Chapter 5

Inferences for Generalized Topp-Leone Distribution under Order Statistics with Application to Polyester Fibers Data

5.1 Introduction

This chapter follows the following structure: In GTL distribution case, we provide expressions for the single moments of order statistics in Section 5.2. A discussion of the lifetimes of coherent systems is provided in Section 5.3 in application of the theoretical results presented in Section 5.2. Section 5.4 is devoted to the product moments of order statistics. We use these moments in Section 5.5 to obtain BLUEs for the location and scale parameters. A real data application is provided in Section 5.6. Finally, in Section 5.7 , we draw a conclusion for the chapter.

Shekhawat and Sharma (2020) proposed two parameter GTL distribution by using the transformation $X = Z^{\alpha}$. The hazard function of this model has increasing and bathtub shape such as Weibull and exponentiated exponential distributions; and so on and also show that GTL model is superior model tissue damage proportions data as compare to the unit-gamma distributions, Beta, Kumaraswamy and unit-Weibull. They also study properties and MLEs of unknown parameters and significance of concentration level of drugs on tissue damage proportion by developing a parameteric regression model.

Shekhawat and Sharma (2020) proposed two parameter GTL distribution with pdf

$$h(z;\kappa,\xi) \;=\; 2\kappa\xi z^{\kappa\xi-1}(1-z^{\kappa})(2-z^{\kappa})^{\xi-1},\ 0<z<1,\ \kappa,\ \xi>0. \tag{5.1}$$

The associated cdf and quantile function are, respectively, given by

$$H(z;\kappa,\xi)=(z^{\kappa}(2-z^{\kappa}))^{\xi},\ 0<z<1,\ \kappa,\ \xi>0. \tag{5.2}$$

and let $H(z_p;\kappa,\xi)=p$, $p\in\{0,1\}$, then quantile function is

$$z_p=\left\{1-\sqrt{1-p^{\frac{1}{\xi}}}\right\}^{\frac{1}{\kappa}},$$

The pth moment can be easily computed as

$$\mu_p^{'}=\xi 2^{2\xi+\frac{p}{\kappa}}\left\{B_{1/2}\left(\xi+\frac{p}{\kappa},\ \xi\right)-2B_{1/2}\left(\xi+1+\frac{p}{\kappa},\ \xi\right)\right\}, \tag{5.3}$$

where $B_z(.,.)$ is defined as $B_z(\kappa,\xi)=\int_0^z t^{\kappa-1}(1-t)^{\xi-1}dt$.

5.2 Relations for Single Moments of Order Statistics

Let $Z_1,Z_2,\ldots,Z_n$ be a random sample from GTL distribution, the pdf of uth order statistic is obtained by using equation (5.1) and (5.2) in (1.1) as follows:

$$h_{Z_{u:n}}(z)=\frac{2n!\kappa\xi\, z^{\kappa\xi-1}}{(u-1)!(n-u)!}\left[z^{\kappa}(2-z^{\kappa})\right]^{\xi(u-1)}\left[1-\{z^{\kappa}(2-z^{\kappa})\}^{\xi}\right]^{n-u}(1-z^{\kappa})(2-z^{\kappa})^{\xi-1} \tag{5.4}$$

The pth moments of uth order statistics can be obtain from (5.4) as

$$\mu_{u:n}^{(p)} = \frac{\xi\, n!}{(u-1)!(n-u)!}\sum_{j=0}^{n-u}\binom{n-u}{j}(-1)^j\, 2^{\frac{p+2\kappa\xi(u+j)}{\kappa}} \times \left\{B_{1/2}\left(\frac{p+\kappa\xi(u+j)}{\kappa},\xi(u+j)\right)-2B_{1/2}\left(\frac{p+\kappa\xi(u+j)+\kappa}{\kappa},\xi(u+j)\right)\right\} \quad (5.5)$$

Note that when $u=n=1$, $\mu_{1:1}^{(p)}=\xi 2^{2\xi+\frac{p}{\kappa}}\left\{B_{1/2}\left(\xi+\frac{p}{\kappa},\,\xi\right)-2B_{1/2}\left(\xi+1+\frac{p}{\kappa},\,\xi\right)\right\}$, which agrees with (5.3). In addition from (5.5), the first and second moments of order statistics are, respectively, given by

$$\mu_{u:n}^{(1)} = \frac{\xi\, n!}{(u-1)!(n-u)!}\sum_{j=0}^{n-u}\binom{n-u}{j}(-1)^j\, 2^{\frac{1+2\kappa\xi(u+j)}{\kappa}} \times \left\{B_{1/2}\left(\frac{1+\kappa\xi(u+j)}{\kappa},\xi(u+j)\right)-2B_{1/2}\left(\frac{1+\kappa\xi(u+j)+\kappa}{\kappa},\xi(u+j)\right)\right\} \quad (5.6)$$

and

$$\mu_{u:n}^{(2)} = \frac{\xi\, n!}{(u-1)!(n-u)!}\sum_{j=0}^{n-u}\binom{n-u}{j}(-1)^j\, 2^{\frac{2+2\kappa\xi(u+j)}{\kappa}} \times \left\{B_{1/2}\left(\frac{2+\kappa\xi(u+j)}{\kappa},\xi(u+j)\right)-2B_{1/2}\left(\frac{2+\kappa\xi(u+j)+\kappa}{\kappa},\xi(u+j)\right)\right\} \quad (5.7)$$

The variance of uth order statistics of GTL distribution can be calculated by using the formula $V_{u:n}=\mu_{u:n}^{(2)}-(\mu_{u:n}^{(1)})^2$.

Theorem 1. For $\frac{p}{\kappa}=m, m\in Z^+$, the pth moment of uth order statistics can be expressed as,

$$\mu_{u:n}^{(p)} = \frac{n!}{(u-1)!(n-u)!}\sum_{j=0}^{n-u}\binom{n-u}{j}\frac{(-1)^j}{(u+j)}\left[\frac{(-1)^{m+1}(-m-\xi(u+j)-1)_{m+1}}{(\xi(u+j))_{m+1}} - 2^{m+2\xi(u+j)-1}(m+2)Be(\xi(u+j),m+\xi(u+j)+1) - \sum_{k=1}^{m}(-1)^k(k-1)\frac{(-m-\xi(u+j))_k}{(\xi(u+j))_k}\right]. \quad (5.8)$$

Proof. By using some relations given below, we will prove the result.

$$(i)\qquad F(\kappa,\xi;1+\xi;u) = \xi u^{-\xi} Be(\xi,1-\kappa;u),$$

$$(ii)\qquad Be(m+v,v,\frac{1}{2}) = \frac{2^{-(m+v)}}{v} F(1-v,m+v;1+m+v;\frac{1}{2}),$$

$$(iii)\qquad F(\kappa,\xi;y;u) = (1-u)^{-\kappa} F(\kappa,y-\xi;y;\frac{-u}{1-u}),$$

$$(iv)\qquad F(\kappa,\xi;y;u) = \sum_{k=0}^{\infty} \frac{(\kappa)_k(\xi)_k u^k}{(\gamma)_k k!}, (a)_k = a(a+1)\dots(a+k-1),$$

$$(v)\qquad F(a,1,l-a,-1) = 2^{l-2a-2}\frac{\Gamma(1-a)\Gamma(l-a)}{\Gamma(l-2a)} - \frac{1}{2}\sum_{k=1}^{l-2}(-1)^k \frac{(1-l+a)_k}{(1-a)_k}.$$

From (i)-(iii) in (5.5) and simplifying, we get

$$\begin{aligned} \mu_{u:n}^{(p)} &= \frac{n!}{(u-1)!(n-u)!}\sum_{j=0}^{n-u}\binom{n-u}{j}\frac{2(-1)^j}{u+j}\big[F(1-\xi(u+j),1,1+m+\xi(u+j),-1) say(I_1) \\ &- F(1-\xi(u+j),1,2+m+\xi(u+j),-1) say(I_2)\big] \end{aligned} \tag{5.9}$$

