Chapter 6

Weibull Marshall-Olkin Lomax Distribution with Applications to Bladder and Head Cancer Data*

6.1 Introduction

This chapter is sketched into the following sections: In section 6.2, we introduce the WMOL distribution and some special cases are presented. We derive two linear representations for the WMOL density which hold for $0 < \alpha < 1$ and $\alpha > 1$ in Sections 6.3. Some mathematical and statistical properties of the WMOL distribution are presented in Section 6.4. Section 6.5 describes the method of maximum likelihood for estimation of the model parameters. A simulation study is investigated in Section 6.6. In Section 6.7, we analyze two real data sets. Finally, in Section 6.8, we offer some concluding remarks.

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The proposal of new families has been worked out by many authors over recent years. Many ways to generate new families have been developed as the methods of addition, linear combination, composition and, one of the newer, the T-X family of distributions. Using this latter method, Korkmaz et al. (2019) proposed a new class called Weibull Marshall-Olkin-G (WMO-G) family. Here, we come up with a distribution, based on the WMO-G family, using the Lomax distribution as baseline, called Weibull Marshall-Olkin Lomax (WMOL) distribution. This distribution can have different shape of hazard rate function, like unimodal, decreasing, increasing, decreasing-increasing-decreasing and bathtub-shaped. Some properties of proposed model are developed. We also find the maximum likelihood estimates of unknown parameters of the WMOL distribution. For the confirmation of asymptotic behaviour of maximum likelihood estimates we provide simulation study and also used two real data sets to check the applicability of model in real life.

Abdul-Moniem and Abdel-Hameed (2012) proposed exponentiated Lomax distribution by generalizing Lomax distributionto analyze failure time data. Also, they proved it may provide better fits than exponential distribution and gave some mathematical properties of the exponentiated Lomax distribution. The statistical literary works contains many extended structures of the lomax distribution. For example, the Exponentiated Weibull-Lomax distribution (Hassan and Abd-Allah (2018)), Kumaraswamy exponentiated Lomax distribution (Elbatal and Kareem (2014)), exponentiated Lomax geometric distribution (Hassan and Abdelghafar (2017)), Weibull-Lomax distribution (Tahir et al. (2015)), Kumaraswamy-generalized Lomax distribution (Shams (2013)), power Lomax distribution (Rady et al. (2016)),transmuted Weibull Lomax distribution (Afify et al. (2015)), Marshall-Olkin power generalized Weibull distribution (Afify et al. (2020a)), Weibull Marshall–Olkin Lindley distribution (Afify et al. (2020b)).

For a baseline G distribution with parameter vector η, Korkmaz et al. (2019) proposed a wider class of continuous distributions called the *Weibull Marshall-Olkin-G* (WMO-G) family. They defined this family based on the T-X generator by choosing $r(t) = \beta t^{\beta-1} \mathrm{e}^{-t^{\beta}}$, $t > 0$, where $\beta > 0$ is a shape parameter and $W[G(z;\eta)] = -\log\left[\frac{\alpha\bar{G}(z;\eta)}{G(z;\eta)+\alpha\bar{G}(z;\eta)}\right]$.

The cdf of the WMO-G family is

$$H(z;\alpha,\beta,\eta)=1-\exp\left(-\left\{-\log\left[\frac{\alpha\bar{G}(z;\eta)}{1-\bar{\alpha}\bar{G}(z;\eta)}\right]\right\}^{\beta}\right). \tag{6.1}$$

The pdf corresponding to (6.1) is

$$\begin{aligned} h(z;\alpha,\beta,\eta) &= \frac{\beta g(z;\eta)}{\bar{G}(z;\eta)\left[1-\bar{\alpha}\bar{G}(z;\eta)\right]}\left\{-\log\left[\frac{\alpha\bar{G}(z;\eta)}{1-\bar{\alpha}\bar{G}(z;\eta)}\right]\right\}^{\beta-1} \\ &\quad\times\exp\left(-\left\{-\log\left[\frac{\alpha\bar{G}(z;\eta)}{1-\bar{\alpha}\bar{G}(z;\eta)}\right]\right\}^{\beta}\right), \end{aligned} \tag{6.2}$$

where $g(z;\eta)$ is the baseline PDF, $\bar{\alpha}=1-\alpha$, and α and β are two extra positive shape parameters.

The hazard rate function (HRF) of the WMO-G family takes the form

$$\tau(z;\alpha,\beta,\eta)=\frac{\beta\, w(z;\eta)}{\left[1-\bar{\alpha}\bar{G}(z;\eta)\right]}\left\{-\log\left[\frac{\alpha\bar{G}(z;\eta)}{1-\bar{\alpha}\bar{G}(z;\eta)}\right]\right\}^{\beta-1},$$

where $w(z;\eta)=g(z;\eta)/\bar{G}(z;\eta)$ is the baseline HRF.

For $\alpha=1$, we obtain the Weibull-X family (Alzaatreh et al. (2013); Cordeiro et al. (2015)) as a special case of the WMO-G family. For $\beta=1$, we obtain the MO-G family (Marshall and Olkin (1997)). For $\alpha=\beta=1$, we have the baseline distribution. Further details on the WMO-G family can be explored in Korkmaz et al. (2019).

6.2 Proposed Model

We propose *Weibull Marshall-Olkin Lomax* (WMOL) distribution with four parameters by setting the Lomax cdf $G(z;\lambda,\theta)=1-(1+\lambda z)^{-\theta}$ in (6.1), we obtain

$$H(z)=1-\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}}\right]\right)^{\beta}\right\}. \tag{6.3}$$

The associated pdf to (6.3) is

$$
\begin{aligned}
h(z) &= \frac{\beta\theta\lambda}{(1+\lambda z)[1-\bar{\alpha}(1+\lambda z)^{-\theta}]}\left(-log\left[\frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}}\right]\right)^{\beta-1} \\
&\times \exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}}\right]\right)^{\beta}\right\}, \qquad (6.4)
\end{aligned}
$$

where $\bar{\alpha} = 1-\alpha$, $\theta > 0$, $\beta > 0$ are shape and $\alpha > 0$, $\lambda > 0$ are scale parameters.

