, $S_1 = [5*1, 0*9]$, and $(n_2 = 11,\ m_2 = 8)$, $T_1 = [3*1, 0*7]$.

$$U : 1.2, 4.8, 5.7, 5.9, 7.0, 7.4, 15.3, 32.6, 38.6, 50.2$$

$$V : 3.3, 10.4, 17.6, 18.2, 22.0, 23.9, 24.6, 32.0$$

Scheme 2 : $(n_1 = 15,\ m_1 = 10)$, $S_2 = [1*1, 0*1, 1*1, 1*0, 1*1, 0*2, 1*1]$, and $(n_2 = 11,\ m_2 = 8)$ $T_2 = [1*1, 0*3, 1*1, 0*2, 1*1]$.

$$U : 1.2, 2.6, 2.7, 2.9, 4.8, 5.9, 7.0, 15.3, 32.6, 38.6$$

$$V : 3.3, 5.5, 5.6, 10.4, 17.6, 22.0, 23.9, 24.6$$

Scheme 3 : $(n_1 = 15,\ m_1 = 10)$, $S_3 = [0*9, 5*1]$, and $(n_2 = 11,\ m_2 = 8)$, $T_3 = [0*7, 3*1]$.

$$U : 1.2, 2.1, 2.6, 2.7, 2.9, 2.9, 4.8, 5.7, 5.9, 7.0$$

$$V : 3.3, 4.7, 5.5, 5.6, 10.4, 17.6, 18.2, 22.0$$

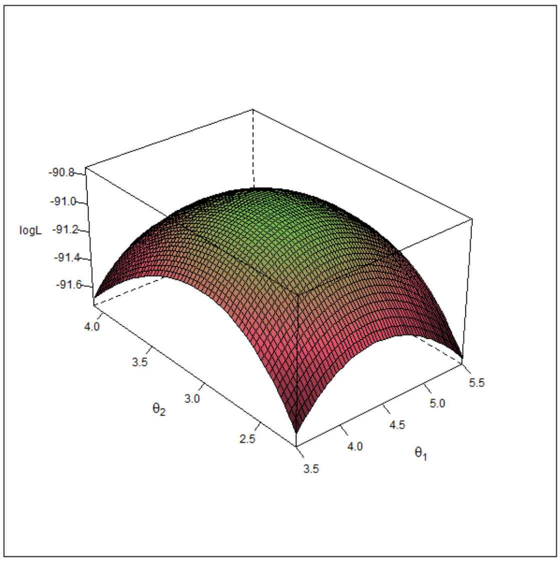
Scheme 4 : $(n_1 = 15,\ m_1 = 15)$, $S_4 = [0*15]$, and $(n_2 = 11,\ m_2 = 11)$, $T_4 = [0*11]$.

$$U : 1.2, 2.1, 2.6, 2.7, 2.9, 2.9, 4.8, 5.7, 5.9, 7.0, 7.4, 15.3, 32.6, 38.6, 50.2$$

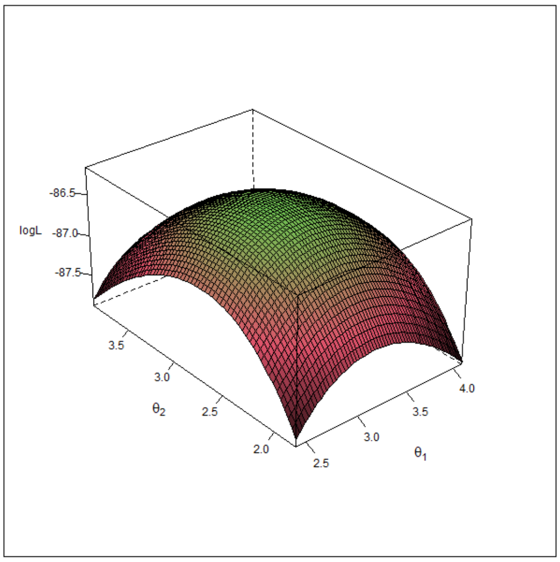
$$V : 3.3, 4.7, 5.5, 5.6, 10.4, 17.6, 18.2, 22.0, 23.9, 24.6, 32.0$$