Using relations (iv) and (v), the terms I_1 and I_2 in (5.9) can be modified as

$$\begin{aligned} I_1 &= 2^{m+2\xi(u+j)-2}\frac{\Gamma(\xi(u+j))\Gamma(1+m+\xi(u+j))}{\Gamma(m+2\xi(u+j))} \\ &- \frac{1}{2}\sum_{k=1}^{m}(-1)^k\frac{(-m-\xi(u+j))_k}{(\xi(u+j))_k} \\ I_2 &= 2^{m+2\xi(u+j)-1}\frac{\Gamma(\xi(u+j))\Gamma(2+m+\xi(u+j))}{\Gamma(1+m+2\xi(u+j))} \\ &- \frac{1}{2}\sum_{k=1}^{m+1}(-1)^k\frac{(-m-\xi(u+j)-1)_k}{(\xi(u+j))_k}. \end{aligned}$$

Substituting the above expressions of I_1 and I_2 in (5.9) we get (5.8).

In this theorem, relation for single moments are derived under the condition that $0<\xi<1$ and $m=1/\xi$ is a positive integer. If $m+1\le u\le n-1$ for all positive integer $n\ge 3$.

Theorem 2. *Let $0 < \xi < 1$ so that $m = 1/\xi$ is a positive integer. If $m+1 \leq u \leq n-1$ for all positive integer $n \geq 3$, then we have the following moment relation for $p = 1, 2, \ldots$.*

$$\mu_{u:n}^{(p-1)} = C\Bigg\{\left(1-\frac{\kappa-1}{p+\kappa-1}\right)\left[\mu_{u-m+1:n-m}^{(p+\kappa-1)}-\mu_{u-m:n-m}^{(p+\kappa-1)}\right] + \left(1-\frac{2\kappa-1}{p+2\kappa-1}\right)\left[\mu_{u-m:n-m}^{(p+2\kappa-1)}-\mu_{u-m+1:n-m}^{(p+2\kappa-1)}\right]\Bigg\},$$

where $p \in N$ and

$$C = \frac{2\kappa n!(u-m)!}{pm(u-1)!(n-m)!}.$$

Proof. We have

$$\mu_{u:n}^{(p-1)} = \frac{n!}{(u-1)!(n-u)!}\int_0^1 z^{p-1}H^{u-1}(z)\left[1-H(z)\right]^{n-u}h(z)dz.$$

Integrating by parts and noting that

$$h^2(z) = 2\kappa\xi(1-z^{\kappa})z^{\kappa-1}H^{1-m}(z)h(z)$$

and

$$h^{'}(z) = 2\kappa(\xi-1)(1-z^{\kappa})z^{\kappa-1}H^{-m}(z)h(z)+2\kappa\xi\left[(\kappa-1)z^{\kappa-2}-(2\kappa-1)z^{2\kappa-2}\right]H^{1-m}(z)$$

we have

$$\mu_{u:n}^{(p-1)} = C_{u:n}\frac{z^p}{p}H^{u-1}(z)\left[1-H(z)\right]^{n-u}h(z)|_0^1 - C_{u:n}\int_0^1\frac{z^p}{p}\Big\{\Big[(u-1)H^{u-2}(z)\left[1-H(z)\right]^{n-u} - (n-u)H^{u-1}(z)\left[1-H(z)\right]^{n-u-1}\Big]h^2(z)+H^{u-1}(z)\left[1-H(z)\right]^{n-u}h^{'}(z)\Big\}dz$$

$$
\begin{aligned}
&= \frac{2\kappa(1-\xi u)C_{u:n}}{p}\int_0^1 z^{p+\kappa-1}(1-z^{\kappa})H^{u-m-1}(z)\,[1-H(z)]^{n-u}\,h(z)dz \\
&+ \frac{2\kappa\xi(n-u)C_{u:n}}{p}\int_0^1 z^{p+\kappa-1}(1-z^{\kappa})H^{u-m}(z)\,[1-H(z)]^{n-u-1}\,h(z)dz \\
&+ \frac{2\kappa\xi C_{u:n}}{p}\int_0^1 z^{p+\kappa-2}\,[(2\kappa-1)z^{\kappa}-(\kappa-1)]\,H^{u-m}(z)\,[1-H(z)]^{n-u}\,dz. \qquad (5.10)
\end{aligned}
$$

After some manipulation the first two integrals above will be

$$
\frac{2\kappa(1-\xi u)n!(u-m-1)!}{p(u-1)!(n-m)!}\left\{\mu_{u-m:n-m}^{(p+\kappa-1)}-\mu_{u-m:n-m}^{(p+2\kappa-1)}\right\}
$$

and

$$
\frac{2\kappa\xi n!(u-m)!}{p(u-1)!(n-m)!}\left\{\mu_{u-m+1:n-m}^{(p+\kappa-1)}-\mu_{u-m+1:n-m}^{(p+2\kappa-1)}\right\},
$$

respectively. The third integral in (5.10) will be

$$
\begin{aligned}
&\frac{2\kappa\xi(1-2\kappa)n!(u-m)!}{p(p+2\kappa-1)(u-1)!(n-m)!}\left\{\mu_{u-m:n-m}^{(p+2\kappa-1)}-\mu_{u-m+1:n-m}^{(p+2\kappa-1)}\right\} \\
&+\frac{2\kappa\xi(\kappa-1)n!(u-m)!}{p(p+\kappa-1)(u-1)!(n-m)!}\left\{\mu_{u-m:n-m}^{(p+\kappa-1)}-\mu_{u-m+1:n-m}^{(p+\kappa-1)}\right\}.
\end{aligned}
$$

after some manipulation and substituting these results into (5.10) completes the proof. □

The recurrence relation without any restriction is presented by the following theorem.

Theorem 3.. *Let $Z_1,\ldots,Z_n$ be a random sample of size n from the GTL distribution, and let $Z_{1:n}\le Z_{2:n}\le\cdots\le Z_{n:n}$ be the corresponding order statistics, then for $1\le u<n$ and $p\in N$, we have*

$$
\mu_{u+1:n}^{(p+\kappa)} = \left(1+\frac{\kappa}{p}\right)\mu_{u+1:n}^{(p)}+\left(\frac{p+\kappa}{2\kappa\xi u}+1\right)\mu_{u:n}^{(p+\kappa)}-\left(\frac{p+\kappa}{\kappa\xi u}+\frac{\kappa}{p}+1\right)\mu_{u:n}^{(p)}. \qquad (5.11)
$$

Proof. We have

$$
\begin{aligned}
2\mu_{u:n}^{(p)} - \mu_{u:n}^{(p+\kappa)} &= \frac{n!}{(u-1)!(n-u)!}\int_0^1 z^{p-1}H^{u-1}(z)\,[1-H(z)]^{n-u}\,(2z-z^{\kappa+1})h(z)dz \\
&= \frac{2\kappa\xi\, n!}{(u-1)!(n-u)!}\int_0^1 z^{p-1}(1-z^{\kappa})H^{u}(z)\,[1-H(z)]^{n-u}\,dz,
\end{aligned}
$$

where the last equality is obtained from (5.1) and (5.2) note that

$$
(2z-z^{\kappa+1})h(z) = 2\kappa\xi(1-z^{\kappa})H(z). \tag{5.12}
$$

Integrating by parts by treating $z^{p-1}(1-z^{\kappa})$ for integration and $H^{u}(z)\,[1-H(z)]^{n-u}$ for differentiation, we have

$$
\begin{aligned}
2\mu_{u:n}^{(p)} - \mu_{u:n}^{(p+\kappa)} &= 2\frac{\kappa\xi\, n!}{(u-1)!(n-u)!}\left\{\left(\frac{1}{p}-\frac{z^{\kappa}}{p+\kappa}\right)z^{p}H^{u}(z)\,[1-H(z)]^{n-u}|_0^1\right. \\
&- u\int_0^1\left(\frac{1}{p}-\frac{z^{\kappa}}{p+\kappa}\right)z^{p}H^{u-1}(z)\,[1-H(z)]^{n-u}\,h(z)dz \\
&+ \left.(n-u)\int_0^1\left(\frac{1}{p}-\frac{z^{\kappa}}{p+\kappa}\right)z^{p}H^{u}(z)\,[1-H(z)]^{n-u-1}\,h(z)dz\right\}.
\end{aligned} \tag{5.13}
$$

