Chapter 3

Estimation of the Parameter for Modified Power Function Distribution based on Order Statistics with Application*

3.1 Introduction

This chapter follows the following structure: In Sections 3.2 and 3.3, we obtain explicit expressions for single and product moments of order statistics. We use these moments to obtain the BLUEs for location and scale parameters δ and φ in section 3.4. Next, we performed a numerical study using R software in Section 3.5 and we see that these expressions provide precise numerical evaluations. We also provided graph and a real data application of the result in section 3.6. The chapter is finalized with concluding remarks in Section 3.7.

Power function distribution is the oldest distribution and very attractive in lifetime literature due to their simplicity, easiness and flexible features to model various types of life time data

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in different fields. The power function distribution is a flexible lifetime model which can be derived from the Pareto distribution using the inverse transformation. Theoretically, the power function distribution is a special case of the beta distribution. Besides, it is a special case of Pearson type I distribution. It is also used to fit the distribution of certain likelihood ratios in statistical tests. If the likelihood ratio is based on n iid random variables, it is often found that a useful goodness-of-fit can be obtained by letting $(likelihoodratio)^{2/n}$ to have a power function distribution (see Ali et al. (2005)). Meniconi and Barry (1996) showed that the power function distribution is the best distribution to check the reliability of any electrical component. Also, they showed from survival and hazard functions of power function distribution is the better than exponential, log-normal and Weibull distributions. Johnson et al. (1994, 1995) provided a comprehensive account of the statistical properties of the power function distribution. Many generalizations of the power function distribution have attempted by researchers. Notable amoung them are: Tahir et al. (2016) proposed the two parameter power function distribution to a more general and flexible four-parameter Weibull-Power function distribution as an adequate distribution for modelling survival data. They provided a comprehensive account of the statistical properties and the bivariate extension was also proposed. Naveed and Asghar (2016) introduced the transmuted Power function distribution. Ibrahim (2017) proposed the Kumaraswamy power function distribution and also discuss its statistical properties. Bursa and Gamze (2017) proposed a exponentiated Kumaraswamy-power function distribution and provided the statistical and mathematical properties.

Kumar and Khan (2014) established the explicit expressions for single and product moments of the generalized order statistics of the three-parameter Power function distribution and discussed their characterization based on the conditional moments of the generalized order statistics. Saran and Pandey (2004) derived and discussed the linear unbiased estimates of the parameters of a three-parameter Power function distribution based on the kth record values. Chang (2007) provides the characterizations of the two-parameter Power function distribution using the independence record values. Lim and Lee (2013) gave some proof of a characterization of the two-parameter Power function distribution using the lower record values and Ahsanullah et al. (2013) presented a new characterization of the two-parameter Power function distribution based on the lower records. Ahsanullah and Alzaatreh (2018) derived and discussed the BLUEs of the parameters of a log-logistic distribution based on order statistics.

The modified power function distribution is an important distribution for analyzing the lifetime data, which is quite flexible and can be used effectively in modeling survival data. It can have increasing, decreasing, upside-down bathtub and bathtub shaped failure rate.

Recently, Okorie et al. (2017) introduced a two parameter modified power function (MPF) distribution with probability density function (pdf)

$$h(z) = \frac{\alpha\beta(1-z)^{\beta-1}}{[1-(1-\alpha)(1-z)^{\beta}]^2},\ 0 < z < 1,\ \alpha,\ \beta > 0, \tag{3.1}$$

and the associated cumulative distribution function (cdf) is

$$H(z) = 1 - \frac{\alpha(1-z)^{\beta}}{[1-(1-\alpha)(1-z)^{\beta}]}\ 0 < z < 1,\ \alpha,\ \beta > 0. \tag{3.2}$$

Quantiles are fundamental for estimation (for example, quantile estimators) and simulation. The p^{th} quantile z_p of the MPF distribution is the root of the equation

$$Z_p = 1 - \left(\frac{1-p}{1-p+p\alpha}\right)^{\frac{1}{\beta}},\ \ 0 < p < 1,\ \alpha,\ \beta > 0. \tag{3.3}$$

Let $u \sim U(0,1)$, then equation (3.3) can be used to simulate a random sample of size n from the MPF distribution as follows

$$z_i = \left(\frac{1-u_i}{1-u_i+u_i\alpha}\right)^{\frac{1}{\beta}}, i = 1,2,\ldots,n.$$

3.2 Relations for Single Moments of Order Statistics

Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from MPF distribution, the pth moment of uth order statistic is obtained by using equation (3.1) and (3.2) in (1.1) as follows:

$$\mu_{u:n}^{(p)} = C_{u:n} \int_{-\infty}^{\infty} z^p H^{u-1}(z) \left[1 - H(z)\right]^{n-u} h(z) dz, \;\; u = 1, 2, \ldots, n, \;\; p = 1, 2, \ldots. \tag{3.4}$$

Expanding $H[(z)]^{u-1}$ and $[1-H(z)]^{n-u}$ in Equation (3.4) binomially, we get

$$\begin{aligned}
\mu_{u:n}^{(p)} &= C_{u:n} \sum_{i=0}^{u-1} \binom{u-1}{i} (-1)^i \int_0^1 z^p [1-H(z)]^{n-u+i} h(z) dz \\
&= \beta C_{u:n} \sum_{i=0}^{u-1} \binom{u-1}{i} (-1)^i \alpha^{n-u+i+1} \int_0^1 z^p \frac{(1-z)^{\beta(n-u+i+1)-1}}{[1-(1-\alpha)(1-z)^{\beta}]^{n-u+i+2}} dz \\
&= \beta C_{u:n} \sum_{i=0}^{u-1} \sum_{j=0}^{\infty} \binom{u-1}{i} (-1)^i \alpha^{n-u+i+1} \frac{(1-\alpha)^j \Gamma(j+n-i+2)}{\Gamma(j+1)\Gamma(n-u+i+2)} \\
&\times \int_0^1 z^p (1-z)^{\beta(n-u+i+j+1)-1} dz \\
&= \beta C_{u:n} \sum_{i=0}^{u-1} \sum_{j=0}^{\infty} \binom{u-1}{i} (-1)^i \alpha^{n-u+i+1} \frac{(1-\alpha)^j \Gamma(j+n-i+2)}{\Gamma(j+1)\Gamma(n-u+i+2)} \\
&\times B(p+1, \beta(n-u+i+j+1)),
\end{aligned} \tag{3.5}$$

where $B(a,\; b)$ denote the beta function and defined by $B(a,\; b) = \int_0^1 z^{a-1}(1-z)^{b-1} dz$. The validity of the single moments of order statistics in Equation (3.5) can be checked by using Arnold et al. (2008).

