Chapter 4

Inference for the Extended Power Lindley Distribution based on Order Statistics with Application

4.1 Introduction

This chapter follows the following structure: In section 4.2, we introduce some lemmas on EPL distribution. We discuss single and product moments of order statistics in Sections 4.3 and 4.4. Section 4.5 describes how these moments can be used to calculate BLUEs for δ and φ. Section 4.6 demonstrate a real data application. This chapter is concluded in Section 4.7.

The Lindley distribution can be viewed as a mixture of $exp(\xi)$ and $gamma(2,\xi)$ distributions, which was introduced by Lindley (1958) in the context of fiducial and Bayesian statistics. Later, Ghitany et al. (2008) investigated the mathematical and statistical properties of this distribution and also demonstrate that this distribution makes better than exponential distribution in a number of ways. The Lindley distribution has a single scale parameter and can model the data with increasing monotonous failure rates and by virtue of this Lindley distribution offers insufficient flexibility to analyze different kinds of lifetime data. In order to increase flexibility for modelling, it will be helpful to consider other alternatives to this distribution. Also, Bouchahed and Zeghdoudi and Zeghdoudi et al. (2018) generalized the the Lindley distribution. Recently the three parameter extended power Lindley distribution was proposed by Alkarni (2015) for the flexibility of purpose.

The extended power Lindley (EPL) distribution is specified by the following pdf

$$h(z;\tau,\xi,\kappa) = \frac{\tau\xi^2}{\xi+\kappa}(1+\kappa z^{\tau})\, z^{\tau-1}\, e^{-\xi z^{\tau}}, \quad z>0; \quad \tau>0, \xi>0, \kappa>0 \tag{4.1}$$

and the associated cdf is

$$H(z;\tau,\xi,\kappa) = 1-\left(1+\frac{\kappa\xi}{\xi+\kappa}z^{\tau}\right)e^{-\xi z^{\tau}}, \quad z>0; \quad \tau>0, \xi>0, \kappa>0. \tag{4.2}$$

For $\kappa = 1$ and $\kappa = 1, \ \tau = 1$, the EPL distribution reduces to power Lindley (PL) and Lindley distributions respectively.

4.2 Technical Lemmas

In this section, we provide and prove some lemmas.

Lemma 1 Let $h(z)$ and $H(z)$ be given by (4.1) and (4.2), respectively. For $a>0$, $b>0$ and $p>0$, let

$$K(a,b,p) = \int_0^{\infty} z^p H^a(z)[1-H(z)]^b h(z)dz.$$

Then,

$$
\begin{aligned}
K(a,b,p) &= \frac{\xi^2}{(\xi+\kappa)^{b+1}} \sum_{i_1=0}^{a} \sum_{i_2=0}^{i_1+b} \sum_{i_3=0}^{i_2+1} (-1)^{i_1} \frac{\xi^{i_2} \kappa^{b+i_1-i_2+i_3}}{(\xi+\kappa)^{i_1}} \binom{a}{i_1} \binom{i_1+b}{i_2} \binom{i_2+1}{i_3} \\
&\times \frac{\Gamma\left(\frac{p+\tau(i_3+1)}{\tau}\right)}{[\xi(i_1+b+1)]^{\frac{p+\tau(i_3+1)}{\tau}}}.
\end{aligned}
$$

Proof. By using (4.1) and (4.2), we get

$$
\begin{aligned}
K(a,b,p) &= \sum_{i_1=0}^{a} (-1)^{i_1} \binom{a}{i_1} \int_0^\infty z^p [1-H(z)]^{i_1+b} h(z) dz. \\
&= \frac{\tau\xi^2}{(\xi+\kappa)} \sum_{i_1=0}^{a} (-1)^{i_1} \binom{a}{i_1} \int_0^\infty z^{p+\tau-1} (1+\kappa z^\tau) e^{-\xi(i_1+b+1)z^\tau} \\
&\times \left[\left(1+\frac{\kappa\xi}{\xi+\kappa} z^\tau\right)\right]^{i_1+b} dz \\
&= \frac{\tau\xi^2}{(\xi+\kappa)} \sum_{i_1=0}^{a} (-1)^{i_1} \binom{a}{i_1} \int_0^\infty z^{p+\tau-1} (1+\kappa z^\tau) e^{-\xi(i_1+b+1)z^\tau} \\
&\times \left[\left(\frac{\xi(1+\kappa z^\tau)+\kappa}{\xi+\kappa}\right)\right]^{i_1+b} dz \\
&= \frac{\tau\xi^2}{(\xi+\kappa)^{b+1}} \sum_{i_1=0}^{a} \sum_{i_2=0}^{i_1+b} \sum_{i_3=0}^{i_2+1} (-1)^{i_1} \frac{\xi^{i_2} \kappa^{b+i_1-i_2+i_3}}{(\xi+\kappa)^{i_1}} \binom{a}{i_1} \binom{i_1+b}{i_2} \binom{i_2+1}{i_3} \\
&\times \int_0^\infty z^{p+\tau(i_3+1)-1} \, e^{-\xi(i_1+b+1)z^\tau} dz \\
&= \frac{\xi^2}{(\xi+\kappa)^{b+1}} \sum_{i_1=0}^{a} \sum_{i_2=0}^{i_1+b} \sum_{i_3=0}^{i_2+1} (-1)^{i_1} \frac{\xi^{i_2} \kappa^{b+i_1-i_2+i_3}}{(\xi+\kappa)^{i_1}} \binom{a}{i_1} \binom{i_1+b}{i_2} \binom{i_2+1}{i_3} \\
&\times \frac{1}{[\xi(i_1+b+1)]^{\frac{p+\tau(i_3+1)}{\tau}}} \int_0^\infty y^{\frac{p+\tau i_3}{\tau}} \, e^{-y} dy,
\end{aligned}
$$

where $y = \xi(i_1+b+1)z^\tau$.

Lemma 2 Let $h(z)$ and $H(z)$ be given by (4.1) and (4.2), respectively. For $a > 0$, $b > 0$, c, $p > 0$ and $q > 0$, let

$$
L(a,b,c,p,q) = \int_0^\infty \int_z^\infty z^p y^q [H(z)]^a [H(y)-H(z)]^b [1-H(y)]^c h(z) h(y) dy dz.
$$

Then,

$$
\begin{aligned}
L(a,b,c,p,q) &= \frac{\xi^4}{(\xi+\kappa)^{b+c+2}} \sum_{i_1=0}^{a} \sum_{i_2=0}^{b} \sum_{i_3=0}^{i_1+b-i_2} \sum_{i_4=0}^{i_2+c} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} \sum_{i_7=0}^{\frac{q+\tau i_6}{\tau}} (-1)^{i_1+i_2} \binom{a}{i_1} \binom{b}{i_2} \\
&\times \binom{i_1+b-i_2}{i_3} \binom{i_2+c}{i_4} \binom{i_3+1}{i_5} \binom{i_4+1}{i_6} \frac{\xi^{i_3+i_4} \kappa^{i_1+b-i_3-i_4+i_5+i_6}}{(\xi+\kappa)^{i_1}} \\
&\times \frac{\left(\frac{q+\tau i_6}{\tau}\right)!}{[\xi(i_2+c+1)]^{\frac{q+\tau(i_6+1)}{\tau}}} \frac{[\xi(i_2+c+1)]^{i_7}}{i_7!} \frac{\Gamma\left(\frac{p+\tau(i_5+i_7+1)}{\tau}\right)}{[\xi(i_1+b+c+2)]^{\frac{p+\tau(i_5+i_7+1)}{\tau}}}.
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
L(a,b,c,p,q) &= \sum_{i_1=0}^{a} \sum_{i_2=0}^{b} (-1)^{i_1+i_2} \binom{a}{i_1} \binom{b}{i_2} \\
&\times \int_0^\infty \int_z^\infty z^p y^q [1-H(z)]^{i_1+b-i_2} [1-H(y)]^{i_2+c} h(z) h(y) \, dy dz \\
&= \frac{\tau^2 \xi^4}{(\xi+\kappa)^2} \sum_{i_1=0}^{a} \sum_{i_2=0}^{b} (-1)^{i_1+i_2} \binom{a}{i_1} \binom{b}{i_2} \int_0^\infty \int_z^\infty z^{p+\tau-1} y^{q+\tau-1} \\
&\times (1+\kappa z^\tau)(1+\kappa y^\tau) e^{-\xi(i_1+b-i_2+1) z^\tau} e^{-\xi(i_2+c+1) y^\tau} \\
&\times \left[\left(1+\frac{\xi\kappa}{\xi+\kappa} z^\tau\right)\right]^{i_1+b-i_2} \left[\left(1+\frac{\xi\kappa}{\xi+\kappa} y^\tau\right)\right]^{i_2+c} dz dy \\
&= \frac{\tau^2 \kappa^4}{(\xi+\kappa)^{b+c+2}} \sum_{i_1=0}^{a} \sum_{i_2=0}^{b} \sum_{i_3=0}^{i_1+b-i_2} \sum_{i_4=0}^{i_2+c} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} (-1)^{i_1+i_2} \frac{\kappa^{i_1+b-i_3-i_4+i_5+i_6}}{(\xi+\kappa)^{i_1}} \\
&\times \binom{a}{i_1} \binom{b}{i_2} \binom{i_1+b-i_2}{i_3} \binom{i_2+c}{i_4} \binom{i_3+1}{i_5} \binom{i_4+1}{i_6} \\
&\times \int_0^\infty \int_z^\infty z^{p+\tau(i_5+1)-1} y^{q+\tau(i_6+1)-1} e^{-\xi(i_1+b-i_2+1) z^\tau} e^{-\xi(i_2+c+1) y^\tau} dz dy \\
&= \frac{\tau^2 \xi^4}{(\xi+\kappa)^{b+c+2}} \sum_{i_1=0}^{a} \sum_{i_2=0}^{b} \sum_{i_3=0}^{i_1+b-i_2} \sum_{i_4=0}^{i_2+c} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} (-1)^{i_1+i_2} \\
&\times \frac{\xi^{i_3+i_4} \kappa^{i_1+b-i_3-i_4+i_5+i_6}}{(\xi+\kappa)^{i_1}} \binom{a}{i_1} \binom{b}{i_2} \binom{i_1+b-i_2}{i_3} \binom{i_2+c}{i_4} \binom{i_3+1}{i_5} \\
&\times \binom{i_4+1}{i_6} \frac{1}{[\xi(i_2+c+1)]^{\frac{q+\tau(i_6+1)}{\tau}}} \int_0^\infty z^{p+\tau(i_5+1)-1} e^{-\xi(i_1+b-i_2+1) z^\tau} \\
&\times \Gamma\left(\frac{q+\tau(i_6+1)}{\tau}, \ \xi(i_2+c+1) z^\tau\right) dz,
\end{aligned}
$$