Furthermore, for the applicability of considered methodology, we analyzed data set I under consideration of the proposed study. The ML, Bayes estimates, and 95% of asymptotic confidence/HPD credible intervals of SSR R are obtained. We further confirm the presence and uniqueness of the MLEs by plotting the log-likelihood function of the parameters θ_1 and θ_2 for four distinct progressively censored samples. These plots for four different censoring schemes are given in Figure 4.2. These plots show that the likelihood surfaces have curvature in both θ_1 and θ_2 directions, indicating that the MLEs $\hat{\theta}_1$ and $\hat{\theta}_2$ exist and are unique.

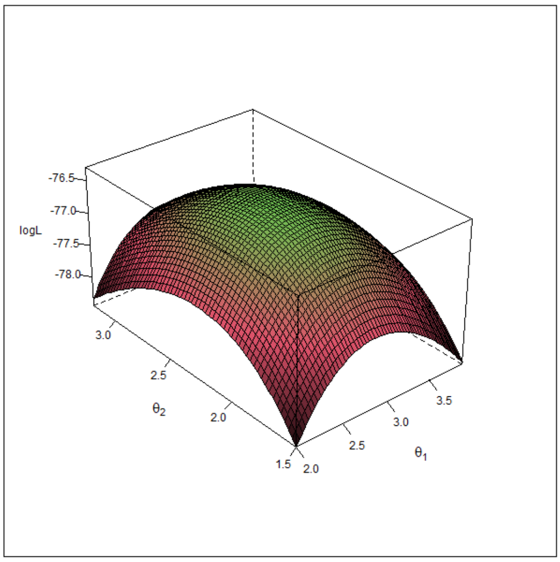
The Bayes estimates of SSR are computed using the importance sampling approach under GELF in the situation of non-informative priors because we do not have prior information. For the importance sampling approach, $M = 10,000$ samples are generated, with the burn-in period accounting for 20% of M. For GELF, we look at three distinct $q = -1, 1, -2$ values. Figure 4.3 shows the trace plots and histograms with posterior density plots based on importance samples for all four progressively censored data sets in Bayesian computations. From Figure 4.3, we observe that the trace plots represent fine mixing of the chains and converge to their stationary distributions. Also, histograms with corresponding density plots are almost symmetrical about their means in all cases. This shows good performance of the importance sampling technique and therefore, we can conclude that the Bayes estimates are good. In the case of real data set I estimation results are reported in Tables 4.7 for all four censoring schemes.



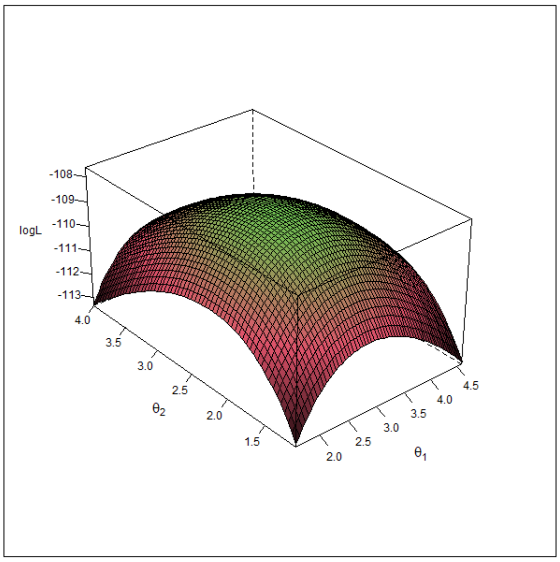
For censoring scheme (S_1, T_1).



For censoring scheme (S_2, T_2).



For censoring scheme (S_3, T_3).



For censoring scheme (S_4, T_4).

FIGURE 4.2: Plots of log-likelihood function of θ_1 and θ_2 for different censoring schemes in case of real data set I.