$$\sum_{u=1}^{n} \mu_{u:n} = nE(Z).$$

In particular, the mean and the variance of order statistic are

$$\begin{aligned}
\mu_{u:n}^{(1)} &= \beta C_{u:n} \sum_{i=0}^{u-1} \sum_{j=0}^{\infty} \binom{u-1}{i} (-1)^i \alpha^{n-u+i+1} \frac{(1-\alpha)^j \Gamma(j+n-u+i+2)}{\Gamma(j+1)\Gamma(n-u+i+2)} \\
&\times B(2, \beta(n-u+i+j+1))
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\sigma^2 &= \mu_{u:n}^{(2)} - (\mu_{u:n}^{(1)})^2 \\ &= \beta C_{u:n} \sum_{i=0}^{u-1} \sum_{j=0}^{\infty} \binom{u-1}{i} (-1)^i \alpha^{n-u+i+1} \frac{(1-\alpha)^j \Gamma(j+n-u+i+2)}{\Gamma(j+1)\Gamma(n-u+i+2)} \\ &\times B(3, \beta(n-u+i+j+1)) - [\mu_{u:n}^{(1)}]^2.\end{aligned}$$

Some special cases from Equation (3.5) are

(1) If $p = n = u = 1$, we get

$$\mu_{1:1} = \alpha\beta \sum_{j=0}^{\infty} (1-\alpha)^j (i+1) B(2, \beta(j+1)) = E(Z),$$

as obtained by Okorie et al. (2017).

(2) If $p = u = 1$, we get

$$\mu_{1:n} = \alpha^n \beta C_{1:n} \sum_{j=0}^{\infty} \frac{(1-\alpha)^j \Gamma(j+n+1)\, B(2, \beta(n+j))}{\Gamma(j+1)\Gamma(n+1)}.$$

(3) If $p = 1, u = n$, we get

$$\mu_{n:n} = \beta C_{n:n} \sum_{i=0}^{n-1} \sum_{j=0}^{\infty} \binom{n-1}{i} (-1)^i \alpha^{i+1} \frac{(1-\alpha)^j \Gamma(j+i+2)\, B(2, \beta(i+j+1))}{\Gamma(j+1)\Gamma(i+2)}.$$

3.3 Relations for Product Moments of Order Statistics

Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from MPF distribution, the product moments of uth and vth order statistic is obtained by using equation (3.1) and (3.2) in (1.9) as follows:

$$\begin{aligned}\mu_{u,v:n}^{(p,q)} &= C_{u,v:n} \int_0^1 \int_z^1 z^p y^q H^{u-1}(z) \left[H(y) - H(z)\right]^{v-1-u} \left[1 - H(y)\right]^{n-v} h(z) h(y) dz dy, \\ & u, v = 1, 2, \ldots, n, \quad u < v. \qquad (3.7)\end{aligned}$$

By using the same argument as in the single moments case, we get

$$\begin{aligned}\mu_{u,v:n}^{(p,q)} &= C_{u,v:n}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{u-1}\sum_{l=0}^{v-u-1}\sum_{r=0}^{q}\sum_{s=0}^{p}\binom{u-1}{k}(-1)^{k+l+r+s}\alpha^{k+n-u+1}\beta^{2}(1-\alpha)^{i+j}\\ &\times \binom{v-u-1}{l}\binom{q}{r}\binom{p}{s}\frac{\Gamma(i+k+v-u-l+1)\,\Gamma(j+l+n-v+2)}{\Gamma(i+1)\Gamma(k+v-u-l+1)\,\Gamma(j+1)\Gamma(l+n-v+2)}\\ &\times \frac{1}{[\beta(l+n+j-v+1)+r][\beta(k+i+n+j-u+1)+r+s]}. \end{aligned} \tag{3.8}$$

The validity of the product moments of order statistics in Equation (3.8) can be checked by using Arnold et al. (2008).

$$\sum_{u=1}^{n-1}\sum_{v=u+1}^{n}\mu_{u,v:n} = \binom{n}{2}\,[E(Z)]^2.$$

The covariance between $Z_{u:n}$ and $Z_{v:n}$ $\psi_{u,v:n} = \mu_{u,v:n} - \mu_{u:n}\mu_{v:n}$ can be obtain from equation (3.8) and equation (3.6).

3.4 BLUEs of the Location and Scale Parameters

In this section we obtain the BLUEs of the location and scale parameters of the MPF distribution based on order statistics.

Let $Z_{1:n} \leq Z_{2:n} \leq \cdots \leq Z_{n:n}$ be the order statistics from the scale-parameter and location-scale parameter MPF distribution, the pdf of the scale-parameter MPF distribution can be obtained after reparametrizing (3.1) as

$$h(z) = \frac{\alpha\beta\left[1-\left(\frac{z}{\varphi}\right)\right]^{\beta-1}}{\varphi\left[1-(1-\alpha)\left\{1-\left(\frac{z}{\varphi}\right)\right\}^{\beta}\right]^{2}},\ 0<x<1,\ \alpha,\ \beta>0,\ \varphi>0, \tag{3.9}$$

while the pdf of the location-scale parameter MPF distribution is

$$h(z)=\frac{\alpha\beta\left[1-\left(\frac{z-\delta}{\varphi}\right)\right]^{\beta-1}}{\varphi\left[1-(1-\alpha)\left\{1-\left(\frac{z-\delta}{\varphi}\right)\right\}^{\beta}\right]^{2}},\ z>\delta, \tag{3.10}$$

The expression for the BLUEs of location and scale parameter are given in (1.18) and also variances and covariance for these parameters are given in eqn (1.21), (1.22) and (1.23).

Tables 3.4 and 3.5 display the coefficient of the BLUEs for type-II right censored sample of various values of $n=7,\ 10$ and censoring cases $c=0(1)([n/2]-1)$. Also, Table 3.6 shows variances and covariances of the BLUEs.