If $u<n$, right hand side of (5.13) vanishes. Therefore, we have

$$
\begin{aligned}
\frac{1}{2\kappa\xi u}\left(2\mu_{u:n}^{(p)} - \mu_{u:n}^{(p+\kappa)}\right) &= \int_0^1\left(\frac{1}{p}-\frac{z^{\kappa}}{p+\kappa}\right)z^{p}h_{Z_{u+1:n}}(z;\kappa,\xi)dz \\
&- \int_0^1\left(\frac{1}{p}-\frac{z^{\kappa}}{p+\kappa}\right)z^{p}h_{Z_{u:n}}(z;\kappa,\xi)dz \\
&= \frac{\mu_{u+1:n}^{(p)}-\mu_{u:n}^{(p)}}{p} - \frac{\mu_{u+1:n}^{(p+\kappa)}-\mu_{u:n}^{(p+\kappa)}}{p+\kappa}
\end{aligned}
$$

we can get result by above equation.

Remark 5.1. Under the assumptions of Theorem 1 and for $p,n\in N$, we have

$$
\mu_{n:n}^{(p+\kappa)} = \left(\frac{1}{p+\kappa}+\frac{1}{2n\kappa\xi}\right)^{-1}\left\{\left(\frac{1}{p}+\frac{1}{n\kappa\xi}\right)\mu_{n:n}^{(p)} - \frac{\kappa}{p(p+\kappa)}\right\}. \tag{5.14}
$$

The proof is similar to that of Theorem 1.

Next, by using the techniques which is introduced by Thomas and Samuel (2008), if we assume that ξ is a positive integer value to obtain the several recurrence relations for single moments of lower sample sizes for GTL distribution.

Theorem 4. Suppose that $p, \xi \in N$, then we have

(i) If $2 \leq u \leq n$, then

$$\mu_{u:n}^{(p)} = \frac{n}{u-1} \sum_{r=0}^{\xi} \binom{\xi}{r} (-1)^r 2^{\xi-r} \mu_{u-1:n-1}^{(p+\kappa(\xi+r))}. \tag{5.15}$$

(ii) If $1 \leq u < n$, then

$$\mu_{u:n}^{(p)} = \frac{n}{n-u} \left\{ \mu_{u:n-1}^{(p)} - \sum_{r=0}^{\xi} \binom{\xi}{r} (-1)^r 2^{\xi-r} \mu_{u:n-1}^{(p+\kappa(\xi+r))} \right\}. \tag{5.16}$$

Proof. Proof of (i) is similar to (ii), so we are proving (ii) only. Since $\xi \in Z^+$, the cdf of the GTL distribution can be expressed as

$$H(z) = \sum_{r=0}^{\xi} \binom{\xi}{r} (-1)^r 2^{\xi-r} z^{\kappa(\xi+r)}, \quad 0 < z < 1. \tag{5.17}$$

Therefore if $1 \leq u \leq n-1$, then we may write

$$\mu_{u:n}^{(p)} = \frac{n!}{(u-1)!(n-u)!} \int_0^1 z^p \left\{ 1 - \sum_{r=0}^{\xi} \binom{\xi}{r} (-1)^r 2^{\xi-r} z^{\kappa(\xi+r)} \right\} H^{u-1}(z) \left[1 - H(z)\right]^{n-u-1} h(z) dz,$$

By simplifying above expression, we get the result. □

5.3 Application to the Lifetime of Coherent Systems

Here, we will derive the expression for coherent systems' expected lifetimes by using relations of previous sections. Coherent systems are beneficial in reliability. Samaniego (1985)

TABLE 5.1: First moments, second moments and variances of the uth order statistics from GTL distribution.

κ	ξ	u	n	$E(Z)$	$E(Z^2)$	$V(Z)$	κ	ξ	u	n	$E(Z)$	$E(Z^2)$	$V(Z)$
0.5	1	1	1	0.166667	0.066667	0.038889	0.5	2	1	1	0.266667	0.119048	0.047936
			2	0.066667	0.014286	0.009841				2	0.146032	0.040115	0.018790
			3	0.035714	0.004762	0.003486				3	0.098968	0.019297	0.009503
			4	0.022222	0.002020	0.001526				4	0.074313	0.011134	0.005612
			5	0.015152	0.000999	0.000769				5	0.059259	0.007173	0.003661
			6	0.010989	0.000550	0.000429				6	0.049158	0.004975	0.002559
			7	0.008333	0.000327	0.000257				7	0.041931	0.003638	0.001880
			8	0.006536	0.000206	0.000164				8	0.036516	0.002768	0.001435
			9	0.005263	0.000137	0.000109				9	0.032312	0.002173	0.001128
			10	0.004329	0.000094	0.000075				10	0.028957	0.001748	0.000909
		2	2	0.266667	0.119048	0.047936			2	2	0.387302	0.197980	0.047977
			3	0.128571	0.033333	0.016803				3	0.240160	0.081752	0.024075
			4	0.076191	0.012987	0.007182				4	0.172933	0.043787	0.013882
			5	0.050505	0.006105	0.003554				5	0.134527	0.026979	0.008881
			6	0.035964	0.003247	0.001953				6	0.109765	0.018161	0.006113
			7	0.026923	0.001885	0.001161				7	0.092518	0.012996	0.004437
			8	0.020915	0.001170	0.000732				8	0.079841	0.009728	0.003353
			9	0.016718	0.000764	0.000484				9	0.070146	0.007535	0.002615
			10	0.013671	0.000520	0.000333				10	0.062500	0.005998	0.002091
		3	3	0.335714	0.161905	0.049201			3	3	0.460873	0.256094	0.043690
			4	0.180952	0.053680	0.020936				4	0.307387	0.119716	0.025229
			5	0.114719	0.023310	0.010150				5	0.230541	0.069000	0.015851
			6	0.079587	0.011822	0.005487				6	0.184053	0.044615	0.010740
			7	0.058566	0.006650	0.003220				7	0.152882	0.031072	0.007699
			8	0.044947	0.004033	0.002012				8	0.130547	0.022803	0.005760
			9	0.035604	0.002589	0.001322				9	0.113775	0.017401	0.004456
			10	0.028909	0.001739	0.000904				10	0.100731	0.013686	0.003540
		4	4	0.387302	0.197980	0.047977			4	4	0.512034	0.301553	0.039374
			5	0.225108	0.073926	0.023252				5	0.358617	0.153526	0.024920
			6	0.149850	0.034799	0.012343				6	0.277030	0.093385	0.016639
			7	0.107615	0.018717	0.007136				7	0.225614	0.062673	0.011771
			8	0.081265	0.011013	0.004409				8	0.190107	0.044854	0.008713
			9	0.063634	0.006919	0.002870				9	0.164091	0.033608	0.006682
			10	0.051224	0.004573	0.001949				10	0.144212	0.026068	0.005270
		5	5	0.427850	0.228993	0.045938			5	5	0.550389	0.338560	0.035632
			6	0.262737	0.093490	0.024459				6	0.399411	0.183597	0.024068
			7	0.181527	0.046860	0.013908				7	0.315592	0.116419	0.016820
			8	0.133964	0.026420	0.008474				8	0.261122	0.080492	0.012307
			9	0.103304	0.016129	0.005458				9	0.222626	0.058911	0.009349
			10	0.082249	0.010440	0.003675				10	0.193910	0.044918	0.007317
		6	6	0.460873	0.256094	0.043690			6	6	0.580584	0.369553	0.032474
			7	0.295221	0.112142	0.024986				7	0.432938	0.210469	0.023033
			8	0.210064	0.059124	0.014997				8	0.348274	0.137975	0.016680
			9	0.158492	0.034653	0.009533				9	0.291918	0.097756	0.012540
			10	0.124359	0.021819	0.006354				10	0.251343	0.072904	0.009731
		7	7	0.488481	0.280086	0.041472			7	7	0.605192	0.396067	0.029809
			8	0.323607	0.129815	0.025093				8	0.461159	0.234633	0.021965
			9	0.235851	0.071359	0.015734				9	0.376452	0.158084	0.016368
			10	0.181248	0.043209	0.010359				10	0.318968	0.114324	0.012583
		8	8	0.512034	0.301553	0.039374			8	8	0.625768	0.419129	0.027543
			9	0.348680	0.146516	0.024938				9	0.485361	0.256504	0.020929
			10	0.259252	0.083423	0.016212				10	0.401088	0.176839	0.015967
		9	9	0.532454	0.320933	0.037426			9	9	0.643319	0.439457	0.025597
			10	0.371038	0.162289	0.024620				10	0.506430	0.276421	0.019950
		10	10	0.550389	0.338560	0.035632			10	10	0.658529	0.457572	0.023911

