EXTENSION OF SOME PROBABILITY DISTRIBUTIONS AND ASSOCIATED ORDERED RANDOM VARIABLES

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in

Statistics

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Dedicated to my Parents.

DECLARATION

(As required under clause I2 of Ordinance IIA of the Central University of Haryana)

This is to certify that the material embodied in the present work, entitled **"*Extension of Some Probability Distributions and Associated Ordered Random Variables*"**, is based on my original research work. The research work was carried out under the supervision of **Dr. Devendra Kumar,** Assistant Professor, Department of Statistics, Central University of Haryana, Mahendergarh. This work has not been submitted, in part or full, for any other diploma or degree of any University/Institution Deemed to be University and College/Institution of National Importance. References from other works have been duly cited at the relevant places.

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Abstract

The present thesis entitled **"Extension of Some Probability Distributions and Associated Ordered Random Variable"** is being submitted for the award of the degree of Doctor of Philosophy (Statistics). It is based on six chapters and deals mainly with exact expressions and some recurrence relations for the single and product moments of order statistics, best linear unbiased estimators (BLUEs) for the location and scale parameters based on type-II right censored samples and real life applications of some specific continuous distributions. Also, we propose a new model, its statistical and mathematical properties and real life applications. All numerical computations and statistical simulation are performed using R software.

Order statistics and its functions play a vital role in a wide range of theoretical and practical problems such as characterization of probability distributions and goodness-of-fit tests, entropy estimation, analysis of censored samples, reliability analysis, quality control and strength of materials. The moments of order statistics finds wide applicability in many areas such as quality control testing, reliability, etc. For instance, when the reliability of an item or product is high, the duration of the failed items will be high which in turn will make the product too dear, both in terms of time and money. This fact might prevent a practitioner from knowing enough about the product in a relatively short time. Therefore, a practioner needs to predict the failure of future items based on the times of a few early failures. These predictions are often based on moments of order statistics.

Since the turn of this century, order statistics and their moments have gotten a lot of attention. Galton (1902) and Pearson (1902) explored the distribution of the difference of successive order statistics. The moments of order statistics have gained a lot of traction in the statistics field, and they've been numerically tabulated for a variety of distributions. For more information, see David and Nagaraja (2003), Sarhān and Greenberg (1962), Arnold and Balakrishnan (2012), and Arnold et al. (2008).

Chapter 1 is expository in nature and provides a brief review of the concepts and results concerning order statistics, distribution of order statistics, moments and recurrence relations and

some estimation methods. We also discussed few methods to generate distributions. Some continuous distributions, which are used in subsequent chapters, have also been discussed.

Chapter 2 deals with exact explicit expressions for the single and product moments of order statistics from the type II exponentiated log-logistic distribution, and then use these results to compute the means, variances, skewness and kurtosis of *u*th order statistics. Besides, best linear unbiased estimators (BLUEs) for the location and scale parameters for the type II exponentiated log-logistic distribution with known shape parameters are studied. Finally, the results are illustrated with a real data set.

Chapter 3 deals with exact explicit expressions for the single and product moments of order statistics from the modified power function distribution. By using these relations, we have tabulated the expected values, second moments, variances and covariances of order statistics from samples of sizes up to 10 for various values of the parameters. Also, we use these moments to obtain the best linear unbiased estimates (BLUEs) of the location and scale parameters based on Type-II right-censored samples. In addition, we carry out some numerical illustrations through Monte Carlo simulations to show the usefulness of the findings. Finally, we apply the findings of the chapter to one real data set.

Chapter 4 deals with the inference procedures for the estimation of the parameters by using order statistics. First, we derive some new expressions for the single and product moments of the order statistics from the extended power Lindley distribution. Then, we use these moments to obtain the best linear unbiased estimates (BLUEs) of the location and scale parameters based on Type-II right-censored samples. A real data set are analyzed to illustrate the flexibility and importance of the model.

Chapter 5 deals with recurrence relations (without any restriction for shape parameter) for moments of order statistics from generalized Topp-Leone distribution. Also derived some relations for integral values of shape parameter. These relations will be fruitful for computational purposes. Also, these moments are used to obtain BLUEs of location and scale parameters for type-II right censored samples. In addition, Monte Carlo simulations is used to find applicability of the findings. Finally, real data set is used to find real life applications.

Chapter 6 deals with a new distribution, based on the WMO-G family, using the Lomax distribution as baseline, called Weibull Marshall-Olkin Lomax (WMOL) distribution. This distribution can have different shape of hazard rate function, like unimodal, decreasing, increasing, decreasing-increasing-decreasing and bathtub-shaped. Some properties of proposed model are developed. We also find the maximum likelihood estimates of unknown parameters of the WMOL distribution. For the confirmation of asymptotic behaviour of maximum likelihood estimates we provide simulation study and also used two real data sets to check the applicability of model in real life.

List of Research Papers

Published

1. Kumar, D., **Kumar, M.**, and Dey, S. (2020). Inferences for the type-II exponentiated loglogistic distribution based on order statistics with application. Journal of Statistical Theory and Applications, 19(3),352-367. (Scopus Indexed Journal)

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3. Kumar, D., **Kumar, M.**, and Joorel, J. S. (2020). Estimation with modified power function distribution based on order statistics with application to evaporation data. Annals of Data Science, DOI: 10.1007/s40745-020-00244-6. (Scopus Indexed Journal)

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Chapter 1

Preliminaries and Basic Concepts

Basic concepts defined in this chapter will be helpful in succeeding chapters.

1.1 Order Statistics

Perhaps the earliest model for ordered random variables is order statistics. If sample observations are ascending in order according to their eminence then we call them ordered values. Let the random variables $\{Z_u\}$, $u = 1,2,\ldots,n$ are written in ascending order of eminence like

$$Z_{1:n} \leq Z_{2:n} \leq \ldots \leq Z_{n:n} ,$$

then, we represents uth-order statistics $Z_{u:n}$. Usually in sampling theory, we assume that $\{Z_u\}$ are identically distributed and statistically independent. But in case of order statics $Z_{u:n}$ are necessarily dependent. Some commonly used order statistics are the extremes $Z_{1:n}$ and $Z_{n:n}$, the range $W = Z_{n:n} - Z_{1:n}$, deviation from sample mean of extremes, $Z_{n:n} - \bar{Z}$, and studentized range,W/S_v from a random sample of $N(\mu,\sigma^2)$, where S_v is estimate of σ with v degrees of freedom. The extremes occures in study of droughts, floods, fatigue failure and fracture toughness. Range is a common tool in quality control to estimate standard deviation. Extreme deviation is an essential tool in outliers detection process. While the studentized range is the basis of many quick tests in small samples, and important in the analysis of variance to rank treatment means.

1.2 Distribution of Order Statistics

Let $Z_1, Z_2, \ldots, Z_n$, a sample of size n, randomly selected from a continuous population with cumulative distribution function (cdf) $H(z)$ and probability density function (pdf) $h(z)$. Then uth order statistics' pdf is

$$h_{Z_{u:n}}(z) = C_{u:n}\, H^{u-1}(z)\ [1 - H(z)]^{n-u}\ h(z); \quad -\infty < z < \infty, \tag{1.1}$$

where

$$C_{u:n} = \frac{n!}{(u-1)!(n-u)!}.$$

Special Cases

The pdf of first order statistics is

$$h_{Z_{1:n}}(z) = n\ [1 - H(z)]^{n-1}\ h(z). \tag{1.2}$$

The pdf of nth order statistics is

$$h_{Z_{n:n}}(z) = n\, H^{n-1}(z) h(z). \tag{1.3}$$

and $Z_{u:n}$ follows the distribution function

$$\begin{aligned} H_{u:n}(z) &= Pr\left[Z_{u:n} \le z\right] \\ &= Pr\left[\text{at least } u \text{ of } Z_1, Z_2, \ldots, Z_n \le z\right] \\ &= \sum_{r=u}^{n} Pr\left[\text{exactly } r \text{ of } Z_1, Z_2, \ldots, Z_n \le z\right] \\ &= \sum_{r=u}^{n} \binom{n}{r} \left[H(z)\right]^r \left[1 - H(z)\right]^{n-r} \qquad (1.4) \\ &= C_{u:n} \int_o^{H(z)} t^{u-1} (1-t)^{n-u} dt \qquad (1.5) \\ &= I_{H(z)}\left(u, n-u+1\right), \qquad (1.6) \end{aligned}$$

where

$$\begin{aligned} I_z(\alpha, \beta) &= \frac{1}{B(\alpha, \beta)} \int_0^z t^{\alpha-1} \, (1-t)^{\beta-1} dt, \\ &\text{and} \\ B(\alpha, \beta) &= \int_0^1 t^{\alpha-1} \, (1-t)^{\beta-1} dt. \end{aligned}$$

The result in RHS of (1.6) is obtained with the help of incomplete beta function and binomial sums. Khan (1991) used negative binomial sums to obtain this result.

$$\begin{aligned} H_{u:n}(z) &= \sum_{r=0}^{n-u} \binom{r+u-1}{u-1} \left[H(z)\right]^u \left[1 - H(z)\right]^r \\ &= \sum_{r=u}^{n} \binom{r-1}{u-1} \left[H(z)\right]^u \left[1 - H(z)\right]^{r-u}. \qquad (1.7) \end{aligned}$$

The pdf of $Z_{u:n}$ can be derived by differentiating (1.6) with respect to z for continuous case.

The pth moment of $Z_{u:n}$ can be calculated by

$$E\left(Z_{u:n}^p\right) = \int_{-\infty}^{\infty} z^p h_{u:n}(z) dz. \qquad (1.8)$$

The uth and vth order statistics' joint pdf is

$$\begin{aligned} h_{Z_{u:n},Z_{v:n}}(z,y) &= C_{u,v:n}\, H^{u-1}(z)\, [H(y)-H(z)]^{v-1-u} \\ &\times \; [1-H(y)]^{n-v} h(z)h(y); \quad -\infty < z < y < \infty, \end{aligned} \tag{1.9}$$

for $z < y$, $u, v = 1, 2, \ldots$, $u < v$ and

$$C_{u,v:n} \; = \; \frac{n!}{(u-1)!\; (v-u-1)!\; (n-v)!}.$$

The cdf of $Z_{u:n},\ \&\ Z_{v:n},\ 1 \leq u < v \leq n$ is

$$\begin{aligned} H_{u,v:n}(z,y) &= Pr\,(Z_{u:n} \leq z, Z_{v:n} \leq y) \\ &= Pr\Big(\text{ at leaset } u \text{ of } Z_1, Z_2, \ldots, Z_n \text{ are at most } z\ \& \\ &\quad \text{at least } v \text{ of } Z_1, Z_2, \ldots, Z_n \text{ are at most } y \Big) \\ &= \sum_{s=v}^{n} \sum_{r=u}^{s} Pr\Big(\text{ exactly } r \text{ of } Z_1, Z_2, \ldots, Z_n \text{ are at most } z\ \& \\ &\quad \text{ exactly } s \text{ of } Z_1, Z_2, \ldots, Z_n \text{ are at most } y \Big) \\ &= \sum_{s=v}^{n} \sum_{r=u}^{s} \frac{n!}{r!\; (s-r)!\; (n-s)!}\; [H(z)]^r\; [H(y)-H(z)]^{s-r}\; [1-H(y)]^{n-s} \end{aligned} \tag{1.10}$$

The cdf of $Z_{u:n}$ and $Z_{v:n}$ can be written as

$$\begin{aligned} H_{u,v:n}(z,\ y) &= C_{u,v:n} \int_0^{H(z)} \int_x^{H(y)} x^{u-1} (t-x)^{v-u-1} (1-t)^{n-v} dx dt \\ &= I_{H(z),H(y)}(u, v-u, n-v+1); \quad -\infty < z < y < \infty. \end{aligned} \tag{1.11}$$

which is incomplete bivariate beta function.

For $z \leq y$

$$H_{u,v:n}(z,y) = H_{v:n}(y). \tag{1.12}$$

The pth and qth order product moment of $Z_{u:n}$ and $Z_{v:n}$ is given by

$$E\left(Z_{u:n}^{p}Z_{v:n}^{q}\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{y}z^{p}y^{q}h_{u,v:n}(z,y)dzdy. \tag{1.13}$$

In general the pdf of $Z_{i_1:n},Z_{i_2:n},\ldots,Z_{i_k:n}$, for $1\leq i_1<i_2<\ldots<i_k\leq n$ and $-\infty<Z_{i_1}<Z_{i_2}<\ldots<Z_{i_k}<\infty$, is given by

$$h_{i_1,i_2,\ldots,i_k:n}(z_{i_1},z_{i_2},\ldots,z_{i_k})=n!\left(\prod_{j=1}^{k}h(z_{i_j})\right)\prod_{j=0}^{k}\left[\frac{\left(H(z_{i_{j+1}})-H(z_{i_j})\right)^{i_{j+1}-i_j-1}}{(i_{j+1}-i_j-1)!}\right], \tag{1.14}$$

where, $z_0=-\infty, z_{k+1}=\infty, i_0=0, i_{k+1}=n+1$.

Also the accuracy of calculation can be checked by the following relation David and Nagaraja (2003),

$$\sum_{u=1}^{n}E\left(Z_{u:n}^{p}\right) = nE\left(Z^{p}\right);\ \ p=1,2,\ldots \tag{1.15}$$

$$\sum_{u=1}^{n}\sum_{v=1}^{n}E\left(Z_{u:n}^{p}Z_{v:n}^{q}\right) = nE\left(Z^{p+q}\right)+n(n-1)E\left(Z^{p}\right)E\left(Z^{q}\right);\ \ p,q=1,2,\ldots \tag{1.16}$$

and

$$\sum_{u=1}^{n}\sum_{v=1}^{n}Cov\left(Z_{u:n},Z_{v:n}\right) = nVar(Z), \tag{1.17}$$

where, $E(Z^{p})=E(Z_{1:1}^{p})$.

1.3 Moments and Recurrence Relations

In the last six decades or so, we see a spur in the efforts of order statistics, which are applied successfully to almost every possible sphere of human activity. Also, the order statistics' moments are fruitful in broad practical and theoretical situations like best linear unbiased estimators (BLUEs) of scale and location parameters for instance of censored or complete samples, entropy estimation, quality control, goodness of fit tests, characterization of probability distributions, reliability etc. It's miles visible that reliability of an object will be high if duration of failure of gadgets is high, which increases the cost of product in terms of time and money. In this situation experimenter is not able to predict failure of products by analyzing them for a short period. So he requires few early failures for prediction and this can be achieved through order statistics' moments. The early applications order statistics were concerned with empirical economic studies and coordination among various projects and efficient utilization of future emergencies.

Since the turn of this century, lot of attention paid to order statistics and their moments. Pearson (1902) and Galton (1902) explored the distribution of the difference of successive order statistics. For more information, see Arnold et al. (2008), Arnold and Balakrishnan (2012), Sarhān and Greenberg (1962), and David and Nagaraja (2003).

Recurrence relations and identities have achieved prominence for three primary reasons:

i. Shorten the time and labour and also lessen the number of direct computation.

ii. They give relationship between higher and lower order moments and hence higher order moments can easily assessed.

iii. Dispense some easy checks to check exactness of order statistics' moments.

For logistic distribution, Tarter (1966) and Shah (1966, 1970) found order statistics' moments. For the gamma distribution, Joshi (1979b) and Krishnaiah and Rizvi (1967) derived recurrence relations for order statistics' moments. Joshi (1982) also discovered several mixed aspects of order statistics recurrence relations for exponential and truncated exponential distributions. For the power function distribution, Malik (1967) developed recurrence relations for order statistics' moments.

Some recurrence relations of generalized Lindley, power Lindley, power gereralize Weibull, Extended exponential, Lindley and complementary exponential-geometric distributions for single and product order statistics' moments are established by Kumar and Goyal (2019a,b), Kumar and Dey (2017a,b),Kumar et al. (2017), Sultan and Al-Thubyani (2016), Balakrishnan et al. (2015), respectively. For extended exponential distribution, Kumar et al. (2017) entrenched order statistics' single and product moments and BLUEs of scale parameter besed on type-II right censored and complete samples. For Log-logistic distribution, Ahsanullah and Alzaatreh (2018) obtained order statistics' moments and estimate of parameters.

Several researchers have worked in the field of order statistics have appeared in the literature, see Kamps (1991), and Mohie El-Din et al. (1991), Childs et al. (2000), Sultan et al. (2000), Mahmoud et al. (2005), Sultan and Al-Thubyani (2016), Genç (2012), Kumar (2015), Balakrishnan et al. (2015), Kumar and Dey (2017b), Kumar and Goyal (2019a,b), Kumar et al. (2020b), Balakrishnan and Cohen (1991), Sanmel and Thomas (1997), Balakrishnan et al. (1996), Sultan et al. (2000), Mahmoud et al. (2005), Jabeen et al. (2013), Sultan and Al-Thubyani (2016), Kumar et al. (2017), Kumar and Dey (2017a,b), Ahsanullah and Alzaatreh (2018), Kumar and Goyal (2019a,b), Kumar et al. (2020a,b), Lieblein (1955), Balakrishnan and Joshi (1981), Saleh et al. (1975), Joshi (1978, 1979a) and many others.