By using the relation

$$\Gamma(p,\ y) = (p-1)!\ e^{-y} \sum_{l=0}^{p-1} \frac{y^l}{l!}.$$

$$\begin{aligned}
L(a,b,c,p,q) &= \frac{\tau^2\xi^4}{(\xi+\kappa)^{b+c+2}} \sum_{i_1=0}^{a} \sum_{i_2=0}^{b} \sum_{i_3=0}^{i_1+b-i_2} \sum_{i_4=0}^{i_2+c} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} \sum_{i_7=0}^{\frac{q+\tau i_6}{\tau}} (-1)^{i_1+i_2} \binom{a}{i_1} \binom{b}{i_2} \\
&\times \frac{\xi^{i_3+i_4}\kappa^{i_1+b-i_3-i_4+i_5+6}}{(\xi+\kappa)^{i_1}} \binom{i_1+b-i_2}{i_3} \binom{i_2+c}{i_4} \binom{i_3+1}{i_5} \binom{i_4+1}{i_6} \\
&\times \frac{\left(\frac{q+\tau i_6}{\tau}\right)!}{[\xi(i_2+c+1)]^{\frac{q+\tau(i_6+1)}{\tau}}} \frac{[\xi(i_2+c+1)]^{i_7}}{i_7!} \\
&\times \int_0^\infty z^{p+\tau(i_5+i_7+1)-1} e^{-\xi(i_1+b+c+2)z^\tau} dz \\
&= \frac{\xi^4}{(\xi+\kappa)^{b+c+2}} \sum_{i_1=0}^{a} \sum_{i_2=0}^{b} \sum_{i_3=0}^{i_1+b-i_2} \sum_{i_4=0}^{i_2+c} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} \sum_{i_7=0}^{\frac{q+\tau i_6}{\tau}} (-1)^{i_1+i_2} \binom{a}{i_1} \binom{b}{i_2} \\
&\times \frac{\xi^{i_3+i_4}\kappa^{i_1+b-i_3-i_4+i_5+i_6}}{(\xi+\kappa)^{i_1}} \binom{i_1+b-i_2}{i_3} \binom{i_2+c}{i_4} \binom{i_3+1}{i_5} \binom{i_4+1}{i_6} \\
&\times \frac{\left(\frac{q+\tau i_6}{\tau}\right)!}{[\xi(i_2+c+1)]^{\frac{q+\tau(i_6+1)}{\tau}}} \frac{[\xi(i_2+c+1)]^{i_7}}{i_7!} \\
&\times \frac{1}{[\xi(i_1+b+c+2)]^{\frac{p+\tau(i_5+i_7+1)}{\tau}}} \int_0^\infty t^{\frac{p+\tau(i_5+i_7)}{\tau}} e^{-t} dt,
\end{aligned}$$

where $t = \xi(i_1+b+c+2)z^\tau$.

4.3 Relations for Single Moments of Order Statistics

Here, we present some new expressions for single moments of uth order statistics, $E\left(Z_{u:n}^{(p)}\right) = \mu_{u:n}^{(p)}$ for the given random sample $Z_1, Z_2, \ldots, Z_n$ from the EPL distribution.

Theorem 1. For the EPL distribution given in (4.1) and for, $1 \leq u \leq n$

$$\mu_{u:n}^{(p)} = \frac{\xi^2 C_{u:n}}{(\xi+\kappa)^{n-u+1}} \sum_{i_1=0}^{u-1} \sum_{i_2=0}^{i_1+n-u} \sum_{i_3=0}^{i_2+1} (-1)^{i_1} \binom{u-1}{i_1} \binom{i_1+n-u}{i_2} \binom{i_2+1}{i_3} \times \frac{\xi^{i_2} \kappa^{i_1+n-u-i_2+i_3}}{(\xi+\kappa)^{i_1}} \frac{\Gamma(\frac{p+\tau(i_3+1)}{\tau})}{[\xi(i_1+n-u+1)]^{\frac{p+\tau(i_3+1)}{\tau}}}. \tag{4.3}$$

Proof. By using (1.1), we get

$$\mu_{u:n}^{(p)} = C_{u:n} \int_0^\infty z^p H^{u-1}(z) [1-H(z)]^{n-u} h(z) dz. \tag{4.4}$$

By lemma 1, we get the result.

Special Cases

(i) For $\tau = 1$ and $\kappa = 1$ in (4.3), we obtain relation for order statistic of Lindley distribution

$$\mu_{u:n}^{(p)} = \frac{\xi^2 C_{u:n}}{(\xi+1)^{n-u+1}} \sum_{i_1=0}^{u-1} \sum_{i_2=0}^{i_1+n-u} \sum_{i_3=0}^{i_2+1} (-1)^{i_1} \binom{u-1}{i_1} \binom{i_1+n-u}{i_2} \binom{i_2+1}{i_3} \times \frac{\xi^{i_2}}{(\xi+1)^{i_1}} \frac{\Gamma(p+i_3+1)}{[\xi(i_1+n-u+1)]^{p+i_3+1}},$$

as obtained by Sultan and Al-Thubyani (2016).

(ii) For $\kappa = 1$ in (4.3), we obtain relation for order statistic of power Lindley distribution

$$\mu_{u:n}^{(p)} = \frac{\xi^2 C_{u:n}}{(\xi+1)^{n-u+1}} \sum_{i_1=0}^{u-1} \sum_{i_2=0}^{i_1+n-u} \sum_{i_3=0}^{i_2+1} (-1)^{i_1} \binom{u-1}{i_1} \binom{i_1+n-u}{i_2} \binom{i_2+1}{i_3} \times \frac{\xi^{i_2}}{(\xi+1)^{i_1}} \frac{\Gamma\frac{(p+\tau(i_3+1))}{\tau}}{[\xi(i_1+n-u+1)]^{\frac{p+\tau(i_3+1)}{\tau}}},$$

as obtained by Kumar and Goyal (2019b).