derived system reliability and coherent systems' expected lifetimes using the concept of signature vector. Let a system having n components with lifetimes represented by independently and identically distributed $Z_1, Z_2, \ldots, Z_n$, random variables. Moreover, let lifetime of whole system be Y and $s_u = Pr(Y = Z_{u:n})$ for $u = 1, \ldots, n$. Then for $p \in N$, we have Samaniego (1985)

$$E(Y^p) = \sum_{u=1}^{n} s_u \mu_{u:u}^{(p)}, \tag{5.18}$$

and the vector $s = (s_1, \ldots, s_n)$ is called the signature vector.

Navarro et al. (2007) gave concept of exchangeable components for a coherent system and can be written as

$$E(Y^p) = \sum_{u=1}^{n} \lambda_u \mu_{1:u}^{(p)} = \sum_{u=1}^{n} \theta_u \mu_{u:u}^{(p)}, \tag{5.19}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is minimal and $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$is maximal signature, respectively.

Suppose, independently and identically distributed $Z_1, Z_2, \ldots, Z_n$ are lifetimes of components of a coherent system and follows generalized Topp-Leone distribution then using relations of previous sections, we find expected system lifetime. For instance, from (5.14) and (5.19) and for $p \in \mathbb{N}$, we have

$$E(Y^{p+\kappa}) = \sum_{u=1}^{n} \theta_u \left(\frac{1}{p+\kappa} + \frac{1}{2u\kappa\xi} \right)^{-1} \left\{ \left(\frac{1}{p} + \frac{1}{u\kappa\xi} \right) \mu_{u:u}^{(p)} - \frac{\kappa}{p(p+\kappa)} \right\},$$

or from (5.16) and (5.18), for $\xi, p \in \mathbb{N}$, we can state

$$E(Y^p) = \sum_{u=1}^{n} \frac{ns_u}{n-u} \left\{ \mu_{u:n-1}^{(p)} - \sum_{r=0}^{\xi} \binom{\xi}{r} (-1)^r 2^{\xi-r} \mu_{u:n-1}^{(p+\kappa(\xi+r))} \right\}.$$

5.4 Relations for Product Moments of Order Statistics

Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from GTL distribution, then under the condition $\xi \in \mathbb{N}$, we can express product moment of uth and vth order statistic of GTL distribution as follows:

Theorem 5. *For $1 \leq u < v \leq n$, $n \in N$ and $\xi \in N$ we have,*

$$\begin{aligned}
\mu_{u,v:n} &= C_{u,v:n} \sum_{r=0}^{n-v} \sum_{s=0}^{v-u-1} \binom{n-v}{r} \binom{v-u-1}{s} (-1)^{r+s} \frac{1}{(s+u)\Delta_{r,s}} \mu_{s+u:s+u} \mu_{\Delta_{r,s}:\Delta_{r,s}} \\
&+ C_{u,v:n} \sum_{r=0}^{n-v} \sum_{s=0}^{v-u-1} \sum_{t=0}^{\xi\Delta_{r,s}-1} \binom{n-v}{r} \binom{v-u-1}{s} \binom{\xi\Delta_{r,s}-1}{t} (-1)^{r+s+t} \frac{2^{\xi\Delta_{r,s}-t}\kappa\xi}{(s+u)} \\
&\times \left\{ \frac{1}{1+\kappa+\kappa t+\kappa\xi\Delta_{r,s}} \mu_{s+u:s+u}^{2+\kappa+\kappa t+\kappa\xi\Delta_{r,s}} - \frac{1}{1+\kappa t+\kappa\xi\Delta_{r,s}} \mu_{s+u:s+u}^{2+\kappa t+\kappa\xi\Delta_{r,s}} \right\}, \quad (5.20)
\end{aligned}$$

where $\Delta_{r,s} = v-u-s+r$.

Proof. From joint pdf of uth and vth order statistics, we get

$$\begin{aligned}
\mu_{u,v:n} &= C_{u,v:n} \int_0^1 \int_z^1 zyH^{u-1}(z) \left[H(y)-H(z)\right]^{v-1-u} \left[1-H(y)\right]^{n-v} h(z)h(y)dzdy \\
&= C_{u,v:n} \sum_{r=0}^{n-v} \sum_{s=0}^{v-u-1} \binom{n-v}{r} \binom{v-u-1}{s} (-1)^{r+s} \int_0^1 zH^{s+u-1}(z)h(z)I(z)dz. (5.21)
\end{aligned}$$

where

$$\begin{aligned}
I(z) &= \int_z^1 y[H(y)]^{v-u-s+r-1} h(y)dy \\
&= \frac{1}{v-u-s+r} \mu_{v-u-s+r:v-u-s+r} - \int_0^z y[H(y)]^{v-u-s+r-1} h(y)dy \\
&= \frac{1}{\Delta_{r,s}} \mu_{\Delta_{r,s}:\Delta_{r,s}} + \kappa\xi 2^{\xi\Delta_{r,s}} \left[\int_0^1 z^{\kappa\xi\Delta_{r,s}+\kappa+1} u^{\kappa\xi\Delta_{r,s}+\kappa} \left[1-\frac{1}{2}(zu)^{\kappa}\right]^{\xi\Delta_{r,s}-1} dz \right. \\
&- \left. \int_0^1 z^{\kappa\xi\Delta_{r,s}+1} u^{\kappa\xi\Delta_{r,s}} \left[1-\frac{1}{2}(zu)^{\kappa}\right]^{\xi\Delta_{r,s}-1} dz \right] \quad (5.22)
\end{aligned}$$

Now substituting the values of $I(z)$ with some manipulations we get the result. $\square$

Next, we elucidate the (p,q)th product moment for unconditional ξ which comprise the simple product moments as well, the result.