1.4 BLUEs of the Location and Scale Parameters

Let $Z_{1:n} \leq Z_{2:n} \leq \cdots \leq Z_{n:n}$ be the order statistics from the continuous population with pdf of the location-scale parameter be $h(z)$. Let δ and φ are the location and scale parameters, respectively. To compute the BLUEs of the location and scale parameters δ and φ, we utilize the single and product moments. There are many applications of the scale-parameter and location-scale parameter distributions, see Arnold et al. (2008), Meyer (1987) and Wasserman (2003). Let $Z_{1:n} \leq Z_{2:n} \leq \cdots \leq Z_{n-c:n}$, $c = 0(1)([n/2]-1)$, denote Type-II right censored sample of $h(z)$. Let us denote $Y_{u:n} = (Z_{u:n} - \delta)/\varphi$, $E(Y_{u:n}) = \delta^{(1)}_{u;n}$, $1 \leq u \leq (n-c)$, and $Cov(Y_{u:n}, Y_{v:n}) = \varphi_{u,v:n} = \delta^{(1,1)}_{u,v:n} - \delta^{(1)}_{u:n}\, \delta^{(1)}_{v:n}$, $1 \leq u < v \leq (n-c)$. We shall use the following notations

$$\begin{aligned}\mathbf{Z} &= (Z_{1:n}, Z_{2:n}, \ldots, Z_{(n-c):n})^T,\\ \delta &= (\delta_{1:n}, \delta_{2:n}, \ldots, \delta_{(n-c):n})^T,\\ \mathbf{1} &= \underbrace{(1,\ 1,\ \ldots,\ 1)^T}_{n-c},\end{aligned}$$

and

$$\Psi = ((\varphi_{u,v}));\ \ 1 \le u,\ \ v \le n-c,$$

where, $\delta_{u:n} = E(Y_{u:n})$, $\varphi_{uu} = Var(Y_{u:n})$ and $\varphi_{uv} = Cov(Y_{u:n}, Y_{v:n})$; $u, v = 1, 2, \ldots (n-c)$. Then the BLUEs of δ and φ are given by Arnold et al. (2008)

$$\delta^* = \sum_{u=1}^{n-c} a_u Z_{u:n} \quad \text{and} \qquad \varphi^* = \sum_{u=1}^{n-c} \varphi_u Z_{u:n}, \tag{1.18}$$

where,

$$a_u = \left\{\frac{\boldsymbol{\delta^T\Psi^{-1}\delta 1^T\Psi^{-1} - \delta^T\Psi^{-1}1\delta^T\Psi^{-1}}}{(\boldsymbol{\delta^T\Psi^{-1}\delta})(\mathbf{1^T\Psi^{-1}1}) - (\boldsymbol{\delta^T\Psi^{-1}1})^2}\right\}, \tag{1.19}$$

$$b_u = \left\{\frac{\mathbf{1^T\Psi^{-1}1}\boldsymbol{\delta^T\Psi^{-1}} - \mathbf{1^T\Psi^{-1}}\boldsymbol{\delta}\mathbf{1^T\Psi^{-1}}}{(\boldsymbol{\delta^T\Psi^{-1}\delta})(\mathbf{1^T\Psi^{-1}1}) - (\boldsymbol{\delta^T\Psi^{-1}1})^2}\right\}. \tag{1.20}$$

Furthermore, the variances and covariance of these BLUEs are given by Arnold et al. (2008)

$$Var(\delta^*) = \varphi^2\left\{\frac{\boldsymbol{\delta^T\Psi^{-1}\delta}}{(\boldsymbol{\delta^T\Psi^{-1}\delta})(\mathbf{1^T\Psi^{-1}1}) - (\boldsymbol{\delta^T\Psi^{-1}1})^2}\right\}, \tag{1.21}$$

$$Var(\varphi^*) = \varphi^2\left\{\frac{\mathbf{1^T\Psi^{-1}1}}{(\boldsymbol{\delta^T\Psi^{-1}\delta})(\mathbf{1^T\Psi^{-1}1}) - (\boldsymbol{\delta^T\Psi^{-1}1})^2}\right\}, \tag{1.22}$$

and

$$Cov(\delta^*,\ \varphi^*) = \varphi^2\left\{\frac{-\boldsymbol{\delta^T\Psi^{-1}1}}{(\boldsymbol{\delta^T\Psi^{-1}\delta})(\mathbf{1^T\Psi^{-1}1}) - (\boldsymbol{\delta^T\Psi^{-1}1})^2}\right\}. \tag{1.23}$$

The values of a_u and b_u can be obtained for different values of sample sizes for example $n = 7,\ 10$, and different censoring cases $c = 0(1)([n/2]-1)$, and for some selected values for parameters. The coefficient of the BLUEs a_u and b_u given by (1.19) and (1.20) respectively, the conditions,

$$\sum_{i=1}^{n-c} a_i = 1$$

and

$$\sum_{i=1}^{n-c} b_i = 0,$$

which are used to check the computations accuracy.

1.5 Method of Maximum Likelihood Estimation

Methods name clearly indicates the way of obtaining estimator at which likelihood function attains its maximum. Let Z be a random variable with density $h(z;\Delta)$, where $\Delta = (\delta_1, \delta_2, \ldots, \delta_k)$ is a k-dimensional parameter vector. Therefore the M.L.E. of Δ usually denoted by $\hat{\Delta}_{mle} = (\hat{\delta}_{mle1}, \hat{\delta}_{mle2}, \ldots, \hat{\delta}_{mlek})$ is obtained by solving the following system of equations

$$\frac{\partial log\,[\ell(\Delta|\underline{z})]}{\partial \delta_i} = 0 \xrightarrow{s.t.} \frac{\partial^2 log\,[\ell(\Delta|\underline{z})]}{\partial \delta_i^2} < 0;\;\; i = 1,2,\ldots,k, \tag{1.24}$$

where, $log\,[\ell(\Delta|\underline{z})] = \sum_{u=1}^{n} log\,[h(z_u,\Delta)]$ and $\underline{z} = (z_1, z_2, \ldots, z_n)$, denotes respectively the log-likelihood function and a random sample of size n. In general, M.L.E.s are not unbiased but consistent estimators. M.L.E.s are also satisfies invariance property i.e. if $\hat{\Delta}_{mle}$ is the M.L.E. of Δ, then the M.L.E. of one-to-one transformation $g(\Delta)$ is $g(\hat{\Delta}_{mle})$.

1.6 Method of Moments

Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from a population with pdf or pmf $H(z, \Delta)$, where $\Delta = (\delta_1, \delta_2, \ldots, \delta_k)$. Then the moment estimator of δ i.e. $\hat{\Delta}_{mm} = (\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_k)$ is obtained by solving the following system of k equation.

$$\sum_{u=1}^{n} \frac{z_u^r}{n} = \int_{-\infty}^{\infty} z^r h(z) dz \quad \text{for continuous}$$
$$\sum_{u=1}^{n} \frac{z_u^r}{n} = \sum_{x} z^r h(z), \quad r = 1, 2, \ldots, k, \quad \text{for discrete} \tag{1.25}$$

where, $\sum_{u=1}^{n} \frac{z_u^r}{n}$ is the rth sample moment and $\int_{-\infty}^{\infty} z^r h(z) dz$ or $\sum_x z^r h(z)$ is the rth population moment.

1.7 Methods of Generating Distribution

The amount of data available for analysis is growing increasingly faster, requiring new probabilistic distributions to better describe each phenomenon or experiment studied. Distributions with more complexity and greater parameters can be with the help of computer softwares.

The literature in the field describes several generalizations and extensions of symmetric, asymmetric, discrete and continuous distributions. The relevance of these new models is that, according to situation, each one of them can better fit the mass of data. We presents several classes of distributions described in literature, their nomenclature and the title of the work where they have been presented.

1. **Exponentiated Generalized:** For constant $\alpha > 0$ Mudholkar et al. (1995) defined exponentiated generalized as

$$G(z) = H^{\alpha}(z)$$

2. **Beta1 Generalized:** Eugene et al. (2002) presents beta1 generalized model as

$$G(z) = \frac{1}{B(\alpha,\beta)} \int_0^{H(z)} t^{\alpha-1}(1-t)^{\beta-1}dt;\;\; \alpha,\beta > 0 \; and \; 0 < t < 1.$$

3. **Beta2 Generalized:** Tahir and Nadarajah (2013) presents beta2 generalized model as

$$G(z) = \frac{1}{B(\alpha,\beta)} \int_0^{H(z)} t^{\alpha-1}(1+t)^{-(\alpha+\beta)}dt;\;\; \alpha,\beta > 0 \; and \; t > 0.$$

4. **Mc1 Generalized:** McDonald (1984) presents Mc1 generalized model as

$$G(z) = \frac{1}{B(\alpha,\beta)} \int_0^{H^{\gamma}(z)} t^{\alpha-1}(1-t)^{\beta-1}dt;\;\; \alpha,\beta,\gamma > 0 \; and \; 0 < t < 1.$$

5. **Mc2 Generalized:** Tahir and Nadarajah (2013) presents Mc2 generalized model as

$$G(z) = \frac{1}{B(\alpha,\beta)} \int_0^{H^{\gamma}(z)} t^{\alpha-1}(1+t)^{-(\alpha+\beta)}dt;\;\; \alpha,\beta,\gamma > 0 \; and \; t > 0.$$

6. **Kumaraswamy G_1:** Cordeiro and de Castro (2011) defined Kumaraswamy G_1 model as

$$G(z) = 1 - (1 - H^{\alpha}(z))^{\beta}$$

7. **Kumaraswamy Type 2:** For $\alpha > 0$ and $\beta > 0$ Tahir and Nadarajah (2013) defined Kumaraswamy type 2 model as

$$G(z) = 1 - \left[1 - (1 - H(z))^{\alpha}\right]^{\beta}$$

8. **Marshall-Olkin:** Marshall and Olkin (1997) presented Masrshall-Olkin model as

$$G(z) = \frac{H(z)}{H(z) + \alpha(1 - H(z))};\;\; \alpha > 0.$$

9. **Marshall-Olkin G_1:** Jayakumar and Mathew (2008) presented Masrshall-Olkin G_1 model as

$$G(z) = 1 - \left[\frac{\alpha(1-H(z))}{H(z)+\alpha(1-H(z))}\right]^{\beta}; \ \alpha, \beta > 0.$$

10. **Marshall-Olkin G_1:** Tahir and Nadarajah (2013) presented a different type of Masrshall-Olkin G_1 model as

$$G(z) = \left[\frac{H(z)}{H(z)+\alpha(1-H(z))}\right]^{\theta}; \ \alpha, \theta > 0.$$

11. **Gamma-Generated:** Zografos and Balakrishnan (2009) defined Gamma-Generated model as

$$G(z) = \frac{\theta^{\gamma}}{\Gamma(\gamma)} \int_0^{-ln(1-H(z))} t^{\gamma-1} e^{-\theta t} dt.$$

12. **Gamma-Generated:** Cordeiro et al. (2017b) also defined a different form of Gamma-Generated model as

$$G(z) = 1 - \frac{\theta^{\gamma}}{\Gamma(\gamma)} \int_0^{-ln(H(z))} t^{\gamma-1} e^{-\theta t} dt.$$

13. **Extended Weibull Distribution:** Silva et al. (2013) defined extended Weibull distribution as

$$G(z) = 1 - \frac{C(\gamma e^{-\beta H(z)})}{C(\gamma)},$$

where $z > 0, \gamma > 0$ and $C(\gamma) = \sum_{n=1}^{\infty} a_n \gamma^n$.

14. **Kumaraswamy-G Poisson:** Ramos (2014) defined Kumaraswamy-G Poisson model as

$$G(z) = \frac{1 - exp(-\theta H(z))}{1 - exp(-\theta)}.$$

15. **Kumaraswamy-G Exponentiated:** Ramos (2014) defined Kumaraswamy-G exponentiated model as

$$G(z) = \left[1-(1-H^{\gamma}(z))^{\alpha}\right]^{\beta};\ \ \alpha,\beta,\gamma > 0.$$

16. **Beta Weibull Poisson Family:** Paixao (2014) defined Beta Weibull Poisson family model as

$$G(z) = \frac{exp(\theta exp(-\gamma H^{\beta}(z))) - exp(\theta)}{1-exp(\theta)}.$$

17. **Beta Kummer Generalized:** Pescim et al. (2012) defined Beta Kummer generalized model as

$$G(z) = \int_0^{H(z)} Kt^{\alpha-1}(1-t)^{\beta-1}e^{-\gamma t}dt;\ \ \alpha,\beta > 0, -\infty < \gamma < \infty.$$

18. **Weibull Gneralized Poisson Distribution:** Paixao (2014) defined Beta Weibull generalized Poisson distribution model as

$$G(z) = \frac{exp\left(-\frac{\theta}{\beta}R(-\beta e^{-\beta})\right) - exp\left(-\frac{\theta}{\beta}R(\xi(z))\right)}{exp\left(-\frac{\theta}{\beta}R(-\beta e^{-\beta})\right) - 1},$$

where, $R(z) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}m^{m-2}}{(m-1)!}z^m$ and $\xi(z) = -\beta exp(-\beta - \gamma z^{\alpha})$.

19. **G-Negative Binomial Family:** Paixao (2014) defined G-Negative Binomial family model as

$$G(z) = \frac{(1-\theta)^{-m} - [1-\theta(1-H(z))]^{-m}}{(1-\theta)^{-m}-1}.$$

20. **Zeta-G:** Paixao (2014) defined Zeta-G model as

$$G(z) = \frac{\xi(t) - Li_t[1-H(z)]}{\xi(t)},$$

where, $Li_t(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^t}$ and $\xi(t) = \sum_{m=1}^{\infty} \frac{1}{m^t}$.

21. **Power Series Distributions Family:** Consul and Famoye (2006) defined Power Series distributions family model as

$$G(z) = \sum_{m=0}^{z} \frac{B^(m)(\alpha)}{m!B(\gamma)}(\gamma-\alpha)^m.$$

22. **Basic Lagrangian:** Consul and Famoye (2006) defined Basic Lagrangian model as

$$G(z) = \sum_{m=1}^{z} \frac{1}{m!}\left[(B(0))^m\right]^{m-1}.$$

23. **Lagrangian Delta:** Consul and Famoye (2006) defined Lagrangian delta model as

$$G(z) = \sum_{m=n}^{z} \frac{n}{(m-n)!m}\left[(B(0))^m\right]^{m-n}.$$

24. **Weibull Marshall-Olkin-G (WMO-G) Family:** Korkmaz et al. (2019) proposed the *Weibull Marshall-Olkin-G* (WMO-G) family as

$$H(z;\alpha,\beta,\eta) = 1-\exp\left(-\left\{-\log\left[\frac{\alpha\bar{G}(z;\eta)}{1-\bar{\alpha}\bar{G}(z;\eta)}\right]\right\}^{\beta}\right).$$

1.8 Some Continuous Distributions

1. **Type-II Exponentiated Log-logistic Distribution**

   Recently, Rao et al. (2012) proposed Type-II exponentiated log-logistic (TIIELL) distribution with pdf

$$h(z;\tau,\varphi,\eta) = \frac{\tau\eta\left(\frac{z}{\varphi}\right)^{\eta-1}}{\varphi\left[1+\left(\frac{z}{\varphi}\right)^{\eta}\right]^{\tau+1}}, \quad z>0, \quad (\tau,\varphi)>0, \eta>1$$

and associated cdf is

$$H(z;\tau,\varphi,\eta) = 1 - \left[1 + \left(\frac{z}{\varphi}\right)^{\eta}\right]^{-\tau}, \quad z > 0, \quad (\tau,\varphi) > 0, \eta > 1$$

where φ is the scale parameter, and η and τ are the shape parameters of the distribution. If $\tau = 1$, then TIIELL distribution becomes log-logistic distribution, and if $\eta = 1$, then TIIELL distribution becomes Pareto type-II distribution.

2. **Log-logistic Distribution**

Let Z, a random variable, is said to follow log-logistic distribution with parameters φ, η denoted by $Z \sim LL(\varphi, \eta)$ if its pdf is

$$h(z;\varphi,\eta) = \frac{\eta \left(\frac{z}{\varphi}\right)^{\eta-1}}{\varphi \left[1 + \left(\frac{z}{\varphi}\right)^{\eta}\right]^{2}}, \quad z > 0, \quad \varphi > 0, \eta > 1. \tag{1.26}$$

and associated cdf is

$$H(z;\varphi,\eta) = 1 - \left[1 + \left(\frac{z}{\varphi}\right)^{\eta}\right]^{-1}, \quad z > 0, \quad \varphi > 0, \eta > 1$$

Log-logistic distribution is closed under scaling, i.e., if $Z \sim LL(\varphi, \eta)$, then for some $p > 0, pZ \sim (p\varphi; \eta)$. If $Z \sim LL(\varphi, \eta)$ then the transformation
$Y = log(Z) \sim logistic\ distribution[L(log(\varphi), 1/\eta)]$.

3. **Modified Power Function Distribution**

Recently, Okorie et al. (2017) introduced a two parameter modified power function (MPF) distribution with pdf

$$h(z;\alpha,\beta) \;=\; \frac{\alpha\, \beta\, (1-z)^{\beta-1}}{[1-(1-\alpha)\, (1-z)^{\beta}]^{2}}, \; 0 < z < 1, \; \alpha, \; \beta > 0,$$

and associated cdf is

$$H(z;\alpha,\beta) \;=\; 1-\frac{\alpha\,(1-z)^{\beta}}{[1-(1-\alpha)\,(1-z)^{\beta}]}\;0<z<1,\;\alpha,\;\beta>0.$$

If $\alpha = 1$, then MPF distribution becomes power function distribution.

4. **Power Function Distribution**

Let Z, a random variable, is said to follow power function distribution with shape parameter β denoted by $Z \sim power(\beta)$ if its pdf is

$$h(z;\beta) \;=\; \beta\,(1-z)^{\beta-1},\;\; z\in(0,1),\; \beta>0. \tag{1.27}$$

and associated cdf is

$$H(z;\beta) \;=\; 1-(1-z)^{\beta},\;\; z\in(0,1),\; \beta>0. \tag{1.28}$$

The density of power function is monotone increasing in nature with global maximum occurring at $z=\lambda$. Power distribution is closed under scaling, i.e., if $Z \sim power(\lambda,\beta)$, then for some $p<0, pZ \sim power(\lambda/p;\beta)$. It is also closed under maximum, i.e., if $Z \sim power(\lambda,\beta)$ and $Z_1 \sim power(\lambda,\beta_1)$, then $max(Z,Z_1) \sim power(\lambda,\beta+\beta_1)$. Power function distribution with $\beta=1$ reduces to $U(0,\lambda)$ distribution. $power(1;\beta)$ is a special case of Kumaraswamy distribution whose density is given by $\theta\beta z^{\beta-1}(1-z^{\beta})^{\theta-1},\;\; 0<z<1$. If $Y \sim exp(\theta)$, then the transformation $Z=(\lambda e^{Y})^{-1} \sim power(\lambda,\beta)$. Inverse of power function random variable follows Pareto distribution.

5. **Extended Power Lindley Distribution**

Recently the three parameter extended power Lindley distribution was proposed by Alkarni (2015) for the flexibility of purpose. A random variable Z said to follow extended power Lindley (EPL) distribution if it has following pdf

$$h(z;\tau,\xi,\kappa)=\frac{\tau\xi^{2}}{\xi+\kappa}(1+\kappa z^{\tau})\,z^{\tau-1}\,e^{-\xi z^{\tau}},\quad z>0;\quad \tau>0,\xi>0,\kappa>0$$

and associated cdf is

$$H(z;\tau,\xi,\kappa) = 1 - \left(1 + \frac{\kappa\xi}{\xi+\kappa} z^{\tau}\right) e^{-\xi z^{\tau}}, \quad z > 0; \quad \tau > 0, \xi > 0, \kappa > 0.$$

For $\kappa = 1$ and $\kappa = 1, \ \tau = 1$, the EPL distribution reduces to power Lindley (PL) and Lindley distributions respectively.

6. **Power Lindley Distribution**

Let Z, a random variable, is said to follow power Lindley distribution with parameters ξ, τ if its pdf is

$$h(z;\xi,\tau) = \frac{\tau\xi^2}{(1+\xi)}(1+z^{\tau})z^{\tau-1}exp(-\xi z^{\tau}), \ z > 0, \xi, \tau > 0. \tag{1.29}$$

and associated cdf is

$$H(z;\tau,\xi) = 1 - \left(1 + \frac{\xi}{\xi+1} z^{\tau}\right) e^{-\xi z^{\tau}}, \quad z > 0; \quad \xi, \tau > 0.$$

7. **Lindley Distribution**

Let Z, a random variable is said to follow Lindley distribution with parameter ξ if its pdf is

$$h(z;\xi) = \frac{\xi^2}{(1+\xi)}(1+z)exp(-\xi z), \ z > 0, \xi > 0. \tag{1.30}$$

and associated cdf is

$$H(z;\xi) = 1 - \left(1 + \frac{\xi}{\xi+1} z\right) e^{-\xi z}, \quad z > 0; \quad \xi > 0.$$

It is also useful in medicine, engineering and biology. Ghitany et al. (2008) used it for modeling in mortality studies. The parameter, $\xi > 0$ can result in either a unimodal or monotone decreasing distribution.