The HRF of Z is

$$
\tau(z) = \frac{\beta\theta\lambda}{(1+\lambda z)[1-\bar{\alpha}(1+\lambda z)^{-\theta}]}\left(-log\left[\frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}}\right]\right)^{\beta-1}.
$$

where $w(z;\eta) = g(z;\eta)/\bar{G}(z;\eta)$ is the baseline HRF.

It is observed that the density function of the new model provides a wide range of shapes based on its additional shape parameter, for example a monotonically decreasing density of exponentiated Lomax (EL) will become monotonically decreasing, decreasing, symmetric, reversed J, right-skewed and left-skewed. The WMOL distribution have decreasing, increasing-decreasing, increasing, constant and upside down bathtub shaped hazard function based on its additional parameter and can be used to provide a good fit for the real data than well-known distributions (see Figure 6.1).

Some mathematical properties of WMOL model can directly obtained from Lehmann type II (LTII) exponentiated Lomax model properties because it can be expressed as the linear combination of LTIIEL and EL densities. Additionally, the new model contains some distributions as special cases, these sub-models being listed in Table 6.1.

TABLE 6.1: Special cases of the WMOL distribution

Parametric values in Eq. (4)	Sub-models
$\beta = 1$	Marshall-Olkin Lomax distribution(α,θ,λ)
$\alpha = 1$	Weibull Lomax distribution(β,θ,λ)
$\alpha = \beta = 1$	Lomax distribution (θ)

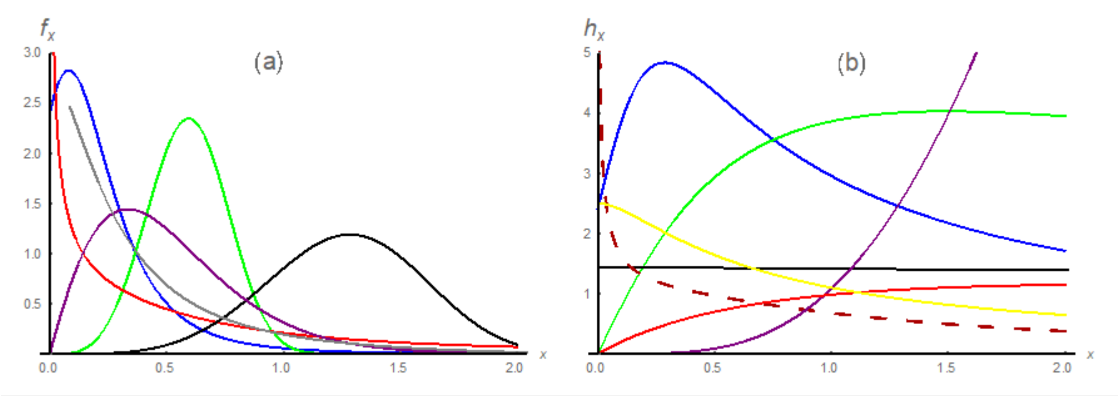

FIGURE 6.1: Plots of the WMOL density and hazard functions. (a) ($\theta = 1.5, \beta = 1, \alpha = 3, \lambda = 5$) (gray), ($\theta = 2, \beta = 5, \alpha = 1, \lambda = 1$) (green), ($\theta = 2, \beta = 5, \alpha = 2, \lambda = 0.8$) (black), ($\theta = 3, \beta = 2, \alpha = 1, \lambda = 0.7$) (purple), ($\theta = 4, \beta = 0.5, \alpha = 5, \lambda = 1$) (red), ($\theta = 4, \beta = 1, \alpha = 5, \lambda = 3$) (blue) (b) ($\theta = 1, \beta = 1, \alpha = 5, \lambda = 1$) (black), ($\theta = 1.5, \beta = 1, \alpha = 3, \lambda = 5$) (yellow), ($\theta = 1.5, \beta = 2, \alpha = 1, \lambda = 0.7$) (red), ($\theta = 2, \beta = 2, \alpha = 2, \lambda = 2$) (green), ($\theta = 2, \beta = 5, \alpha = 2, \lambda = 0.8$) (purple), ($\theta = 4, \beta = 0.5, \alpha = 5, \lambda = 1$) (dashes-red), ($\theta = 4, \beta = 1, \alpha = 5, \lambda = 3$) (blue).

6.3 Linear Representation

In this section, we provide two linear representations for the WMOL density depending on α. By using the power series

$$e^{z} = \sum_{i=0}^{\infty} \frac{z^{i}}{i!},$$

the cdf in (6.3) can be expressed as

$$H(z) = \sum_{h=1}^{\infty} \frac{(-1)^h}{\Gamma(h+1)} \left(-log \left[1 - \left(1 - \frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}} \right) \right] \right)^{h\beta}. \tag{6.5}$$

For a real number d and $z \in (0,1)$, we have

$$[-\log(1-z)]^{d} = z^{d} + \sum_{i=0}^{\infty} \psi_i(d)\, z^{i+d+1}, \tag{6.6}$$

where

$$\psi_0(d) = \frac{1}{2}d, \;\; \psi_1(d) = \frac{1}{24}[d(3d+5)], \;\; \psi_2(d) = \frac{1}{48}[d(d^2+5d+6)],$$
$$\psi_3(d) = \frac{1}{5760}[d(15d^3+150d^2+485d+502)], \ldots,$$

are Stirling polynomials. The proof is given in Theorem 3A of Flajolet and Odlyzko (1990) and in Theorem VI.2 of Flajolet and Sedgewick (2009). The previous results have been used by Cordeiro et al. (2017a). We can write

$$[-\log(1-z)]^{h\beta} = \sum_{i=0}^{\infty} \psi_{i-1}(h\beta)\, z^{i+h\beta}, \tag{6.7}$$

where $\psi_{-1}(h\beta) = 0$ by convention and $\psi_i(h\beta)$ for $i \geq 0$ can be obtained from (6.6). Then, the cdf (6.5) can be expressed using (6.7) as

$$H(z) = \sum_{h=1}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^h}{\Gamma(h+1)} \psi_{i-1}(h\beta) \left[1 - \frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}}\right]^{i+h\beta}.$$