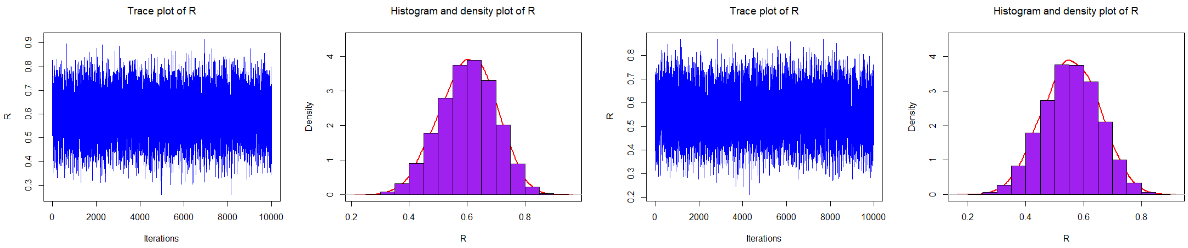
4.6.2 Real Data Set II

Here, we consider breakdown times (in minutes) of an insulating fluid between electrodes at different voltages 34 kV and 36 kV, respectively. These data sets are reported in (Nelson, 1982, p. 105). The breakdown times at two different electrodes, respectively, are as follows:

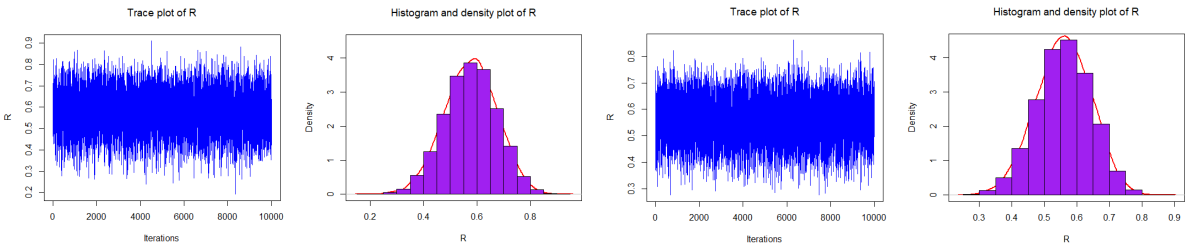
34 kV (U): 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06,

TABLE 4.7: The ML, Bayes estimates and 95% ACIs and HPD credible intervals of SSR in case of real data set I.

	ACI		q=-1		q=1		q=-2	
Schemes	$\hat{R}$	CI	$\hat{R}_B$	HPD	$\hat{R}_B$	HPD	$\hat{R}_B$	HPD
Scheme 1	0.3379	(0.1785,0.4973)	0.3470	(0.1381,0.5962)	0.3185	(0.1387,0.5980)	0.3600	(0.1310,0.5944)
Scheme 2	0.3451	(0.1946,0.4956)	0.3532	(0.1329,0.5913)	0.3287	(0.1369,0.5947)	0.3650	(0.1330,0.5904)
Scheme 3	0.3204	(0.1813,0.4596)	0.3295	(0.1180,0.5609)	0.3059	(0.1218,0.5629)	0.3409	(0.1194,0.5606)
Scheme 4	0.3326	(0.1899,0.4752)	0.3405	(0.1640,0.5500)	0.3177	(0.1578,0.5439)	0.3523	(0.1599,0.5461)



For censoring scheme (S_1, T_1). For censoring scheme (S_2, T_2).



For censoring scheme (S_3, T_3). For censoring scheme (S_4, T_4).

FIGURE 4.3: Trace plots and histogram with density plots of R for different censoring schemes in case of real data set I.

31.75, 32.52, 33.91, 36.71, 72.89.

36 kV (V): 0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.90, 3.67, 3.99, 5.35, 13.77, 25.50.

A similar procedure is followed in this sub-section as discussed in the case of real data sets I for fitting the real data sets. We perform KS and AD goodness of fit tests to check whether the given data sets follow the IP lifetime model or not. The ML estimates of the unknown parameters, KS, and AD test statistics with their corresponding p-values for these data sets are reported in Table 4.8. From Table 4.8, it is clear that these data sets follow the IP lifetime model at a 5% level of significance. Now, four progressively censored samples are generated from the above complete sample data sets based on following censoring schemes:

Scheme 1 : $(n_1 = 19,\ m_1 = 15),\ S_1 = [4*1, 0*14],$ and $(n_2 = 15,\ m_2 = 10),\ T_1 = [5*1, 0*$

TABLE 4.8: Fitting of real data set II for IP lifetime model.