8. **Generalized Topp-Leone Distribution**

Recently, Shekhawat and Sharma (2020) proposed a generalization of the Topp-Leone distribution called generalized Topp-Leone (GTL) distribution. A random variable Z said to follow generalized Topp-Leone (GTL) distribution if it has following pdf

$$h(z;\kappa,\xi) \;=\; 2\kappa\xi z^{\kappa\xi-1}(1-z^{\kappa})(2-z^{\kappa})^{\xi-1},\; 0<z<1,\; \kappa,\; \xi>0.$$

and associated cdf is

$$H(z;\kappa,\xi) = (z^{\kappa}(2-z^{\kappa}))^{\xi},\; 0<z<1,\; \kappa,\; \xi>0.$$

9. **Topp-Leone Distribution**

The single parameter (ξ) Topp-Leone distribution is defined by the pdf

$$h(z;\xi) \;=\; 2\xi z^{\xi-1}(1-z)(2-z)^{\xi-1},\; 0<z<1,\; \xi>0.$$

and associated cdf is

$$H(z;\xi) = (z(2-z))^{\xi},\; 0<z<1,\; \xi>0.$$

This distribution has J-shaped frequency curve for $\xi<1$. Topp-Leone distribution is also effective for the generation of new flexible families of distributions.

10. **Weibull Marshall-Olkin Lomax (WMOL) Distribution**

Let Z, a random variable is said to follow *Weibull Marshall-Olkin Lomax* (WMOL) distribution if its pdf is

$$\begin{aligned} h(z;\beta,\theta,\lambda,\alpha) \;=\; & \frac{\beta\theta\lambda}{(1+\lambda z)[1-\bar{\alpha}(1+\lambda z)^{-\theta}]}\left(-log\left[\frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}}\right]\right)^{\beta-1} \\ & \times\; \exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}}\right]\right)^{\beta}\right\},\;\; z\geq 0,\; \theta,\lambda,\beta,\alpha>0. \end{aligned}$$

The associated cdf is

$$H(z;\beta,\theta,\lambda,\alpha)=1-\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}}\right]\right)^{\beta}\right\},\ z\geq 0,\ \theta,\lambda,\beta,\alpha>0.$$

where, $\theta > 0$ and $\beta > 0$ are two shape parameters and $\alpha > 0$, $\lambda > 0$ are the scale parameters.

Additionally, the new model contains some distributions as special cases, these sub-models being listed in Table 1.

TABLE 1.1: Special cases of the WMOL distribution

Parametric values in WMOL distribution	Sub-models
$\beta=1$	Marshall-Olkin Lomax distribution(α,θ,λ)
$\alpha=1$	Weibull Lomax distribution(β,θ,λ)
$\alpha=\beta=1$	Lomax distribution (θ)

11. **Marshall-Olkin Lomax Distribution**

The pdf corresponding to MOL distribution is

$$\begin{aligned} h(z;\theta,\lambda,\alpha) &= \frac{\theta\lambda}{(1+\lambda z)[1-\bar{\alpha}(1+\lambda z)^{-\theta}]} \\ &\times \exp\left\{-\left(-log\left[\frac{\alpha\ (1+\lambda z)^{-\theta}}{1-\bar{\alpha}\ (1+\lambda z)^{-\theta}}\right]\right)\right\},\ z\geq 0,\ \theta,\lambda,\alpha>0. \end{aligned}$$

and associated cdf is

$$H(z;\theta,\lambda,\alpha)=1-\exp\left\{-\left(-log\left[\frac{\alpha\ (1+\lambda z)^{-\theta}}{1-\bar{\alpha}\ (1+\lambda z)^{-\theta}}\right]\right)\right\},\ z\geq 0,\ \theta,\lambda,\alpha>0.$$

12. **Weibull Lomax Distribution**

The pdf corresponding to WL distribution is

$$\begin{aligned} h(z;\theta,\lambda,\beta) &= \frac{\beta\theta\lambda}{(1+\lambda z)}\left(\theta log(1+\lambda z)\right)^{\beta-1} \\ &\times \exp\left\{-\left(\theta log(1+\lambda z)\right)^{\beta}\right\},\ z\geq 0,\ \theta,\lambda,\beta>0. \end{aligned} \tag{1.31}$$

and associated cdf is

$$H(z;\theta,\lambda,\beta) = 1-\exp\left\{-\left(-log\left[(1+\lambda z)^{-\theta}\right]\right)^{\beta}\right\},\ \ z\geq 0,\ \theta,\lambda,\beta>0.$$

13. **Lomax Distribution**

The pdf corresponding to Lomax distribution is

$$h(z;\theta,\lambda) \quad = \quad \frac{\theta\lambda}{(1+\lambda z)^{\theta+1}},\ \ z\geq 0,\ \theta,\lambda>0. \tag{1.32}$$

and associated cdf is

$$H(z;\theta,\lambda) = 1-\exp\left\{\ -\left(\ -log\left[\ (1+\lambda z)^{-\theta}\ \right]\ \right)\right\},\ \ z\geq 0,\ \theta,\lambda>0.$$

Chapter 2

Inferences for Type-II Exponentiated Log-logistic Distribution based on Order Statistics with Application*

2.1 Introduction

This chapter follows the following structure: In Sections 2.2 and 2.3, we derive the exact expressions for the single and product moments of order statistics from TIIELLD. In Section 2.4, we obtain BLUEs for δ and φ by using these moments. These BLUEs are then used in Section 2.5 to obtain $(1-\alpha)100\%$ confidence intervals (CIs) for the location and scale parameters of the BLUEs based on the pivotal quantities. Besides, lower and upper percentage points of pivotal quantities through Edgeworth approximations are obtained and compare the results with simulated percentage points. A real data application is provided in Section 2.6. Finally, in Section 2.7 , we draw a conclusion for the chapter.

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In recent past several authors have tabulated the moments of order statistics quite extensively for several distributions and also obtained MLEs and BLUEs for the scale and location parameters of the distributions based on complete and type-II censored samples. Further, they developed point prediction and goodness-of-fit tests. In this regard, readers may refer to the works of Balakrishnan and Cohen (1991), Balakrishnan and Sultan (1998), Saran and Pushkarna (2000), Childs et al. (2000), Sultan et al. (2000), Jabeen et al. (2013), Balakrishnan et al. (2015), Sultan and Al-Thubyani (2016), Kumar et al. (2017), Kumar and Dey (2017a,b), Ahsanullah and Alzaatreh (2018), Kumar and Goyal (2019a,b) Kumar et al. (2020b) and many others.

Rao et al. (2012) suggested a generalization of the log-logistic distribution called Type-II exponentiated log-logistic (TIIELL) distribution with pdf

$$h(z) = \frac{\tau\eta\left(\frac{z}{\varphi}\right)^{\eta-1}}{\varphi\left[1+\left(\frac{z}{\varphi}\right)^{\eta}\right]^{\tau+1}}, \quad z>0, \ \ (\tau,\varphi)>0, \eta>1. \tag{2.1}$$

The associated cdf and quantile function are, respectively given by

$$H(z) = 1-\left[1+\left(\frac{z}{\varphi}\right)^{\eta}\right]^{-\tau}, \quad z>0, \ \ (\tau,\varphi)>0, \eta>1. \tag{2.2}$$

and

$$H^{-1}(z) = \varphi\left(\left(\frac{1}{1-z}\right)^{\frac{1}{\tau}}-1\right)^{\frac{1}{\eta}}. \tag{2.3}$$

where, φ is the scale parameter, and η and τ are the shape parameters of the distribution. If $\tau = 1$, then Eq. (2.1) becomes log-logistic distribution, and if $\eta = 1$, then TIIELL distribution becomes Pareto type-II distribution. The pth moments of the TIIELL distribution in (2.1) can be easily computed as

$$E(Z^p) = \varphi^p \tau B\left(\tau-\frac{p}{\eta}, 1+\frac{p}{\eta}\right). \tag{2.4}$$

where $B(.,.)$ is the beta function. Note that the pth moment exists iff $\eta > \max\{1, p/\tau\}$. A more compact form of (2.4) can be derived using the fact that $\Gamma(z)\Gamma(1-z) = \pi csc(\pi z)$ [see

Abramowitz and Stegun (1964)] as follows

$$E(Z^p) = \frac{\varphi^p \frac{p}{\eta}\pi csc\frac{p\pi}{\eta}}{\Gamma\tau} \prod_{i=1}^{\tau-1}\left(\tau - \frac{p}{\eta} - i\right). \tag{2.5}$$

Therefore,

$$E(Z) = \frac{\varphi \frac{\pi}{\eta} csc\frac{\pi}{\eta}}{\Gamma\tau} \prod_{i=1}^{\tau-1}\left(\tau - \frac{1}{\eta} - i\right),$$

$$E(Z^2) = \frac{\varphi^2 \frac{2\pi}{\eta} csc\frac{2\pi}{\eta}}{\Gamma\tau} \prod_{i=1}^{\tau-1}\left(\tau - \frac{2}{\eta} - i\right)$$

and

$$Var(Z) = \left[\frac{\varphi^2 \frac{2\pi}{\eta} csc\frac{2\pi}{\eta}}{\Gamma\tau} \prod_{i=1}^{\tau-1}\left(\tau - \frac{2}{\eta} - i\right) - \left(\frac{\varphi \frac{\pi}{\eta} csc\frac{\pi}{\eta}}{\Gamma\tau} \prod_{i=1}^{\tau-1}\left(\tau - \frac{1}{\eta} - i\right)\right)^2\right].$$

2.2 Relations for Single Moments of Order Statistics

Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from TIIELL distribution, the pdf of uth order statistic is obtained by using equation (2.1) and (2.2) in (1.1) as follows:

$$h_{Z_{u:n}}(z) = \frac{\tau\eta}{\varphi} C_{u:n} \sum_{i=0}^{u-1} \binom{u-1}{i} (-1)^i \frac{\left(\frac{z}{\varphi}\right)^{\eta-1}}{\left[1+\left(\frac{z}{\varphi}\right)^{\eta}\right]^{\tau i+\tau+1+\tau(n-u)}}, \quad z \geq 0. \tag{2.6}$$

The pth moments of $Z_{u:n}$, $\mu_{u:n}^{(p)} = E(Z_{u:n}^p)$ can be derived from (2.6) as

$$\mu_{u:n}^{(p)} = \varphi^p \tau C_{u:n} \sum_{i=0}^{u-1} \binom{u-1}{i} (-1)^i B\left(\tau(i+n-u+1) - \frac{p}{\eta}, 1 + \frac{p}{\eta}\right). \tag{2.7}$$

Similarly as in (2.5), one can show that

$$\mu_{u:n}^{(p)} = \frac{\varphi^p p\tau n! \pi csc\frac{p\pi}{\eta}}{\eta(u-1)!(n-u)!} \sum_{i=0}^{u-1} \binom{u-1}{i} (-1)^i \prod_{j=1}^{\tau(i+n-u+1)-1} \frac{\tau(i+n-u+1) - j - \frac{p}{\eta}}{\Gamma(\tau(i+n-u+1)-1)}. \tag{2.8}$$

Note that from (2.8), the first and second moments of $Z_{u:n}$ are, respectively, given by

$$\mu_{u:n}^{(1)} = \frac{\varphi\tau n!\pi csc\frac{\pi}{\eta}}{\eta(u-1)!(n-u)!}\sum_{i=0}^{u-1}\binom{u-1}{i}(-1)^{i}\prod_{j=1}^{\tau(i+n-u+1)-1}\frac{\tau(i+n-u+1)-j-\frac{1}{\eta}}{\Gamma(\tau(i+n-u+1)-1)},$$

and

$$\mu_{u:n}^{(2)} = \frac{\varphi^2\tau n!\pi csc\frac{2\pi}{\eta}}{\eta(u-1)!(n-u)!}\sum_{i=0}^{u-1}\binom{u-1}{i}(-1)^{i}\prod_{j=1}^{\tau(i+n-u+1)-1}\frac{\tau(i+n-u+1)-j-\frac{2}{\eta}}{\Gamma(\tau(i+n-u+1)-1)},$$

Some special cases from equation (2.8) are

1. For $\tau = 1$, in (2.8), we get the explicit expression for order statistic of log-logistic distribution

$$\mu_{u:n}^{(p)} = \frac{\varphi^p pn!\pi csc\frac{p\pi}{\eta}}{\eta(u-1)!(n-u)!}(-1)^{u}\prod_{j=1}^{n}\frac{n-u+1-j-\frac{p}{\eta}}{\Gamma((n-u+1)-1)}.$$

2. If $u = n = 1$, we get

$$\mu_{1:1}^{(p)} = \varphi^p\tau B\left(\tau-\frac{p}{\eta}, 1+\frac{p}{\eta}\right)$$

which agrees with (2.4).

3. If $p = u = 1$ in (2.8), we get

$$\mu_{1:n}^{(1)} = \frac{\varphi\tau n\pi csc\frac{\pi}{\eta}}{\eta}\prod_{j=1}^{\tau n-1}\frac{\left[\tau n-j-\frac{1}{\eta}\right]}{\Gamma(\tau n-1)},$$

4. If $p = 1$, $u = n$ in (2.8), we get

$$\mu_{n:n}^{(1)} = \frac{\varphi\tau n\pi csc\frac{\pi}{\eta}}{\eta}\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^{i}\prod_{j=1}^{\tau(i+1)-1}\frac{\left[\tau(i+1)-j-\frac{1}{\eta}\right]}{\Gamma(\tau(i+1)-1)},$$

5. If $p = u = n = 1$ in (2.8), we get

$$\mu_{1:1}^{(1)} = \frac{\varphi\tau\pi csc\frac{\pi}{\eta}}{\eta}\prod_{j=1}^{\tau-1}\frac{\left[\tau-j-\frac{1}{\eta}\right]}{\Gamma(\tau-1)}, \tag{2.9}$$

which agree with equation (2.5) for $p = 1$.

It is interesting to note that (2.8) can be used easily to derive several recurrence relations for the moments of order statistics. Some of these recurrence relations already exist in the literature. Below, we provide some of these recurrence relations.

I. From equation (2.8) we can write

$$\mu_{u:n}^{(p)} = \frac{\varphi^p \, \tau \, p n(n-1)! \, \pi \csc\frac{p\pi}{\eta}}{\eta(u-1)(u-2)!(n-u)!} \sum_{i=0}^{u-1} \binom{u-1}{i} (-1)^{u-1+i} \prod_{j=1}^{\tau(i+n-u+1)-1} \frac{\tau(i+n-u+1)-j-\frac{p}{\eta}}{\Gamma(\tau(i+n-u+1)-1)}.$$

Let us define $\Delta(\tau,\eta) = \frac{\varphi^p \, \tau \, p\,(n-1)! \, \pi \csc\frac{p\pi}{\eta}}{\eta(u-2)!(n-u)!}$, we have

$$\begin{aligned}
\mu_{u-1:n-1}^{(p)} &= \Delta(\tau,\eta) \sum_{i=0}^{u-2} \binom{u-2}{i} (-1)^{i} \prod_{j=1}^{\tau(i+n-u+1)-1} \frac{\tau(i+n-u+1)-j-\frac{p}{\eta}}{\Gamma(\tau(i+n-u+1)-1)} \\
&= \Delta(\tau,\eta) \binom{u-2}{0} (-1)^{0} \prod_{j=1}^{\tau(n-u+1)-1} \frac{\tau(n-u+1)-j-\frac{p}{\eta}}{\Gamma(\tau(i+n-u+1)-1)} \\
&+ \Delta(\tau,\eta) \binom{u-2}{1} (-1)^{1} \prod_{j=1}^{\tau(n-u+2)-1} \frac{\tau(n-u+2)-j-\frac{p}{\eta}}{\Gamma(\tau(n-u+2)-1)} \\
&\vdots \\
&+ \Delta(\tau,\eta) \binom{u-2}{u-2} (-1)^{u-2} \prod_{j=1}^{\tau(n-1)-1} \frac{\tau(n-1)-j-\frac{p}{\eta}}{\Gamma(\tau(n-1)-1)},
\end{aligned}$$

which can be written in vector form as

$$\mu_{u-1:n-1}^{(p)} = \mathbf{1}' \, {}_i\mu_{u-1:n-1}^{(p)},$$

where $\mathbf{1}' = (1, 1, \ldots, 1)$ and ${}_i\mu_{u-1:n-1}^{(p)}$ denotes a vector of order $(1 \times u-2)$ and $((u-2) \times 1)$, respectively, where

$${}_i\mu_{u-1:n-1}^{(p)} = \begin{pmatrix} \Delta(\tau,\eta) \binom{u-2}{0} (-1)^{0} \prod_{j=1}^{\tau(n-u+1)-1} \frac{\tau(n-u+1)-j-\frac{p}{\eta}}{\Gamma(\tau(n-u+1)-1)} \\ \vdots \\ \vdots \\ \Delta(\tau,\eta) \binom{u-2}{u-2} (-1)^{u-2} \prod_{j=1}^{\tau(n-1)-1} \frac{\tau(n-1)-j-\frac{p}{\eta}}{\Gamma(\tau(n-1)-1)}. \end{pmatrix}$$

Therefore, we can write $\mu_{u:n}^{(p)}$ as

$$\begin{aligned}
\mu_{u:n}^{(p)} &= \frac{n}{u-1}\Delta(\tau,\eta)\left[\binom{u-1}{u-1}(-1)^{u-1}\prod_{j=1}^{\tau n-1}\frac{\tau n-j-\frac{p}{\eta}}{\Gamma(\tau n-1)}\right.\\
&+ \left.\sum_{i=0}^{u-2}\frac{u-1}{u-1-i}\binom{u-2}{i}(-1)^{i}\prod_{j=1}^{\tau(i+n-u+1)-1}\frac{\tau(i+n-u+1)-j-\frac{p}{\eta}}{\Gamma(\tau(i+n-u+1)-1)}\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{n}{u-1}\Delta(\tau,\eta)(-1)^{u-1}\prod_{j=1}^{\tau n-1}\frac{\tau n-j-\frac{p}{\eta}}{\Gamma(\tau n-1)}\\
&+ n\Delta(\tau,\eta)\sum_{i=0}^{u-2}\frac{1}{(u-1-i)}\binom{u-2}{i}(-1)^{i}\prod_{j=1}^{\tau(i+n-u+1)-1}\frac{\tau(i+n-u+1)-j-\frac{p}{\eta}}{\Gamma(\tau(i+n-u+1)-1)}\\
&= \frac{n}{u-1}\Delta(\tau,\eta)(-1)^{u-1}\prod_{j=1}^{\tau n-1}\frac{\tau n-j-\frac{p}{\eta}}{\Gamma(\tau n-1)}+n\,v'\,{}_{i}\mu_{u-1:n-1}^{(p)},
\end{aligned}$$

where $v' = \left(\frac{1}{u-1},\frac{1}{u-2},\frac{1}{u-3},\ldots,1\right)$ is vector of order $(1\times(u-2))$