Theorem 6. For $1 \le u < v \le n$ and $n,p,q \in \mathbb{N}$, we have

$$\begin{aligned}
\mu_{u,v:n}^{(p,q)} &= \frac{\xi^2 n!}{(u-1)!(v-u-1)!(n-v)!}\sum_{r=0}^{n-v}\sum_{s=0}^{v-u-1}\sum_{l=0}^{\infty}\binom{n-v}{r}\binom{v-u-1}{s}\binom{\xi(s+u)-1}{l} \\
&\times (-1)^{r+s+l}2^{\frac{q+p+2\kappa\xi(v+r)}{\kappa}}\left\{\frac{1}{\eta_{u,p,\kappa,\xi}(s,l)}B_{1/2}(\eta^*_{v,p,q,\kappa,\xi}(r,l),\xi\Delta_{r,s})\right. \\
&- \frac{2(2\eta_{u,p,\kappa,\xi}(s,l)+1)}{\eta_{u,p,\kappa,\xi}(s,l)(\eta_{u,p,\kappa,\xi}(s,l)+1)}B_{1/2}(\eta^*_{v,p,q,\kappa,\xi}(r,l)+1,\xi\Delta_{r,s}) \\
&+ \left.\frac{4}{\eta_{u,p,\kappa,\xi}(s,l)+1}B_{1/2}(\eta^*_{v,p,q,\kappa,\xi}(r,l)+2,\xi\Delta_{r,s})\right\}. \qquad (5.23)
\end{aligned}$$

where, $\eta_{u,p,\kappa,\xi}(s,l) = \frac{p}{\kappa} + \xi(s+u) + l$ and $\eta^*_{v,p,q,\kappa,\xi}(r,l) = \frac{q+p}{\kappa} + \xi(v+r) + l$. If $\xi \in \mathbb{N}$, then the third summation stops at $l = \xi(s+u) - 1$.

Proof. Using (1.9), the (p,q)-th product moment of (u,v)th order statistics can be written as

$$\begin{aligned}
\mu_{u,v:n}^{(p,q)} &= C_{u,v:n}\int_0^1\int_0^y z^p y^q H^{u-1}(z)\,[H(y)-H(z)]^{v-1-u}\,[1-H(y)]^{n-v}\,h(z)h(y)dzdy \\
&= C_{u,v:n}\sum_{r=0}^{n-v}\sum_{s=0}^{v-u-1}\binom{n-v}{r}\binom{v-u-1}{s}(-1)^{r+s}\int_0^1 y^q h(y)[H(y)]^{r,s-1}I(y)dy \qquad (5.24)
\end{aligned}$$

where

$$\begin{aligned}
I(y) &= \int_0^y z^p [H(z)]^{s+u-1}h(z)dz \\
&= 2\kappa\xi\int_0^y z^{p+\kappa\xi(s+u)-1}(2-z^\kappa)^{\xi(s+u)-1}(1-z^\kappa)dz \\
&= \xi 2^{\frac{p}{\kappa}+2\xi(s+u)}\int_0^{y^\kappa/2} t^{\frac{p}{\kappa}+\xi(s+u)-1}(1-t)^{\xi(s+u)-1}(1-2t)dt \\
&= \xi 2^{\frac{p}{\kappa}+2\xi(s+u)}\sum_{l=0}^{\infty}\binom{\xi(s+u)-1}{l}(-1)^l\int_0^{y^\kappa/2} t^{\eta_{u,p,\kappa,\xi}(s,l)-1}(1-2t)dt \\
&= \xi\sum_{l=0}^{\infty}2^{\xi(s+u)-l}\binom{\xi(s+u)-1}{l}(-1)^l\left(\frac{1}{\eta_{u,p,\kappa,\xi}(s,l)} - \frac{y^\kappa}{\eta_{u,p,\kappa,\xi}(s,l)+1}\right)y^{\kappa\eta_{u,p,\kappa,\xi}(s,l)}
\end{aligned}$$

TABLE 5.2: Covariances of order statistics.

κ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$	κ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$
0.5	1	2	1	2	0.027778	0.010001	0.5	2	2	1	2	0.071111	0.014553
				3	0.008730	0.004138					3	0.032354	0.008586
				4	0.003608	0.001914					4	0.018197	0.005346
				5	0.001758	0.000992					5	0.011550	0.003578
				6	0.000957	0.000562					6	0.007933	0.002537
				7	0.000566	0.000341					7	0.005761	0.001882
				8	0.000356	0.000219					8	0.004361	0.001445
				9	0.000235	0.000147					9	0.003408	0.001141
				10	0.000161	0.000102					10	0.002732	0.000922
		3	1	3	0.015873	0.003883			3	1	3	0.052117	0.006505
				4	0.006118	0.002097					4	0.027582	0.004739
				5	0.002886	0.001148					5	0.017013	0.003351
				6	0.001543	0.000668					6	0.011489	0.002441
				7	0.000901	0.000412					7	0.008250	0.001839
				8	0.000561	0.000267					8	0.006194	0.001427
				9	0.000368	0.000180					9	0.004811	0.001135
				10	0.000251	0.000126					10	0.003838	0.000921
			2	3	0.058730	0.015567				2	3	0.128862	0.018179
				4	0.021587	0.007801					4	0.065441	0.012284
				5	0.009879	0.004085					5	0.039322	0.008307
				6	0.005173	0.002310					6	0.026078	0.005876
				7	0.002973	0.001396					7	0.018478	0.004333
				8	0.001831	0.000891					8	0.013731	0.003308
				9	0.001190	0.000594					9	0.010579	0.002598
				10	0.000807	0.000411					10	0.008384	0.002088
		4	1	4	0.010476	0.001870			4	1	4	0.041590	0.003539
				5	0.004607	0.001196					5	0.024186	0.002935
				6	0.002382	0.000735					6	0.015880	0.002262
				7	0.001363	0.000466					7	0.011213	0.001752
				8	0.000838	0.000307					8	0.008324	0.001382
				9	0.000545	0.000210					9	0.006413	0.001111
				10	0.000369	0.000148					10	0.005084	0.000908
			2	4	0.036421	0.006913				2	4	0.097705	0.009158
				5	0.015607	0.004237					5	0.055513	0.007269
				6	0.007923	0.002534					6	0.035848	0.005440
				7	0.004472	0.001575					7	0.025001	0.004127
				8	0.002722	0.001022					8	0.018381	0.003203
				9	0.001753	0.000689					9	0.014053	0.002542
				10	0.001181	0.000481					10	0.011072	0.002059
			3	4	0.088456	0.018373				3	4	0.176151	0.018759
				5	0.036286	0.010462					5	0.096526	0.013850
				6	0.017917	0.005990					6	0.060923	0.009934
				7	0.009922	0.003619					7	0.041819	0.007327
				8	0.005954	0.002301					8	0.030392	0.005574
				9	0.003795	0.001529					9	0.023028	0.004358
				10	0.002535	0.001054					10	0.018015	0.003488
		5	1	5	0.007511	0.001028			5	1	5	0.034785	0.002169
				6	0.003627	0.000740					6	0.021599	0.001964
				7	0.002009	0.000496					7	0.014844	0.001611
				8	0.001212	0.000336					8	0.010843	0.001308
				9	0.000777	0.000233					9	0.008262	0.001069
				10	0.000522	0.000166					10	0.006499	0.000884

TABLE 5.2: Continued.