Illustrative example: In this example, we show the usefulness of the coefficients of the BLUEs in Tables 3.4 and 3.5 by simulate a random sample of order statistics from the MPF distribution of size $n=10$ when $\delta=0$, $\varphi=1$, $\alpha=0.5$ and $\beta=1.0$ as: 0.040628, 0.043592, 0.113282, 0.157260, 0.172874, 0.224061, 0.337366, 0.484587, 0.853319, 0.917235. By using the enterers of Tables 3.4 and 3.5, we calculate the BLUEs of δ and φ, to be

$$\begin{aligned}
\delta^* &= \sum_{u=1}^{n} a_u Z_{u:n} \\
&= (1.095146\times 0.040628)+(-0.00898\times 0.043592)+(-0.00802\times 0.113282) \\
&+ (-0.00713\times 0.157260)+(-0.00631\times 0.172874)+(-0.00557\times 0.224061) \\
&+ (-0.00488\times 0.337366)+(-0.00427\times 0.484587)+(-0.00371\times 0.853319) \\
&+ (-0.04627\times 0.917235) \\
&= -0.009500
\end{aligned}$$

and

$$\begin{aligned}\varphi^* &= \sum_{u=1}^{n} b_u Z_{u:n} \\ &= (-1.41637 \times 0.040628) + (0.043404 \times 0.043592) + (0.038667 \times 0.113282) \\ &+ (0.034292 \times 0.157260) + (0.03026 \times 0.172874) + (0.026576 \times 0.224061) \\ &+ (0.023221 \times 0.337366) + (0.020191 \times 0.484587) + (0.01747 \times 0.853319) \\ &+ (1.182291 \times 0.917235) \\ &= 1.082200.\end{aligned}$$

3.5 Numerical Results

The relations obtained in the preceding sections allow us to evaluate the expected values, second moments, variances, product moment and covariances of order statistics from samples of sizes up to 10 for various values of the parameters. The relation in (3.5) can be used to compute the expected values, second order moments and variances of all order statistics for sample sizes $n = 1, 2, 3, \ldots, 10$. In Tables 3.1 and 3.2, we have presented expected values, second order moments and variances of the uth order statistic from MPF distribution for $n = 1, 2, \ldots, 10$ and $\beta = 0.5,\ 1.0$ and $\alpha = 0.5$. One can see that the means and variances decreasing both with respect to n and β. In Table 3.3 we have reported the product moments and covariances of the uth and vth order statistic from MPF distribution for $n = 1, 2, \ldots, 10$ and $\alpha = 0.5$ and $\beta = 0.5$. From Table 3.3, one can observe that product moments are decreasing with respect to n.

Tables 3.4 and 3.5 display the coefficients of the BLUEs for Type-II right-censored samples of sample sizes $n = 7,\ 10$, $\beta = 0.5, 1.0, 1,5$ and different censoring cases $c = 0(1)([n/2] - 1)$. The coefficients of the BLUEs in Tables 3.4 and 3.5. The variances and covariances of the BLUEs are presented in Table 3.6. We see the variance of the BLUEs increases as the censoring level increases while the variance of the BLUEs decreases when the sample size increases and increase as β increases. In addition, we see the covariances of the BLUEs decrease as the censoring level increases while the covariances of the BLUEs increase when the sample size increases and decrease as α increases. All computations here were performed using R software.

3.6 Real Data Application

To demonstrate how the proposed methods can be used in practice, we consider the following real-life data set (see Okorie et al. (2017)) is presented the Evaporation data which was extracted from the monthly publication of climatological data of the National Oceanic and Atmospheric Administration. The data is on the daily pan evaporation in hundredths of inches that was recorded in September 2016 in San Joaquin Drainage 05 Friant Government Camp, California, USA.