(iii) If $p = u = 1$ in (4.3), we obtain

$$\mu_{1:n}^{(1)} = \frac{n!}{(n-1)!}\frac{\xi^2}{(\xi+\kappa)^n}\sum_{i_2=0}^{n-1}\sum_{i_3=0}^{i_2+1}\binom{n-1}{i_2}\binom{i_2+1}{i_3}\xi^{i_2}\kappa^{n-1-i_2+i_3} \times \frac{\Gamma(\frac{\tau(i_3+1)+1}{\tau})}{(\xi n)^{\frac{\tau(i_3+1)+1}{\tau}}}.$$

(iv) If $p = 1$, $u = n$ in (4.3), we obtain

$$\mu_{n:n}^{(1)} = \frac{n!}{(n-1)!}\frac{\xi^2}{(\xi+\kappa)}\sum_{i_1=0}^{n-1}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_2+1}(-1)^{i_1}\binom{n-1}{i_1}\binom{i_1}{i_2}\binom{i_2+1}{i_3} \times \frac{\xi^{i_2}\kappa^{i_1-i_2+i_3}}{(\xi+\kappa)^{i_1}}\frac{\Gamma(\frac{\tau(i_3+1)+1}{\tau})}{[\xi(i_1+1)]^{\frac{\tau(i_3+1)+1}{\tau}}}.$$

(v) If $p = n = u = 1$ in (4.3), we obtain unordered mean of the extended power Lindley distribution

$$\mu_{1:1}^{(1)} = \frac{\xi^2}{(\xi+\kappa)}\sum_{i_3=0}^{1}\frac{\kappa^{i_3}\,\Gamma(\frac{\tau(i_3+1)+1}{\tau})}{\xi^{\frac{\tau(i_3+1)+1}{\tau}}} = E(Z),$$

as obtained by Alkarni (2015).

Specially, the first moment (mean) of the uth order statistic is

$$\mu_{u:n} = \mu_{u:n}^{(1)} = \vartheta(\tau,\xi,\kappa,u,n,1),$$

where

$$\vartheta(\tau,\xi,\kappa,u,n,p) = \frac{\xi^2 C_{u:n}}{(\xi+\kappa)^{n-u+1}}\sum_{i_1=0}^{u-1}\sum_{i_2=0}^{i_1+n-u}\sum_{i_3=0}^{i_2+1}(-1)^{i_1}\binom{u-1}{i_1}\binom{i_1+n-u}{i_2}\binom{i_2+1}{i_3} \times \frac{\xi^{i_2}\kappa^{i_1+n-u-i_2+i_3}}{(\xi+\kappa)^{i_1}}\frac{\Gamma(\frac{p+\tau(i_3+1)}{\tau})}{[\xi(i_1+n-u+1)]^{\frac{p+\tau(i_3+1)}{\tau}}}.$$

In addition, the variance of $Z_{u:n}$ is found to be

$$\sigma^2_{u:n} = \mu^{(2)}_{u:n} - [\mu^{(1)}_{u:n}]^2 = \vartheta(\tau,\xi,\kappa,u,n,2) - [\vartheta(\tau,\xi,\kappa,u,n,1)]^2.$$

4.4 Relations for Product Moments of Order Statistics

Here, we present some new expressions for product moments of uth and vth order statistics, $E\left(Z^{(p,q)}_{u,v:n}\right) = \mu^{(p,q)}_{u,v:n}$ for the given random sample $Z_1, Z_2, \ldots, Z_n$ from the EPL distribution.

Theorem 2. For the EPL distribution given in (4.1) and for $1 \le u < v \le n$

$$\begin{aligned}
\mu^{(p,q)}_{u,v:n} &= C_{u,v:n}\frac{\xi^4}{(\xi+\kappa)^{n-u+1}}\sum_{i_1=0}^{u-1}\sum_{i_2=0}^{v-u-1}\sum_{i_3=0}^{i_1+v-u-i_2-1}\sum_{i_4=0}^{i_2+n-v}\sum_{i_5=0}^{i_3+1}\sum_{i_6=0}^{i_4+1}\sum_{i_7=0}^{\frac{q+\tau i_6}{\tau}}(-1)^{i_1+i_2} \\
&\times \frac{\xi^{i_3+i_4}\kappa^{i_1+v-u-1-i_3-i_4+i_5+i_6}}{(\xi+\kappa)^{i_1}}\binom{u-1}{i_1}\binom{v-u-1}{i_2}\binom{i_1+v-u-i_2-1}{i_3} \\
&\times \binom{i_2+n-v}{i_4}\binom{i_3+1}{i_5}\binom{i_4+1}{i_6}\frac{\left(\frac{q+\tau i_6}{\tau}\right)!}{[\xi(i_2+n-v+1)]^{\frac{q+\tau(i_6+1)}{\tau}}} \\
&\times \frac{[\xi(i_2+n-v+1)]^{i_7}}{i_7!}\frac{\Gamma\left(\frac{p+\tau(i_5+i_7+1)}{\tau}\right)}{[\xi(i_1+n-u+1)]^{\frac{p+\tau(i_5+i_7+1)}{\tau}}}.
\end{aligned}$$

Proof. By using (1.9), we obtain

$$\mu^{(p,q)}_{u,v:n} = C_{u,v:n}\int_0^\infty\int_z^\infty z^p\, y^q\, H^{u-1}(z)\,[H(y)-H(z)]^{v-1-u}\,[1-H(y)]^{n-v}\,h(z)h(y)dzdy \quad (4.5)$$

The result follow by using lemma 2.

Special Cases

(i) For $\tau = 1$ and $\kappa = 1$ in Therom 2, we obtain the relation for order statistic of Lindley distribution

$$\begin{aligned}\mu_{u,v:n}^{(p,q)} &= \frac{\xi^4 C_{u,v:n}}{(\xi+1)^{n-u+1}} \sum_{i_1=0}^{u-1} \sum_{i_2=0}^{v-u-1} \sum_{i_3=0}^{i_1+v-u-i_2-1} \sum_{i_4=0}^{i_2+n-v} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} \sum_{i_7=0}^{q+i_6} (-1)^{i_1+i_2} \frac{\xi^{i_3+i_4}}{(\xi+1)^{i_1}} \\ &\times \binom{u-1}{i_1} \binom{v-u-1}{i_2} \binom{i_1+v-u-i_2-1}{i_3} \binom{i_2+n-v}{i_4} \binom{i_3+1}{i_5} \binom{i_4+1}{i_6} \\ &\times \frac{(q+i_6)!}{[\xi(i_2+n-v+1)]^{q+i_6+1}} \frac{[\xi(i_2+n-v+1)]^{i_7}}{i_7!} \frac{\Gamma(p+i_5+i_7+1)}{[\xi(i_1+n-u+1)]^{p+i_5+i_7+1}}.\end{aligned}$$

as obtained by Sultan and Al-Thubyani (2016).

(ii) For $\kappa = 1$ in Therom 2, we obtain the relation for order statistic of power Lindley distribution

$$\begin{aligned}\mu_{u,v:n}^{(p,q)} &= \frac{\xi^4 C_{u,v:n}}{(\xi+1)^{n-u+1}} \sum_{i_1=0}^{u-1} \sum_{i_2=0}^{v-u-1} \sum_{i_3=0}^{i_1+v-u-i_2-1} \sum_{i_4=0}^{i_2+n-v} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} \sum_{i_7=0}^{\frac{q+\tau i_6}{\tau}} (-1)^{i_1+i_2} \frac{\xi^{i_3+i_4}}{(\xi+1)^{i_1}} \\ &\times \binom{u-1}{i_1} \binom{v-u-1}{i_2} \binom{i_1+v-u-i_2-1}{i_3} \binom{i_2+n-v}{i_4} \binom{i_3+1}{i_5} \binom{i_4+1}{i_6} \\ &\times \frac{\left(\frac{q+\tau i_6}{\tau}\right)!}{[\xi(i_2+n-v+1)]^{\frac{q+\tau(i_6+1)}{\tau}}} \frac{[\xi(i_2+n-v+1)]^{i_7}}{i_7!} \frac{\Gamma\left(\frac{p+\tau(i_5+i_7+1)}{\tau}\right)}{[\xi(i_1+n-u+1)]^{\frac{p+\tau(i_5+i_7+1)}{\tau}}}.\end{aligned}$$

as obtained by Kumar and Goyal (2019b).

Let $\mu_{u,v:n}$ represents (1, 1)-th moment of $Z_{u:n}$ and $Z_{v:n}$ order statistics. So, covariance term can be written as

$$\sigma_{u,v:n} = Cov(Z_{u:n}, Z_{v:n}) = \mu_{u,v:n} - \mu_{u:n}\mu_{v:n}.$$

Tables 4.1 and 4.2 contains first order moments, second order moments and variances for various values of parameters and sample size n, of uth order statistic from EPL disribution. Table 4.3, shows product moments and covariance of uth and vth order statistic from EPL distribution.