κ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$	κ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$
			2	5	0.025241	0.003633				2	5	0.079409	0.005367
				6	0.011993	0.002544					6	0.048563	0.004721
				7	0.006559	0.001672					7	0.032991	0.003793
				8	0.003918	0.001116					8	0.023878	0.003030
				9	0.002492	0.000765					9	0.018062	0.002446
				10	0.001664	0.000540					10	0.014122	0.002003
			3	5	0.058009	0.008926				3	5	0.137098	0.010211
				6	0.026904	0.005994					6	0.082128	0.008615
				7	0.014463	0.003832					7	0.054979	0.006731
				8	0.008528	0.002507					8	0.039359	0.005271
				9	0.005372	0.001694					9	0.029521	0.004192
				10	0.003559	0.001181					10	0.022925	0.003392
			4	5	0.115995	0.019683				4	5	0.215710	0.018332
				6	0.051660	0.012288					6	0.125061	0.014412
				7	0.027063	0.007528					7	0.082007	0.010805
				8	0.015676	0.004789					8	0.057876	0.008235
				9	0.009747	0.003173					9	0.042955	0.006424
				10	0.006393	0.002180					10	0.033089	0.005124
		6	1	6	0.005684	0.000620			6	1	6	0.029980	0.001440
				7	0.002947	0.000487					7	0.019544	0.001390
				8	0.001722	0.000349					8	0.013913	0.001196
				9	0.001083	0.000249					9	0.010439	0.001006
				10	0.000718	0.000180					10	0.008125	0.000847
			2	6	0.018701	0.002126				2	6	0.067187	0.003459
				7	0.009586	0.001638					7	0.043326	0.003271
				8	0.005550	0.001156					8	0.030576	0.002769
				9	0.003465	0.000816					9	0.022779	0.002302
				10	0.002285	0.000585					10	0.017627	0.001918
			3	6	0.041676	0.004997				3	6	0.113165	0.006307
				7	0.021036	0.003746					7	0.071990	0.005802
				8	0.012036	0.002594					8	0.050282	0.004816
				9	0.007447	0.001804					9	0.037158	0.003945
				10	0.004874	0.001279					10	0.028566	0.003248
			4	6	0.079265	0.010203				4	6	0.171378	0.010538
				7	0.039110	0.007340					7	0.106985	0.009308
				8	0.022016	0.004945					8	0.073730	0.007521
				9	0.013458	0.003372					9	0.053944	0.006043
				10	0.008727	0.002357					10	0.041152	0.004905
			5	6	0.141262	0.020173				5	6	0.249456	0.017565
				7	0.067095	0.013505					7	0.151106	0.014474
				8	0.036865	0.008724					8	0.102175	0.011233
				9	0.022160	0.005787					9	0.073779	0.008790
				10	0.014195	0.003966					10	0.055740	0.007002

TABLE 5.2: Continued.

κ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$	κ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$
		7	1	7	0.004470	0.000399			7	1	7	0.026389	0.001013
				8	0.002451	0.000335					8	0.017866	0.001026
				9	0.001495	0.000253					9	0.013080	0.000916
				10	0.000973	0.000188					10	0.010030	0.000794
			2	7	0.014494	0.001342				2	7	0.058373	0.002382
				8	0.007880	0.001111					8	0.039196	0.002376
				9	0.004773	0.000830					9	0.028502	0.002095
				10	0.003090	0.000612					10	0.021734	0.001798
			3	7	0.031674	0.003065				3	7	0.096746	0.004223
				8	0.017036	0.002490					8	0.064334	0.004131
				9	0.010230	0.001833					9	0.046420	0.003589
				10	0.006577	0.001337					10	0.035174	0.003044
			4	7	0.058559	0.005991				4	7	0.143309	0.006769
				8	0.031038	0.004740					8	0.094118	0.006449
				9	0.018431	0.003422					9	0.067269	0.005497
				10	0.011746	0.002462					10	0.050596	0.004597
			5	7	0.099657	0.010985				5	7	0.201510	0.010516
				8	0.051693	0.008341					8	0.130045	0.009626
				9	0.030227	0.005863					9	0.091801	0.007992
				10	0.019045	0.004137					10	0.068410	0.006559
			6	7	0.164416	0.020206				6	7	0.278715	0.016705
				8	0.082268	0.014289					8	0.174891	0.014281
				9	0.047015	0.009634					9	0.121288	0.011395
				10	0.029155	0.006616					10	0.089283	0.009113
		8	1	8	0.003617	0.000271			8	1	8	0.023594	0.000744
				9	0.002075	0.000240					9	0.016466	0.000783
				10	0.001311	0.000189					10	0.012335	0.000720
			2	8	0.011606	0.000896				2	8	0.051684	0.001722
				9	0.006615	0.000786					9	0.035837	0.001790
				10	0.004158	0.000614					10	0.026698	0.001630
			3	8	0.025021	0.002006				3	8	0.084684	0.002992
				9	0.014148	0.001734					9	0.058289	0.003067
				10	0.008834	0.001340					10	0.043164	0.002762
			4	8	0.045423	0.003813				4	8	0.123631	0.004668
				9	0.025422	0.003234					9	0.084338	0.004694
				10	0.015744	0.002464					10	0.062007	0.004166
			5	8	0.075291	0.006697				5	8	0.170366	0.006964
				9	0.041552	0.005532					9	0.114878	0.006824
				10	0.025459	0.004136					10	0.083720	0.005945
			6	8	0.118997	0.011436				6	8	0.228260	0.010322
				9	0.064336	0.009073					9	0.151410	0.009724
				10	0.038844	0.006604					10	0.109066	0.008256

TABLE 5.2: Continued.

κ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$	κ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$
			7	8	0.185677	0.019979				7	8	0.304429	0.015850
				9	0.097007	0.014771					9	0.196672	0.013956
				10	0.057307	0.010318					10	0.139328	0.011394
		9	1	9	0.002994	0.000191			9	1	9	0.021352	0.000565
				10	0.001783	0.000177					10	0.015278	0.000613
			2	9	0.009527	0.000625				2	9	0.046418	0.001292
				10	0.005646	0.000574					10	0.033042	0.001391
			3	9	0.020337	0.001379				3	9	0.075405	0.002211
				10	0.011979	0.001252					10	0.053358	0.002345
			4	9	0.036452	0.002570				4	9	0.108949	0.003386
				10	0.021307	0.002301					10	0.076590	0.003557
			5	9	0.059395	0.004390				5	9	0.148136	0.004917
				10	0.034376	0.003859					10	0.103251	0.005050
			6	9	0.091576	0.007186				6	9	0.194801	0.007005
				10	0.052295	0.006154					10	0.134318	0.007030
			7	9	0.137247	0.011667				7	9	0.252222	0.010043
				10	0.076847	0.009597					10	0.171222	0.009687
			8	9	0.205262	0.019606				8	9	0.327281	0.015038
				10	0.111231	0.015038					10	0.216691	0.013568
		10	1	10	0.002522	0.000139			10	1	10	0.019510	0.000441
			2	10	0.007976	0.000452				2	10	0.042156	0.000998
			3	10	0.016897	0.000985				3	10	0.068024	0.001690
			4	10	0.030003	0.001809				4	10	0.097514	0.002546
			5	10	0.048301	0.003031				5	10	0.131332	0.003637
			6	10	0.073274	0.004828				6	10	0.170556	0.005040
			7	10	0.107274	0.007517				7	10	0.217004	0.006954
			8	10	0.154439	0.011750				8	10	0.273856	0.009728
			9	10	0.223369	0.019154				9	10	0.347783	0.014284

If $\xi \in \mathbb{N}$, summation stops at $\xi(s+u)-1$. Substituting the value of $I(y)$ in (5.24) and after simplification, we get the result given in (5.23). □

Next, we present a recurrence relation for the product moments of order statistics for GTL distribution.