Data Set	MLE	KS Test		AD Test	
		KS	p-value	AD	p-value
34 kV (U)	2.8327	0.2267	0.2433	1.5718	0.1605
36 kV (V)	2.2371	0.1937	0.5623	0.5227	0.7209

9].

$$U: 0.19, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89$$

$$V: 0.35, 2.07, 2.58, 2.71, 2.90, 3.67, 3.99, 5.35, 13.77, 25.50$$

Scheme 2 : $(n_1 = 19,\ m_1 = 15)$, $S_2 = [1*1, 0*3, 1*1, 0*4, 1*1, 0*4, 1*1]$, and $(n_2 = 15,\ m_2 = 10)$ $T_2 = [1*1, 0*1, 1*1, 0*1, 1*1, 0*1, 1*1, 0*2, 1*1]$.

$$U: 0.19, 0.96, 1.31, 2.78, 3.16, 4.67, 4.85, 6.50, 7.35, 8.01, 12.06, 31.75, 32.52, 33.91, 36.71$$

$$V: 0.35, 0.96, 0.99, 1.97, 2.07, 2.71, 2.90, 3.99, 5.35, 13.77$$

Scheme 3 : $(n_1 = 19,\ m_1 = 15)$, $S_3 = [0*14, 4*1]$, and $(n_2 = 15,\ m_2 = 10)$, $T_3 = [0*9, 5*1]$.

$$U: 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75$$

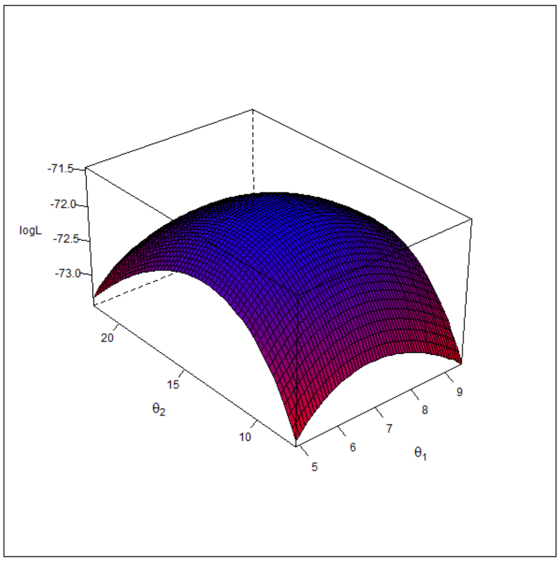
$$V: 0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.90$$

Scheme 4 : $(n_1 = 19,\ m_1 = 19)$, $S_4 = [0*19]$, and $(n_2 = 15,\ m_2 = 15)$, $T_4 = [0*15]$.

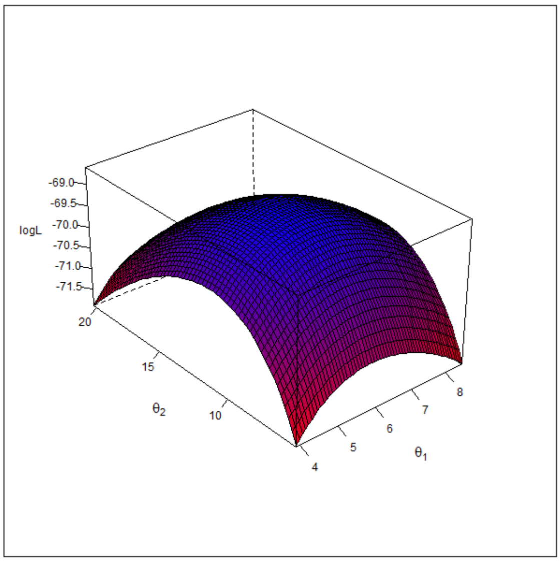
U : 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89

$$V: 0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.90, 3.67, 3.99, 5.35, 13.77, 25.50$$