II If $u=1$ in equation (2.8), we get

$$\mu_{1:n}^{(p)} = \prod_{j=1}^{\tau}\frac{n\left(j-\frac{p}{\eta}\right)}{\eta(n-1)\prod_{h=1}^{\tau+1}[\tau(n)-h]}\mu_{1:n-1}^{(p)}.$$

2.3 Relations for Product Moments of Order Statistics

Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from TIIELL distribution, the joint pdf of uth and vth order statistic is obtained by using equation (2.1) and (2.2) in (1.9) as follows:

$$\begin{aligned}
h_{u,v:n}(z,y) &= \frac{\tau^2\eta^2}{\varphi^2}C_{u,v:n}\sum_{i=0}^{u-1}\sum_{j=0}^{v-u-1}\binom{u-1}{i}\binom{v-u-1}{j}(-1)^{i+j}\\
&\times \frac{\left(\frac{z}{\varphi}\right)^{\eta-1}\left(\frac{y}{\varphi}\right)^{\eta-1}}{\left(1+\left(\frac{z}{\varphi}\right)^{\eta}\right)^{\tau(i+v-u-j)+1}\left(1+\left(\frac{y}{\varphi}\right)^{\eta}\right)^{\tau(n-v+1+j)+1}}
\end{aligned}$$

Therefore the product moments, $\mu_{u,v:n} = E(Z_{u:n}Z_{v:n})$, can be written as

$$\begin{aligned}
\mu_{u,v:n} &= \tau^2\eta^2 C_{u,v:n}\sum_{i=0}^{u-1}\sum_{j=0}^{v-u-1}\binom{u-1}{i}\binom{v-u-1}{j}(-1)^{i+j} \\
&\times \int_0^\infty\int_0^y \frac{\left(\frac{z}{\varphi}\right)^\eta\left(\frac{y}{\varphi}\right)^\eta}{\left(1+\left(\frac{z}{\varphi}\right)^\eta\right)^{\tau(i+v-u-j)+1}\left(1+\left(\frac{y}{\varphi}\right)^\eta\right)^{\tau(n-v+1+j)+1}}dzdy \\
&= \varphi\tau^2\eta^2 C_{u,v:n}\sum_{i=0}^{u-1}\sum_{j=0}^{v-u-1}\binom{u-1}{i}\binom{v-u-1}{j}(-1)^{i+j} \\
&\times \int_0^\infty \frac{\left(\frac{y}{\varphi}\right)^\eta}{\left(1+\left(\frac{y}{\varphi}\right)^\eta\right)^{\tau(n-v+1+j)+1}}\left(\frac{1}{\eta}\int_0^{\left(\frac{y}{\varphi}\right)^\eta}\frac{x^{\frac{1}{\eta}}}{(1+x)^{\tau(i+v-u-j)+1}}dx\right)dy \\
&= \varphi\tau^2\eta^2 C_{u,v:n}\sum_{i=0}^{u-1}\sum_{j=0}^{v-u-1}\binom{u-1}{i}\binom{v-u-1}{j}(-1)^{i+j} \\
&\times \int_0^\infty \frac{\left(\frac{y}{\varphi}\right)^\eta}{\left(1+\left(\frac{y}{\varphi}\right)^\eta\right)^{\tau(n-v+1+j)+1}}\underbrace{\left(\frac{1}{\eta}\int_{\frac{1}{1+\left(\frac{y}{\varphi}\right)^\eta}}^1 (1-t)^{\frac{1}{\eta}}t^{\tau(i+v-u-j)-\frac{1}{\eta}-1}dt\right)}_{I}dy,
\end{aligned} \tag{2.10}$$

where $z^\eta = x$ and $t = \frac{1}{x+1}$, it is not difficult to show that I can be simplified

$$\begin{aligned}
I &= B\left(\tau(i+v-u-j)-\frac{1}{\eta},\frac{1}{\eta}+1\right) - \frac{\left(\frac{1}{1+\left(\frac{y}{\varphi}\right)^\eta}\right)^{\tau(i+v-u-j)+1-\frac{1}{\eta}}}{\tau(i+v-u-j)+1-\frac{1}{\eta}} \\
&\times {}_2F_1\left[\tau(i+v-u-j)-\frac{1}{\eta},\frac{1}{\eta},\tau(i+v-u-j)-\frac{1}{\eta}+1,\frac{1}{1+\left(\frac{y}{\varphi}\right)^\eta}\right],
\end{aligned} \tag{2.11}$$

where ${}_pF_q$ is the generalized hypergeometric function defined as

$${}_pF_q(r_1,\ldots,s_1,\ldots,s_q;z) = \sum_{k=0}^\infty \frac{(r_1)_k\ldots(r_p)_k}{(s_1)_k\ldots(s_q)_k}\frac{z^k}{n!}$$

Using (2.11) and (2.10), we get the

$$\begin{aligned}
\mu_{u,v:n} &= \varphi^2\tau^2 C_{u,v:n}\sum_{i=0}^{u-1}\sum_{j=0}^{v-u-1}\binom{u-1}{i}\binom{v-u-1}{j}(-1)^{i+j}\Bigg[B\left(\tau(i+v-u-j)-\frac{1}{\eta},\frac{1}{\eta}+1\right)\\
&\times\ B\left(\frac{1}{\eta}+1,\tau(n-v+1+j)-\frac{1}{\eta}\right)-\psi(\tau,\eta)\\
&\times\ \sum_{k=0}^{\infty}B\left(\frac{1}{\eta}+1,\tau(i+n-u+1)-\frac{2}{\eta}+k+1\right)\Bigg], \qquad (2.12)
\end{aligned}$$

where

$$\psi(\tau,\eta) = \frac{\left[\tau(i+v-u-j)-\frac{1}{\eta}\right]_k\left[\frac{1}{\eta}\right]_k}{\left[\tau(i+v-u-j)-\frac{1}{\eta}+1\right]_k\left[\tau(i+v-u-j)-\frac{1}{\eta}+1\right]k!}.$$

2.4 BLUEs of the Location and Scale Parameters

Here, we study parameter estimation for the TIIELL distribution based on order statistics. Let $Z_1 \leq Z_2 \leq \cdots \leq Z_n$ be a random sample of size n from TIIELL distribution, the pdf of the scale-parameter TIIELL distribution is

$$h(z)=\frac{\tau\eta\left(\frac{z}{\varphi}\right)^{\eta-1}}{\varphi\left(1+\left(\frac{z}{\varphi}\right)^{\eta}\right)^{\tau+1}},\quad z>0,\ \ (\tau,\varphi)>0,\ \ \eta>1 \qquad (2.13)$$

and the pdf of the location-scale parameter is

$$h(z)=\frac{\tau\eta\left(\frac{z-\delta}{\varphi}\right)^{\eta-1}}{\varphi\left(1+\left(\frac{z-\delta}{\varphi}\right)^{\eta}\right)^{\tau+1}},\quad z>\delta,\ \ (\tau,\varphi)>0,\ \ \eta>1. \qquad (2.14)$$

The expression for the BLUEs of location and scale parameter are given in (1.18) and also variances and covariance for these parameters are given in eqn (1.21), (1.22) and (1.23).

Tables 2.3 and 2.4 display the coefficient of the BLUEs for type-II right censored sample of various values of $n = 7,\ 10$ and censoring cases $c = 0(1)([n/2]-1)$. Also, Table 2.5 shows variances and covariances of the BLUEs.

2.5 Approximate Inference

Here, we derive the $(1-\alpha)100\%$ confidence intervals for the location and scale parameters of the BLUEs δ^* and φ^* based on the pivotal quantities

$$U_1 = \frac{\delta^* - \delta}{\varphi\sqrt{W_1}}, \quad U_2 = \frac{\varphi^* - \varphi}{\varphi\sqrt{W_2}}, \quad U_3 = \frac{\delta^* - \delta}{\varphi^*\sqrt{W_1}}, \tag{2.15}$$

where δ^* and φ^* are the BLUEs of δ and φ with variances $\varphi^2 W_1$ and $\varphi^2 W_2$, respectively. U_1 used to draw inference for δ when φ is known, while U_3 can be used to draw inference for δ when φ is unknown. Similarly, U_2 can be used to draw inference for φ.

To derive the CIs of the location and scale parameters based on the pivotal quantities in Eqn. (2.15), the moments presented in Section 2.2, are used.

Hence U_1 and U_2 can be rewritten as

$$U_1 = \frac{1}{\sqrt{W_1}}\left(\sum_{u=1}^{n-c} p_u Z_{u:n}\right) = \frac{U_1^*}{\sqrt{W_1}}, \qquad U_2 = \frac{1}{\sqrt{W_2}}\left(\sum_{u=1}^{n-c} q_u Z_{u:n} - 1\right) = \frac{U_2^* - 1}{\sqrt{W_2}}, \tag{2.16}$$

where $Z_{u:n} = (Y_{u:n} - \delta)/\varphi$, $u = 1, 2, \ldots, n-c$, is the standardized form of the available Type-II right-censored sample $Y_{u:n}$, $u = 1, 2, \ldots, n-c$. Now, we consider to find the approximate distribution by using Edgeworth approximation for a statistic S (with mean 0 and variance 1) as

$$H(s) \approx \Phi(s) - \phi(s)\left[\frac{\sqrt{\tau_1}}{6}(s^2-1) + \frac{\tau_2 - 3}{24}(s^3 - 3s) + \frac{\tau_1}{72}(s^5 - 10s^2 + 15s)\right], \tag{2.17}$$

where $\sqrt{\tau_1}$ and τ_2 are the coefficients of skewness and kurtosis of S, respectively and $\Phi(s)$, $\phi(s)$ are the cdf and pdf of the standard normal distribution, respectively.

To obtain the the coefficients of skewness and kurtosis of linear functions of order statistics, single moments $E(Z^{p}_{u:n})$ denoted by $\mu^{p}_{u:n}$, the double moments $Z^{p}_{u:n}Z^{q}_{v:n}$, denoted by $\mu^{(p;q)}_{u,v:n}$ of the TIIELL distribution for $1 \leq u < v \leq (n-c)$ are required.

Table 2.9 displays the values of the mean, variance, coefficients of skewness and kurtosis ($\sqrt{\tau_1}$ and τ_2) of U_1^* and U_2^*. From Table 2.9 it is observed that the distributions of U_1^* and U_2^* and hence of U_1 and U_2 are positively skewed and heavier tailed than normal. Also we can see that $\sqrt{\tau_1}$ of U_1^* and U_2^* increases as η increases and decreases as n increases and decreases as c increases. τ_2 of U_1^* decreases as n increases and increases as η increases and decreases as c increases, while τ_2 of U_2^* increases as n and η increases and decreases as c increases.

We also obtained the lower and upper $1\%,\ 2.5\%,\ 5\%,$ *and* 10% points of U_1 and U_2 through Edgeworth approximation (see Tables 2.6 and 2.7). From Tables 2.6 and 2.7, we can observe that the percentage points of U_1 and U_2 increases as η increases for $n = 7$ and decreases for $n = 10$ in most of the cases and increases as n increases in most of the cases for $\eta = 2$, while decreases as n increases for $\eta = 3$ and decreases as c increases. Similarly the percentage points of U_3 increases as n increases.

The performance of the developed inference can be shown from the simulated average width of confidence intervals in Table 2.10. We observe that the Edgeworth approximations of the distributions of U_1 and U_2 both work quite satisfactory; this is also clear from the average width of the confidence intervals based on U_1 and U_2 which are presented in Table 2.10. In addition we can see that average width decreases as η increases for most of the cases.

TABLE 2.1: Expected values, second moments, variances, skewness and kurtosis of the uth order statistic from TIIELL distribution for $n = 1, 2, ..., 10$, $\tau = 2.5$, $\eta = 2$ and $\varphi = 0.25$

u	n	$E(Z)$	$E(Z^2)$	$V(Z)$	τ_1	τ_2	γ_1	γ_2
1	1	0.166667	0.041667	0.013889	6.124244	24.00119	2.474721	21.00119
	2	0.107379	0.015625	0.004095	1.480527	5.834856	1.216769	2.834856
	3	0.085248	0.009615	0.002348	0.970243	4.645654	0.985009	1.645654
	4	0.072834	0.006944	0.001639	0.775930	4.337018	0.880869	1.337018
	5	0.064627	0.005435	0.001258	0.690985	4.060399	0.831255	1.060399
	6	0.058687	0.004464	0.001020	0.653052	3.734868	0.808116	0.734868
	7	0.054132	0.003788	0.000858	0.577341	4.350134	0.759829	1.350134
	8	0.050495	0.003289	0.000739	0.576750	3.853975	0.759440	0.853975
	9	0.047505	0.002907	0.000650	0.534243	4.254706	0.730919	1.254706
	10	0.044990	0.002604	0.000580	0.479173	3.332768	0.692223	0.332768
2	2	0.225955	0.067708	0.016652	6.669762	26.89585	2.582588	23.89585
	3	0.151640	0.027644	0.004649	1.311287	5.810189	1.145115	2.810189
	4	0.122490	0.017628	0.002624	0.742933	4.483691	0.861936	1.483691
	5	0.105661	0.012983	0.001819	0.537407	4.080492	0.733081	1.080492
	6	0.094328	0.010287	0.001389	0.445519	3.949799	0.667472	0.949799
	7	0.086020	0.008523	0.001124	0.391159	3.375693	0.625427	0.375693
	8	0.079588	0.007277	0.000943	0.335838	3.046048	0.579516	0.046048
	9	0.074417	0.006349	0.000811	0.307304	3.904693	0.554350	0.904693
	10	0.070140	0.005632	0.000712	0.278758	3.848412	0.527975	0.848412
3	3	0.263112	0.087740	0.018512	7.184192	28.97034	2.680334	25.97034
	4	0.180790	0.037660	0.004975	1.322886	5.987013	1.150168	2.987013
	5	0.147734	0.024596	0.002771	0.707746	4.471739	0.841277	1.471739
	6	0.128326	0.018375	0.001907	0.492489	3.942891	0.701776	0.942891
	7	0.115099	0.014699	0.001451	0.380277	4.011283	0.616666	1.011283
	8	0.105315	0.012261	0.001170	0.312036	3.453741	0.558602	0.453741
	9	0.097688	0.010522	0.000979	0.302289	3.434712	0.549808	0.434712
	10	0.091521	0.009218	0.000842	0.251679	2.882750	0.501676	0.399884
4	4	0.290553	0.104434	0.020013	7.578064	30.47057	2.752828	27.47057
	5	0.202827	0.046370	0.005231	1.363788	6.098496	1.167813	3.098496
	6	0.167141	0.030817	0.002881	0.708859	4.536456	0.841938	1.536456
	7	0.145963	0.023276	0.001971	0.489885	3.966303	0.699918	0.966303
	8	0.131407	0.018762	0.001494	0.363963	3.895946	0.603293	0.895946
	9	0.120569	0.015738	0.001201	0.301321	3.393636	0.548927	0.393636
	10	0.112075	0.013565	0.001004	0.272530	3.072031	0.522044	0.072031
5	5	0.312484	0.118950	0.021304	7.887611	31.61464	2.808489	28.61464
	6	0.220670	0.054146	0.005451	1.404782	6.200086	1.185235	3.200086
	7	0.183024	0.036473	0.002975	0.717814	4.548271	0.847239	1.548271
	8	0.160518	0.027791	0.002025	0.477320	4.057296	0.690884	1.057296
	9	0.144955	0.022541	0.001529	0.368251	3.717102	0.606837	0.717102
	10	0.133308	0.018998	0.001227	0.293470	3.286510	0.541729	0.286501
6	6	0.330847	0.131910	0.022450	8.139275	32.53265	2.852941	29.53265
	7	0.235729	0.061216	0.005648	1.443160	6.280985	1.201316	3.280985
	8	0.196528	0.041682	0.003059	0.735319	4.529456	0.857508	1.529456
	9	0.172968	0.031990	0.002072	0.488891	4.037825	0.699208	1.037825
	10	0.156602	0.026085	0.001561	0.363625	3.532627	0.603013	0.532627
7	7	0.346701	0.143693	0.023492	8.346127	33.28143	2.888966	30.28143
	8	0.248795	0.067727	0.005828	1.477872	6.381171	1.215677	3.381171
	9	0.208308	0.046527	0.003135	0.750557	4.606563	0.866347	1.606563
	10	0.183879	0.035927	0.002116	0.499695	4.026693	0.706891	1.026693
8	8	0.360687	0.154545	0.024450	8.522843	33.91907	2.919391	30.91907
	9	0.260363	0.073784	0.005995	1.505565	6.457144	1.227015	3.457144
	10	0.218777	0.051071	0.003207	0.757008	4.664468	0.870062	1.664468
9	9	0.373227	0.164640	0.025342	8.674506	34.46545	2.945251	31.46545
	10	0.270760	0.079463	0.006152	1.531512	6.518105	1.237543	3.518105
10	10	0.384613	0.174104	0.026177	8.808048	34.94389	2.967835	31.94389