For a real non-integer d and $|z| < 1$, we have

$$(1-z)^d = \sum_{k=0}^{\infty} (-1)^k \binom{d}{k} z^k.$$

Hence, we can write

$$H(z) = \sum_{h=1}^{\infty}\sum_{i,k=0}^{\infty} \frac{(-1)^{h+k}\alpha^k}{\Gamma(h+1)} \psi_{i-1}(h\beta) \binom{i+h\beta}{k} (1+\lambda z)^{-\theta k} \times \left[1-\bar{\alpha}(1+\lambda z)^{-\theta}\right]^{-k}. \tag{6.8}$$

For a positive integer ϑ and $|z| < 1$, a convergent power series can be expressed as

$$(1-z)^{-\vartheta} = \sum_{l=0}^{\infty} (-1)^l \binom{-\vartheta}{l} z^l,$$

For $\alpha \in (0,1)$, $H(z)$ can be written as

$$H(z) = \sum_{k,l=0}^{\infty} \upsilon_{k,l}\,\bar{G}(z;\theta,\lambda)^{k+l}, \tag{6.9}$$

where

$$\upsilon_{k,l} = \sum_{h=1}^{\infty}\sum_{i,k=0}^{\infty} \frac{(-1)^{i+k+l}\,\alpha^{k}\;\phi_{i-1}(h\beta)}{\Gamma(h+1)\,(1-\alpha)^{-l}} \binom{h\beta+i}{k}\binom{-k}{l}$$

and $\bar{G}(z;\theta,\lambda) = 1 - G(z;\theta,\lambda)$, is the exponentiated Lomax survival function.

For a baseline $G(z)$ and power parameter e, $\Pi_e(z) = 1 - \{1 - G(z)\}^e$ Lehmann (1953) is known as LTII cdf. Thus, the LTII density is given by $\pi_e(z) = e\,\bar{G}(z)^{e-1}\,g(z)$, where $g(z) = dG(z)/dz$.

Let $J = \{(k,l); k,l = 0,1,2,\ldots; k+l \geq 1\}$ be a set of non-negative integers. By differentiating the last equation for $H(z)$, the pdf of Z is

$$f(z) = \sum_{(k,l)\in J} \upsilon_{k,l}\,\pi_{k+l}(z;\theta,\lambda), \tag{6.10}$$

where $\pi_{k+l}(z) = (k+l)\,\bar{G}(z;\theta,\lambda)^{k+l-1}\,g(z;a)$ is known as LTII exponentiated Lomax density function with power parameter $k+l$.

If $\alpha > 1$, (6.8) can be written as

$$\begin{aligned} H(z) &= \sum_{h=1}^{\infty}\sum_{i,k=0}^{\infty} \frac{(-1)^{h+k}\alpha^{k}}{\Gamma(h+1)}\phi_{i-1}(h\beta)\binom{i+h\beta}{k}[(1+\lambda z)^{-\theta}]^{k} \\ &\times\; \alpha^{-k}\left[1-(1-\alpha^{-1})(1+\lambda z)^{-\theta}\right]^{-k}. \end{aligned}$$

Using series expansion, we have

$$H(z) = \sum_{k,l=0}^{\infty} \nu_{k,l}\,\bar{G}(z;\theta,\lambda)^{k+l},$$

where,

$$\nu_{k,l} = \sum_{h=1}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{i+k+l} \quad \phi_{i-1}(h\beta)}{\Gamma(h+1)\,(1-\alpha)^{-l}} \binom{h\beta+i}{k}\binom{-k}{l}$$

So, pdf of Z is

$$h(z) = \sum_{(k,l)\in J} \nu_{k,l}\, \pi_{k+l}(z;\theta,\lambda), \tag{6.11}$$

From (6.10) and (6.11), we find that WMOL density function can be expressed as linear combination of EL density function and LTII exponentiated Lomax densities for both cases.

Every LTII Lindley can be a linear combination of EL densities. By expanding $\Pi_e(z) = 1 - \{1-G(z)\}^e$ (for e real), the power series converges everywhere

$$\Pi_e(z) = \sum_{r=1}^{\infty}(-1)^{r+1}\binom{e}{r} G(z)^r.$$

Differentiating last equation, we get

$$\pi_e(z) = \sum_{r=0}^{\infty}(-1)^{r}\binom{e}{r+1} \rho_{r+1}(z), \tag{6.12}$$

where $\rho_{r+1}(z) = (r+1)\,G(z)^r\,g(z)$ represents EL density with $r+1$ as power parameter. Sum lasts at e, if e is a positive integer.

6.4 Properties of the WMOL Distribution

This section deals with statistical properties of WMOL distribution. We will use weights $\upsilon_{k,l}$ and $\nu_{k,l}$ according to $0<\alpha<1$ and $\alpha>1$, respectively, to derive properties.

6.4.1 Quantiles Function

Quantiles are fruitful in estimation and simulation. The root of the equation given below will give the pth quantile for WMOL distribution.

$$\xi_p = \frac{1}{\lambda}\left[\left(\frac{\alpha-\bar{\alpha}\, e^{-(-log(1-p))^{\frac{1}{\beta}}}}{e^{-(-log(1-p))^{\frac{1}{\beta}}}}\right)^{\frac{1}{\theta}}-1\right],\ 0<p<1,\ \lambda,\ \theta>0. \tag{6.13}$$

A random sample of size n can be generated with the help of uniform distribution and equation (6.13) for WMOL distribution as follows

$$\xi_i = \frac{1}{\lambda}\left[\left(\frac{\alpha-\bar{\alpha}\, e^{-(-log(1-u_i))^{\frac{1}{\beta}}}}{e^{-(-log(1-u_i))^{\frac{1}{\beta}}}}\right)^{\frac{1}{\theta}}-1\right].$$

In particular, the first three quantiles, Q_1, Q_2 and Q_3, can be derived for specific values of p.