TABLE 4.1: Moments, variances skewness and kurtosis of order statistic from EPL distribution $\tau = 2$, $\kappa = 0.5$ and $\xi = 5$

u	n	$E(Z)$	$E(Z^2)$	$V(Z)$	δ_1	δ_2	γ_1	γ_2
1	1	0.414348	0.218182	0.046498	0.379571	3.207079	0.616093	0.207079
	2	0.293422	0.109504	0.023408	0.387238	3.219388	0.622284	0.219388
	3	0.239709	0.073106	0.015646	0.389946	3.229529	0.624457	0.229529
	4	0.207652	0.054870	0.011751	0.391543	3.234472	0.625734	0.234472
	5	0.185762	0.043916	0.009408	0.392307	3.236857	0.626344	0.236857
	6	0.169597	0.036608	0.007845	0.394587	3.229235	0.628161	0.229235
	7	0.157029	0.031385	0.006727	0.395586	3.221588	0.628956	0.221588
	8	0.146897	0.027467	0.005888	0.395741	3.229525	0.629079	0.229525
	9	0.138503	0.024418	0.005235	0.394610	3.235330	0.628180	0.235330
	10	0.131400	0.021978	0.004712	0.396862	3.226521	0.629970	0.226521
2	2	0.535273	0.326860	0.040343	0.241593	3.215795	0.491521	0.215795
	3	0.400850	0.182301	0.021620	0.189444	3.115636	0.435251	0.115636
	4	0.335877	0.127814	0.015001	0.174957	3.088871	0.418279	0.088871
	5	0.295213	0.098686	0.011535	0.170039	3.070969	0.412358	0.070969
	6	0.266589	0.080456	0.009386	0.167111	3.066828	0.408792	0.066828
	7	0.245001	0.067944	0.007919	0.165394	3.065728	0.406687	0.065728
	8	0.227956	0.058815	0.006851	0.164916	3.061962	0.406099	0.061962
	9	0.214051	0.051856	0.006038	0.164358	3.069852	0.405411	0.069852
	10	0.202423	0.046373	0.005398	0.164208	3.048848	0.405226	0.048848
3	3	0.602485	0.399139	0.036151	0.225634	3.273408	0.475009	0.273408
	4	0.465823	0.236788	0.019797	0.145361	3.125628	0.381262	0.125628
	5	0.396873	0.171505	0.013997	0.123096	3.072745	0.350850	0.072745
	6	0.352462	0.135146	0.010917	0.112742	3.054180	0.335770	0.054180
	7	0.320559	0.111736	0.008978	0.108337	3.051939	0.329145	0.051939
	8	0.296135	0.095332	0.007636	0.105760	3.035578	0.325207	0.035578
	9	0.276626	0.083171	0.006649	0.105379	3.008423	0.324621	0.008423
	10	0.260564	0.073785	0.005891	0.104265	3.025598	0.322900	0.025598
4	4	0.648039	0.453256	0.033301	0.229320	3.321976	0.478874	0.321976
	5	0.511790	0.280311	0.018382	0.130464	3.147984	0.361198	0.147984
	6	0.441284	0.207865	0.013133	0.102281	3.090069	0.319815	0.090069
	7	0.394999	0.166359	0.010335	0.090404	3.056551	0.300673	0.056551
	8	0.361267	0.139076	0.008562	0.084264	3.040724	0.290282	0.040724
	9	0.335151	0.119652	0.007326	0.081505	3.010495	0.285491	0.010495
	10	0.314105	0.105073	0.006411	0.078213	3.023921	0.279667	0.023921
5	5	0.682101	0.496492	0.031230	0.237912	3.362265	0.487762	0.362265
	6	0.547043	0.316534	0.017278	0.124926	3.165975	0.353448	0.165975
	7	0.475998	0.238994	0.012420	0.092794	3.093716	0.304621	0.093716
	8	0.428730	0.193642	0.009833	0.078295	3.075215	0.279812	0.075215
	9	0.393912	0.163355	0.008188	0.071074	3.053238	0.266597	0.053238
	10	0.366721	0.141522	0.007038	0.067297	3.019226	0.259417	0.019226
6	6	0.709112	0.532483	0.029643	0.247927	3.393989	0.497922	0.393989
	7	0.575461	0.347549	0.016394	0.123722	3.179939	0.351742	0.179939
	8	0.504359	0.266206	0.011828	0.087950	3.098604	0.296564	0.098604
	9	0.456584	0.217871	0.009402	0.072183	3.067347	0.268669	0.067347
	10	0.421103	0.185187	0.007859	0.063867	3.048346	0.252719	0.048346
7	7	0.731387	0.563306	0.028379	0.257507	3.423281	0.507451	0.423281
	8	0.599162	0.374664	0.015668	0.123781	3.197735	0.351825	0.197735
	9	0.528246	0.290374	0.011330	0.081133	3.116019	0.284838	0.116019
	10	0.480238	0.239660	0.009031	0.067515	3.077528	0.259836	0.077528
8	8	0.750277	0.590255	0.027339	0.267491	3.444852	0.517195	0.444852
	9	0.619424	0.398746	0.015060	0.125692	3.209343	0.354531	0.209343
	10	0.548821	0.312108	0.010904	0.083717	3.116898	0.289339	0.116898
9	9	0.766633	0.614193	0.026467	0.276388	3.468726	0.525726	0.468726
	10	0.637074	0.420406	0.014543	0.127287	3.217234	0.356773	0.217234
10	10	0.781029	0.635725	0.025719	0.285348	3.482843	0.534180	0.482843

TABLE 4.2: Moments, variances skewness and kurtosis of order statistic from EPL distribution for $\tau = 2$, $\kappa = 0.5$ and $\xi = 10$

u	n	$E(Z)$	$E(Z^2)$	$V(Z)$	δ_1	δ_2	γ_1	γ_2
1	1	0.286922	0.104762	0.022438	0.392513	3.233671	0.626509	0.233671
	2	0.202969	0.052438	0.011242	0.395237	3.235008	0.628679	0.235008
	3	0.165748	0.034972	0.007500	0.395367	3.241712	0.628782	0.241712
	4	0.143552	0.026234	0.005627	0.395894	3.257377	0.629201	0.257377
	5	0.128403	0.020990	0.004503	0.397574	3.250389	0.630535	0.250389
	6	0.117219	0.017493	0.003753	0.396251	3.264481	0.629485	0.264481
	7	0.108526	0.014995	0.003217	0.399772	3.270879	0.632275	0.270879
	8	0.101518	0.013121	0.002815	0.399592	3.202086	0.632133	0.202086
	9	0.095714	0.011663	0.002502	0.396203	3.258955	0.629447	0.258955
	10	0.090803	0.010497	0.002252	0.403750	3.146694	0.635413	0.146694
2	2	0.370875	0.157086	0.019538	0.252992	3.233830	0.502984	0.233830
	3	0.277412	0.087369	0.010412	0.195887	3.130170	0.442592	0.130170
	4	0.232333	0.061185	0.007206	0.180837	3.092581	0.425249	0.092581
	5	0.204150	0.047211	0.005534	0.173430	3.076871	0.416449	0.076871
	6	0.184323	0.038474	0.004499	0.171049	3.090499	0.413580	0.090499
	7	0.169376	0.032482	0.003794	0.165917	3.090481	0.407329	0.090481
	8	0.157579	0.028111	0.003280	0.166400	3.110573	0.407921	0.110573
	9	0.147956	0.024781	0.002890	0.166052	3.079916	0.407494	0.079916
	10	0.139911	0.022158	0.002583	0.163607	3.108243	0.404484	0.108243
3	3	0.417607	0.191945	0.017549	0.236218	3.294003	0.486023	0.294003
	4	0.322490	0.113553	0.009553	0.151694	3.141847	0.389479	0.141847
	5	0.274609	0.082146	0.006736	0.127620	3.077971	0.357239	0.077971
	6	0.243803	0.064685	0.005245	0.117231	3.069729	0.342389	0.069729
	7	0.221690	0.053456	0.004310	0.111750	3.067425	0.334290	0.067425
	8	0.204769	0.045593	0.003663	0.108367	3.027919	0.329192	0.027919
	9	0.191258	0.039767	0.003187	0.107573	2.974637	0.327983	0.025360
	10	0.180137	0.035272	0.002823	0.107969	3.008948	0.328587	0.008948
4	4	0.449313	0.218075	0.016193	0.239672	3.345870	0.489563	0.345870
	5	0.354410	0.134491	0.008885	0.136586	3.148446	0.369575	0.148446
	6	0.305414	0.099607	0.006329	0.106752	3.095144	0.326730	0.095144
	7	0.273287	0.079658	0.004972	0.094570	3.035524	0.307522	0.035524
	8	0.249893	0.066561	0.004114	0.086248	3.032981	0.293680	0.032981
	9	0.231790	0.057244	0.003517	0.082227	3.081452	0.286752	0.081452
	10	0.217207	0.050255	0.003076	0.078999	3.035468	0.281068	0.035468
5	5	0.473039	0.238971	0.015205	0.248785	3.378142	0.498784	0.378142
	6	0.378909	0.151933	0.008361	0.131396	3.170584	0.362486	0.170584
	7	0.329508	0.114569	0.005993	0.096047	3.099872	0.309914	0.099872
	8	0.296682	0.092756	0.004736	0.081285	3.081199	0.285105	0.081199
	9	0.272522	0.078207	0.003939	0.073627	3.027151	0.271344	0.027151
	10	0.253665	0.067728	0.003382	0.069769	2.985835	0.264139	0.014160
6	6	0.491865	0.256379	0.014448	0.258173	3.415604	0.508108	0.415604
	7	0.398669	0.166879	0.007942	0.128823	3.206365	0.358919	0.206365
	8	0.349204	0.127657	0.005714	0.090977	3.107751	0.301623	0.107751
	9	0.316010	0.104395	0.004533	0.074676	3.078239	0.273269	0.078239
	10	0.291379	0.088686	0.003784	0.065202	3.049056	0.255346	0.049056
7	7	0.507398	0.271295	0.013842	0.268815	3.439121	0.518474	0.439121
	8	0.415157	0.179953	0.007598	0.130030	3.189014	0.360597	0.189014
	9	0.365801	0.139288	0.005478	0.087421	3.153352	0.295671	0.153352
	10	0.332431	0.114868	0.004358	0.071826	3.031533	0.268003	0.031533
8	8	0.520575	0.284344	0.013346	0.277922	3.467195	0.527183	0.467195
	9	0.429259	0.191572	0.007309	0.131214	3.202192	0.362235	0.202192
	10	0.380103	0.149753	0.005275	0.087888	3.122100	0.296459	0.122100
9	9	0.531990	0.295940	0.012927	0.287613	3.487318	0.536295	0.487318
	10	0.441548	0.202027	0.007062	0.132007	3.237307	0.363328	0.237307
10	10	0.542039	0.306375	0.012569	0.296542	3.497892	0.544557	0.497892