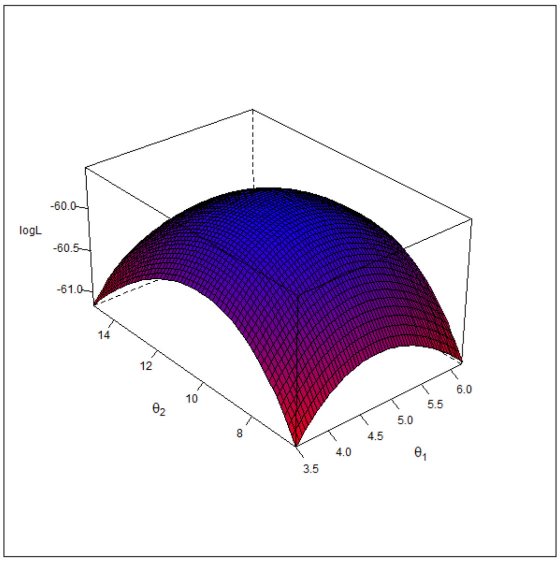
Similarly as we have discussed in case of real data set I, we analyze data set II for the applicability of considered methodology. The ML, Bayes estimates, and 95% of ACIs and HPD credible intervals of SSR R are obtained. To confirm the existence and uniqueness of the MLEs, we display the log-likelihood function of the parameters θ_1 and θ_2 for four distinct progressively censored samples. Figure 4.4 shows these graphs for four distinct censoring schemes. In the



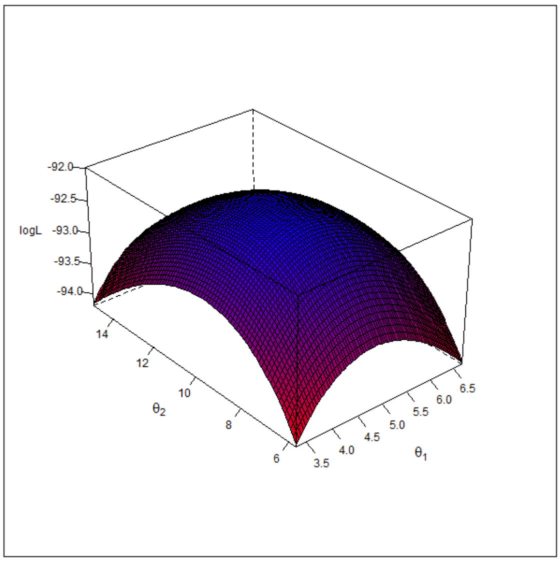
For censoring scheme (S_1, T_1).



For censoring scheme (S_2, T_2).



For censoring scheme (S_3, T_3).



For censoring scheme (S_4, T_4).

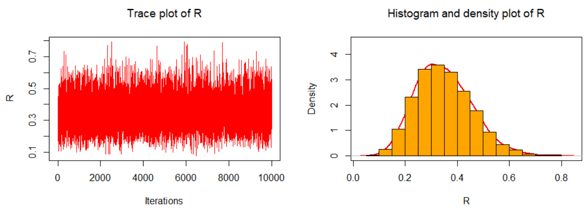
FIGURE 4.4: Plots of log-likelihood function of θ_1 and θ_2 for different censoring schemes in case of real data set II.

case of real data set II, these graphs demonstrate that the likelihood surfaces exhibit curvature in both θ_1 and θ_2 directions, suggesting that the ML estimates $\hat{\theta}_1$ and $\hat{\theta}_2$ exist and are unique.