6.4.2 Moments and Generating Functions

Moments tell us about important features and characteristics of a distribution. Here, we derive raw moments and moment generating function (MGF) of WMOL distribution.

Now, the nth raw moment of the WMOL can be written as

$$\begin{aligned}\mu_n^{'} &= E[Z^n]=\int_0^\infty z^n h(z)dz=\sum_{(k,l)\in J}\sum_{r=0}^{k+l}(-1)^r\binom{k+l}{r+1}\upsilon_{k,l}\lambda\theta(r+1)\\ &\times \int_0^\infty z^n[1-(1+\lambda z)^{-\theta}]^{(r+1)-1}(1+\lambda z)^{-\theta-1}dz\\ &= \frac{1}{\lambda^n}\sum_{(k,l)\in J}\sum_{r=0}^{k+l}\sum_{m=0}^{n}(-1)^{r+m}\binom{k+l}{r+1}\binom{n}{m}\upsilon_{k,l}(r+1)\\ &\times B\left(1-\frac{1}{\theta}(n-m),r+1\right),\end{aligned} \tag{6.14}$$

where, $B(a,\ b)=\int_0^1 z^{a-1}(1-z)^{b-1}dz$.

The nth central moments μ_n and cumulants k_n of Z can be determined from (6.14) as

$$\mu_n = E(Z-\mu)^n = \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} \mu_1'^n \mu_{n-k}',$$

and

$$k_n = \mu_n' - \sum_{k=1}^{n-1} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} k_n \mu_{n-k}',$$

where $k_1 = \mu_1'$. Cumulants are useful to calculate moments, skewness and kurtosis. The MGF of Z easily follows from (6.10) as

$$\begin{aligned} M(t) &= \frac{1}{\lambda^p} \sum_{(k,l)\in J} \sum_{r=0}^{k+l} \sum_{m=0}^{n} \sum_{p=0}^{\infty} \frac{t^p (-1)^{r+m}}{p!} \binom{k+l}{r+1} \binom{p}{m} \upsilon_{k,l}(r+1) \\ &\times B\left(1 - \frac{1}{\theta}(p-m), r+1\right). \end{aligned}$$

6.4.3 Conditional Moments, Mean Residual Life and Mean Deviations

Conditional moments, $E(Z^n|Z>z)$, of WMOL distribution can be derived as

$$E(Z^n|Z>z) = \frac{1}{S(z)} J_n(z)$$

where,

$$\begin{aligned} J_n(z) &= \int_z^{\infty} y^n f(y) dy = \frac{1}{\lambda^n} \sum_{(k,l)\in J} \sum_{r=0}^{k+l} \sum_{m=0}^{n} \sum_{p=0}^{\infty} (-1)^{r+m} \binom{k+l}{r+1} \binom{n}{m} \\ &\times \upsilon_{k,l}(r+1) \frac{(1-\{r+1\})_p \, (1+\lambda z)^{n-m-\theta(p+1)}}{p! \left[\frac{1}{\theta}(m-n)+p+1\right]}, \end{aligned} \tag{6.15}$$

where, $S(z) = 1 - H(z)$, defined in (6.3).

Conditional moments are helpful in deriving mean residual life (MRL). MRL is expected residual life of an item with the condition that it has survived for time z. Using the conditional moment, the MRL function can be expressed as

$$m_Z(z) = E(Z - z|Z > z) = \frac{1}{S(z)} J_1(z) - z.$$

where, $J_1(z)$ can be derived from (6.15) where $n = 1$.

Also, conditional moments can be used to derive the mean deviation about mean and median. Let M and μ represents median and mean, then mean deviations can be expressed as

$$\begin{aligned}
\delta_\mu &= \int_0^\infty |z - \mu|\, h(z) dz = 2\mu H(\mu) - 2\mu + 2J_1(\mu) \\
\delta_M &= \int_0^\infty |z - M|\, h(z) dz = 2J_1(M) - \mu
\end{aligned}$$

respectively. Where $J_1(\mu)$ and $J_1(M)$ are derived from (6.15). Also, $H(\mu)$ and $H(M)$ are calculated from (6.3).

6.4.4 Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves evaluate disparity of the distribution of a random variable and they are applicable in economics, reliability, medical and demography, among other areas. For a probability p, these curves are given by

$$B(p) = \frac{1}{p\mu_1'} \int_0^q z h(z) dz \quad \text{and} \quad L(p) = pB(p),$$

respectively, where $\mu_1' = E(Z)$ and $q = F^{-1}(p)$.

Bonferroni and Lorenz curves for the WMOL distribution can be expressed as

$$\begin{aligned} B(p) &= \frac{1}{p} - \frac{1}{\lambda p \mu_1'} \sum_{(k,l)\in J} \sum_{r=0}^{k+l} \sum_{m=0}^{n} \sum_{s=0}^{\infty} (-1)^{r+m} \binom{k+l}{r+1} \binom{n}{m} \upsilon_{k,l}(r+1) \\ &\times \frac{(1-\{r+1\})_s \, (1+\lambda q)^{1-m-\theta(s+1)}}{s! \left[\frac{1}{\theta}(m-1)+s+1\right]} \end{aligned}$$

and $L(p) = pB(p)$, respectively.