TABLE 4.3: Covariances of order statistics.

			$\tau=2,\xi=5,\kappa=0.5$				$\tau=2,\xi=10,\kappa=0.5$			
v	u	n	$\mu_{u,v:n}$	$E(Z_{u:n})$	$E(Z_{v:n})$	$\sigma_{u,v:n}$	$\mu_{u,v:n}$	$E(Z_{u:n})$	$E(Z_{v:n})$	$\sigma_{u,v:n}$
2	1	2	0.125100	0.293422	0.535273	-0.03196	0.059192	0.202969	0.370875	-0.01608
		3	0.073220	0.239709	0.400850	-0.02287	0.034402	0.165748	0.277412	-0.01158
		4	0.052133	0.207652	0.335877	-0.01761	0.024407	0.143552	0.232333	-0.00895
		5	0.040552	0.185762	0.295213	-0.01429	0.018943	0.128403	0.204150	-0.00727
		6	0.033203	0.169597	0.266589	-0.01201	0.015488	0.117219	0.184323	-0.00612
		7	0.028119	0.157029	0.245001	-0.01035	0.013102	0.108526	0.169376	-0.00528
		8	0.024389	0.146897	0.227956	-0.00910	0.011355	0.101518	0.157579	-0.00464
		9	0.021534	0.138503	0.214051	-0.00811	0.010020	0.095714	0.147956	-0.00414
		10	0.019279	0.131400	0.202423	-0.00732	0.008966	0.090803	0.139911	-0.00374
3	1	3	0.146966	0.239709	0.602485	0.002545	0.066242	0.165748	0.417607	-0.00298
		4	0.100791	0.207652	0.465823	0.004062	0.044251	0.143552	0.322490	-0.00204
		5	0.079630	0.185762	0.396873	0.005906	0.034146	0.128403	0.274609	-0.00112
		6	0.067164	0.169597	0.352462	0.007387	0.028186	0.117219	0.243803	-0.00039
		7	0.058882	0.157029	0.320559	0.008545	0.024224	0.108526	0.221690	0.000165
		8	0.052962	0.146897	0.296135	0.009461	0.021392	0.101518	0.204769	0.000604
		9	0.048512	0.138503	0.276626	0.010199	0.019262	0.095714	0.191258	0.000956
		10	0.045043	0.131400	0.260564	0.010805	0.017601	0.090803	0.180137	0.001244
3	2	3	0.155112	0.400850	0.602485	-0.08639	0.076932	0.277412	0.417607	-0.03892
		4	0.087823	0.335877	0.465823	-0.06864	0.044543	0.232333	0.322490	-0.03038
		5	0.059381	0.295213	0.396873	-0.05778	0.031058	0.204150	0.274609	-0.02500
		6	0.043331	0.266589	0.352462	-0.05063	0.023522	0.184323	0.243803	-0.02142
		7	0.032948	0.245001	0.320559	-0.04559	0.018679	0.169376	0.221690	-0.01887
		8	0.025656	0.227956	0.296135	-0.04185	0.015295	0.157579	0.204769	-0.01697
		9	0.020245	0.214051	0.276626	-0.03897	0.012794	0.147956	0.191258	-0.01550
		10	0.016067	0.202423	0.260564	-0.03668	0.010868	0.139911	0.180137	-0.01434
4	1	4	0.157123	0.207652	0.648039	0.022556	0.067875	0.143552	0.449313	0.003375
		5	0.116281	0.185762	0.511790	0.021210	0.048633	0.128403	0.354410	0.003126
		6	0.096719	0.169597	0.441284	0.021879	0.039214	0.117219	0.305414	0.003413
		7	0.084955	0.157029	0.394999	0.022929	0.033425	0.108526	0.273287	0.003766
		8	0.077118	0.146897	0.361267	0.024049	0.029472	0.101518	0.249893	0.004104
		9	0.071578	0.138503	0.335151	0.025159	0.026598	0.095714	0.231790	0.004413
		10	0.067511	0.131400	0.314105	0.026238	0.024416	0.090803	0.217207	0.004693
	2	4	0.172829	0.335877	0.648039	-0.04483	0.084966	0.232333	0.449313	-0.01942
		5	0.112133	0.295213	0.511790	-0.03895	0.055696	0.204150	0.354410	-0.01666
		6	0.082851	0.266589	0.441284	-0.03479	0.041888	0.184323	0.305414	-0.01441
		7	0.064709	0.245001	0.394999	-0.03207	0.033554	0.169376	0.273287	-0.01273
		8	0.052012	0.227956	0.361267	-0.03034	0.027892	0.157579	0.249893	-0.01149
		9	0.042428	0.214051	0.335151	-0.02931	0.023756	0.147956	0.231790	-0.01054
		10	0.034802	0.202423	0.314105	-0.02878	0.020579	0.139911	0.217207	-0.00981
	3	4	0.179898	0.465823	0.648039	-0.12197	0.089110	0.322490	0.449313	-0.05579
		5	0.104111	0.396873	0.511790	-0.09900	0.052452	0.274609	0.354410	-0.04487
		6	0.071720	0.352462	0.441284	-0.08382	0.036947	0.243803	0.305414	-0.03751
		7	0.053410	0.320559	0.394999	-0.07321	0.028192	0.221690	0.273287	-0.03239
		8	0.041644	0.296135	0.361267	-0.06534	0.022532	0.204769	0.249893	-0.02864
		9	0.033502	0.276626	0.335151	-0.05921	0.018569	0.191258	0.231790	-0.02576
		10	0.027592	0.260564	0.314105	-0.05425	0.015641	0.180137	0.217207	-0.02349
5	1	5	0.165311	0.185762	0.682101	0.038602	0.068527	0.128403	0.473039	0.007787
		6	0.128166	0.169597	0.547043	0.035389	0.051311	0.117219	0.378909	0.006896
		7	0.110052	0.157029	0.475998	0.035306	0.042581	0.108526	0.329508	0.006820
		8	0.099086	0.146897	0.428730	0.036107	0.037070	0.101518	0.296682	0.006952
		9	0.091832	0.138503	0.393912	0.037274	0.033237	0.095714	0.272522	0.007153
		10	0.086817	0.131400	0.366721	0.038630	0.030414	0.090803	0.253665	0.007380
	2	5	0.173400	0.295213	0.682101	-0.02796	0.085162	0.204150	0.473039	-0.01141
		6	0.119750	0.266589	0.547043	-0.02609	0.059439	0.184323	0.378909	-0.01040
		7	0.092013	0.245001	0.475998	-0.02461	0.046480	0.169376	0.329508	-0.00933
		8	0.073911	0.227956	0.428730	-0.02382	0.038296	0.157579	0.296682	-0.00846

TABLE 4.3: Continued.