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0.28 0.29 0.29 0.27 0.17 0.33 0.26 0.26 0.32 0.24 0.28 0.26 0.30
0.29 0.21 0.23 0.25 0.27 0.26 0.28 0.32 0.28 0.18 0.18 0.26 0.42
0.11 0.34 0.32 0.36
```

Now a random sample of size 10 is selected from the given data set are 0.34, 0.29, 0.18, 0.28, 0.30, 0.26, 0.32, 0.21, 0.32, 0.36. By using the MPF distribution in Eq. (3.1) for the given sample, we have the maximum likelihood estimate of $\alpha_{ML} = 3351.0206$ and $\beta_{ML} = 23.4902$. Figure 3.11 shows ecdf and QQ plot of the sample. This conclusion is also supported by the Kolmogorov-Smirnov (K-S) tests, KS statistic is 0.1298 and p value is 0.996. This shows the suitability of the modified power function distribution for this real data set.

3.7 Conclusion

In this chapter, we have considered mdified power function distribution. We first presented expressions for the single and double moments in explicit forms, and by making use of them we

TABLE 3.1: Expected values, second moments and variances of the uth order statistic from MPF distribution for $n = 1,2,...,10$, $\alpha = 0.5$ and $\beta = 0.5$ (sim.=simulated)

n	u	$E(Z)$	$Sim.E(Z)$	$E(Z^2)$	$Sim.E(Z^2)$	$V(Z)$	$Sim.V(Z)$
1	1	0.545177	0.545373	0.395602	0.394743	0.098384	0.097312
2	1	0.364468	0.362797	0.202700	0.202114	0.069863	0.070493
	2	0.725887	0.725130	0.588504	0.589248	0.061592	0.063434
3	1	0.271065	0.270508	0.120836	0.120153	0.047360	0.046979
	2	0.551274	0.551012	0.366427	0.365801	0.062524	0.062187
	3	0.813194	0.814045	0.699543	0.699807	0.038259	0.037138
4	1	0.214892	0.215320	0.079469	0.079931	0.033290	0.033568
	2	0.439582	0.440096	0.244937	0.244762	0.051705	0.051078
	3	0.662965	0.664027	0.487917	0.487130	0.048394	0.046199
	4	0.863270	0.863235	0.770084	0.770401	0.024849	0.025226
5	1	0.177662	0.178002	0.055954	0.055813	0.024390	0.024129
	2	0.363814	0.364499	0.173527	0.173551	0.041166	0.040692
	3	0.553233	0.553238	0.352053	0.351382	0.045986	0.045309
	4	0.736120	0.735030	0.578494	0.578923	0.036621	0.038654
	5	0.895058	0.894940	0.817982	0.817766	0.016853	0.016849
6	1	0.151274	0.150900	0.041411	0.041517	0.018527	0.018746
	2	0.309602	0.310018	0.128670	0.128915	0.032817	0.032803
	3	0.472240	0.472914	0.263241	0.263948	0.040230	0.040302
	4	0.634227	0.633639	0.440865	0.440731	0.038621	0.039233
	5	0.787067	0.786317	0.647308	0.647432	0.027834	0.029138
	6	0.916656	0.916521	0.852117	0.851970	0.011859	0.011959
7	1	0.131635	0.130852	0.031830	0.031753	0.014502	0.014631
	2	0.269105	0.268968	0.098899	0.098266	0.026481	0.025923
	3	0.410843	0.410805	0.203098	0.203402	0.034306	0.034642
	4	0.554102	0.554408	0.343432	0.343523	0.036403	0.036155
	5	0.694320	0.694669	0.513940	0.514422	0.031860	0.031857
	6	0.824165	0.824515	0.700655	0.702136	0.021407	0.022311
	7	0.932071	0.932025	0.877361	0.877533	0.008605	0.008863
8	1	0.116469	0.116617	0.025201	0.025226	0.011636	0.011627
	2	0.237797	0.238244	0.078233	0.077837	0.021686	0.021078
	3	0.363029	0.362852	0.160896	0.160069	0.029106	0.028408
	4	0.490534	0.489730	0.273433	0.271920	0.032809	0.032084
	5	0.617671	0.616995	0.413431	0.413351	0.031914	0.032669
	6	0.740310	0.740304	0.574245	0.575577	0.026186	0.027527
	7	0.852117	0.852236	0.742792	0.742265	0.016689	0.015959
	8	0.943493	0.943479	0.896585	0.896725	0.006406	0.006572
9	1	0.104414	0.103916	0.020433	0.020252	0.009531	0.009454
	2	0.212914	0.212434	0.063349	0.063699	0.018017	0.018571
	3	0.324888	0.324554	0.130324	0.130673	0.024772	0.025338
	4	0.439311	0.439420	0.222040	0.220889	0.029046	0.027799
	5	0.554561	0.554663	0.337675	0.337012	0.030137	0.029361
	6	0.668158	0.668675	0.474036	0.473758	0.027601	0.026632
	7	0.776386	0.777208	0.624350	0.624517	0.021575	0.020466
	8	0.873754	0.873348	0.776632	0.776071	0.013186	0.013334
	9	0.952211	0.952708	0.911579	0.911223	0.004873	0.003569
10	1	0.094606	0.094210	0.016892	0.016974	0.007942	0.008098
	2	0.192685	0.193123	0.052298	0.052522	0.015170	0.015226
	3	0.293828	0.293520	0.107553	0.107825	0.021218	0.021671
	4	0.397361	0.397470	0.183457	0.182247	0.025561	0.024265
	5	0.502237	0.501979	0.279914	0.280440	0.027672	0.028458
	6	0.606886	0.606632	0.395435	0.393637	0.027124	0.025636
	7	0.709006	0.708881	0.526437	0.527053	0.023747	0.024541
	8	0.805263	0.804815	0.666313	0.666905	0.017865	0.019178
	9	0.890877	0.891297	0.804212	0.804178	0.010550	0.009768
	10	0.959025	0.959336	0.923509	0.923843	0.003780	0.003517

TABLE 3.2: Expected values, second moments and variances of the uth order statistic from MPF distribution for $n = 1, 2, ..., 10$, $\alpha = 0.5$ and $\beta = 1.0$ (sim.=simulated)

n	u	$E(Z)$	$Sim.E(Z)$	$E(Z^2)$	$Sim.E(Z^2)$	$V(Z)$	$Sim.V(Z)$