6.4.5 Residuals Life Function

Let Z follows the pdf $h(z)$ given by (6.4). The conditional random variable $R_{(t)} = Z - t | Z > t$, $t \geq 0$ describes the residual life. Using (6.3), the survival function of residual lifetime $R_{(t)}$ is given by

$$S_{R_{(t)}}(z) = \frac{S(z+t)}{S(t)} = \frac{\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda(z+t))^{-\theta}}{1-\bar{\alpha}(1+\lambda(z+t))^{-\theta}}\right]\right)^{\beta}\right\}}{\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda t)^{-\theta}}{1-\bar{\alpha}(1+\lambda t)^{-\theta}}\right]\right)^{\beta}\right\}}, \quad z > 0.$$

The associated cdf is given by

$$H_{R_{(t)}}(z) = \frac{\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda t)^{-\theta}}{1-\bar{\alpha}(1+\lambda t)^{-\theta}}\right]\right)^{\beta}\right\} - \exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda(z+t))^{-\theta}}{1-\bar{\alpha}(1+\lambda(z+t))^{-\theta}}\right]\right)^{\beta}\right\}}{\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda t)^{-\theta}}{1-\bar{\alpha}(1+\lambda t)^{-\theta}}\right]\right)^{\beta}\right\}}.$$

Then, the associated pdf is given by

$$\begin{aligned} h_{R_{(t)}}(z) &= \beta\,\theta\,\lambda \, \frac{\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda(z+t))^{-\theta}}{1-\bar{\alpha}(1+\lambda(z+t))^{-\theta}}\right]\right)^{\beta}\right\}}{\exp\left\{-\left(-\log\left[\frac{\alpha(1+\lambda t)^{-\theta}}{1-\bar{\alpha}(1+\lambda t)^{-\theta}}\right]\right)^{\beta}\right\}} \\ &\times \frac{\left(-\log\left[\frac{\alpha(1+\lambda(z+t))^{-\theta}}{1-\bar{\alpha}(1+\lambda(z+t))^{-\theta}}\right]\right)^{\beta-1}}{[1-\bar{\alpha}(1+\lambda(z+t))^{-\theta}][1+\lambda(z+t)]}, \quad z > 0. \end{aligned}$$

The associated hazard rate function is given by

$$\tau_{R_{(t)}}(z) \;=\; \frac{\beta\theta\lambda\left(-\log\left[\frac{\alpha(1+\lambda(z+t))^{-\theta}}{1-\bar{\alpha}(1+\lambda(z+t))^{-\theta}}\right]\right)^{\beta-1}}{[1-\bar{\alpha}(1+\lambda(z+t))^{-\theta}][1+\lambda(z+t)]},\; z>0.$$

The nth moments of residual life of Z, $m_n(t) = E[(Z-t)|Z>t]$ for $n = 1,2,\ldots,$ uniquely determines $H(z)$, we have

$$\begin{aligned} m_n(t) &= E(R_{(t)}) = E[(Z-t)^n|Z>t] = \frac{1}{S(t)}\int_t^{\infty} z^n dH(z) - t \\ &= \frac{1}{S(t)}\left(E(Z^n) - \int_0^t z^n dH(z)\right) - t. \end{aligned} \qquad (6.16)$$

On the other hand, the variance residual life is given by

$$\begin{aligned} V(t) &= Var(R_{(t)}) = Var[Z-t|Z>t] = \frac{2}{S(t)}\int_t^{\infty} zS(z)dz - 2tm_1(t) - [m_1(t)]^2 \\ &= \frac{1}{S(t)}\left(E(Z^2) - \int_0^t z^2 h(z)dz\right) - t^2 - 2tm_1(t) - [m_1(t)]^2. \end{aligned}$$

The conditional random variable $\bar{R}_{(t)} = t - Z|Z \leq t,\, t \geq 0$ describes reverse residual life. Using the cdf (6.3), the survival function of the reverse residual lifetime $\bar{R}_{(t)}$ is given by

$$S_{\bar{R}_{(t)}}(z) = \frac{1-\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda(t-z))^{-\theta}}{1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}}\right]\right)^{\beta}\right\}}{1-\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda t)^{-\theta}}{1-\bar{\alpha}(1+\lambda t)^{-\theta}}\right]\right)^{\beta}\right\}},\;\; 0 \leq z \leq t.$$

The associated cdf is given by

$$H_{\bar{R}_{(t)}}(z) = \frac{\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda(t-z))^{-\theta}}{1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}}\right]\right)^{\beta}\right\} - \exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda t)^{-\theta}}{1-\bar{\alpha}(1+\lambda t)^{-\theta}}\right]\right)^{\beta}\right\}}{1-\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda t)^{-\theta}}{1-\bar{\alpha}(1+\lambda t)^{-\theta}}\right]\right)^{\beta}\right\}}.$$

Therefore, the associated pdf is given by

$$h_{\bar{R}_{(t)}}(z) = \beta\theta\lambda \frac{\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda(t-z))^{-\theta}}{1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}}\right]\right)^{\beta}\right\}}{1-\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda t)^{-\theta}}{1-\bar{\alpha}(1+\lambda t)^{-\theta}}\right]\right)^{\beta}\right\}} \times \frac{\left(-log\left[\frac{\alpha(1+\lambda(t-z))^{-\theta}}{1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}}\right]\right)^{\beta-1}}{[1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}][1+\lambda(t-z)]}.$$

The associated hazard rate is given by

$$\tau_{\bar{R}_{(t)}}(z) = \beta\theta\lambda \frac{\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda(t-z))^{-\theta}}{1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}}\right]\right)^{\beta}\right\}}{1-\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda(t-z))^{-\theta}}{1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}}\right]\right)^{\beta}\right\}} \times \frac{\left(-log\left[\frac{\alpha(1+\lambda(t-z))^{-\theta}}{1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}}\right]\right)^{\beta-1}}{[1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}][1+\lambda(t-z)]}.$$

In the similar manner, Navarro et al. (1998) prove that the nth moment of the reversed residual life, say $M_n(t) = E[(t-Z)^n | Z \le t]$ for $t > 0$ and $n = 1, 2, \ldots,$ uniquely determines $H(z)$. We obtain

$$M_n(t) = \frac{1}{H(t)} \int_0^t (t-z)^n dH(z).$$

The mean reversed residual life is define as

$$M(t) = E(\bar{R}_{(t)}) = E(t-z|z \le t) = t - \frac{1}{H(t)} \int_0^t zh(z)dz,$$