v	u	n	$\mu_{u,v:n}$	$E(Z_{u:n})$	$E(Z_{v:n})$	$\sigma_{u,v:n}$	$\mu_{u,v:n}$	$E(Z_{u:n})$	$E(Z_{v:n})$	$\sigma_{u,v:n}$
			$\tau=2,\xi=5,\ \kappa=0.5$				$\tau=2,\xi=10,\ \kappa=0.5$			
		9	0.060674	0.214051	0.393912	-0.02364	0.032550	0.147956	0.272522	-0.00777
		10	0.050264	0.202423	0.366721	-0.02397	0.028244	0.139911	0.253665	-0.00725
	3	5	0.202606	0.396873	0.682101	-0.06810	0.099405	0.274609	0.473039	-0.03050
		6	0.133799	0.352462	0.547043	-0.05901	0.065762	0.243803	0.378909	-0.02662
		7	0.100903	0.320559	0.475998	-0.05168	0.049799	0.221690	0.329508	-0.02325
		8	0.080899	0.296135	0.428730	-0.04606	0.040135	0.204769	0.296682	-0.02062
		9	0.067311	0.276626	0.393912	-0.04166	0.033574	0.191258	0.272522	-0.01855
		10	0.057471	0.260564	0.366721	-0.03808	0.028798	0.180137	0.253665	-0.01690
	4	5	0.197590	0.511790	0.682101	-0.15150	0.097898	0.354410	0.473039	-0.06975
		6	0.115810	0.441284	0.547043	-0.12559	0.058351	0.305414	0.378909	-0.05737
		7	0.080302	0.394999	0.475998	-0.10772	0.041419	0.273287	0.329508	-0.04863
		8	0.059934	0.361267	0.428730	-0.09495	0.031757	0.249893	0.296682	-0.04238
		9	0.046659	0.335151	0.393912	-0.08536	0.025457	0.231790	0.272522	-0.03771
		10	0.037334	0.314105	0.366721	-0.07786	0.021014	0.217207	0.253665	-0.03408
6	1	6	0.173540	0.169597	0.709112	0.053277	0.069056	0.117219	0.491865	0.011400
		7	0.138937	0.157029	0.575461	0.048573	0.053325	0.108526	0.398669	0.010059
		8	0.122004	0.146897	0.504359	0.047915	0.045185	0.101518	0.349204	0.009735
		9	0.111775	0.138503	0.456584	0.048537	0.039958	0.095714	0.316010	0.009712
		10	0.105094	0.131400	0.421103	0.049761	0.036274	0.090803	0.291379	0.009816
	2	6	0.169539	0.266589	0.709112	-0.01950	0.083624	0.184323	0.491865	-0.00704
		7	0.121311	0.245001	0.575461	-0.01968	0.060716	0.169376	0.398669	-0.00681
		8	0.095139	0.227956	0.504359	-0.01983	0.048693	0.157579	0.349204	-0.00633
		9	0.077345	0.214051	0.456584	-0.02039	0.040853	0.147956	0.316010	-0.00590
		10	0.063859	0.202423	0.421103	-0.02138	0.035208	0.139911	0.291379	-0.00556
	3	6	0.206018	0.352462	0.709112	-0.04392	0.100332	0.243803	0.491865	-0.01959
		7	0.145146	0.320559	0.575461	-0.03932	0.070483	0.221690	0.398669	-0.01790
		8	0.114479	0.296135	0.504359	-0.03488	0.055437	0.204769	0.349204	-0.01607
		9	0.095232	0.276626	0.456584	-0.03107	0.045952	0.191258	0.316010	-0.01449
		10	0.081919	0.260564	0.421103	-0.02781	0.039327	0.180137	0.291379	-0.01316
	4	6	0.223267	0.441284	0.709112	-0.08965	0.110002	0.305414	0.491865	-0.04022
		7	0.148164	0.394999	0.575461	-0.07914	0.073256	0.273287	0.398669	-0.03569
		8	0.111855	0.361267	0.504359	-0.07035	0.055703	0.249893	0.349204	-0.03156
		9	0.089461	0.335151	0.456584	-0.06356	0.045006	0.231790	0.316010	-0.02824
		10	0.073972	0.314105	0.421103	-0.05830	0.037697	0.217207	0.291379	-0.02559
	5	6	0.211616	0.547043	0.709112	-0.17630	0.104759	0.378909	0.491865	-0.08161
		7	0.125475	0.475998	0.575461	-0.14844	0.063092	0.329508	0.398669	-0.06827
		8	0.087690	0.428730	0.504359	-0.12854	0.045094	0.296682	0.349204	-0.05851
		9	0.065827	0.393912	0.456584	-0.11403	0.034744	0.272522	0.316010	-0.05138
		10	0.051487	0.366721	0.421103	-0.10294	0.027951	0.253665	0.291379	-0.04596
7	1	7	0.182405	0.157029	0.731387	0.067556	0.069687	0.108526	0.507398	0.014621
		8	0.149550	0.146897	0.599162	0.061535	0.055063	0.101518	0.415157	0.012917
		9	0.133567	0.138503	0.528246	0.060403	0.047410	0.095714	0.365801	0.012398
		10	0.124002	0.131400	0.480238	0.060899	0.042442	0.090803	0.332431	0.012256
	2	7	0.163819	0.245001	0.731387	-0.01537	0.081635	0.169376	0.507398	-0.00431
		8	0.119737	0.227956	0.599162	-0.01685	0.060918	0.157579	0.415157	-0.00450
		9	0.094876	0.214051	0.528246	-0.01820	0.049741	0.147956	0.365801	-0.00438
		10	0.077366	0.202423	0.480238	-0.01984	0.042276	0.139911	0.332431	-0.00423
	3	7	0.205092	0.320559	0.731387	-0.02936	0.099057	0.221690	0.507398	-0.01343
		8	0.150629	0.296135	0.599162	-0.02680	0.072334	0.204769	0.415157	-0.01268
		9	0.122355	0.276626	0.528246	-0.02377	0.058356	0.191258	0.365801	-0.01161
		10	0.104276	0.260564	0.480238	-0.02086	0.049294	0.180137	0.332431	-0.01059
	4	7	0.227234	0.320559	0.731387	-0.00722	0.111556	0.273287	0.507398	-0.02711
		8	0.160109	0.296135	0.599162	-0.01732	0.078663	0.249893	0.415157	-0.02508
		9	0.125850	0.276626	0.528246	-0.02028	0.062003	0.231790	0.365801	-0.02279
		10	0.103939	0.260564	0.480238	-0.02119	0.051445	0.217207	0.332431	-0.02076

TABLE 4.3: Continued.