1	1	0.384524	0.386569	0.227411	0.227868	0.079552	0.078432
2	1	0.221187	0.227643	0.090355	0.090317	0.041431	0.038496
	2	0.545177	0.545709	0.364468	0.364431	0.067250	0.066633
3	1	0.158883	0.157504	0.046701	0.047021	0.021457	0.022213
	2	0.364468	0.365468	0.177662	0.178248	0.044825	0.044681
	3	0.635532	0.634216	0.457871	0.457989	0.053970	0.055758
4	1	0.121489	0.121790	0.028086	0.027757	0.013326	0.012924
	2	0.271065	0.271523	0.102547	0.103349	0.029071	0.029624
	3	0.457871	0.457786	0.252776	0.253339	0.043130	0.043772
	4	0.694753	0.694951	0.526236	0.526079	0.043554	0.043123
5	1	0.098138	0.098192	0.018615	0.018638	0.008984	0.008996
	2	0.214890	0.215062	0.065970	0.066337	0.019792	0.020085
	3	0.355323	0.354062	0.157413	0.157421	0.031159	0.032062
	4	0.526236	0.526138	0.316351	0.317775	0.039427	0.040954
	5	0.736882	0.737877	0.578707	0.577119	0.035712	0.032656
6	1	0.082234	0.082714	0.013194	0.013395	0.006432	0.006554
	2	0.177662	0.177043	0.045722	0.045601	0.014158	0.014257
	3	0.289353	0.288804	0.106467	0.106211	0.022742	0.022803
	4	0.421293	0.420719	0.208360	0.208521	0.030872	0.031517
	5	0.578707	0.579260	0.370347	0.370214	0.035445	0.034672
	6	0.768517	0.768605	0.620379	0.621239	0.029761	0.030486
7	1	0.070727	0.070741	0.009819	0.009960	0.004817	0.004956
	2	0.151274	0.151549	0.033442	0.033236	0.010558	0.010269
	3	0.243632	0.243303	0.076420	0.076672	0.017063	0.017475
	4	0.350316	0.350495	0.146529	0.146779	0.023808	0.023933
	5	0.474527	0.474278	0.254733	0.254134	0.029557	0.029194
	6	0.620379	0.619972	0.416592	0.416918	0.031722	0.032553
	7	0.793207	0.793798	0.654343	0.654674	0.025166	0.024558
8	1	0.062026	0.062230	0.007583	0.007706	0.003736	0.003834
	2	0.131635	0.131995	0.025473	0.025463	0.008145	0.008041
	3	0.210189	0.209715	0.057349	0.057253	0.013170	0.013273
	4	0.299369	0.300009	0.108204	0.108429	0.018582	0.018424
	5	0.401262	0.401059	0.184854	0.184849	0.023843	0.024001
	6	0.518485	0.518873	0.296661	0.296003	0.027834	0.026774
	7	0.654343	0.653942	0.456569	0.456295	0.028404	0.028656
	8	0.813045	0.813754	0.682596	0.683177	0.021554	0.020981
9	1	0.055221	0.055014	0.006028	0.005985	0.002979	0.002958
	2	0.116469	0.116777	0.020025	0.020085	0.006460	0.006449
	3	0.184716	0.184474	0.044544	0.044777	0.010424	0.010747
	4	0.261136	0.260462	0.082961	0.083428	0.014769	0.015587
	5	0.347160	0.347165	0.139759	0.139925	0.019239	0.019401
	6	0.444544	0.445129	0.220930	0.220832	0.023311	0.022693
	7	0.555456	0.554200	0.334526	0.333689	0.025995	0.026551
	8	0.682596	0.682710	0.491439	0.492183	0.025502	0.026089
	9	0.829351	0.829503	0.706491	0.706493	0.018668	0.018419
10	1	0.049755	0.049694	0.004904	0.004917	0.002428	0.002447
	2	0.104414	0.104116	0.016142	0.016085	0.005240	0.005244
	3	0.164691	0.163870	0.035555	0.035498	0.008432	0.008645
	4	0.231440	0.231469	0.065519	0.065359	0.011955	0.011782
	5	0.305680	0.305999	0.109123	0.109344	0.015683	0.015709
	6	0.388640	0.388656	0.170395	0.170954	0.019354	0.019901
	7	0.481813	0.482272	0.254620	0.254469	0.022476	0.021883
	8	0.587018	0.587175	0.368772	0.368149	0.024182	0.023375
	9	0.706491	0.706978	0.522105	0.522652	0.022975	0.022833
	10	0.843002	0.842981	0.726979	0.727268	0.016327	0.016652

TABLE 3.3: Covariances of order statistics for $n = 2, 3, \ldots, 10$, $\alpha = 0.5$ and $\beta = 0.5$

v	u	n	$\mu_{u,v:n}$	$Sim.\mu_{u,v:n}$	$Cov(Z_{u,v:n})$	$Sim.Cov(Z_{u,v:n})$
2	1	2	0.297218	0.297092	0.032656	0.034017
		3	0.180806	0.179939	0.031374	0.030886
		4	0.119855	0.120238	0.025392	0.025477
		5	0.084670	0.085426	0.020034	0.020544
		6	0.062747	0.062971	0.015913	0.016189
		7	0.048250	0.048072	0.012826	0.012876
		8	0.038201	0.038234	0.010505	0.010451
		9	0.030965	0.031032	0.008734	0.008956
		10	0.025591	0.025507	0.007362	0.007313
3	1	3	0.234487	0.234142	0.014059	0.013936
		4	0.158801	0.158238	0.016335	0.015260
		5	0.113246	0.113614	0.014958	0.015136
		6	0.084299	0.084292	0.012861	0.012929
		7	0.064961	0.064753	0.010879	0.010998
		8	0.051480	0.051480	0.009199	0.009166
		9	0.041744	0.041676	0.007821	0.007950
		10	0.034499	0.034484	0.006702	0.006832
	2	3	0.476362	0.475522	0.028069	0.026973
		4	0.324713	0.324832	0.033285	0.032597
		5	0.232017	0.232362	0.030743	0.030707
		6	0.172732	0.172437	0.026525	0.025826
		7	0.133020	0.132942	0.022461	0.022449
		8	0.105315	0.104768	0.018988	0.018321
		9	0.085306	0.086028	0.016133	0.017082
		10	0.070426	0.070531	0.013809	0.013846
4	1	4	0.192603	0.192904	0.007093	0.007032
		5	0.140184	0.139840	0.009403	0.009003
		6	0.105379	0.104738	0.009437	0.009122
		7	0.081614	0.081570	0.008675	0.009025
		8	0.064851	0.064492	0.007719	0.007381
		9	0.052659	0.052635	0.006789	0.006973