The variance reversed residual life can be derived as

$$\begin{aligned} w(t) &= Var(\bar{R}_{(t)}) = Var(t-z|z \le t) = 2tM(t) - (M(t))^2 - \frac{2}{H(t)} \int_0^t zH(z)dz \\ &= 2tM(t) - (M(t))^2 - t^2 + \frac{1}{H(t)} \int_0^t z^2 h(z)dz. \end{aligned}$$

6.5 Maximum Likelihood Estimation

Let a random sample $z_1,\dots,z_n$ of size n be selected from WMOL distribution. Using (6.4), log-likelihood function can be written as

$$\begin{aligned}\ell &\propto n\log(\beta)+n\log(\theta)+n\log(\lambda)-\sum_{i=1}^{n}\log(1+\lambda z_i)-\sum_{i=1}^{n}\log[1-\bar{\alpha}(1+\lambda z_i)^{-\theta}]\\ &- \sum_{i=1}^{n}\left\{-\log\left[\frac{\alpha(1+\lambda z_i)^{-\theta}}{1-\bar{\alpha}(1+\lambda z_i)^{-\theta}}\right]\right\}^{\beta}+(\beta-1)\sum_{i=1}^{n}\log\left\{-\log\left[\frac{\alpha(1+\lambda z_i)^{-\theta}}{1-\bar{\alpha}(1+\lambda z_i)^{-\theta}}\right]\right\}.\end{aligned}$$

The MLEs of α, β, λ and θ, denoted by $\hat{\alpha}_{MLE}$, $\hat{\beta}_{MLE}$, $\hat{\lambda}_{MLE}$ and $\hat{\theta}_{MLE}$, can be obtained numerically by maximizing the log-likelihood function ℓ or by solving the nonlinear equations:

$$\begin{aligned}\frac{\partial\ell}{\partial\alpha} &= -\sum_{i=1}^{n}\frac{(1+\lambda z_i)^{-\theta}}{1-\bar{\alpha}(1+\lambda z_i)^{-\theta}}+\sum_{i=1}^{n}\left[\beta(-\log(\xi_i))^{\beta-1}+\frac{(\beta-1)}{\log(\xi_i)}\right]\\ &\times\left(\frac{1}{\alpha}-\frac{(1+\lambda z_i)^{-\theta}}{1-\bar{\alpha}(1+\lambda z_i)^{-\theta}}\right)=0,\\ \frac{\partial\ell}{\partial\beta} &= \frac{n}{\beta}+\sum_{i=1}^{n}\log(-\log(\xi_i))[1-(-\log(\xi_i))^{\beta}]=0,\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\lambda} &= \frac{n}{\lambda}-\sum_{i=1}^{n}\left[\beta(-\log(\xi_i))^{\beta-1}+\frac{(\beta-1)}{\log(\xi_i)}\right]\frac{\theta z_i}{(1+\lambda z_i)[1-\bar{\alpha}(1+\lambda z_i)^{-\theta}]}\\ &- \sum_{i=1}^{n}\frac{z_i}{(1+\lambda z_i)}-\sum_{i=1}^{n}\frac{\xi_i\theta z_i}{(1+\lambda z_i)}=0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\ell}{\partial\theta} &= \frac{n}{\theta}-\sum_{i=1}^{n}\left[\beta(-\log(\xi_i))^{\beta-1}+\frac{(\beta-1)}{\log(\xi_i)}\right]\frac{\log(1+\lambda z_i)}{[1-\bar{\alpha}(1+\lambda z_i)^{-\theta}]}\\ &- \sum_{i=1}^{n}\xi_i\log(1+\lambda z_i)=0,\end{aligned}$$

where,

$$\xi_i=\xi(\alpha,a;z_i)=\frac{\alpha(1+\lambda z_i)^{-\theta}}{1-\bar{\alpha}(1+\lambda z_i)^{-\theta}}. \tag{6.17}$$

6.6 Simulation Study

This section deals with simulation study to verify asymptotic properties of MLE's. We perform a Monte Carlo simulation, with 1000 replications and using the R software. To do this, we choose some scenarios:

- 1. $(\alpha, \beta, \lambda, \theta) = (1, 2, 1, 2)$, $n = 50, 100, 150$ and uncensored;
- 2. Some parameter value, $n = 50, 100, 150$ with 10% censorship;
- 3. Some parameter value, $n = 50, 100, 150$ with 20% censorship;

and we calculate the average estimates (AEs) of the MLEs and the mean squared errors (MSEs), for each parameter point.

The results are present in Table 6.2 and indicates that the AEs become closer to the true parameter values and MSEs of MLEs of the model parameters approach to zero when n increases.

TABLE 6.2: Simulation study

		0% censured		10% censured		20% censured	
n	Parameter	AE	MSE	AE	MSE	AE	MSE
50	α	1.0019	0.0143	1.1523	0.0429	1.1706	0.0503
	β	2.0346	0.0500	1.9525	0.0562	1.9421	0.0541
	λ	1.0103	0.0093	0.9013	0.0171	0.8904	0.0198
	θ	2.0173	0.0209	1.8489	0.0400	1.8315	0.0467
100	α	1.0000	0.0065	1.1210	0.0231	1.1757	0.0413
	β	2.0129	0.0226	1.9591	0.0278	1.9223	0.0308
	λ	1.0050	0.0041	0.9164	0.0105	0.8818	0.0176
	θ	2.0079	0.0093	0.9164	0.0105	1.8174	0.0419
150	α	1.0003	0.0048	1.0881	0.0141	1.2100	0.0513
	β	2.0153	0.0155	1.9625	0.0161	1.9054	0.0265
	λ	1.0039	0.0030	0.9374	0.0068	0.8597	0.0219
	θ	2.0068	0.0068	1.9049	0.0156	1.7830	0.0524

6.7 Real Data Application

By making use of two practical data sets, we illustrate the applicability of the WMOL distribution among a set of classical and recent models containing beta exponentiated Lomax, transmuted Weibull Lomax, exponentiated Weibull Lomax, beta Marshall-Olkin Lomax, Gompertz Lomax and Kumaraswamy generalized Lomax , based on a set of goodness-of-fit statistics. ML method is used to estimate model parameters and compared with the help of K-S statistic, p-value, Cramer-von Mises (W^*) and Anderson Darling (A^*). Generally larger p-value and smaller values of these statistics indicates a better fit to data.