			$\tau=2,\xi=5,\ \kappa=0.5$				$\tau=2,\xi=10,\ \kappa=0.5$			
v	u	n	$\mu_{u,v:n}$	$E(Z_{u:n})$	$E(Z_{v:n})$	$\sigma_{u,v:n}$	$\mu_{u,v:n}$	$E(Z_{u:n})$	$E(Z_{v:n})$	$\sigma_{u,v:n}$
	5	7	0.240059	0.475998	0.731387	-0.10808	0.118411	0.329508	0.507398	-0.04878
		8	0.160346	0.428730	0.599162	-0.09653	0.079330	0.296682	0.415157	-0.04384
		9	0.121653	0.393912	0.528246	-0.08643	0.060576	0.272522	0.365801	-0.03911
		10	0.097711	0.366721	0.480238	-0.07840	0.049095	0.253665	0.332431	-0.03523
	6	7	0.223156	0.575461	0.731387	-0.19773	0.110362	0.398669	0.507398	-0.09192
		8	0.133614	0.504359	0.599162	-0.16858	0.067043	0.349204	0.415157	-0.07793
		9	0.094015	0.456584	0.528246	-0.14717	0.048208	0.316010	0.365801	-0.06739
		10	0.070923	0.421103	0.480238	-0.13131	0.037307	0.291379	0.332431	-0.05956
8	1	8	0.192134	0.146897	0.750277	0.081920	0.070480	0.101518	0.520575	0.017632
		9	0.160463	0.138503	0.619424	0.074671	0.056695	0.095714	0.429259	0.015609
		10	0.145243	0.131400	0.548821	0.073128	0.049442	0.090803	0.380103	0.014928
	2	8	0.156885	0.227956	0.750277	-0.01415	0.079554	0.157579	0.520575	-0.00248
		9	0.116038	0.214051	0.619424	-0.01655	0.060572	0.147956	0.429259	-0.00294
		10	0.092204	0.202423	0.548821	-0.01889	0.050124	0.139911	0.380103	-0.00306
	3	8	0.203197	0.296135	0.750277	-0.01899	0.097193	0.204769	0.520575	-0.00940
		9	0.153848	0.276626	0.619424	-0.01750	0.072961	0.191258	0.429259	-0.00914
		10	0.127820	0.260564	0.548821	-0.01518	0.059979	0.180137	0.380103	-0.00849
	4	8	0.225773	0.361267	0.750277	-0.04528	0.110449	0.249893	0.520575	-0.01964
		9	0.165186	0.335151	0.619424	-0.04241	0.080817	0.231790	0.429259	-0.01868
		10	0.133147	0.314105	0.548821	-0.03924	0.065255	0.217207	0.380103	-0.01731
	5	8	0.245110	0.428730	0.750277	-0.07656	0.120610	0.296682	0.520575	-0.03384
		9	0.173431	0.393912	0.619424	-0.07057	0.085377	0.272522	0.429259	-0.03161
		10	0.136810	0.366721	0.548821	-0.06445	0.067482	0.253665	0.380103	-0.02894
	6	8	0.253982	0.504359	0.750277	-0.12443	0.125347	0.349204	0.520575	-0.05644
		9	0.170589	0.456584	0.619424	-0.11223	0.084412	0.316010	0.429259	-0.05124
		10	0.129956	0.421103	0.548821	-0.10115	0.064702	0.291379	0.380103	-0.04605
	7	8	0.232942	0.504359	0.750277	-0.14547	0.115084	0.415157	0.520575	-0.10104
		9	0.140652	0.456584	0.619424	-0.14217	0.070427	0.365801	0.429259	-0.08660
		10	0.099571	0.421103	0.548821	-0.13154	0.050909	0.332431	0.380103	-0.07545
9	1	9	0.202844	0.138503	0.766633	0.096663	0.071447	0.095714	0.531990	0.020528
		10	0.171939	0.131400	0.637074	0.088228	0.058309	0.090803	0.441548	0.018215
	2	9	0.148835	0.214051	0.766633	-0.01526	0.077490	0.147956	0.531990	-0.00122
		10	0.110573	0.202423	0.637074	-0.01839	0.059909	0.139911	0.441548	-0.00187
	3	9	0.201458	0.276626	0.766633	-0.01061	0.095239	0.191258	0.531990	-0.00651
		10	0.156264	0.260564	0.637074	-0.00973	0.073008	0.180137	0.441548	-0.00653
	4	9	0.222547	0.335151	0.766633	-0.03439	0.108528	0.231790	0.531990	-0.01478
		10	0.167173	0.314105	0.637074	-0.03294	0.081537	0.217207	0.441548	-0.01437
	5	9	0.244147	0.393912	0.766633	-0.05784	0.119779	0.272522	0.531990	-0.02520
		10	0.179218	0.366721	0.637074	-0.05441	0.087879	0.253665	0.441548	-0.02413
	6	9	0.260024	0.456584	0.766633	-0.09001	0.128154	0.316010	0.531990	-0.03996
		10	0.184607	0.421103	0.637074	-0.08367	0.091033	0.291379	0.441548	-0.03762
	7	9	0.265867	0.528246	0.766633	-0.13910	0.131234	0.365801	0.531990	-0.06337
		10	0.179461	0.480238	0.637074	-0.12649	0.088775	0.332431	0.441548	-0.05801
	8	9	0.241423	0.619424	0.766633	-0.23345	0.119156	0.429259	0.531990	-0.10921
		10	0.146845	0.548821	0.637074	-0.20279	0.073382	0.380103	0.441548	-0.09445
10	1	10	0.214615	0.131400	0.781029	0.111988	0.072587	0.090803	0.542039	0.023369
	2	10	0.139577	0.202423	0.781029	-0.01852	0.075465	0.139911	0.542039	-0.00037
	3	10	0.200369	0.260564	0.781029	-0.00314	0.093369	0.180137	0.542039	-0.00427
	4	10	0.218699	0.314105	0.781029	-0.02663	0.106389	0.217207	0.542039	-0.01135
	5	10	0.241199	0.366721	0.781029	-0.04522	0.117950	0.253665	0.542039	-0.01955
	6	10	0.259491	0.421103	0.781029	-0.06940	0.127622	0.291379	0.542039	-0.03032
	7	10	0.272860	0.480238	0.781029	-0.10222	0.134607	0.332431	0.542039	-0.04558
	8	10	0.276212	0.548821	0.781029	-0.15243	0.136336	0.380103	0.542039	-0.06969
	9	10	0.248896	0.637074	0.781029	-0.24868	0.122731	0.441548	0.542039	-0.11661

4.5 BLUEs of the Location and Scale Parameters

Let $Y_{1:n} \leq Y_{2:n} \leq \cdots \leq Y_{n-c:n},\ \ c = 0(1)([n/2]-1)$ denote Type-II right-censored sample from the location-scale parameter extended power Lindley distribution in Equation (4.1).

Let us consider scale-parameter pdf of EPL distribution as

$$h(z;\tau,\kappa,\xi,\varphi) = \frac{\tau\xi^2}{\varphi(\xi+\kappa)}\left[1+\kappa\left(\frac{z}{\varphi}\right)^{\tau}\right]\left(\frac{z}{\varphi}\right)^{\tau-1} e^{-\xi\left(\frac{z}{\varphi}\right)^{\tau}},\ z>0;\ \tau>0,\kappa>0,\xi>0,\varphi>0, \tag{4.6}$$

and location-scale parameter pdf of EPL distribution is

$$\begin{aligned} h(z;\tau,\kappa,\xi,\delta,\varphi) &= \frac{\tau\xi^2}{\varphi(\xi+\kappa)}\left[1+\kappa\left(\frac{z-\delta}{\varphi}\right)^{\tau}\right]\left(\frac{z-\delta}{\varphi}\right)^{\tau-1} \\ &\times\ e^{-\xi\left(\frac{z-\delta}{\varphi}\right)^{\tau}},\ z>\delta;\ \tau>0,\kappa>0,\xi>0,\varphi>0,\delta>0, \end{aligned} \tag{4.7}$$

The expression for the BLUEs of location and scale parameter are given in (1.18) and also variances and covariance for these parameters are given in eqn (1.21), (1.22) and (1.23).

Tables 4.4 and 4.5 display the coefficient of the BLUEs for type-II right censored sample of various values of $n = 4,\ 8,\ 10,\ \tau = 2,3$ and censoring cases $c = 0(1)([n/2]-1)$. Also, Table 4.6 shows variances and covariances of the BLUEs.

4.6 Real Data Application

The data set refers to a study on the vinyl chloride from clean upgradient monitoring wells in mg/L which studied by Bhaumik et al. (2009). The data are:

```
5.1 1.2 1.3 0.6 0.5 2.4 0.5 1.1 8.0 0.8 0.4 0.6 0.9 0.4 2.0 0.5
5.3 3.2 2.7 2.9 2.5 2.3 1.0 0.2 0.1 0.1 1.8 0.9 2.0 4.0 6.8 1.2
0.4 0.2
```

TABLE 4.4: Coefficients of the BLUEs of the location parameter for $\tau = 0.5$, $\kappa = 2$.

ξ	n	c	$a_u, u = 1,2,3,\ldots,(n-c)$						
0.5	4	0	0.60056	0.52054	-0.0249	-0.0962			
		1	1.14951	-0.0022	-0.1473				
		2	1.46179	-0.4618					
	8	0	0.94978	-0.0079	-0.0355	0.11756	0.04068	-0.0357	-0.0211
			-0.0079						
		1	0.99500	-0.0542	-0.0168	0.13125	0.02120	-0.0520	-0.0245
		2	1.10008	-0.1932	0.02847	0.16721	-0.0256	-0.0769	
		3	-0.1442	3.22795	-1.6960	-1.0836	0.69588		
		4	1.22957	-5E-05	-0.2371	0.00762			
		5	1.20093	0.03644	-0.2374				
		6	1.55875	-0.5587					
	10	0	0.84633	0.46665	-0.1943	0.02331	0.04968	-0.1587	-0.0944
			0.02940	0.02612	0.00600				
		1	0.87109	0.40252	-0.1787	0.02622	0.03714	-0.1427	-0.0747
			0.03345	0.02564					
		2	0.92580	0.23990	-0.1220	0.04079	0.00905	-0.0958	-0.0266
			0.02881						
		3	1.14784	-0.0594	-0.0458	0.05472	-0.0716	-0.0521	0.02645
		4	0.97357	0.11809	-0.0719	0.05459	-0.0127	-0.0617	
		5	1.28460	-0.3778	0.12364	0.10493	-0.1354		
		6	1.19440	0.06579	-0.2574	-0.0028			
1	4	0	3.49685	-2.3573	-0.4864	0.34688			
		1	1.11260	0.02270	-0.1353				
		2	1.41595	-0.4159					
	8	0	0.95517	0.00129	-0.0557	0.09979	0.04894	-0.0232	-0.0185
			-0.0078						
		1	1.00829	-0.0616	-0.0298	0.11856	0.02769	-0.0411	-0.0220
		2	1.12022	-0.2225	0.02545	0.16109	-0.0176	-0.0667	
		3	0.33441	1.88369	-0.9937	-0.5809	0.35650		
		4	1.21720	-0.0211	-0.2102	0.01419			
		5	1.15723	0.05438	-0.2116				
		6	1.49807	-0.4981					
	10	0	0.88677	0.32334	-0.1486	0.03122	0.03628	-0.1013	-0.0611
			0.01480	0.01489	0.00371				
		1	0.91353	0.27429	-0.1386	0.03356	0.02542	-0.0940	-0.0482
			0.01886	0.01508					
		2	0.97087	0.15491	-0.1026	0.04442	0.00171	-0.0720	-0.0174
			0.02004						
		3	1.27311	-0.2207	-0.0148	0.06681	-0.1006	-0.0450	0.04122
		4	0.98785	0.08119	-0.0667	0.05692	-0.0081	-0.0512	
		5	1.32640	-0.4955	0.17390	0.13470	-0.1395		
		6	1.17694	0.04512	-0.2260	0.00389			
		7	1.15952	0.06823	-0.2278				
		8	1.51118	-0.5112					

TABLE 4.5: Coefficients of the BLUEs of the scale parameter for $\tau = 0.5$, $\kappa = 2$.