		10	0.043551	0.043610	0.005958	0.006164
	2	4	0.393942	0.394204	0.014464	0.014298
		5	0.287143	0.288069	0.019332	0.020152
		6	0.215822	0.215984	0.019464	0.019544
		7	0.167022	0.167316	0.017910	0.018199
		8	0.132580	0.133082	0.015932	0.016407
		9	0.107539	0.107529	0.014004	0.014182
		10	0.088843	0.089191	0.012277	0.012430
	3	4	0.593397	0.593089	0.021079	0.019879
		5	0.436193	0.437042	0.028947	0.030395
		6	0.329042	0.329614	0.029535	0.029958
		7	0.255010	0.255112	0.027361	0.027358
		8	0.202503	0.201782	0.024425	0.024083
		9	0.164231	0.164732	0.021504	0.022116
		10	0.135619	0.135266	0.018864	0.018601
5	1	5	0.163002	0.163316	0.003984	0.004014
		6	0.124886	0.125488	0.005823	0.006833
		7	0.097671	0.097874	0.006274	0.006975
		8	0.078024	0.078014	0.006085	0.006062
		9	0.063548	0.063613	0.005644	0.005975
		10	0.052647	0.052686	0.005132	0.005395
	2	5	0.333828	0.334538	0.008193	0.008334
		6	0.255689	0.255558	0.012011	0.011785
		7	0.199797	0.199978	0.012952	0.013134

TABLE 3.3: Continued.

v	u	n	$\mu_{u,v:n}$	$Sim.\mu_{u,v:n}$	$Cov(Z_{u,v:n})$	$Sim.Cov(Z_{u,v:n})$
		8	0.159439	0.159872	0.012559	0.012877
		9	0.129715	0.129814	0.011641	0.011985
		10	0.107349	0.106969	0.010575	0.010026
	3	5	0.507456	0.507554	0.012280	0.012439
		6	0.389918	0.390507	0.018234	0.018647
		7	0.305048	0.305068	0.019792	0.019695
		8	0.243487	0.243230	0.019255	0.019352
		9	0.198047	0.197806	0.017876	0.017788
		10	0.163820	0.163034	0.016249	0.015693
	4	5	0.674446	0.673905	0.015576	0.016097
		6	0.523052	0.521850	0.023873	0.023609
		7	0.411072	0.410618	0.026348	0.025488
		8	0.328861	0.329074	0.025872	0.026913
		9	0.267775	0.268748	0.024150	0.025018
		10	0.221589	0.220846	0.022020	0.021325
6	1	6	0.141084	0.140601	0.002418	0.002298
		7	0.112300	0.112199	0.003812	0.004310
		8	0.090569	0.091106	0.004346	0.004774
		9	0.074170	0.073912	0.004405	0.004426
		10	0.061647	0.061408	0.004232	0.004257
	2	6	0.288787	0.289856	0.004988	0.005718
		7	0.229656	0.229662	0.007869	0.007894
		8	0.185014	0.184476	0.008971	0.008104
		9	0.151345	0.151285	0.009085	0.009236
		10	0.125657	0.125293	0.008719	0.008138
	3	6	0.440458	0.441634	0.007577	0.008199
		7	0.350629	0.350569	0.012027	0.011854
		8	0.282509	0.283372	0.013755	0.014751
		9	0.231028	0.230798	0.013952	0.013777
		10	0.191718	0.191488	0.013398	0.013429
	4	6	0.591299	0.590084	0.009932	0.009341
		7	0.472692	0.472835	0.016020	0.015717
		8	0.381635	0.381659	0.018488	0.019110
		9	0.312380	0.312087	0.018851	0.018259
		10	0.259310	0.259394	0.018157	0.018276
	5	6	0.733081	0.732447	0.011612	0.011771
		7	0.591632	0.591381	0.019397	0.018616
		8	0.480089	0.480398	0.022821	0.023635
		9	0.394067	0.393973	0.023533	0.023084
		10	0.327624	0.327149	0.022823	0.022633
7	1	7	0.124250	0.124977	0.001557	0.003020
		8	0.101851	0.101798	0.002606	0.002413
		9	0.084179	0.084565	0.003113	0.003801
		10	0.070350	0.070708	0.003274	0.003924
	2	7	0.254039	0.254464	0.003214	0.003779
		8	0.208009	0.207742	0.005378	0.004702
		9	0.171724	0.171852	0.006420	0.006747
		10	0.143360	0.143906	0.006746	0.007005
	3	7	0.387848	0.388970	0.004914	0.006089
		8	0.317591	0.318025	0.008248	0.008790
		9	0.262099	0.261550	0.009860	0.009304
		10	0.218691	0.218729	0.010365	0.010658
	4	7	0.523011	0.522597	0.006549	0.005875
		8	0.429081	0.428626	0.011089	0.011261
		9	0.354400	0.354529	0.013325	0.013009
		10	0.295779	0.295479	0.014048	0.013721

TABLE 3.3: Continued.

v	u	n	$\mu_{u,v:n}$	$Sim.\mu_{u,v:n}$	$Cov(Z_{u,v:n})$	$Sim.Cov(Z_{u,v:n})$
	5	7	0.655096	0.655111	0.007940	0.007662
		8	0.540025	0.539712	0.013697	0.013887
		9	0.447194	0.447169	0.016641	0.016081
		10	0.373751	0.373467	0.017662	0.017624
	6	7	0.776968	0.776666	0.008787	0.008197
		8	0.646570	0.646241	0.015739	0.015327
		9	0.538281	0.538594	0.019532	0.018895
		10	0.451285	0.451676	0.020999	0.021647
8	1	8	0.110937	0.111259	0.001050	0.001234
		9	0.093077	0.093538	0.001845	0.002784
		10	0.078475	0.078646	0.002292	0.002824
	2	8	0.226526	0.226706	0.002166	0.001928
		9	0.189840	0.190425	0.003806	0.004896
		10	0.159885	0.159884	0.004723	0.004456
	3	8	0.345837	0.347017	0.003322	0.004674
		9	0.289717	0.289256	0.005845	0.005807
		10	0.243866	0.243807	0.007257	0.007578
	4	8	0.467283	0.467003	0.004467	0.004952
		9	0.391750	0.391017	0.007900	0.007250
		10	0.329817	0.329537	0.009837	0.009647
	5	8	0.588291	0.588312	0.005522	0.006190
		9	0.494420	0.494491	0.009870	0.010077
		10	0.416803	0.416871	0.012370	0.012871
	6	8	0.704832	0.704369	0.006355	0.005908
		9	0.595399	0.595409	0.011594	0.011423
		10	0.503417	0.503202	0.014714	0.014976
	7	8	0.810724	0.811121	0.006757	0.007054
		9	0.691195	0.691247	0.012825	0.012474