Theorem 7. For the GTL distribution $1 \le u \le v-2, v \le n$ and $p,q \in \mathbb{N}$, we have

$$\mu_{u+1,v:n}^{(p+\kappa,q)} = \left(1+\frac{\kappa}{p}\right)\mu_{u+1,v:n}^{(p,q)} + \left(\frac{p+\kappa}{2\kappa\xi u}+1\right)\mu_{u,v:n}^{(p+\kappa,q)} - \left(\frac{p+\kappa}{\kappa\xi u}+\frac{\kappa}{p}+1\right)\mu_{u,v:n}^{(p,q)} \quad (5.25)$$

Proof. Form (1.9) and (5.12), we get

$$2\mu_{u,v:n}^{(p,q)} - \mu_{u,v:n}^{(p+\kappa,q)} = \frac{n!}{(u-1)!(v-u-1)!(n-v)!} \int_0^1 y^q h(y)[1-H(y)]^{n-v} G(y)dy$$

where

$$\begin{aligned} G(y) &= \int_0^y z^{p-1}[H(z)]^{u-1}[H(y)-H(z)]^{v-u-1} z(2-z^\kappa)h(z)dz \\ &= 2\kappa\xi \int_0^y z^{p-1}(1-z^\kappa)[H(z)]^u[H(y)-H(z)]^{v-u-1}dz \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} G(y) &= 2\kappa\xi \left\{ \left(\frac{1}{p} - \frac{z^\kappa}{p+\kappa}\right) z^p H^u(z)\,[H(y)-H(z)]^{v-u-1}|_0^y \right. \\ &- u \int_0^y \left(\frac{1}{p} - \frac{z^\kappa}{p+\kappa}\right) z^p H^{u-1}(z)\,[H(y)-H(z)]^{v-u-1} h(z)dz \\ &+ \left. (v-u-1) \int_0^y \left(\frac{1}{p} - \frac{z^\kappa}{p+\kappa}\right) z^p H^u(z)\,[H(y)-H(z)]^{v-u-2} h(z)dz \right\}. \end{aligned} \tag{5.26}$$

If $v > u-1$, then the right hand side of (5.26) is vanishes and as a effect we get

$$\begin{aligned} \frac{1}{2\kappa\xi u}\left(2\mu_{u,v:n}^{(p,q)} - \mu_{u,v:n}^{(p+\kappa,q)}\right) &= \int_0^1 \int_0^y \left(\frac{1}{p} - \frac{z^\kappa}{p+\kappa}\right) z^p y^q h_{Z_{u+1,v:n}}(z,y;\kappa,\xi)dzdy \\ &- \int_0^1 \int_0^y \left(\frac{1}{p} - \frac{z^\kappa}{p+\kappa}\right) z^p y^q h_{Z_{u,v:n}}(z,y;\kappa,\xi)dzdy \\ &= \frac{\mu_{u+1,v:n}^{(p,q)} - \mu_{u,v:n}^{(p,q)}}{p} - \frac{\mu_{u+1,v:n}^{(p+\kappa,q)} - \mu_{u,v:n}^{(p+\kappa,q)}}{p+\kappa} \end{aligned}$$

and hence the result follows. □

Remark 5.2. For $1 \le u \le n-1$ and $p,q \in \mathbb{N}$, we have

$$\mu_{u,u+1:n}^{(p+\kappa,q)} = -\left(\frac{p+\kappa}{2\kappa\xi u}+1\right)^{-1} \left\{ \left(1+\frac{\kappa}{p}\right)\mu_{u+1:n}^{(p+q)} - \mu_{u+1:n}^{(p+\kappa+q)} - \left(\frac{p+\kappa}{\kappa\xi u} + \frac{\kappa}{p} + 1\right)\mu_{u,u+1:n}^{(p,q)} \right\}.$$

Next, by using the techniques which is introduced by Thomas and Samuel (2008), if we assume that ξ is a positive integer value to prove the several recurrence relations for product moments of lower sample sizes for GTL distribution.

Theorem 8. Let $\xi \in \mathbb{N}$, then we have

(i) If $2 \leq u < v \leq n$, then

$$\mu_{u,v:n}^{(p,q)} = \frac{n}{u-1} \sum_{r=0}^{\xi} \binom{\xi}{r} (-1)^r 2^{\xi-r} \mu_{u-1,v-1:n-1}^{(p+\kappa(\xi+r),q)} \tag{5.27}$$

(ii) If $1 \leq u < v-1 \leq n-1$, then

$$\mu_{u,v:n}^{(p,q)} = \frac{n}{v-u-1} \sum_{r=0}^{\xi} \binom{\xi}{r} (-1)^r 2^{\xi-r} \left\{ \mu_{u,v-1:n-1}^{(p,q+\kappa(\xi+r))} - \mu_{u,v-1:n-1}^{(p+\kappa(\xi+r),q)} \right\} \tag{5.28}$$

(iii) If $1 \leq u < v \leq n-1$, then

$$\mu_{u,v:n}^{(p,q)} = \frac{n}{n-v} \left\{ \mu_{u,v:n-1}^{(p,q)} - \sum_{r=0}^{\xi} \binom{\xi}{r} (-1)^r 2^{\xi-r} \mu_{u,v:n-1}^{(p,q+\kappa(\xi+r))} \right\} \tag{5.29}$$

Proof. Proof of (i) and (iii) are similar to (ii), so we proved only (ii). If $1 \leq u < v-1 \leq n-1$, then using the equation (5.11) and expanding the term $H(y) - H(z)$, we may write

$$\begin{aligned}
\mu_{u,v:n}^{(p,q)} &= C_{u,v:n} \int_0^1 \int_0^y z^p y^q H^{u-1}(z) \left[H(y) - H(z)\right]^{v-1-u} \left[1 - H(y)\right]^{n-v} h(z)h(y)dzdy \\
&= C_{u,v:n} \sum_{r=0}^{\xi} \binom{\xi}{r} (-1)^r 2^{\xi-r} \int_0^1 \int_0^y z^p y^{q+\kappa(\xi+r)} \\
&\times H^{u-1}(z) \left[H(y) - H(z)\right]^{v-u-2} \left[1 - H(y)\right]^{n-v} h(z)h(y)dzdy \\
&- C_{u,v:n} \sum_{r=0}^{\xi} \binom{\xi}{r} (-1)^r 2^{\xi-r} \int_0^1 \int_0^y z^{p+\kappa(\xi+r)} y^q \\
&\times H^{u-1}(z) \left[H(y) - H(z)\right]^{v-u-2} \left[1 - H(y)\right]^{n-v} h(z)h(y)dzdy
\end{aligned}$$

After some manipulations of above expression, we get the result. □

5.5 BLUEs of the Location and Scale Parameters

Let $Z_1 \leq Z_2 \leq \cdots \leq Z_n$ be a random sample of size n from GTL distribution with the pdf of scale-parameter GTL distribution is

$$h(z;\kappa,\xi,\varphi) = \frac{2\kappa\xi}{\varphi}\left(\frac{z}{\varphi}\right)^{\kappa\xi-1}\left[1-\left(\frac{z}{\varphi}\right)^{\kappa}\right]\left[2-\left(\frac{z}{\varphi}\right)^{\kappa}\right]^{\xi-1}, \qquad (5.30)$$

where, $0 < z < 1$, κ, $\xi, \varphi > 0$. The pdf of the location-scale parameter is

$$h(z;\kappa,\xi,\delta,\varphi) = \frac{2\kappa\xi}{\varphi}\left(\frac{z-\delta}{\varphi}\right)^{\kappa\xi-1}\left[1-\left(\frac{z-\delta}{\varphi}\right)^{\kappa}\right]\left[2-\left(\frac{z-\delta}{\varphi}\right)^{\kappa}\right]^{\xi-1}, \quad (5.31)$$

where $0 < z < 1$, κ, $\xi, \delta, \varphi > 0$.

The expression for the BLUEs of location and scale parameter are given in (1.18) and also variances and covariance for these parameters are given in eqn (1.21), (1.22) and (1.23).

Tables 5.3 and 5.4 display the coefficient of the BLUEs for type-II right censored sample of various values of $n = 7,\ 10$ and censoring cases $c = 0(1)([n/2]-1)$. Also, Table 5.5 shows variances and covariances of the BLUEs.