TABLE 2.2: Expected values, second moments, variances, skewness and kurtosis of the uth order statistic from TIIELL distribution for $n = 1,2,...,10$, $\tau = 5$, $\eta = 2$ and $\varphi = 0.5$

u	n	$E(Z)$	$E(Z^2)$	$V(Z)$	τ_1	τ_2	γ_1	γ_2
1	1	0.214757	0.062501	0.016379	1.482629	5.832059	1.217633	2.832059
	2	0.145668	0.027778	0.006559	0.782284	4.113691	0.884468	1.113691
	3	0.117375	0.017857	0.004080	0.626158	3.790735	0.791301	0.790735
	4	0.100991	0.013158	0.002959	0.560859	3.566212	0.748905	0.566212
	5	0.089980	0.010417	0.002321	0.525832	3.442966	0.725143	0.442966
	6	0.081930	0.008621	0.001908	0.500077	3.502186	0.707161	0.502186
	7	0.075714	0.007353	0.001620	0.495238	3.237636	0.703732	0.237636
	8	0.070728	0.006410	0.001408	0.478453	3.227514	0.691703	0.227514
	9	0.066612	0.005682	0.001245	0.456279	3.397797	0.675484	0.397797
	10	0.063141	0.005102	0.001115	0.451449	3.303742	0.671901	0.303742
2	2	0.283846	0.097222	0.016653	1.400052	6.014629	1.183238	3.014629
	3	0.202256	0.047619	0.006712	0.563894	4.001324	0.750929	1.001324
	4	0.166526	0.031955	0.004224	0.391693	3.627183	0.625854	0.627183
	5	0.145032	0.024123	0.003089	0.322388	3.480211	0.567792	0.480211
	6	0.130231	0.019397	0.002437	0.284753	3.357070	0.533623	0.357070
	7	0.119225	0.016227	0.002012	0.265080	3.246232	0.514859	0.246232
	8	0.110621	0.013952	0.001715	0.245458	3.320959	0.495437	0.320959
	9	0.103652	0.012238	0.001494	0.230641	3.469610	0.480251	0.469610
	10	0.097858	0.010901	0.001324	0.233139	2.927740	0.482845	0.218501
3	3	0.324642	0.122024	0.016632	1.466340	6.256670	1.210925	3.256670
	4	0.237985	0.063283	0.006646	0.526732	4.039598	0.725763	1.039598
	5	0.198768	0.043703	0.004194	0.342041	3.552107	0.584842	0.552107
	6	0.174635	0.033575	0.003078	0.263913	3.429646	0.513725	0.429646
	7	0.157746	0.027320	0.002436	0.222956	3.410097	0.472182	0.410097
	8	0.145037	0.023054	0.002018	0.193353	3.346856	0.439719	0.346856
	9	0.135010	0.019951	0.001723	0.176914	3.377494	0.420612	0.377494
	10	0.126829	0.017590	0.001504	0.175914	2.934875	0.419421	0.192158
4	4	0.353527	0.141604	0.016623	1.540551	6.459855	1.241189	3.459855
	5	0.264129	0.076337	0.006573	0.522017	4.094437	0.722507	1.094437
	6	0.222901	0.053831	0.004146	0.322694	3.617912	0.568062	0.617912
	7	0.197152	0.041916	0.003047	0.241729	3.378629	0.491659	0.378629
	8	0.178928	0.034431	0.002416	0.197982	3.283768	0.444952	0.283768
	9	0.165092	0.029259	0.002004	0.179578	3.085929	0.423766	0.085929
	10	0.154098	0.025460	0.001714	0.161099	3.196057	0.401371	0.196057
5	5	0.375877	0.157921	0.016637	1.608648	6.618557	1.268325	3.618557
	6	0.284744	0.087590	0.006511	0.529866	4.110246	0.727919	1.110246
	7	0.242213	0.062767	0.004101	0.317573	3.609708	0.563536	0.609708
	8	0.215376	0.049401	0.003014	0.232813	3.364292	0.482507	0.364292
	9	0.196223	0.040895	0.002392	0.189503	3.326950	0.435320	0.326950
	10	0.181583	0.034958	0.001986	0.166596	3.238532	0.408162	0.238532
6	6	0.394104	0.171987	0.016669	1.668102	6.757334	1.291550	3.757334
	7	0.301756	0.097519	0.006462	0.540625	4.142795	0.735272	1.142795
	8	0.258315	0.070787	0.004060	0.313873	3.650085	0.560244	0.650085
	9	0.230698	0.056205	0.002983	0.229549	3.451351	0.479112	0.451351
	10	0.210864	0.046832	0.002368	0.180792	3.353638	0.425196	0.353638
7	7	0.409495	0.184398	0.016712	1.719701	6.877223	1.311373	3.877223
	8	0.316236	0.106430	0.006425	0.551682	4.167832	0.742753	1.167832
	9	0.272123	0.078077	0.004026	0.318976	3.677382	0.564780	0.677382
	10	0.243921	0.062454	0.002957	0.228410	3.391861	0.477923	0.391861
8	8	0.422818	0.195537	0.016762	1.764809	6.979383	1.328461	3.979383
	9	0.328840	0.114530	0.006394	0.562789	4.217233	0.750193	1.217233
	10	0.284209	0.084773	0.003998	0.320279	3.693316	0.565932	0.693316
9	9	0.434565	0.205663	0.016816	1.805957	7.064301	1.343859	4.064301
	10	0.339998	0.121970	0.006371	0.572928	4.244680	0.756920	1.244680
10	10	0.445072	0.214962	0.016873	1.842618	7.141398	1.357431	4.141398

TABLE 2.3: Cofficient of the BLUEs of the a_u for $\tau = 1.5$

η	n	c										
2	7	0	0.013963	0.101597	0.558978	0.263992	0.051795	0.009676	-5.27E-07			
		1	-0.34963	0.833878	0.401217	0.108408	0.006114	1.22E-05				
		2	-0.11492	0.812707	0.290103	0.012084	2.69E-05					
	10	0	0.026562	0.067891	0.131564	0.228105	0.328165	0.132287	0.060663	0.023448	0.001311	3.53E-06
		1	0.039625	0.100778	0.188777	0.284606	0.265189	0.082232	0.036252	0.002535	4.15E-06	
		2	0.047991	0.127099	0.250121	0.374234	0.143044	0.053643	0.003868	1.68E-06		
		3	0.029534	0.131456	0.511993	0.235001	0.087684	0.004320	1.37E-05			
		4	-0.25036	0.761078	0.334894	0.145439	0.008938	9.13E-06				
3	7	0	-0.46467	-0.74389	1.449078	0.545318	0.204818	0.009332	1.17E-05			
		1	-0.29246	-0.60828	1.444277	0.453126	0.003324	1.28E-05				
		2	-1.58802	1.838166	0.728422	0.021388	4.29E-05					
	10	0	-0.01692	-0.03763	-0.06087	-0.04818	0.118261	0.764299	0.199978	0.078079	0.002988	1.05E-06
		1	0.002353	0.007739	0.019326	0.051598	0.170786	0.611624	0.131185	0.005383	5.88E-06	
		2	-0.01192	-0.03172	-0.03954	0.060255	0.846969	0.166772	0.009176	7.58E-06		
		3	-0.06968	-0.16761	-0.15093	1.136136	0.240728	0.011335	2.05E-05			
		4	-0.29882	-0.61856	1.540259	0.349781	0.027336	8.17E-06				

TABLE 2.4: Cofficient of the BLUEs of the b_u for $\tau = 1.5$

η	n	c										
2	7	0	-1.42624	-2.69332	2.763219	1.092811	0.224505	0.039025	-3.63E-06			
		1	-5.05483	3.111726	1.515761	0.403758	0.023545	4.36E-05				
		2	-4.45187	3.226687	1.177263	0.047808	0.000117					
	10	0	-0.40703	-0.92138	-1.29380	-0.64137	2.361500	0.514100	0.274523	0.107245	0.006194	1.47E-05
		1	-0.28225	-0.68770	-1.08679	-0.84100	2.287784	0.409765	0.185349	0.014828	1.58E-05	
		2	-0.56606	-1.31300	-1.47408	2.431089	0.650346	0.253884	0.017819	7.43E-06		
		3	-1.34466	-2.59115	2.584963	0.962910	0.369273	0.018615	5.26E-05			
		4	-4.68322	2.837779	1.270177	0.540726	0.034506	3.06E-05				
3	7	0	-2.17816	-3.85459	4.049251	1.422057	0.537002	0.024409	3.09E-05			
		1	-1.65382	-3.79265	4.181823	1.255093	0.009517	3.47E-05				
		2	-6.66135	4.733542	1.872912	0.054788	0.000108					
	10	0	-0.21816	-0.51608	-0.92486	-1.24654	-0.76193	2.846231	0.570023	0.242188	0.009133	2.23E-06
		1	-0.12748	-0.32472	-0.62717	-1.01443	-0.95517	2.569284	0.460068	0.019596	1.77E-05	
		2	-0.25649	-0.69056	-1.27040	-1.44404	3.097968	0.533783	0.029706	2.53E-05		
		3	-0.60690	-1.57047	-2.26163	3.697287	0.707580	0.034080	5.81E-05			
		4	-1.67530	-3.83228	4.469613	0.962392	0.075552	2.27E-05				

TABLE 2.5: Variances and covariance of the BLUEs when $\tau = 1.5$ $\delta = 0$ and $\varphi = 1$

η	n	c	$Var(\delta)$	$Var(\varphi)$	$Cov(\delta,\varphi)$
2	7	0	0.074821	0.970035	0.253528
		1	0.093972	1.102865	0.310651
		2	0.125613	1.597531	0.426752
	10	0	0.044879	0.616469	0.152306
		1	0.052941	0.848929	0.185043
		2	0.060478	0.878651	0.209287
		3	0.073563	0.955970	0.248770
		4	0.087637	1.020583	0.287920
3	7	0	0.153237	0.952101	0.377268
		1	0.214149	1.459957	0.549955
		2	0.248820	1.525382	0.610006
	10	0	0.106367	0.810697	0.284275
		1	0.132891	1.206524	0.379329
		2	0.150195	1.238222	0.414675
		3	0.182625	1.375398	0.488490
		4	0.211345	1.424032	0.539068

TABLE 2.6: Edgeworth approximate and the simulated (*) values of the distribution of U_1 when $\tau = 1.5$, $\delta = 0$ and $\varphi = 1$

η	n	c	1%	2.50%	5%	10%	90%	95%	97.50%	99%
2	7	0	-0.77841	-0.76196	-0.73504	-0.68271	1.083601	1.925702	1.854568	4.255494
			-1.03231*	-0.97438*	-0.90792*	-0.81937*	1.083648*	1.768060*	2.560536*	3.758601*
		1	-2.27628	-2.32369	-0.67909	-0.62752	0.208921	1.889575	1.822893	4.241948
			-1.47652*	-1.41945*	-1.35722*	-1.26558*	0.682143*	1.351194*	2.104918*	3.327379*
		2	-2.34755	-2.41664	-2.56902	-2.86364	0.324397	0.432937	1.791339	3.749634
			-2.06661*	-2.01487*	-1.95338*	-1.86370*	0.126588*	0.818807*	1.556711*	2.749675*
	10	0	-0.63703	-0.62091	-0.59465	-0.54415	1.968333	1.825218	3.499767	4.188481
			-1.19869*	-1.11596*	-1.03013*	-0.90966*	1.176293*	1.816772*	2.499808*	3.605541*
		1	-2.56341	-1.08063	-1.02533	-0.92635	0.773829	1.106087	1.429446	3.896156
			-1.68973*	-1.60117*	-1.51202*	-1.38262*	0.766383*	1.428123*	2.120552*	3.137602*
		2	-2.51757	-2.77431	-2.97307	-2.89647	0.704014	0.993738	1.354706	3.585678
			-2.19815*	-2.10884*	-2.02115*	-1.89652*	0.271168*	0.924273*	1.627683*	2.585719*
		3	-2.53477	-3.16427	-2.98272	-2.98269	0.732411	1.042713	1.382714	1.382703
			-2.72401*	-2.63791*	-2.55189*	-2.42756*	-0.25733*	0.390673*	1.068124*	2.103971*
		4	-3.83866	-3.30443	-2.96218	-2.96219	0.181199	0.835414	1.163553	1.314588
			-3.25567*	-3.17872*	-3.09859*	-2.98289*	-0.81876*	-0.16451*	0.504467*	1.485206*
3	7	0	-0.38025	-0.37311	-0.36123	-0.33769	1.936298	1.888275	1.865233	4.491297
			-0.95660*	-0.90678*	-0.85151*	-0.77025*	1.032663*	1.716643*	2.505057*	3.743581*
		1	-2.28575	-2.32319	-0.66079	-0.61545	0.161453	1.927139	1.872661	4.263209
			-1.45259*	-1.40138*	-1.34743*	-1.26801*	0.586968*	1.278905*	2.080461*	3.263277*
		2	-2.30645	-2.35543	-2.45118	-2.81325	0.226085	0.306988	1.837418	3.764807
			-1.99110*	-1.94309*	-1.89224*	-1.81329*	0.111492*	0.792065*	1.608480*	2.764848*
	10	0	-0.83021	-0.81163	-0.78146	-0.72326	1.296581	1.916373	1.828839	4.202645
			-1.08730*	-1.01830*	-0.94733*	-0.84676*	1.096046*	1.772315*	2.542991*	3.727137*
		1	-2.18033	-2.21922	-2.29146	-2.39668	0.156439	1.849359	1.797111	4.137826
			-1.65246*	-1.57954*	-1.50311*	-1.39675*	0.607943*	1.262664*	2.027025*	3.137897*
		2	-2.41444	-2.52619	-2.92543	-2.92543	0.480104	0.644207	1.703217	3.567014
			-2.28619*	-2.21093*	-2.12713*	-2.01619*	0.051319*	0.725112*	1.482041*	2.567054*
		3	-2.44641	-2.58122	-2.94305	-2.94304	0.424358	0.740532	0.873089	1.518757
			-2.94442*	-2.86884*	-2.78695*	-2.67421*	-0.57559*	0.095727*	0.809759*	1.841936*
		4	-3.94473	-3.48833	-2.95145	-2.95145	-0.14498	0.510946	0.891869	1.527782
			-3.48544*	-3.41859*	-3.34497*	-3.23734*	-1.14494*	-0.48899*	0.243865*	1.308830*

TABLE 2.7: Edgeworth approximate and the simulated (*) values of the distribution of U_2 when $\tau = 1.5$, $\delta = 0$ and $\varphi = 1$

η	n	c	1%	2.50%	5%	10%	90%	95%	97.50%	99%
2	7	0	-0.68625	-0.67387	-0.65342	-0.61301	1.120145	1.963351	1.917835	4.386462
			-0.89515*	-0.85571*	-0.81026*	-0.74149*	1.016280*	1.700985*	2.516495*	3.864885*
		1	-2.23387	-2.26766	-0.58962	-0.54756	0.133227	1.900113	1.852216	4.326887
			-1.37336*	-1.33204*	-1.28475*	-1.21241*	0.661287*	1.334092*	2.137424*	3.366250*
		2	-2.28594	-2.32889	-2.41048	-2.68876	0.190544	0.261264	1.847944	3.857855
			-1.90759*	-1.87523*	-1.83573*	-1.77180*	0.101144*	0.809872*	1.640061*	2.857924*
	10	0	-0.43146	-0.42262	-0.40801	-0.37914	1.928908	1.865193	1.835121	4.403044
			-1.03058*	-0.97355*	-0.91151*	-0.82102*	1.084735*	1.761624*	2.523320*	3.710513*
		1	-2.39482	-0.86625	-0.83388	-0.77177	0.366606	1.928228	1.824014	4.173098
			-1.50745*	-1.44968*	-1.38733*	-1.29935*	0.641782*	1.332129*	2.118734*	3.318632*
		2	-2.25389	-2.31073	-2.42521	-2.81980	0.242052	0.315944	1.774996	3.702989
			-2.09084*	-2.03112*	-1.96779*	-1.87474*	0.139231*	0.812459*	1.597538*	2.703031*
		3	-2.39193	-2.48068	-2.72223	-2.91628	0.413258	0.553318	0.636243	1.679891
			-2.62983*	-2.57057*	-2.50472*	-2.40972*	-0.38170*	0.297366*	1.054344*	2.169122*
		4	-3.95033	-3.49826	-2.94629	-2.94627	0.159258	0.734727	0.863918	0.968469
			-3.14328*	-3.08945*	-3.02804*	-2.93252*	-0.84070*	-0.15407*	0.568380*	1.601999*
3	7	0	-0.37383	-0.36704	-0.35582	-0.33353	1.947225	1.903340	1.882234	4.526831
			-0.89584*	-0.8538*	-0.80822*	-0.73897*	1.003720*	1.702315*	2.507245*	3.772574*
		1	-2.25553	-0.60699	-0.58794	-0.55025	0.094725	1.941317	1.901147	4.364091
			-1.33671*	-1.30106*	-1.26096*	-1.19843*	0.562082*	1.280316*	2.118576*	3.364137*
		2	-2.28995	-2.33181	-2.41096	-2.66945	0.186123	0.257073	1.855683	3.870927
			-1.88758*	-1.85427*	-1.81481*	-1.74995*	0.119499*	0.832150*	1.645907*	2.871001*
	10	0	-0.72809	-0.71409	-0.69103	-0.64574	1.167925	1.949229	1.893921	4.330301
			-0.95512*	-0.90828*	-0.85568*	-0.77823*	1.042522*	1.747703*	2.533224*	3.812363*
		1	-2.12219	-2.13917	-2.16865	-2.23302	0.037221	1.888342	1.864152	4.247133
			-1.43782*	-1.39501*	-1.34838*	-1.27871*	0.511022*	1.219516*	2.011160*	3.247213*
		2	-2.33214	-2.38482	-2.48992	-2.88514	0.254338	0.346561	1.841463	3.647901
			-2.11302*	-2.06494*	-2.01162*	-1.93031*	-0.02184*	0.681566*	1.499523*	2.647942*
		3	-2.42043	-2.50059	-3.24692	-2.94216	0.379866	0.576944	0.665016	1.735106
			-2.76610*	-2.71757*	-2.66356*	-2.58041*	-0.62009*	0.089863*	0.870971*	2.015355*
		4	-4.07898	-3.68032	-2.72291	-2.94198	-0.13232	0.561205	0.679897	1.720768
			-3.29861*	-3.25072*	-3.19745*	-3.11502*	-1.13227*	-0.43873*	0.318227*	1.532725*

TABLE 2.8: Simulated values of the distribution of U_3 when $\tau = 1.5$, $\delta = 0$ and $\varphi = 1$

η	n	c	1%	2.50%	5%	10%	90%	95%	97.50%	99%
2	7	0	-8.58565	-6.00463	-4.36124	-2.92522	0.577351	0.740279	0.891971	1.127982
		1	-22.0015	-14.7982	-10.3426	-7.00606	0.490966	0.754123	0.961019	1.207395
		2	-64.3401	-40.6360	-27.6904	-17.7657	0.128992	0.633082	1.003714	1.452917
	10	0	-6.71847	-4.88497	-3.67234	-2.55740	0.671541	0.860460	1.024841	1.245431
		1	-14.4160	-10.4421	-7.84722	-5.55788	0.624766	0.977244	1.314201	1.778569
		2	-25.1951	-18.3949	-14.0381	-10.1702	0.274881	0.778888	1.175830	1.693537
		3	-43.5958	-31.0430	-23.3486	-16.8049	-0.31343	0.390987	0.902069	1.484084
		4	-80.3106	-55.2902	-40.0114	-27.9420	-1.15339	-0.18127	0.463265	1.118295
3	7	0	-8.28370	-5.77654	-4.18322	-2.83497	0.526657	0.656970	0.744066	0.828474
		1	-25.6916	-17.0079	-12.0546	-8.10941	0.431894	0.703867	0.882413	1.061739
		2	-68.5725	-41.7708	-28.1831	-17.9228	0.104877	0.574845	0.890182	1.167284
	10	0	-7.14072	-5.19687	-3.88199	-2.69139	0.589233	0.735267	0.842694	0.964767
		1	-18.5664	-13.4642	-10.1707	-7.19092	0.482042	0.788716	1.012966	1.290519
		2	-34.5303	-25.0795	-19.0127	-13.4866	0.055358	0.591351	0.969128	1.348443
		3	-60.0887	-43.3305	-32.6716	-23.4520	-0.75955	0.100226	0.669017	1.197819
		4	-104.838	-72.0204	-52.8649	-36.8763	-1.75691	-0.58387	0.226488	0.958274