		10	0.587589	0.587706	0.016653	0.017188
9	1	9	0.100158	0.100705	0.000734	0.001704
		10	0.085628	0.085797	0.001345	0.001828
	2	9	0.204253	0.203607	0.001514	0.001219
		10	0.174430	0.174888	0.002772	0.002758
	3	9	0.311687	0.311876	0.002325	0.002670
		10	0.266024	0.265905	0.004259	0.004291
	4	9	0.421460	0.420854	0.003144	0.002216
		10	0.359773	0.359611	0.005774	0.005347
	5	9	0.531988	0.532590	0.003929	0.004158
		10	0.454693	0.454852	0.007262	0.007440
	6	9	0.640846	0.641018	0.004617	0.003966
		10	0.549303	0.549156	0.008642	0.008467
	7	9	0.744399	0.743753	0.005117	0.003301
		10	0.641426	0.641677	0.009789	0.009854
	8	9	0.837274	0.836571	0.005276	0.004525
		10	0.727907	0.726732	0.010516	0.009403
10	1	10	0.091259	0.091135	0.000529	0.000756
	2	10	0.185881	0.186612	0.001091	0.001342
	3	10	0.283465	0.283552	0.001677	0.001968
	4	10	0.383352	0.383133	0.002273	0.001826
	5	10	0.484517	0.483960	0.002859	0.002394
	6	10	0.585424	0.585534	0.003405	0.003571
	7	10	0.683814	0.683729	0.003860	0.003673
	8	10	0.776421	0.775737	0.004153	0.003649
	9	10	0.858552	0.859339	0.004178	0.004286

TABLE 3.4: Coefficients of the BLUEs of the location parameter for $\alpha = 0.5$.

β	n	c	$a_u, u = 1,2,3,\ldots,(n-c)$							
0.5	7	0	1.194764	-0.01428	-0.01332	-0.01229	-0.01068	-0.00623	-0.13798	
		1	1.213593	-0.01437	-0.01250	-0.00972	-0.00397	-0.17303		
		2	1.251611	-0.01455	-0.01092	-0.00473	-0.22142			
	10	0	1.136302	-0.00783	-0.00750	-0.00720	-0.00694	-0.00668	-0.00635	-0.00562
			-0.00306	-0.08513						
		1	1.142073	-0.00795	-0.00750	-0.00703	-0.00648	-0.00572	-0.00434	-0.00106
			-0.10199							
		2	1.153030	-0.00817	-0.00750	-0.00671	-0.00566	-0.00397	-0.00070	-0.12032
		3	1.171009	-0.00850	-0.00750	-0.00622	-0.00435	-0.00121	-0.14323	
		4	1.199250	-0.00901	-0.00749	-0.00546	-0.00237	-0.17493		
1	7	0	1.137575	-0.01683	-0.01429	-0.01205	-0.01010	-0.00842	-0.07588	
		1	1.168959	-0.02070	-0.01757	-0.01482	-0.01242	-0.10346		
		2	1.214326	-0.02600	-0.02208	-0.01863	-0.14762			
	10	0	1.095146	-0.00898	-0.00802	-0.00713	-0.00631	-0.00557	-0.00488	-0.00427
			-0.00371	-0.04627						
		1	1.109445	-0.01044	-0.00932	-0.00828	-0.00733	-0.00646	-0.00567	-0.00495
			-0.05701							
		2	1.126908	-0.01213	-0.01083	-0.00963	-0.00852	-0.00751	-0.00658	-0.07172
		3	1.149408	-0.01422	-0.01270	-0.01129	-0.00999	-0.00880	-0.09242	
		4	1.180259	-0.01698	-0.01516	-0.01348	-0.01194	-0.12269		
1.5	7	0	1.127859	-0.02042	-0.01693	-0.01403	-0.01182	-0.01076	-0.05390	
		1	1.162506	-0.02608	-0.02185	-0.01844	-0.01609	-0.08005		
		2	1.208192	-0.03318	-0.02804	-0.02401	-0.12296			
	10	0	1.089154	-0.01082	-0.00949	-0.00829	-0.00721	-0.00628	-0.00552	-0.00502
			-0.00507	-0.03147						
		1	1.106252	-0.01300	-0.01145	-0.01005	-0.00882	-0.00779	-0.00700	-0.00663
			-0.04151							
		2	1.124736	-0.01526	-0.01348	-0.01189	-0.01050	-0.00936	-0.00857	-0.05568
		3	1.147426	-0.01792	-0.01587	-0.01405	-0.01249	-0.01124	-0.07585	
		4	1.177902	-0.02138	-0.01898	-0.01687	-0.01508	-0.10560		
2	7	0	1.126869	-0.02348	-0.01924	-0.01578	-0.01323	-0.01216	-0.04298	
		1	1.161922	-0.02992	-0.02479	-0.02071	-0.01798	-0.06852		
		2	1.206867	-0.03778	-0.03159	-0.02680	-0.11070			
	10	0	1.089517	-0.01246	-0.01084	-0.00939	-0.00811	-0.00701	-0.00615	-0.00563
			-0.00578	-0.02415						
		1	1.107111	-0.01493	-0.01305	-0.01137	-0.00990	-0.00868	-0.00778	-0.00737
			-0.03403							
		2	1.125451	-0.01740	-0.01525	-0.01335	-0.01171	-0.01037	-0.00944	-0.04793
		3	1.147758	-0.02029	-0.01784	-0.01567	-0.01383	-0.01236	-0.06777	
		4	1.177681	-0.02403	-0.02119	-0.01870	-0.01659	-0.09717		
2.5	7	0	1.127986	-0.02592	-0.02108	-0.01714	-0.01427	-0.01301	-0.03657	
		1	1.162612	-0.03270	-0.02687	-0.02224	-0.01909	-0.06171		
		2	1.206728	-0.04093	-0.03393	-0.02849	-0.10338			
	10	0	1.091185	-0.01378	-0.01194	-0.01028	-0.00883	-0.00760	-0.00663	-0.00604
			-0.00615	-0.01993						
		1	1.108564	-0.01635	-0.01422	-0.01232	-0.01067	-0.00929	-0.00826	-0.00774
			-0.02971							
		2	1.126538	-0.01891	-0.01649	-0.01434	-0.01250	-0.01099	-0.00991	-0.04341
		3	1.148431	-0.02190	-0.01915	-0.01673	-0.01465	-0.01299	-0.06301	
		4	1.177887	-0.02579	-0.02262	-0.01983	-0.01748	-0.09216		

TABLE 3.5: Coefficients of the BLUEs of the scale parameter for $\alpha = 0.5$.

β	n	c	$b_u, u = 1,2,3,\ldots,(n-c)$							
0.5	7	0	-1.25759	0.014642	0.010491	0.003327	-0.01274	-0.06285	1.304718	
		1	-1.43565	0.015537	0.002798	-0.02094	-0.07622	1.514474		
		2	-1.76840	0.017055	-0.01108	-0.06463	1.827043			
	10	0	-1.16678	0.008524	0.007517	0.006240	0.004393	0.001299	-0.00485	-0.01981
			-0.06832	1.231788						
		1	-1.25028	0.010265	0.007536	0.003732	-0.00220	-0.01264	-0.03386	-0.08577