5.6 Real Data Application

To see the practical utility of the model a real data set of 30 observations from Quesenberry and Hales (1980) is taken. The data are:

```
0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148,
0.169, 0.188, 0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395,
0.432, 0.463, 0.481, 0.519, 0.529, 0.567, 0.642, 0.674, 0.752,
0.823, 0.887, 0.926
```

TABLE 5.3: Coefficient of the BLUEs of the a_u

κ	ξ	n	c	$a_u,\ u=1,2,\ldots,(n-c)$						
0.5	1	7	0	1.122123	-0.075374	-0.023245	-0.009345	-0.004594	-0.002852	-0.006712
			1	1.129701	-0.076824	-0.024211	-0.010194	-0.005477	-0.012996	
			2	1.141753	-0.07901	-0.025653	-0.011471	-0.025621		
		10	0	1.120081	-0.075032	-0.023505	-0.009452	-0.004447	-0.002353	-0.001384
				-0.000921	-0.000737	-0.002249				
			1	1.122372	-0.075425	-0.023748	-0.009646	-0.004623	-0.002525	-0.001566
				-0.001129	-0.003709					
			2	1.125214	-0.075894	-0.024035	-0.009874	-0.00483	-0.002731	-0.001783
				-0.006068						
			3	1.129322	-0.076545	-0.024429	-0.010186	-0.005113	-0.003011	-0.010038
			4	1.135917	-0.077546	-0.025026	-0.010658	-0.005544	-0.017143	
	2	7	0	1.129977	-0.025254	-0.016827	-0.013254	-0.012016	-0.012837	-0.049791
			1	1.162885	-0.029873	-0.021133	-0.017699	-0.017052	-0.077129	
			2	1.205879	-0.035752	-0.026621	-0.023376	-0.120132		
		10	0	1.095827	-0.018305	-0.011527	-0.008401	-0.006652	-0.005841	-0.005352
				-0.005636	-0.006409	-0.027705				
			1	1.111367	-0.019731	-0.013508	-0.009195	-0.008641	-0.007048	-0.007175
				-0.007635	-0.038435					
			2	1.128081	-0.021833	-0.014442	-0.011367	-0.009619	-0.008913	-0.008861
				-0.053046						
			3	1.148688	-0.023804	-0.016537	-0.013022	-0.011512	-0.010857	-0.072956
			4	1.176809	-0.026692	-0.018993	-0.015504	-0.013931	-0.101689	

In this data set, we select a random sample of size 10 and data are: 0.105, 0.432, 0.642, 0.529, 0.069, 0.361, 0.887, 0.674, 0.216, 0.081. By using the GTL in Eq. (5.1) for the given sample, we have the maximumlikelihood estimate of $\kappa_{ML} = 2.9967$ and $\xi_{ML} = 0.32063$. The K-S statistic and corresponding p-value are 0.15679 and 0.9357, respectively, which conforms the suitability of GTL distribution. Figure 5.1 shows ecdf and Q–Q plot of the sample.

Then by using the BLUEs coefficients in Tables 5.3 and 5.4, we have

$$\delta^* = \sum_{u=1}^{n} a_u Z_{u:n} = .030836 \quad \text{and} \quad \varphi^* = \sum_{u=1}^{n} b_u Z_{u:n} = 1.329179$$

TABLE 5.4: Coefficient of the BLUEs of the b_u

κ	ξ	n	c	$b_u, u = 1, 2, \ldots, (n-c)$						
0.5	1	7	0	-3.46878	0.549522	0.341474	0.287862	0.289566	0.344431	1.655924
			1	-5.33835	0.907121	0.579507	0.497268	0.507453	2.846996	
			2	-7.97863	1.386104	0.895646	0.776871	4.920006		
		10	0	-3.40004	0.483125	0.278286	0.213224	0.188383	0.181781	0.188651
				0.212461	0.271219	1.382916				
			1	-4.80901	0.725179	0.427867	0.332547	0.296316	0.287677	0.300279
				0.340695	2.098449					
			2	-6.41672	0.990501	0.590117	0.461615	0.413331	0.403162	0.423261
				3.134731						
			3	-8.53949	1.326667	0.793374	0.622812	0.559831	0.548687	4.688121
			4	-11.6192	1.793964	1.072399	0.843379	0.760811	7.148644	
	2	7	0	-2.46947	0.220001	0.198251	0.199872	0.222026	0.280561	1.348755
			1	-3.36093	0.345122	0.314918	0.320282	0.358451	2.022157	
			2	-4.48814	0.499256	0.458759	0.469136	3.060991		
		10	0	-2.3293	0.155376	0.131581	0.126299	0.121686	0.131588	0.141512
				0.168971	0.222404	1.129882				
			1	-2.96309	0.213516	0.212391	0.158725	0.202792	0.180793	0.215848
				0.250522	1.528504					
			2	-3.62776	0.297134	0.249521	0.245132	0.241661	0.254961	0.282914
				2.056436						
			3	-4.42664	0.373541	0.330768	0.309272	0.315052	0.330328	2.767681
			4	-5.49345	0.483106	0.423914	0.403443	0.406811	3.776179	

TABLE 5.5: Variances and covariance of the BLUEs

κ	ξ	n	c	$var(\delta^*)$	$var(\varphi^*)$	$cov(\delta^*, \varphi^*)$
0.5	1	7	0	0.000251	0.167441	-0.000853
			1	0.000252	0.284615	-0.001328
			2	0.000255	0.430011	-0.001992
		10	0	0.000071	0.109877	-0.000236
			1	0.000072	0.171915	-0.000337
			2	0.000072	0.236574	-0.000451
			3	0.000072	0.313967	-0.000601
			4	0.000073	0.414729	-0.000817
	2	7	0	0.002146	0.086711	-0.005036
			1	0.002217	0.138541	-0.006951
			2	0.002309	0.201735	-0.009361
		10	0	0.000991	0.054401	-0.002226
			1	0.001006	0.080332	-0.002861
			2	0.001022	0.106462	-0.003519
			3	0.001043	0.137329	-0.004315
			4	0.001071	0.177438	-0.005372

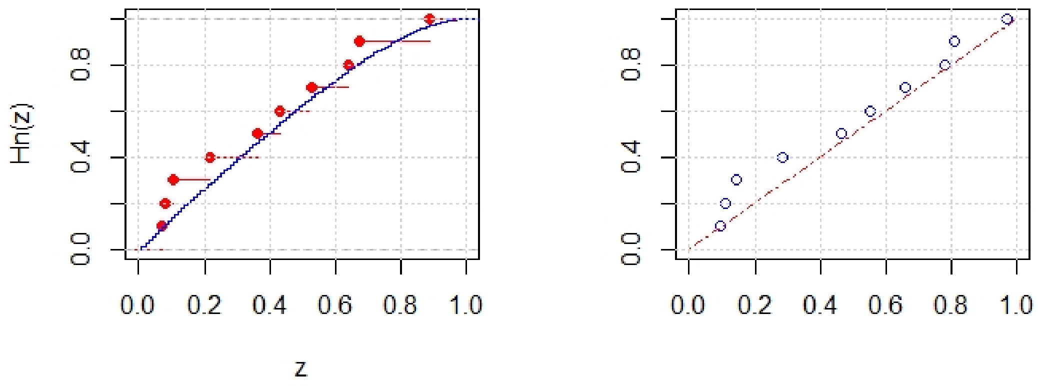

FIGURE 5.1: ECDF-plot and QQ-plot

5.7 Conclusion

Here, we come up with expressions of single and product moments of order statistics from GTL distribution. The BLUEs for the corresponding parameters have been derived using the moments of the order statistics. Also, the performance of the BLUEs is checked by variances and covariances. Finally, for illustration, a real data set is analysed. Based on our results we can claim for well behaviour of moments of order statistics.