TABLE 2.9: Mean, Variance and coefficients of skewness and kurtosis of U_1^* and U_2^* when $\tau = 1.5$, $\delta = 0$ and $\varphi = 1$

			U_1				U_2			
η	n	c	Mean	V_1	$\sqrt{\tau_1}$	τ_2	Mean	V_1	$\sqrt{\tau_1}$	τ_2
2	7	0	0.000963	0.074656	1.904050	32.19869	1.005801	0.982068	2.114422	46.51046
		1	-0.08758	0.041760	1.864872	32.90590	0.714386	0.465816	1.986304	42.78576
		2	-0.15359	0.023106	1.763638	25.11762	0.498366	0.258819	1.922601	35.45901
	10	0	0.000786	0.046427	1.746194	31.89380	1.001825	0.652555	2.067376	60.53579
		1	-0.07138	0.026898	1.489195	12.70246	0.722236	0.351057	1.801017	24.78170
		2	-0.12518	0.017567	1.485461	13.29467	0.554844	0.192955	1.758955	28.75834
		3	-0.16467	0.012416	1.478552	12.92351	0.448749	0.127834	1.699910	21.34141
		4	-0.19641	0.009247	1.512846	14.19956	0.374720	0.093584	1.610215	17.27234
3	7	0	0.002024	0.157938	2.234621	78.61409	1.004534	0.982242	2.312447	86.37732
		1	-0.13231	0.077857	1.991401	39.44096	0.676460	0.499386	2.126234	50.97791
		2	-0.21177	0.044925	1.879843	32.08649	0.503454	0.265125	1.939696	36.16978
	10	0	-0.00042	0.105689	1.831395	27.65306	0.999542	0.796568	2.021134	39.48200
		1	-0.11958	0.052235	1.851664	37.89021	0.669847	0.405057	2.214512	75.30156
		2	-0.19250	0.029811	1.632681	18.67683	0.502932	0.196356	1.870876	30.37217
		3	-0.24453	0.019317	1.595066	16.93230	0.398287	0.118517	1.764992	22.59934
		4	-0.28135	0.014526	1.606979	17.01038	0.331043	0.086028	1.748858	22.01017

Table 2.10: Average width of the Edgeworth and simulated(*) C.I.'s

		U_1				U_2				U_3	
n	c	90%		95%		90%		95%		90%	95%
7	0	2.660745	2.675980*	2.616533	3.534921*	2.616774	2.511241*	2.591710	3.372206*	5.101516*	6.896599*
	1	2.568667	2.708410*	4.146580	3.524364*	2.489716	2.618837*	4.119881	3.469468*	11.09670*	15.75922*
	2	3.001957	2.772187*	4.207975	3.571577*	2.671747	2.645604*	4.176834	3.515287*	28.32346*	41.63974*
10	0	2.419866	2.846901*	4.120680	3.615770*	2.273196	2.673121*	2.257739	3.496869*	4.532798*	5.909810*
	1	2.131419	2.940145*	2.510081	3.721726*	2.762113	2.719455*	2.690262	3.568417*	8.824464*	11.75634*
	2	3.966812	2.945421*	4.129015	3.736519*	2.741151	2.780245*	4.085728	3.628634*	14.81699*	19.57076*
	3	4.025414	2.942563*	4.546983	3.706037*	3.275550	2.802063*	3.116921	3.624910*	23.73962*	31.94510*
	4	3.797594	2.934083*	4.467987	3.683186*	3.681019	2.873962*	4.362180	3.657831*	39.83013*	55.75350*
7	0	2.249506	2.568142*	2.238341	3.411832*	2.259165	2.510530*	2.249274	3.361044*	4.840185*	6.520606*
	1	2.587927	2.626336*	4.195851	3.481836*	2.529257	2.541276*	2.508136	3.419633*	12.75846*	17.89027*
	2	2.758169	2.684303*	4.192849	3.551572*	2.668035	2.646963*	4.187487	3.500173*	28.75793*	42.66099*
10	0	2.697834	2.719643*	2.640473	3.561290*	2.640264	2.603388*	2.608009	3.441503*	4.617257*	6.039560*
	1	4.140821	2.765767*	4.016333	3.606564*	4.056995	2.567898*	4.003325	3.406160*	10.95945*	14.47721*
	2	3.569634	2.852243*	4.229410	3.692967*	2.836481	2.693163*	4.226280	3.564463*	19.60404*	26.04864*
	3	3.683578	2.882681*	3.454311	3.678602*	3.823864	2.753427*	3.165611	3.588544*	32.77182*	43.99951*
	4	3.462393	2.855982*	4.380202	3.662456*	3.284114	2.758720*	4.360213	3.568945*	52.28107*	72.24690*

2.6 Real Data Application

To demonstrate how the proposed methods can be used in practice, we consider the following real-life data set (see Bhaumik et al. (2009)). The data set represents vinyl chloride from clean upgradient monitoring wells in mg/L. The data are:

```
5.1 1.2 1.3 0.6 0.5 2.4 0.5 1.1 8.0 0.8 0.4 0.6 0.9 0.4 2.0 0.5
5.3 3.2 2.7 2.9 2.5 2.3 1.0 0.2 0.1 0.1 1.8 0.9 2.0 4.0 6.8 1.2
0.4 0.2
```

Now a random sample of size 10 is selected from the given data set and data are: 0.1, 1.1, 0.9, 2.3, 1.3, 2.5, 0.4, 2.0, 0.5, 3.2. Figure 2.1 shows ecdf and QQ-plot of the sample. The Kolmogorov-Smirnov (K-S) statistic is 0.17491 and the corresponding p-value is 0.8697. This shows the suitability of the TIIELL distribution for this data set.

Then by using the BLUEs coefficients in Tables 2.3 and 2.4, we have

$$\delta^* = \sum_{u=1}^{n} a_u Z_{u:n} = 1.012389 \quad \text{and} \quad \varphi^* = \sum_{u=1}^{n} b_u Z_{u:n} = 2.443834$$

By using Table 2.8, the CIs for the location parameter δ as

	90% CI	95% CI
Edgeworth	(0.6257, 1.13836)	(0.27096, 1.14393)
Simulated	(0.6275, 1.23062)	(0.48281, 1.24881)

By using Table 2.9, the CIs for the scale parameter φ as

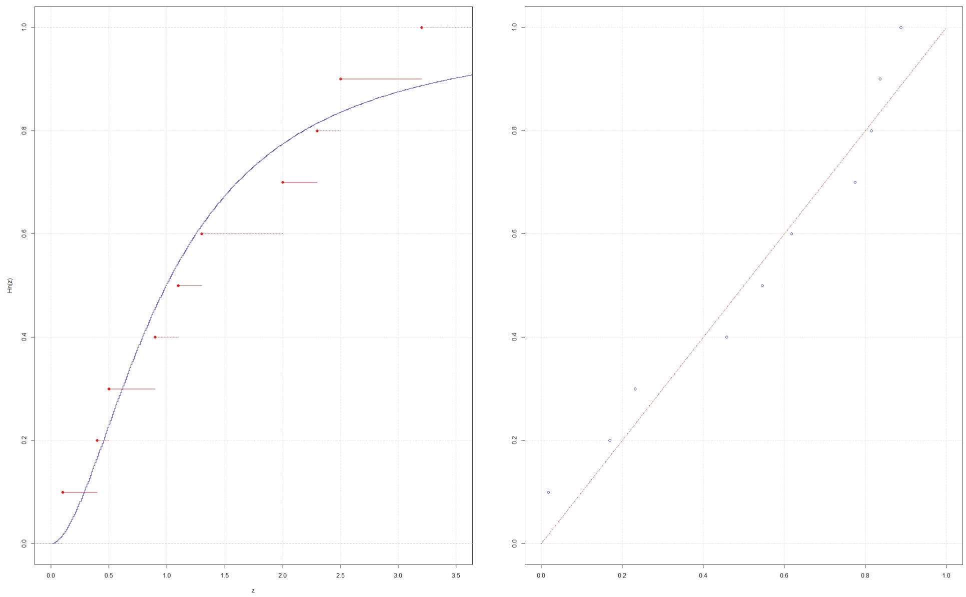

FIGURE 2.1: ECDF-plot and QQ-plot

	90% CI	95% CI
Edgeworth	(0.99163, 3.59568)	(1.00122, 3.65746)
simulated	(1.02546, 3.59603)	(0.91974, 3.67228)

By using Table 2.10, the CIs for the location parameter δ when φ is unknown as

	90% CI	95% CI
simulated	(0.56691, 2.91362)	(0.48181, 3.54143)

We note that the average width of the CIs increase as the level of significant increases.

2.7 Conclusion

In this chapter, the single and product moments of the order statistics from the type II exponentiated log-logistic distribution are derived in explicit forms. The single and product moments are used to obtain the BLUEs of the location and scale parameters of TIIELL distribution. The variances and covariances are calculated to show the performance of the BLUEs. Next, we calculate mean, variance, coefficient of skewness and kurtosis for some linear pivotal quantities. The distributions of the pivotal quantities are calculated in terms of Edgeworth approximation based on BLUEs which in turn can be used to develop confidence intervals. Hence, the distributions of the pivotal quantities are used to construct the interval estimation for the location and scale parameters. The accuracy of the estimated confidence intervals is investigated in terms of the average width. Finally, one real data set has been used to obtain the MLEs of the model parameters, BLUEs of δ and φ and CIs of δ and φ.

Chapter 3

Estimation of the Parameter for Modified Power Function Distribution based on Order Statistics with Application*

3.1 Introduction

This chapter follows the following structure: In Sections 3.2 and 3.3, we obtain explicit expressions for single and product moments of order statistics. We use these moments to obtain the BLUEs for location and scale parameters δ and φ in section 3.4. Next, we performed a numerical study using R software in Section 3.5 and we see that these expressions provide precise numerical evaluations. We also provided graph and a real data application of the result in section 3.6. The chapter is finalized with concluding remarks in Section 3.7.

Power function distribution is the oldest distribution and very attractive in lifetime literature due to their simplicity, easiness and flexible features to model various types of life time data

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in different fields. The power function distribution is a flexible lifetime model which can be derived from the Pareto distribution using the inverse transformation. Theoretically, the power function distribution is a special case of the beta distribution. Besides, it is a special case of Pearson type I distribution. It is also used to fit the distribution of certain likelihood ratios in statistical tests. If the likelihood ratio is based on n iid random variables, it is often found that a useful goodness-of-fit can be obtained by letting $(likelihoodratio)^{2/n}$ to have a power function distribution (see Ali et al. (2005)). Meniconi and Barry (1996) showed that the power function distribution is the best distribution to check the reliability of any electrical component. Also, they showed from survival and hazard functions of power function distribution is the better than exponential, log-normal and Weibull distributions. Johnson et al. (1994, 1995) provided a comprehensive account of the statistical properties of the power function distribution. Many generalizations of the power function distribution have attempted by researchers. Notable amoung them are: Tahir et al. (2016) proposed the two parameter power function distribution to a more general and flexible four-parameter Weibull-Power function distribution as an adequate distribution for modelling survival data. They provided a comprehensive account of the statistical properties and the bivariate extension was also proposed. Naveed and Asghar (2016) introduced the transmuted Power function distribution. Ibrahim (2017) proposed the Kumaraswamy power function distribution and also discuss its statistical properties. Bursa and Gamze (2017) proposed a exponentiated Kumaraswamy-power function distribution and provided the statistical and mathematical properties.

Kumar and Khan (2014) established the explicit expressions for single and product moments of the generalized order statistics of the three-parameter Power function distribution and discussed their characterization based on the conditional moments of the generalized order statistics. Saran and Pandey (2004) derived and discussed the linear unbiased estimates of the parameters of a three-parameter Power function distribution based on the kth record values. Chang (2007) provides the characterizations of the two-parameter Power function distribution using the independence record values. Lim and Lee (2013) gave some proof of a characterization of the two-parameter Power function distribution using the lower record values and Ahsanullah et al. (2013) presented a new characterization of the two-parameter Power function distribution based on the lower records. Ahsanullah and Alzaatreh (2018) derived and discussed the BLUEs of the parameters of a log-logistic distribution based on order statistics.

The modified power function distribution is an important distribution for analyzing the lifetime data, which is quite flexible and can be used effectively in modeling survival data. It can have increasing, decreasing, upside-down bathtub and bathtub shaped failure rate.

Recently, Okorie et al. (2017) introduced a two parameter modified power function (MPF) distribution with probability density function (pdf)

$$h(z) = \frac{\alpha\beta(1-z)^{\beta-1}}{[1-(1-\alpha)(1-z)^{\beta}]^2},\ 0<z<1,\ \alpha,\ \beta>0, \tag{3.1}$$

and the associated cumulative distribution function (cdf) is

$$H(z) = 1-\frac{\alpha(1-z)^{\beta}}{[1-(1-\alpha)(1-z)^{\beta}]}\ 0<z<1,\ \alpha,\ \beta>0. \tag{3.2}$$

Quantiles are fundamental for estimation (for example, quantile estimators) and simulation. The p^{th} quantile z_p of the MPF distribution is the root of the equation

$$Z_p = 1-\left(\frac{1-p}{1-p+p\alpha}\right)^{\frac{1}{\beta}},\ \ 0<p<1,\ \alpha,\ \beta>0. \tag{3.3}$$

Let $u \sim U(0,1)$, then equation (3.3) can be used to simulate a random sample of size n from the MPF distribution as follows

$$z_i = \left(\frac{1-u_i}{1-u_i+u_i\alpha}\right)^{\frac{1}{\beta}},i=1,2,\ldots,n.$$

3.2 Relations for Single Moments of Order Statistics

Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from MPF distribution, the pth moment of uth order statistic is obtained by using equation (3.1) and (3.2) in (1.1) as follows:

$$\mu_{u:n}^{(p)} = C_{u:n} \int_{-\infty}^{\infty} z^p H^{u-1}(z) \left[1 - H(z)\right]^{n-u} h(z) dz, \;\; u = 1, 2, \ldots, n, \;\; p = 1, 2, \ldots. \tag{3.4}$$

Expanding $H[(z)]^{u-1}$ and $[1-H(z)]^{n-u}$ in Equation (3.4) binomially, we get

$$\begin{aligned}
\mu_{u:n}^{(p)} &= C_{u:n} \sum_{i=0}^{u-1} \binom{u-1}{i} (-1)^i \int_0^1 z^p [1-H(z)]^{n-u+i} h(z) dz \\
&= \beta C_{u:n} \sum_{i=0}^{u-1} \binom{u-1}{i} (-1)^i \alpha^{n-u+i+1} \int_0^1 z^p \frac{(1-z)^{\beta(n-u+i+1)-1}}{[1-(1-\alpha)(1-z)^{\beta}]^{n-u+i+2}} dz \\
&= \beta C_{u:n} \sum_{i=0}^{u-1} \sum_{j=0}^{\infty} \binom{u-1}{i} (-1)^i \alpha^{n-u+i+1} \frac{(1-\alpha)^j \Gamma(j+n-i+2)}{\Gamma(j+1)\Gamma(n-u+i+2)} \\
&\times \int_0^1 z^p (1-z)^{\beta(n-u+i+j+1)-1} dz \\
&= \beta C_{u:n} \sum_{i=0}^{u-1} \sum_{j=0}^{\infty} \binom{u-1}{i} (-1)^i \alpha^{n-u+i+1} \frac{(1-\alpha)^j \Gamma(j+n-i+2)}{\Gamma(j+1)\Gamma(n-u+i+2)} \\
&\times B(p+1, \beta(n-u+i+j+1)),
\end{aligned} \tag{3.5}$$

where $B(a,\ b)$ denote the beta function and defined by $B(a,\ b) = \int_0^1 z^{a-1}(1-z)^{b-1} dz$. The validity of the single moments of order statistics in Equation (3.5) can be checked by using Arnold et al. (2008).

$$\sum_{u=1}^{n} \mu_{u:n} = nE(Z).$$

In particular, the mean and the variance of order statistic are

$$\begin{aligned}
\mu_{u:n}^{(1)} &= \beta C_{u:n} \sum_{i=0}^{u-1} \sum_{j=0}^{\infty} \binom{u-1}{i} (-1)^i \alpha^{n-u+i+1} \frac{(1-\alpha)^j \Gamma(j+n-u+i+2)}{\Gamma(j+1)\Gamma(n-u+i+2)} \\
&\times B(2, \beta(n-u+i+j+1))
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\sigma^2 &= \mu_{u:n}^{(2)} - (\mu_{u:n}^{(1)})^2 \\ &= \beta C_{u:n} \sum_{i=0}^{u-1} \sum_{j=0}^{\infty} \binom{u-1}{i} (-1)^i \alpha^{n-u+i+1} \frac{(1-\alpha)^j \Gamma(j+n-u+i+2)}{\Gamma(j+1)\Gamma(n-u+i+2)} \\ &\times\; B(3, \beta(n-u+i+j+1)) - [\mu_{u:n}^{(1)}]^2.\end{aligned}$$

Some special cases from Equation (3.5) are

(1) If $p = n = u = 1$, we get

$$\mu_{1:1} = \alpha\beta \sum_{j=0}^{\infty} (1-\alpha)^j (i+1) B(2, \beta(j+1)) = E(Z),$$

as obtained by Okorie et al. (2017).

(2) If $p = u = 1$, we get

$$\mu_{1:n} = \alpha^n \beta C_{1:n} \sum_{j=0}^{\infty} \frac{(1-\alpha)^j \Gamma(j+n+1)\, B(2, \beta(n+j))}{\Gamma(j+1)\Gamma(n+1)}.$$

(3) If $p = 1, u = n$, we get

$$\mu_{n:n} = \beta C_{n:n} \sum_{i=0}^{n-1} \sum_{j=0}^{\infty} \binom{n-1}{i} (-1)^i \alpha^{i+1} \frac{(1-\alpha)^j \Gamma(j+i+2)\, B(2, \beta(i+j+1))}{\Gamma(j+1)\Gamma(i+2)}.$$

3.3 Relations for Product Moments of Order Statistics

Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from MPF distribution, the product moments of uth and vth order statistic is obtained by using equation (3.1) and (3.2) in (1.9) as follows:

$$\begin{aligned}\mu_{u,v:n}^{(p,q)} &= C_{u,v:n} \int_0^1 \int_z^1 z^p y^q H^{u-1}(z) \left[H(y) - H(z)\right]^{v-1-u} \left[1 - H(y)\right]^{n-v} h(z)h(y)\,dz\,dy, \\ & \quad u, v = 1, 2, \ldots, n, \;\; u < v. \end{aligned} \tag{3.7}$$

By using the same argument as in the single moments case, we get

$$\mu_{u,v:n}^{(p,q)} = C_{u,v:n}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{u-1}\sum_{l=0}^{v-u-1}\sum_{r=0}^{q}\sum_{s=0}^{p}\binom{u-1}{k}(-1)^{k+l+r+s}\alpha^{k+n-u+1}\beta^{2}(1-\alpha)^{i+j}$$
$$\times \binom{v-u-1}{l}\binom{q}{r}\binom{p}{s}\frac{\Gamma(i+k+v-u-l+1)\,\Gamma(j+l+n-v+2)}{\Gamma(i+1)\Gamma(k+v-u-l+1)\,\Gamma(j+1)\Gamma(l+n-v+2)}$$
$$\times \frac{1}{[\beta(l+n+j-v+1)+r][\beta(k+i+n+j-u+1)+r+s]}. \tag{3.8}$$

The validity of the product moments of order statistics in Equation (3.8) can be checked by using Arnold et al. (2008).

$$\sum_{u=1}^{n-1}\sum_{v=u+1}^{n}\mu_{u,v:n} = \binom{n}{2}\,[E(Z)]^{2}.$$

The covariance between $Z_{u:n}$ and $Z_{v:n}$ $\psi_{u,v:n} = \mu_{u,v:n} - \mu_{u:n}\mu_{v:n}$ can be obtain from equation (3.8) and equation (3.6).