ξ	n	c	$b_u,\ u = 1,2,3,\ldots,(n-c)$						
0.5	4	0	-0.19072	0.15597	0.04744	-0.01269			
		1	-0.11835	0.08705	0.03130				
		2	-0.18472	0.18472					
	8	0	-0.11319	0.09364	0.00620	-0.02865	0.01231	0.02046	0.00619
			0.00305						
		1	-0.13061	0.11148	-0.00102	-0.03392	0.01981	0.02676	0.00750
		2	-0.16282	0.15410	-0.01488	-0.04495	0.03417	0.03438	
		3	0.39367	-1.37593	0.75633	0.51444	-0.28851		
		4	-0.17592	-0.03759	0.15149	0.06202			
		5	-0.40913	0.25959	0.14954				
		6	-0.63456	0.63456					
	10	0	-0.05656	-0.13237	0.10417	0.01245	-0.02274	0.06273	0.04769
			-0.00559	-0.00908	-0.00070				
		1	-0.05946	-0.12487	0.10235	0.01211	-0.02127	0.06086	0.04538
			-0.00606	-0.00902					
		2	-0.07871	-0.06765	0.08238	0.00698	-0.01139	0.04436	0.02846
			-0.00443						
		3	-0.11285	-0.02163	0.07068	0.00484	0.00101	0.03764	0.02030
		4	-0.24665	0.11466	0.05071	0.00474	0.04621	0.03032	
		5	-0.39958	0.35850	-0.04542	-0.02001	0.10651		
		6	-0.32861	0.00947	0.25437	0.06478			
1	4	0	2.94543	-3.06133	-0.39787	0.51376			
		1	-0.58588	0.46372	0.12215				
		2	-0.85975	0.85975					
	8	0	-0.57040	0.50408	0.01908	-0.14321	0.05548	0.09304	0.02875
			0.01319						
		1	-0.66076	0.61109	-0.02508	-0.17515	0.09163	0.12358	0.03469
		2	-0.83696	0.86427	-0.11200	-0.24208	0.16300	0.16378	
		3	1.09377	-4.31054	2.39211	1.58093	-0.75627		
		4	-0.77897	-0.26961	0.73003	0.31855			
		5	-2.12522	1.42604	0.69918				
		6	-3.25139	3.25139					
	10	0	-0.24289	-0.51009	0.44639	0.05160	-0.11371	0.20398	0.18654
			0.00347	-0.02489	-0.00040				
		1	-0.24574	-0.50487	0.44532	0.05135	-0.11255	0.20320	0.18517
			0.00304	-0.02491					
		2	-0.34047	-0.30765	0.38585	0.03342	-0.07339	0.16686	0.13428
			0.00109						
		3	-0.32406	-0.32804	0.39062	0.03463	-0.07894	0.16833	0.13747
		4	-1.27553	0.67903	0.21763	0.00163	0.22956	0.14769	
		5	-2.25242	2.34293	-0.47649	-0.22280	0.60877		
		6	-1.60032	-0.01563	1.26804	0.34791			
		7	-3.15724	2.05023	1.10701				
		8	-4.86649	4.86649					

TABLE 4.6: Variances and covariance of the BLUEs when $\tau = 0.5$, $\kappa = 2$ and $\delta = 0$ and $\varphi = 1$.

ξ	n	c	$Var(\delta^*)$	$Var(\varphi^*)$	$Cov(\delta^*,\ \varphi^*)$
0.5	4	0	17.65257	1.410936	-0.566364
		1	14.02154	1.347831	-1.045046
		2	4.033713	0.896757	1.077511
	8	0	2.337863	0.846369	-0.520900
		1	2.167946	0.821161	-0.455453
		2	1.696970	0.776917	-0.311100
		3	20.75401	4.588457	-8.833814
		4	4.587047	1.809407	-2.130921
		5	4.594606	2.310717	-2.069361
		6	0.286717	0.600882	0.644635
	10	0	2.750427	1.010938	-1.077970
		1	2.628375	1.009270	-1.063700
		2	2.206944	0.957081	-0.915396
		3	1.727925	0.945757	-0.841744
		4	1.848615	1.016895	-0.749085
		5	0.544039	0.701471	-0.107607
		6	3.260254	2.383097	-2.244815
		7	3.261429	3.000961	-2.271760
		8	0.140384	0.562046	0.487219
1	4	0	0.053382	0.191120	-1.066744
		1	0.641031	1.480219	-0.196377
		2	0.306267	1.207343	0.105863
	8	0	0.086850	0.871200	-0.080502
		1	0.078748	0.847759	-0.066722
		2	0.059150	0.799190	-0.035870
		3	0.444527	3.125664	-0.982743
		4	0.146996	1.786670	-0.351560
		5	0.148080	2.333096	-0.327222
		6	0.027926	1.021373	0.069779
	10	0	0.076264	0.962850	-0.152530
		1	0.073788	0.962822	-0.152266
		2	0.065425	0.939994	-0.138449
		3	0.048548	0.939945	-0.139365
		4	0.056977	1.033724	-0.111248
		5	0.005818	0.607766	0.036372
		6	0.094559	2.297031	-0.350807
		7	0.094639	2.932194	-0.343704
		8	0.013512	1.015521	0.050624

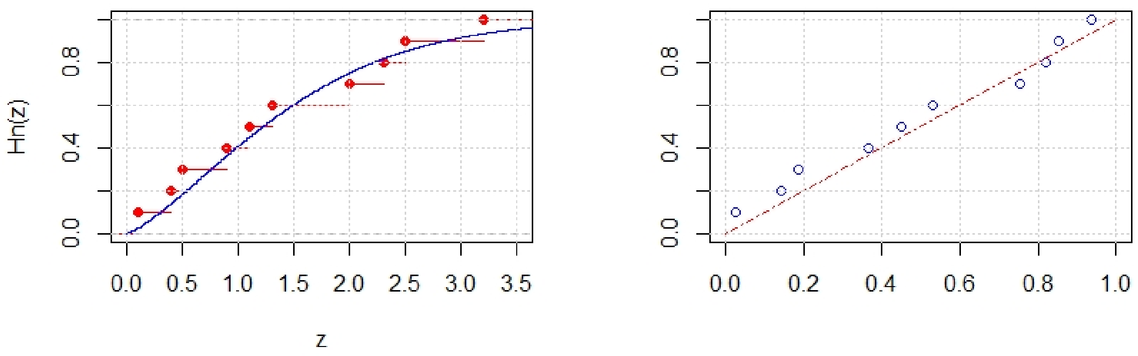

FIGURE 4.1: ECDF-plot and QQ-plot

In this data set, we select a random sample of size 10 and data are: 0.1, 1.1, 0.9, 2.3, 1.3, 2.5, 0.4, 2.0, 0.5, 3.2. $\tau_{ML} = 1.2945$, $\kappa_{ML} = 0.5514$ and $\xi_{ML} = 0.8099$ are the MLEs of EPL distribution for the given sample. The K-S statistic and corresponding p-value are 0.1525 and 0.9476, respectively, which conforms the suitability of the EPL distribution. Figure 4.1 shows ecdf and Q-Q plot of the sample.

By using Tables 4.4 and 4.5, we have

$$\delta^* = \sum_{u=1}^{n} a_u Z_{u:n} = 0.20081 \quad \text{and} \quad \varphi^* = \sum_{u=1}^{n} b_u Z_{u:n} = 0.36501.$$

4.7 Conclusion

Here, we come up with new expressions for moments of order statistics. Also, these results are reduced for special cases of power Lindley and Lindley distribution which is obtained by Kumar and Anju (2019a) and Sultan and AL-Thubyani (2016) respectively. The BLUEs for the corresponding parameters have been derived using moments of the order statistics and simulation is used for the case of Type-II right-censoring. Finally, for illustration, a real data set is analysed