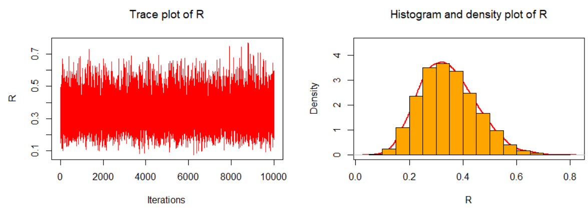
The Bayes estimate of SSR are obtained in case of non-informative priors as we do not have prior information, using the IS procedure under GELF. For the IS technique, $M = 10,000$ observations are generated and first 20% observations are considered as burn-in-period. Again here, we consider three different values of $q = -2, \ -1, \ 1$ for GELF. In Bayesian computations, the trace plots and histograms with posterior density plots based on importance samples are plotted for all four progressively censored data sets and are given in Figure 4.5. From this Figure we observe that the trace plots represent fine mixing of the chains and converge to their stationary distributions. Also, histograms with corresponding density plots are almost symmetrical about their means in all cases. This shows good performance of the IS technique and therefore, we can conclude that the Bayes estimates are good. In case of real data set II estimation results are reported in Tables 4.9 for all four pairs of progressively censored samples.

TABLE 4.9: The ML, Bayes estimates and 95% asymptotic confidence/HPD credible intervals of SSR in case of real data set II.

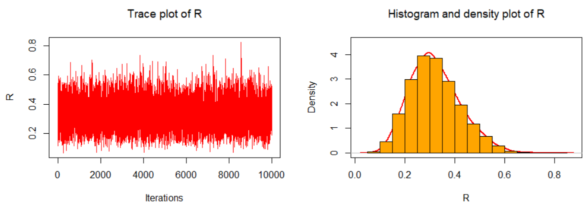
	MLE		q=-1		q=1		q=-2	
Schemes	$\hat{R}$	ACI	$\hat{R}_B$	HPD	$\hat{R}_B$	HPD	$\hat{R}_B$	HPD
Scheme 1	0.5923	(0.3667,0.8179)	0.5890	(0.3839,0.8050)	0.5737	(0.3863,0.8059)	0.5960	(0.3869,0.8069)
Scheme 2	0.5515	(0.3507,0.7523)	0.5775	(0.3832,0.8050)	0.5644	(0.3830,0.8021)	0.5830	(0.3849,0.8061)
Scheme 3	0.5205	(0.3314,0.7095)	0.5903	(0.3933,0.8113)	0.5775	(0.3967,0.8135)	0.5971	(0.3973,0.8138)
Scheme 4	0.5188	(0.3303,0.7074)	0.5922	(0.4073,0.7674)	0.5797	(0.4083,0.7696)	0.5971	(0.4029,0.7692)



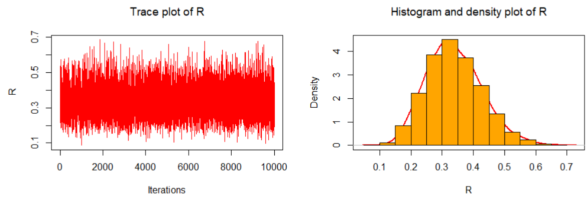
For censoring scheme (S_1, T_1).



For censoring scheme (S_2, T_2).



For censoring scheme (S_3, T_3).



For censoring scheme (S_4, T_4).

FIGURE 4.5: Trace plots and histogram with density plots of R for different censoring schemes in case of real data set II.

4.7 Concluding Remarks

In this chapter, we discussed the problem of estimation of SSR $R = P(V < U)$ for the IP lifetime model using progressively censored data. We derived ML estimate and 95% of asymptotic confidence interval with corresponding coverage probability of SSR. We computed Bayes estimates in case of both informative and non-informative priors under generalized entropy loss function using importance sampling technique. Also, 95% HPD credible interval of SSR was constructed. The performance of ML and Bayes estimators of SSR were examined by computational analysis using a Monte Carlo simulation. The computational results suggested that the Bayes estimator is more precise than the ML estimator and these can be used for all practical purposes when the prior information is available. Two pairs of real data sets were also discussed for practical applicability of considered methodology developed in this chapter. The methodology and estimation results studied in this article will be beneficial to reliability practitioners in real life situations. In this chapter, iterative and approximation methods were used for ML and Bayesian computations, respectively. In future work exact estimation procedures can be developed. Also, we can obtain optimum censoring plans to achieve the optimum accuracy of the estimators. More work is needed along with these directions as future scope.