3.4 BLUEs of the Location and Scale Parameters

In this section we obtain the BLUEs of the location and scale parameters of the MPF distribution based on order statistics.

Let $Z_{1:n} \leq Z_{2:n} \leq \cdots \leq Z_{n:n}$ be the order statistics from the scale-parameter and location-scale parameter MPF distribution, the pdf of the scale-parameter MPF distribution can be obtained after reparametrizing (3.1) as

$$h(z) = \frac{\alpha\beta\left[1-\left(\frac{z}{\varphi}\right)\right]^{\beta-1}}{\varphi\left[1-(1-\alpha)\left\{1-\left(\frac{z}{\varphi}\right)\right\}^{\beta}\right]^{2}},\ 0<x<1,\ \alpha,\ \beta>0,\ \varphi>0, \tag{3.9}$$

while the pdf of the location-scale parameter MPF distribution is

$$h(z) = \frac{\alpha\beta\left[1-\left(\frac{z-\delta}{\varphi}\right)\right]^{\beta-1}}{\varphi\left[1-(1-\alpha)\left\{1-\left(\frac{z-\delta}{\varphi}\right)\right\}^{\beta}\right]^{2}},\ z>\delta, \tag{3.10}$$

The expression for the BLUEs of location and scale parameter are given in (1.18) and also variances and covariance for these parameters are given in eqn (1.21), (1.22) and (1.23).

Tables 3.4 and 3.5 display the coefficient of the BLUEs for type-II right censored sample of various values of $n = 7,\ 10$ and censoring cases $c = 0(1)([n/2]-1)$. Also, Table 3.6 shows variances and covariances of the BLUEs.

Illustrative example: In this example, we show the usefulness of the coefficients of the BLUEs in Tables 3.4 and 3.5 by simulate a random sample of order statistics from the MPF distribution of size $n = 10$ when $\delta = 0$, $\varphi = 1$, $\alpha = 0.5$ and $\beta = 1.0$ as: 0.040628, 0.043592, 0.113282, 0.157260, 0.172874, 0.224061, 0.337366, 0.484587, 0.853319, 0.917235. By using the enterers of Tables 3.4 and 3.5, we calculate the BLUEs of δ and φ, to be

$$\begin{aligned}
\delta^* &= \sum_{u=1}^{n} a_u Z_{u:n} \\
&= (1.095146\times 0.040628)+(-0.00898\times 0.043592)+(-0.00802\times 0.113282) \\
&+ (-0.00713\times 0.157260)+(-0.00631\times 0.172874)+(-0.00557\times 0.224061) \\
&+ (-0.00488\times 0.337366)+(-0.00427\times 0.484587)+(-0.00371\times 0.853319) \\
&+ (-0.04627\times 0.917235) \\
&= -0.009500
\end{aligned}$$

and

$$
\begin{aligned}
\varphi^* &= \sum_{u=1}^{n} b_u Z_{u:n} \\
&= (-1.41637 \times 0.040628) + (0.043404 \times 0.043592) + (0.038667 \times 0.113282) \\
&+ (0.034292 \times 0.157260) + (0.03026 \times 0.172874) + (0.026576 \times 0.224061) \\
&+ (0.023221 \times 0.337366) + (0.020191 \times 0.484587) + (0.01747 \times 0.853319) \\
&+ (1.182291 \times 0.917235) \\
&= 1.082200.
\end{aligned}
$$

3.5 Numerical Results

The relations obtained in the preceding sections allow us to evaluate the expected values, second moments, variances, product moment and covariances of order statistics from samples of sizes up to 10 for various values of the parameters. The relation in (3.5) can be used to compute the expected values, second order moments and variances of all order statistics for sample sizes $n = 1, 2, 3, \ldots, 10$. In Tables 3.1 and 3.2, we have presented expected values, second order moments and variances of the uth order statistic from MPF distribution for $n = 1, 2, \ldots, 10$ and $\beta = 0.5,\ 1.0$ and $\alpha = 0.5$. One can see that the means and variances decreasing both with respect to n and β. In Table 3.3 we have reported the product moments and covariances of the uth and vth order statistic from MPF distribution for $n = 1, 2, \ldots, 10$ and $\alpha = 0.5$ and $\beta = 0.5$. From Table 3.3, one can observe that product moments are decreasing with respect to n.

Tables 3.4 and 3.5 display the coefficients of the BLUEs for Type-II right-censored samples of sample sizes $n = 7,\ 10$, $\beta = 0.5, 1.0, 1, 5$ and different censoring cases $c = 0(1)([n/2] - 1)$. The coefficients of the BLUEs in Tables 3.4 and 3.5. The variances and covariances of the BLUEs are presented in Table 3.6. We see the variance of the BLUEs increases as the censoring level increases while the variance of the BLUEs decreases when the sample size increases and increase as β increases. In addition, we see the covariances of the BLUEs decrease as the censoring level increases while the covariances of the BLUEs increase when the sample size increases and decrease as α increases. All computations here were performed using R software.

3.6 Real Data Application

To demonstrate how the proposed methods can be used in practice, we consider the following real-life data set (see Okorie et al. (2017)) is presented the Evaporation data which was extracted from the monthly publication of climatological data of the National Oceanic and Atmospheric Administration. The data is on the daily pan evaporation in hundredths of inches that was recorded in September 2016 in San Joaquin Drainage 05 Friant Government Camp, California, USA.

```
0.28 0.29 0.29 0.27 0.17 0.33 0.26 0.26 0.32 0.24 0.28 0.26 0.30
0.29 0.21 0.23 0.25 0.27 0.26 0.28 0.32 0.28 0.18 0.18 0.26 0.42
0.11 0.34 0.32 0.36
```

Now a random sample of size 10 is selected from the given data set are 0.34, 0.29, 0.18, 0.28, 0.30, 0.26, 0.32, 0.21, 0.32, 0.36. By using the MPF distribution in Eq. (3.1) for the given sample, we have the maximum likelihood estimate of $\alpha_{ML} = 3351.0206$ and $\beta_{ML} = 23.4902$. Figure 3.11 shows ecdf and QQ plot of the sample. This conclusion is also supported by the Kolmogorov-Smirnov (K-S) tests, KS statistic is 0.1298 and p value is 0.996. This shows the suitability of the modified power function distribution for this real data set.

3.7 Conclusion

In this chapter, we have considered mdified power function distribution. We first presented expressions for the single and double moments in explicit forms, and by making use of them we

TABLE 3.1: Expected values, second moments and variances of the uth order statistic from MPF distribution for $n = 1, 2, ..., 10$, $\alpha = 0.5$ and $\beta = 0.5$ (sim.=simulated)

n	u	$E(Z)$	$Sim.E(Z)$	$E(Z^2)$	$Sim.E(Z^2)$	$V(Z)$	$Sim.V(Z)$
1	1	0.545177	0.545373	0.395602	0.394743	0.098384	0.097312
2	1	0.364468	0.362797	0.202700	0.202114	0.069863	0.070493
	2	0.725887	0.725130	0.588504	0.589248	0.061592	0.063434
3	1	0.271065	0.270508	0.120836	0.120153	0.047360	0.046979
	2	0.551274	0.551012	0.366427	0.365801	0.062524	0.062187
	3	0.813194	0.814045	0.699543	0.699807	0.038259	0.037138
4	1	0.214892	0.215320	0.079469	0.079931	0.033290	0.033568
	2	0.439582	0.440096	0.244937	0.244762	0.051705	0.051078
	3	0.662965	0.664027	0.487917	0.487130	0.048394	0.046199
	4	0.863270	0.863235	0.770084	0.770401	0.024849	0.025226
5	1	0.177662	0.178002	0.055954	0.055813	0.024390	0.024129
	2	0.363814	0.364499	0.173527	0.173551	0.041166	0.040692
	3	0.553233	0.553238	0.352053	0.351382	0.045986	0.045309
	4	0.736120	0.735030	0.578494	0.578923	0.036621	0.038654
	5	0.895058	0.894940	0.817982	0.817766	0.016853	0.016849
6	1	0.151274	0.150900	0.041411	0.041517	0.018527	0.018746
	2	0.309602	0.310018	0.128670	0.128915	0.032817	0.032803
	3	0.472240	0.472914	0.263241	0.263948	0.040230	0.040302
	4	0.634227	0.633639	0.440865	0.440731	0.038621	0.039233
	5	0.787067	0.786317	0.647308	0.647432	0.027834	0.029138
	6	0.916656	0.916521	0.852117	0.851970	0.011859	0.011959
7	1	0.131635	0.130852	0.031830	0.031753	0.014502	0.014631
	2	0.269105	0.268968	0.098899	0.098266	0.026481	0.025923
	3	0.410843	0.410805	0.203098	0.203402	0.034306	0.034642
	4	0.554102	0.554408	0.343432	0.343523	0.036403	0.036155
	5	0.694320	0.694669	0.513940	0.514422	0.031860	0.031857
	6	0.824165	0.824515	0.700655	0.702136	0.021407	0.022311
	7	0.932071	0.932025	0.877361	0.877533	0.008605	0.008863
8	1	0.116469	0.116617	0.025201	0.025226	0.011636	0.011627
	2	0.237797	0.238244	0.078233	0.077837	0.021686	0.021078
	3	0.363029	0.362852	0.160896	0.160069	0.029106	0.028408
	4	0.490534	0.489730	0.273433	0.271920	0.032809	0.032084
	5	0.617671	0.616995	0.413431	0.413351	0.031914	0.032669
	6	0.740310	0.740304	0.574245	0.575577	0.026186	0.027527
	7	0.852117	0.852236	0.742792	0.742265	0.016689	0.015959
	8	0.943493	0.943479	0.896585	0.896725	0.006406	0.006572
9	1	0.104414	0.103916	0.020433	0.020252	0.009531	0.009454
	2	0.212914	0.212434	0.063349	0.063699	0.018017	0.018571
	3	0.324888	0.324554	0.130324	0.130673	0.024772	0.025338
	4	0.439311	0.439420	0.222040	0.220889	0.029046	0.027799
	5	0.554561	0.554663	0.337675	0.337012	0.030137	0.029361
	6	0.668158	0.668675	0.474036	0.473758	0.027601	0.026632
	7	0.776386	0.777208	0.624350	0.624517	0.021575	0.020466
	8	0.873754	0.873348	0.776632	0.776071	0.013186	0.013334
	9	0.952211	0.952708	0.911579	0.911223	0.004873	0.003569
10	1	0.094606	0.094210	0.016892	0.016974	0.007942	0.008098
	2	0.192685	0.193123	0.052298	0.052522	0.015170	0.015226
	3	0.293828	0.293520	0.107553	0.107825	0.021218	0.021671
	4	0.397361	0.397470	0.183457	0.182247	0.025561	0.024265
	5	0.502237	0.501979	0.279914	0.280440	0.027672	0.028458
	6	0.606886	0.606632	0.395435	0.393637	0.027124	0.025636
	7	0.709006	0.708881	0.526437	0.527053	0.023747	0.024541
	8	0.805263	0.804815	0.666313	0.666905	0.017865	0.019178
	9	0.890877	0.891297	0.804212	0.804178	0.010550	0.009768
	10	0.959025	0.959336	0.923509	0.923843	0.003780	0.003517

TABLE 3.2: Expected values, second moments and variances of the uth order statistic from MPF distribution for $n = 1, 2, ..., 10$, $\alpha = 0.5$ and $\beta = 1.0$ (sim.=simulated)

n	u	$E(Z)$	$Sim.E(Z)$	$E(Z^2)$	$Sim.E(Z^2)$	$V(Z)$	$Sim.V(Z)$
1	1	0.384524	0.386569	0.227411	0.227868	0.079552	0.078432
2	1	0.221187	0.227643	0.090355	0.090317	0.041431	0.038496
	2	0.545177	0.545709	0.364468	0.364431	0.067250	0.066633
3	1	0.158883	0.157504	0.046701	0.047021	0.021457	0.022213
	2	0.364468	0.365468	0.177662	0.178248	0.044825	0.044681
	3	0.635532	0.634216	0.457871	0.457989	0.053970	0.055758
4	1	0.121489	0.121790	0.028086	0.027757	0.013326	0.012924
	2	0.271065	0.271523	0.102547	0.103349	0.029071	0.029624
	3	0.457871	0.457786	0.252776	0.253339	0.043130	0.043772
	4	0.694753	0.694951	0.526236	0.526079	0.043554	0.043123
5	1	0.098138	0.098192	0.018615	0.018638	0.008984	0.008996
	2	0.214890	0.215062	0.065970	0.066337	0.019792	0.020085
	3	0.355323	0.354062	0.157413	0.157421	0.031159	0.032062
	4	0.526236	0.526138	0.316351	0.317775	0.039427	0.040954
	5	0.736882	0.737877	0.578707	0.577119	0.035712	0.032656
6	1	0.082234	0.082714	0.013194	0.013395	0.006432	0.006554
	2	0.177662	0.177043	0.045722	0.045601	0.014158	0.014257
	3	0.289353	0.288804	0.106467	0.106211	0.022742	0.022803
	4	0.421293	0.420719	0.208360	0.208521	0.030872	0.031517
	5	0.578707	0.579260	0.370347	0.370214	0.035445	0.034672
	6	0.768517	0.768605	0.620379	0.621239	0.029761	0.030486
7	1	0.070727	0.070741	0.009819	0.009960	0.004817	0.004956
	2	0.151274	0.151549	0.033442	0.033236	0.010558	0.010269
	3	0.243632	0.243303	0.076420	0.076672	0.017063	0.017475
	4	0.350316	0.350495	0.146529	0.146779	0.023808	0.023933
	5	0.474527	0.474278	0.254733	0.254134	0.029557	0.029194
	6	0.620379	0.619972	0.416592	0.416918	0.031722	0.032553
	7	0.793207	0.793798	0.654343	0.654674	0.025166	0.024558
8	1	0.062026	0.062230	0.007583	0.007706	0.003736	0.003834
	2	0.131635	0.131995	0.025473	0.025463	0.008145	0.008041
	3	0.210189	0.209715	0.057349	0.057253	0.013170	0.013273
	4	0.299369	0.300009	0.108204	0.108429	0.018582	0.018424
	5	0.401262	0.401059	0.184854	0.184849	0.023843	0.024001
	6	0.518485	0.518873	0.296661	0.296003	0.027834	0.026774
	7	0.654343	0.653942	0.456569	0.456295	0.028404	0.028656
	8	0.813045	0.813754	0.682596	0.683177	0.021554	0.020981
9	1	0.055221	0.055014	0.006028	0.005985	0.002979	0.002958
	2	0.116469	0.116777	0.020025	0.020085	0.006460	0.006449
	3	0.184716	0.184474	0.044544	0.044777	0.010424	0.010747
	4	0.261136	0.260462	0.082961	0.083428	0.014769	0.015587
	5	0.347160	0.347165	0.139759	0.139925	0.019239	0.019401
	6	0.444544	0.445129	0.220930	0.220832	0.023311	0.022693
	7	0.555456	0.554200	0.334526	0.333689	0.025995	0.026551
	8	0.682596	0.682710	0.491439	0.492183	0.025502	0.026089
	9	0.829351	0.829503	0.706491	0.706493	0.018668	0.018419
10	1	0.049755	0.049694	0.004904	0.004917	0.002428	0.002447
	2	0.104414	0.104116	0.016142	0.016085	0.005240	0.005244
	3	0.164691	0.163870	0.035555	0.035498	0.008432	0.008645
	4	0.231440	0.231469	0.065519	0.065359	0.011955	0.011782
	5	0.305680	0.305999	0.109123	0.109344	0.015683	0.015709
	6	0.388640	0.388656	0.170395	0.170954	0.019354	0.019901
	7	0.481813	0.482272	0.254620	0.254469	0.022476	0.021883
	8	0.587018	0.587175	0.368772	0.368149	0.024182	0.023375
	9	0.706491	0.706978	0.522105	0.522652	0.022975	0.022833
	10	0.843002	0.842981	0.726979	0.727268	0.016327	0.016652

TABLE 3.3: Covariances of order statistics for $n = 2, 3, \ldots, 10$, $\alpha = 0.5$ and $\beta = 0.5$

v	u	n	$\mu_{u,v:n}$	$Sim.\mu_{u,v:n}$	$Cov(Z_{u,v:n})$	$Sim.Cov(Z_{u,v:n})$
2	1	2	0.297218	0.297092	0.032656	0.034017
		3	0.180806	0.179939	0.031374	0.030886
		4	0.119855	0.120238	0.025392	0.025477
		5	0.084670	0.085426	0.020034	0.020544
		6	0.062747	0.062971	0.015913	0.016189
		7	0.048250	0.048072	0.012826	0.012876
		8	0.038201	0.038234	0.010505	0.010451
		9	0.030965	0.031032	0.008734	0.008956
		10	0.025591	0.025507	0.007362	0.007313
3	1	3	0.234487	0.234142	0.014059	0.013936
		4	0.158801	0.158238	0.016335	0.015260
		5	0.113246	0.113614	0.014958	0.015136
		6	0.084299	0.084292	0.012861	0.012929
		7	0.064961	0.064753	0.010879	0.010998
		8	0.051480	0.051480	0.009199	0.009166
		9	0.041744	0.041676	0.007821	0.007950
		10	0.034499	0.034484	0.006702	0.006832
	2	3	0.476362	0.475522	0.028069	0.026973
		4	0.324713	0.324832	0.033285	0.032597
		5	0.232017	0.232362	0.030743	0.030707
		6	0.172732	0.172437	0.026525	0.025826
		7	0.133020	0.132942	0.022461	0.022449
		8	0.105315	0.104768	0.018988	0.018321
		9	0.085306	0.086028	0.016133	0.017082
		10	0.070426	0.070531	0.013809	0.013846
4	1	4	0.192603	0.192904	0.007093	0.007032
		5	0.140184	0.139840	0.009403	0.009003
		6	0.105379	0.104738	0.009437	0.009122
		7	0.081614	0.081570	0.008675	0.009025
		8	0.064851	0.064492	0.007719	0.007381
		9	0.052659	0.052635	0.006789	0.006973
		10	0.043551	0.043610	0.005958	0.006164
	2	4	0.393942	0.394204	0.014464	0.014298
		5	0.287143	0.288069	0.019332	0.020152
		6	0.215822	0.215984	0.019464	0.019544
		7	0.167022	0.167316	0.017910	0.018199
		8	0.132580	0.133082	0.015932	0.016407
		9	0.107539	0.107529	0.014004	0.014182
		10	0.088843	0.089191	0.012277	0.012430
	3	4	0.593397	0.593089	0.021079	0.019879
		5	0.436193	0.437042	0.028947	0.030395
		6	0.329042	0.329614	0.029535	0.029958
		7	0.255010	0.255112	0.027361	0.027358
		8	0.202503	0.201782	0.024425	0.024083
		9	0.164231	0.164732	0.021504	0.022116
		10	0.135619	0.135266	0.018864	0.018601
5	1	5	0.163002	0.163316	0.003984	0.004014
		6	0.124886	0.125488	0.005823	0.006833
		7	0.097671	0.097874	0.006274	0.006975
		8	0.078024	0.078014	0.006085	0.006062
		9	0.063548	0.063613	0.005644	0.005975
		10	0.052647	0.052686	0.005132	0.005395
	2	5	0.333828	0.334538	0.008193	0.008334
		6	0.255689	0.255558	0.012011	0.011785
		7	0.199797	0.199978	0.012952	0.013134