			1.363223							
		2	-1.39673	0.013128	0.007537	-0.00049	-0.01326	-0.03600	-0.08252	1.508328
		3	-1.62211	0.017315	0.007510	-0.00674	-0.02962	-0.07060	1.704243	
		4	-1.95813	0.023307	0.007450	-0.01577	-0.05322	1.996362		
1	7	0	-1.55076	0.070941	0.060069	0.050499	0.042155	0.034955	1.292140	
		1	-2.08519	0.136841	0.115961	0.097574	0.081539	1.653269		
		2	-2.81016	0.221644	0.188031	0.158419	2.242063			
	10	0	-1.41637	0.043404	0.038667	0.034292	0.030260	0.026576	0.023221	0.020191
			0.01747	1.182291						
		1	-1.78174	0.080590	0.071815	0.063708	0.056238	0.049411	0.043195	0.037578
			1.379208							
		2	-2.20424	0.121571	0.108386	0.096202	0.084975	0.074712	0.065366	1.653029
		3	-2.72286	0.169751	0.151419	0.134475	0.118864	0.104588	2.043758	
		4	-3.40512	0.230814	0.206000	0.183057	0.161922	2.623330		
1.5	7	0	-2.11222	0.190593	0.164104	0.145031	0.137081	0.152897	1.322513	
		1	-2.96227	0.329592	0.284825	0.253220	0.241774	1.852861		
		2	-4.01972	0.493952	0.428071	0.382326	2.715365			
	10	0	-1.97233	0.128749	0.115126	0.103345	0.093725	0.086936	0.084380	0.089644
			0.114472	1.155952						
		1	-2.60039	0.208947	0.187076	0.168228	0.152952	0.142399	0.138986	0.148932
			1.452871							
		2	-3.24741	0.288009	0.258106	0.232402	0.211692	0.197627	0.193713	1.865863
		3	-4.00775	0.377242	0.338369	0.305032	0.278322	0.260489	2.448295	
		4	-4.99148	0.488715	0.438727	0.395958	0.361889	3.306189		
2	7	0	-2.79920	0.347695	0.297377	0.261368	0.245794	0.270690	1.376277	
		1	-3.92165	0.553889	0.475089	0.419507	0.397669	2.075492		
		2	-5.28307	0.791861	0.680977	0.603921	3.206309			
	10	0	-2.68531	0.246202	0.219062	0.195691	0.176663	0.163126	0.157412	0.165247
			0.204499	1.157408						
		1	-3.52865	0.364624	0.324722	0.290441	0.262669	0.243179	0.235602	0.248895
			1.558519							
		2	-4.36866	0.477687	0.425749	0.381222	0.345319	0.320464	0.311656	2.106559
		3	-5.34909	0.604611	0.539299	0.483427	0.438606	0.408037	2.875108	
		4	-6.61859	0.763498	0.681577	0.611658	0.555880	4.005982		
2.5	7	0	-3.55154	0.526327	0.446567	0.388514	0.359968	0.385857	1.444309	
		1	-4.91925	0.794147	0.675388	0.589877	0.550291	2.309546		
		2	-6.57040	1.101990	0.939450	0.823867	3.705096			
	10	0	-3.47904	0.381947	0.338184	0.300383	0.269307	0.246455	0.234808	0.241494
			0.288064	1.178397						
		1	-4.50683	0.534029	0.473170	0.420688	0.377694	0.346369	0.331079	0.342338
			1.681463							
		2	-5.52403	0.678433	0.601532	0.535327	0.481301	0.442343	0.424304	2.360792
		3	-6.71466	0.841141	0.746337	0.664876	0.598688	0.551531	3.312088	
		4	-8.26303	1.045876	0.928714	0.828256	0.747032	4.713156		

TABLE 3.6: Variances and covariance of the BLUEs when $\alpha = 0.5,\ \delta = 0$ and $\varphi = 1$.

β	n	c	$Var(\delta^*)$	$Var(\varphi^*)$	$Cov(\delta^* \varphi^*)$
0.5	7	0	0.019258	0.031146	-0.020295
		1	0.019568	0.058868	-0.023227
		2	0.020193	0.106736	-0.028696
	10	0	0.009667	0.014220	-0.009934
		1	0.009717	0.024793	-0.010665
		2	0.009813	0.041887	-0.011944
		3	0.009970	0.066520	-0.013909
		4	0.010215	0.101337	-0.016835
1	7	0	0.005682	0.050930	-0.008022
		1	0.005858	0.102060	-0.011025
		2	0.006111	0.166716	-0.015071
	10	0	0.002710	0.027506	-0.003592
		1	0.002749	0.053000	-0.004590
		2	0.002796	0.080777	-0.005738
		3	0.002857	0.113123	-0.007141
		4	0.002940	0.153808	-0.008981
1.5	7	0	0.002714	0.078499	-0.005501
		1	0.002812	0.136979	-0.007885
		2	0.002939	0.205042	-0.010825
	10	0	0.001266	0.047450	-0.002432
		1	0.001288	0.077737	-0.003256
		2	0.001312	0.107307	-0.004101
		3	0.001342	0.140410	-0.005089
		4	0.001381	0.181499	-0.006362
2	7	0	0.001593	0.100866	-0.004425
		1	0.001652	0.161101	-0.006306
		2	0.001726	0.229589	-0.008567
	10	0	0.000733	0.063908	-0.001967
		1	0.000747	0.094983	-0.002616
		2	0.000761	0.124390	-0.003258
		3	0.000778	0.157157	-0.004003
		4	0.000800	0.197936	-0.004964
2.5	7	0	0.001048	0.118123	-0.003776
		1	0.001087	0.178267	-0.005298
		2	0.001136	0.246417	-0.007119
	10	0	0.000478	0.076564	-0.001690
		1	0.000487	0.107209	-0.002208
		2	0.000496	0.136061	-0.002718
		3	0.000507	0.168343	-0.003311
		4	0.000522	0.208741	-0.004080

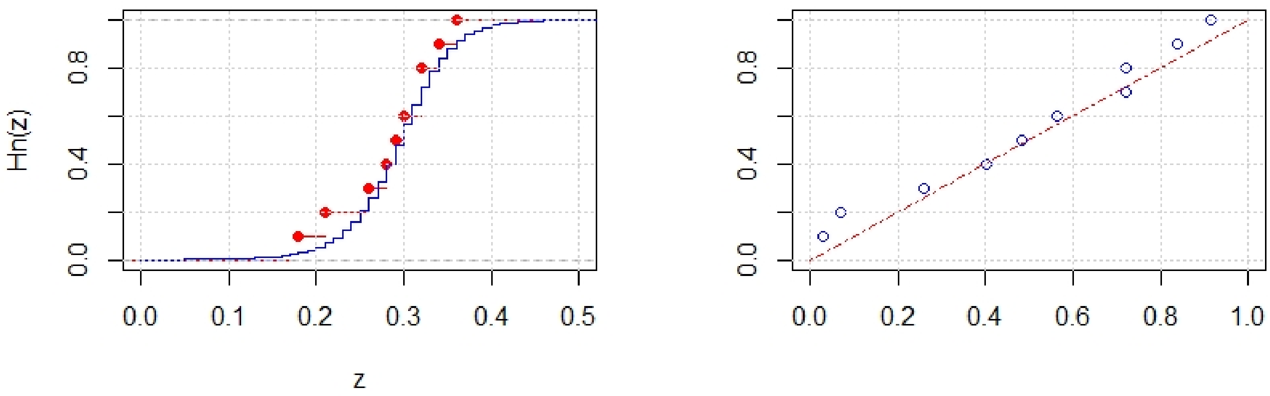

FIGURE 3.1: ECDF-plot and QQ-plot

have obtained means, variance and covariances for the order statistics. In simulation study we observed that means variance and covariances do decrease when a high number of *n* are taken into account however, the behaviour is found opposite when a large value of shape parameter is considered correspond to a fixed value of scale parameter. In practice the reported values can also be used to obtain the BLUEs of the location and scale parameters.