TABLE 6.3: Some competitive models to the WMOL distribution.

Distribution	Author(s)
Beta exponentiated Lomax (BEL)	Mead (2016)
Transmuted Weibull Lomax (TWL)	Afify et al. (2015)
exponentiated Weibull-Lomax (EWL)	Hassan and Abd-Allah (2018)
Beta Marshall-Olkin Lomax (BMOL)	Tablada and Cordeiro (2019)
Gompertz-Lomax (GL)	Oguntunde et al. (2017)
Kumaraswamy generalized Lomax (KGL)	Shams (2013)

Description of the data is as follows: The bladder cancer patient's data: The first data set is remission time (in months) of a group of 128 bladder cancer patients taken from Lee and Wang (2003).

```
0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23,
3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09,
9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24,
25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81,
2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32,
7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66,
15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01,
1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33,
5.49, 7.66, 11.25, 2.07, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87,
11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46,
```

4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 3.36, 6.93, 8.65, 12.63, 22.69.

Survival times of patients treated using RT: The second real data represents the survival time of head cancer patients, who treated using radiotherapy (RT). The data were initially reported by Efron (1988). These data consists of 58 observations:

6.53, 7, 10.42, 14.48, 16.1, 22.7, 34, 41.55, 42, 45.28, 49.4, 53.62, 63, 64, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 146, 149, 154, 157, 160, 160, 165, 173, 176, 218, 225, 241, 248, 273, 277, 297, 405, 417, 420, 440, 523, 583, 594, 1101, 1146, 1417.

Table 6.4 gives some descriptive statistics for both data sets and using it we note that the two data sets have positive skewness and kurtosis.

On the other side, comparing the WMOL distribution with other classical and recent distributions is done as follows. For the two data sets, ML method is used to estimate the parameters of models and by these estimates, we provide the statistics K-S, p-value, W^* and A^*. The obtained results are reported in Tables 6.5-6.8. From these tables, the smallest values of the K-S, W^*,A^*, and the largest p-value is obtained for the WMOL distribution. Hence, we infer that WMOL distribution provides the best fit among the compared distributions.

TABLE 6.4: Descriptive statistics of both data sets (MD:= Mean deviation, Kr:= kurtosis, SK:= skewness, SE:= Shannon entropy).

Data	Mean	Median	SD	SK	Kr	MD-mean	MD-median	SE
First data	9.36561	6.395	10.5081	3.2737	15.338	6.72060	6.12812	2.083
Second data	226.174	151.5	273.943	2.6999	7.5399	172.048	145.068	1.649

Figure 6.2 shows the TTT plot (see Aarset (1987)) for both data sets. Note that the TTT plot for the first data set indicates a bathtub hazard rate function, while the second one indicates increasing-decreasing-increasing-decreasing hazard.

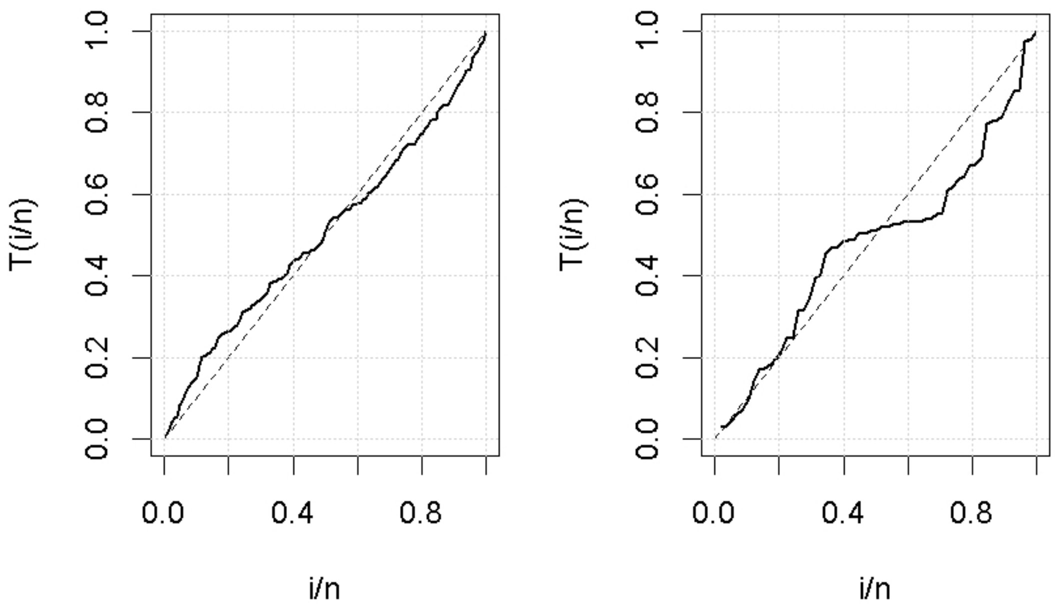

FIGURE 6.2: TTT plot for both data sets.

TABLE 6.5: The MLEs of the parameters of some models fitted to the bladder cancer patient's data.

Distribution		Estimates			
BEL(a,b,λ,θ,β)	0.71556	5.18863	0.08855	0.80847	2.10461
TWL(α,β,λ,a,b)	0.19165	9.60751	0.68149	12.7477	1.48715
EWL(a,α,β,θ,λ)	57.9147	1.20253	1.27849	0.07811	11.0033
BMOL(a,b,c,α)	1.06537	1.43179	46.2985	1.74832	-
GL(θ,γ,α,β)	1.22647	1.04043	0.87032	0.10565	-
KGL(a,b,α,λ)	1.51371	23.9726	0.22322	11.1227	-
WMOL(α,β,λ,θ)	15.6523	1.13049	0.55624	1.90448	-

The empirical and fitted densities are demonstrated in figure 6.3 for this data set. We are comparing only two models WMOL and BEL because of the smallest values of the statistics and goodness of fit measures and according to figure WMOL distribution fits better.

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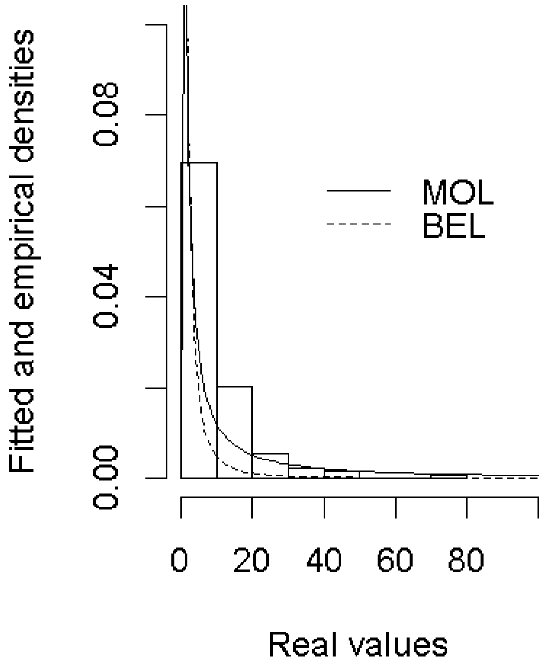

FIGURE 6.3: Fitted and empirical densities for the first data set

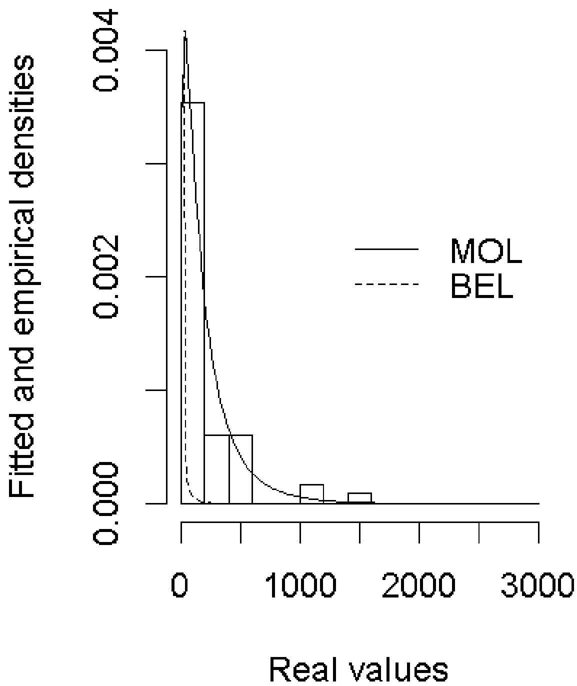

FIGURE 6.4: Fitted and empirical densities for the second data set

TABLE 6.6: The values of K-S, p- value, W^* and A^* statistics for some models fitted to the bladder cancer patient's data.

Distribution	K-S	p-value	(W^*)	(A^*)
BEL(a,b,λ,θ,β)	0.03911	0.88959	0.02330	0.1594
TWL(α,β,λ,a,b)	0.03827	0.87194	0.02160	0.1494
EWL(a,α,β,θ,λ)	0.03990	0.88694	0.02566	0.1755
BMOL(a,b,c,α)	0.02965	0.86980	0.01456	0.0921
GL(θ,γ,α,β)	0.09264	0.22180	0.20280	1.3288
KGL(a,b,α,λ)	0.16266	0.14182	0.02380	0.1635
WMOL(α,β,λ,θ)	0.02922	0.99990	0.01410	0.0903

TABLE 6.7: The MLEs of the parameters of some models fitted to the survival times of patients treated using RT data.

Distribution		Estimates			
BEL(a,b,λ,θ,β)	0.73642	15.2639	0.00504	0.29914	1.86758
TWL(α,β,λ,a,b)	0.64257	2.57857	-1.0000	0.09276	1.02731
EWL(a,α,β,θ,λ)	3.40069	0.90215	3.07405	0.13213	4.63291
BMOL(a,b,c,α)	4.25699	10.5517	23.9284	0.47569	-
GL(θ,γ,α,β)	14.7599	1.009×10^{-6}	0.45312	0.00078	-
WMOL(α,β,λ,θ)	74.9829	0.61133	0.00545	6.88208	-

TABLE 6.8: The values of K-S, p- value,(W^*) and (A^*) statistics for some models fitted to survival times of patients treated using RT data.

Distribution	K-S	p-value	(W^*)	(A^*)
BEL(a,b,λ,θ,β)	0.13473	0.24303	0.18937	0.9141
TWL(α,β,λ,a,b)	0.13581	0.23501	0.21332	1.0387
EWL(a,α,β,θ,λ)	0.14371	0.18208	0.22394	1.0982
BMOL(a,b,c,α)	0.16727	0.07786	0.25599	1.2571
GL(θ,γ,α,β)	0.14522	0.17306	0.25671	1.2410
WMOL(α,β,λ,θ)	0.11319	0.44717	0.12648	0.6496

6.8 Conclusion

Here, we come up with a new lifetime model christended the Weibull Marshall-Olkin Lomax (WMOL) distribution, which has two shape and two scale parameters. It can be reduced to Weibull-Lomax, Marshall-Olkin Lomax and Lomax distributions. The failure rate function of WMOL model can have decreasing, increasing, upside down bathtub and bathtub curve according to its shape parameters. Therefore, WMOL model can be used quite effectively as an alternative to some extended form of Lomax and Weibull distributions and works better than the cited models. Maximum likelihood method is used to estimate the parameters of model and to check the efficiency of estimators we did a simulation study. We hope that the new distribution can be widely used in many different fields.