TABLE 3.3: Continued.

v	u	n	$\mu_{u,v:n}$	$Sim.\mu_{u,v:n}$	$Cov(Z_{u,v:n})$	$Sim.Cov(Z_{u,v:n})$
		8	0.159439	0.159872	0.012559	0.012877
		9	0.129715	0.129814	0.011641	0.011985
		10	0.107349	0.106969	0.010575	0.010026
	3	5	0.507456	0.507554	0.012280	0.012439
		6	0.389918	0.390507	0.018234	0.018647
		7	0.305048	0.305068	0.019792	0.019695
		8	0.243487	0.243230	0.019255	0.019352
		9	0.198047	0.197806	0.017876	0.017788
		10	0.163820	0.163034	0.016249	0.015693
	4	5	0.674446	0.673905	0.015576	0.016097
		6	0.523052	0.521850	0.023873	0.023609
		7	0.411072	0.410618	0.026348	0.025488
		8	0.328861	0.329074	0.025872	0.026913
		9	0.267775	0.268748	0.024150	0.025018
		10	0.221589	0.220846	0.022020	0.021325
6	1	6	0.141084	0.140601	0.002418	0.002298
		7	0.112300	0.112199	0.003812	0.004310
		8	0.090569	0.091106	0.004346	0.004774
		9	0.074170	0.073912	0.004405	0.004426
		10	0.061647	0.061408	0.004232	0.004257
	2	6	0.288787	0.289856	0.004988	0.005718
		7	0.229656	0.229662	0.007869	0.007894
		8	0.185014	0.184476	0.008971	0.008104
		9	0.151345	0.151285	0.009085	0.009236
		10	0.125657	0.125293	0.008719	0.008138
	3	6	0.440458	0.441634	0.007577	0.008199
		7	0.350629	0.350569	0.012027	0.011854
		8	0.282509	0.283372	0.013755	0.014751
		9	0.231028	0.230798	0.013952	0.013777
		10	0.191718	0.191488	0.013398	0.013429
	4	6	0.591299	0.590084	0.009932	0.009341
		7	0.472692	0.472835	0.016020	0.015717
		8	0.381635	0.381659	0.018488	0.019110
		9	0.312380	0.312087	0.018851	0.018259
		10	0.259310	0.259394	0.018157	0.018276
	5	6	0.733081	0.732447	0.011612	0.011771
		7	0.591632	0.591381	0.019397	0.018616
		8	0.480089	0.480398	0.022821	0.023635
		9	0.394067	0.393973	0.023533	0.023084
		10	0.327624	0.327149	0.022823	0.022633
7	1	7	0.124250	0.124977	0.001557	0.003020
		8	0.101851	0.101798	0.002606	0.002413
		9	0.084179	0.084565	0.003113	0.003801
		10	0.070350	0.070708	0.003274	0.003924
	2	7	0.254039	0.254464	0.003214	0.003779
		8	0.208009	0.207742	0.005378	0.004702
		9	0.171724	0.171852	0.006420	0.006747
		10	0.143360	0.143906	0.006746	0.007005
	3	7	0.387848	0.388970	0.004914	0.006089
		8	0.317591	0.318025	0.008248	0.008790
		9	0.262099	0.261550	0.009860	0.009304
		10	0.218691	0.218729	0.010365	0.010658
	4	7	0.523011	0.522597	0.006549	0.005875
		8	0.429081	0.428626	0.011089	0.011261
		9	0.354400	0.354529	0.013325	0.013009
		10	0.295779	0.295479	0.014048	0.013721

TABLE 3.3: Continued.

v	u	n	$\mu_{u,v:n}$	$Sim.\mu_{u,v:n}$	$Cov(Z_{u,v:n})$	$Sim.Cov(Z_{u,v:n})$
	5	7	0.655096	0.655111	0.007940	0.007662
		8	0.540025	0.539712	0.013697	0.013887
		9	0.447194	0.447169	0.016641	0.016081
		10	0.373751	0.373467	0.017662	0.017624
	6	7	0.776968	0.776666	0.008787	0.008197
		8	0.646570	0.646241	0.015739	0.015327
		9	0.538281	0.538594	0.019532	0.018895
		10	0.451285	0.451676	0.020999	0.021647
8	1	8	0.110937	0.111259	0.001050	0.001234
		9	0.093077	0.093538	0.001845	0.002784
		10	0.078475	0.078646	0.002292	0.002824
	2	8	0.226526	0.226706	0.002166	0.001928
		9	0.189840	0.190425	0.003806	0.004896
		10	0.159885	0.159884	0.004723	0.004456
	3	8	0.345837	0.347017	0.003322	0.004674
		9	0.289717	0.289256	0.005845	0.005807
		10	0.243866	0.243807	0.007257	0.007578
	4	8	0.467283	0.467003	0.004467	0.004952
		9	0.391750	0.391017	0.007900	0.007250
		10	0.329817	0.329537	0.009837	0.009647
	5	8	0.588291	0.588312	0.005522	0.006190
		9	0.494420	0.494491	0.009870	0.010077
		10	0.416803	0.416871	0.012370	0.012871
	6	8	0.704832	0.704369	0.006355	0.005908
		9	0.595399	0.595409	0.011594	0.011423
		10	0.503417	0.503202	0.014714	0.014976
	7	8	0.810724	0.811121	0.006757	0.007054
		9	0.691195	0.691247	0.012825	0.012474
		10	0.587589	0.587706	0.016653	0.017188
9	1	9	0.100158	0.100705	0.000734	0.001704
		10	0.085628	0.085797	0.001345	0.001828
	2	9	0.204253	0.203607	0.001514	0.001219
		10	0.174430	0.174888	0.002772	0.002758
	3	9	0.311687	0.311876	0.002325	0.002670
		10	0.266024	0.265905	0.004259	0.004291
	4	9	0.421460	0.420854	0.003144	0.002216
		10	0.359773	0.359611	0.005774	0.005347
	5	9	0.531988	0.532590	0.003929	0.004158
		10	0.454693	0.454852	0.007262	0.007440
	6	9	0.640846	0.641018	0.004617	0.003966
		10	0.549303	0.549156	0.008642	0.008467
	7	9	0.744399	0.743753	0.005117	0.003301
		10	0.641426	0.641677	0.009789	0.009854
	8	9	0.837274	0.836571	0.005276	0.004525
		10	0.727907	0.726732	0.010516	0.009403
10	1	10	0.091259	0.091135	0.000529	0.000756
	2	10	0.185881	0.186612	0.001091	0.001342
	3	10	0.283465	0.283552	0.001677	0.001968
	4	10	0.383352	0.383133	0.002273	0.001826
	5	10	0.484517	0.483960	0.002859	0.002394
	6	10	0.585424	0.585534	0.003405	0.003571
	7	10	0.683814	0.683729	0.003860	0.003673
	8	10	0.776421	0.775737	0.004153	0.003649
	9	10	0.858552	0.859339	0.004178	0.004286

TABLE 3.4: Coefficients of the BLUEs of the location parameter for $\alpha = 0.5$.

β	n	c	$a_u, u = 1,2,3,\ldots,(n-c)$							
0.5	7	0	1.194764	-0.01428	-0.01332	-0.01229	-0.01068	-0.00623	-0.13798	
		1	1.213593	-0.01437	-0.01250	-0.00972	-0.00397	-0.17303		
		2	1.251611	-0.01455	-0.01092	-0.00473	-0.22142			
	10	0	1.136302	-0.00783	-0.00750	-0.00720	-0.00694	-0.00668	-0.00635	-0.00562
			-0.00306	-0.08513						
		1	1.142073	-0.00795	-0.00750	-0.00703	-0.00648	-0.00572	-0.00434	-0.00106
			-0.10199							
		2	1.153030	-0.00817	-0.00750	-0.00671	-0.00566	-0.00397	-0.00070	-0.12032
		3	1.171009	-0.00850	-0.00750	-0.00622	-0.00435	-0.00121	-0.14323	
		4	1.199250	-0.00901	-0.00749	-0.00546	-0.00237	-0.17493		
1	7	0	1.137575	-0.01683	-0.01429	-0.01205	-0.01010	-0.00842	-0.07588	
		1	1.168959	-0.02070	-0.01757	-0.01482	-0.01242	-0.10346		
		2	1.214326	-0.02600	-0.02208	-0.01863	-0.14762			
	10	0	1.095146	-0.00898	-0.00802	-0.00713	-0.00631	-0.00557	-0.00488	-0.00427
			-0.00371	-0.04627						
		1	1.109445	-0.01044	-0.00932	-0.00828	-0.00733	-0.00646	-0.00567	-0.00495
			-0.05701							
		2	1.126908	-0.01213	-0.01083	-0.00963	-0.00852	-0.00751	-0.00658	-0.07172
		3	1.149408	-0.01422	-0.01270	-0.01129	-0.00999	-0.00880	-0.09242	
		4	1.180259	-0.01698	-0.01516	-0.01348	-0.01194	-0.12269		
1.5	7	0	1.127859	-0.02042	-0.01693	-0.01403	-0.01182	-0.01076	-0.05390	
		1	1.162506	-0.02608	-0.02185	-0.01844	-0.01609	-0.08005		
		2	1.208192	-0.03318	-0.02804	-0.02401	-0.12296			
	10	0	1.089154	-0.01082	-0.00949	-0.00829	-0.00721	-0.00628	-0.00552	-0.00502
			-0.00507	-0.03147						
		1	1.106252	-0.01300	-0.01145	-0.01005	-0.00882	-0.00779	-0.00700	-0.00663
			-0.04151							
		2	1.124736	-0.01526	-0.01348	-0.01189	-0.01050	-0.00936	-0.00857	-0.05568
		3	1.147426	-0.01792	-0.01587	-0.01405	-0.01249	-0.01124	-0.07585	
		4	1.177902	-0.02138	-0.01898	-0.01687	-0.01508	-0.10560		
2	7	0	1.126869	-0.02348	-0.01924	-0.01578	-0.01323	-0.01216	-0.04298	
		1	1.161922	-0.02992	-0.02479	-0.02071	-0.01798	-0.06852		
		2	1.206867	-0.03778	-0.03159	-0.02680	-0.11070			
	10	0	1.089517	-0.01246	-0.01084	-0.00939	-0.00811	-0.00701	-0.00615	-0.00563
			-0.00578	-0.02415						
		1	1.107111	-0.01493	-0.01305	-0.01137	-0.00990	-0.00868	-0.00778	-0.00737
			-0.03403							
		2	1.125451	-0.01740	-0.01525	-0.01335	-0.01171	-0.01037	-0.00944	-0.04793
		3	1.147758	-0.02029	-0.01784	-0.01567	-0.01383	-0.01236	-0.06777	
		4	1.177681	-0.02403	-0.02119	-0.01870	-0.01659	-0.09717		
2.5	7	0	1.127986	-0.02592	-0.02108	-0.01714	-0.01427	-0.01301	-0.03657	
		1	1.162612	-0.03270	-0.02687	-0.02224	-0.01909	-0.06171		
		2	1.206728	-0.04093	-0.03393	-0.02849	-0.10338			
	10	0	1.091185	-0.01378	-0.01194	-0.01028	-0.00883	-0.00760	-0.00663	-0.00604
			-0.00615	-0.01993						
		1	1.108564	-0.01635	-0.01422	-0.01232	-0.01067	-0.00929	-0.00826	-0.00774
			-0.02971							
		2	1.126538	-0.01891	-0.01649	-0.01434	-0.01250	-0.01099	-0.00991	-0.04341
		3	1.148431	-0.02190	-0.01915	-0.01673	-0.01465	-0.01299	-0.06301	
		4	1.177887	-0.02579	-0.02262	-0.01983	-0.01748	-0.09216		

TABLE 3.5: Coefficients of the BLUEs of the scale parameter for $\alpha = 0.5$.

β	n	c	$b_u, u = 1, 2, 3, \ldots, (n-c)$							
0.5	7	0	-1.25759	0.014642	0.010491	0.003327	-0.01274	-0.06285	1.304718	
		1	-1.43565	0.015537	0.002798	-0.02094	-0.07622	1.514474		
		2	-1.76840	0.017055	-0.01108	-0.06463	1.827043			
	10	0	-1.16678	0.008524	0.007517	0.006240	0.004393	0.001299	-0.00485	-0.01981
			-0.06832	1.231788						
		1	-1.25028	0.010265	0.007536	0.003732	-0.00220	-0.01264	-0.03386	-0.08577
			1.363223							
		2	-1.39673	0.013128	0.007537	-0.00049	-0.01326	-0.03600	-0.08252	1.508328
		3	-1.62211	0.017315	0.007510	-0.00674	-0.02962	-0.07060	1.704243	
		4	-1.95813	0.023307	0.007450	-0.01577	-0.05322	1.996362		
1	7	0	-1.55076	0.070941	0.060069	0.050499	0.042155	0.034955	1.292140	
		1	-2.08519	0.136841	0.115961	0.097574	0.081539	1.653269		
		2	-2.81016	0.221644	0.188031	0.158419	2.242063			
	10	0	-1.41637	0.043404	0.038667	0.034292	0.030260	0.026576	0.023221	0.020191
			0.01747	1.182291						
		1	-1.78174	0.080590	0.071815	0.063708	0.056238	0.049411	0.043195	0.037578
			1.379208							
		2	-2.20424	0.121571	0.108386	0.096202	0.084975	0.074712	0.065366	1.653029
		3	-2.72286	0.169751	0.151419	0.134475	0.118864	0.104588	2.043758	
		4	-3.40512	0.230814	0.206000	0.183057	0.161922	2.623330		
1.5	7	0	-2.11222	0.190593	0.164104	0.145031	0.137081	0.152897	1.322513	
		1	-2.96227	0.329592	0.284825	0.253220	0.241774	1.852861		
		2	-4.01972	0.493952	0.428071	0.382326	2.715365			
	10	0	-1.97233	0.128749	0.115126	0.103345	0.093725	0.086936	0.084380	0.089644
			0.114472	1.155952						
		1	-2.60039	0.208947	0.187076	0.168228	0.152952	0.142399	0.138986	0.148932
			1.452871							
		2	-3.24741	0.288009	0.258106	0.232402	0.211692	0.197627	0.193713	1.865863
		3	-4.00775	0.377242	0.338369	0.305032	0.278322	0.260489	2.448295	
		4	-4.99148	0.488715	0.438727	0.395958	0.361889	3.306189		
2	7	0	-2.79920	0.347695	0.297377	0.261368	0.245794	0.270690	1.376277	
		1	-3.92165	0.553889	0.475089	0.419507	0.397669	2.075492		
		2	-5.28307	0.791861	0.680977	0.603921	3.206309			
	10	0	-2.68531	0.246202	0.219062	0.195691	0.176663	0.163126	0.157412	0.165247
			0.204499	1.157408						
		1	-3.52865	0.364624	0.324722	0.290441	0.262669	0.243179	0.235602	0.248895
			1.558519							
		2	-4.36866	0.477687	0.425749	0.381222	0.345319	0.320464	0.311656	2.106559
		3	-5.34909	0.604611	0.539299	0.483427	0.438606	0.408037	2.875108	
		4	-6.61859	0.763498	0.681577	0.611658	0.555880	4.005982		
2.5	7	0	-3.55154	0.526327	0.446567	0.388514	0.359968	0.385857	1.444309	
		1	-4.91925	0.794147	0.675388	0.589877	0.550291	2.309546		
		2	-6.57040	1.101990	0.939450	0.823867	3.705096			
	10	0	-3.47904	0.381947	0.338184	0.300383	0.269307	0.246455	0.234808	0.241494
			0.288064	1.178397						
		1	-4.50683	0.534029	0.473170	0.420688	0.377694	0.346369	0.331079	0.342338
			1.681463							
		2	-5.52403	0.678433	0.601532	0.535327	0.481301	0.442343	0.424304	2.360792
		3	-6.71466	0.841141	0.746337	0.664876	0.598688	0.551531	3.312088	
		4	-8.26303	1.045876	0.928714	0.828256	0.747032	4.713156		

TABLE 3.6: Variances and covariance of the BLUEs when $\alpha = 0.5,\ \delta = 0$ and $\varphi = 1$.

β	n	c	$Var(\delta^*)$	$Var(\varphi^*)$	$Cov(\delta^* \varphi^*)$
0.5	7	0	0.019258	0.031146	-0.020295
		1	0.019568	0.058868	-0.023227
		2	0.020193	0.106736	-0.028696
	10	0	0.009667	0.014220	-0.009934
		1	0.009717	0.024793	-0.010665
		2	0.009813	0.041887	-0.011944
		3	0.009970	0.066520	-0.013909
		4	0.010215	0.101337	-0.016835
1	7	0	0.005682	0.050930	-0.008022
		1	0.005858	0.102060	-0.011025
		2	0.006111	0.166716	-0.015071
	10	0	0.002710	0.027506	-0.003592
		1	0.002749	0.053000	-0.004590
		2	0.002796	0.080777	-0.005738
		3	0.002857	0.113123	-0.007141
		4	0.002940	0.153808	-0.008981
1.5	7	0	0.002714	0.078499	-0.005501
		1	0.002812	0.136979	-0.007885
		2	0.002939	0.205042	-0.010825
	10	0	0.001266	0.047450	-0.002432
		1	0.001288	0.077737	-0.003256
		2	0.001312	0.107307	-0.004101
		3	0.001342	0.140410	-0.005089
		4	0.001381	0.181499	-0.006362
2	7	0	0.001593	0.100866	-0.004425
		1	0.001652	0.161101	-0.006306
		2	0.001726	0.229589	-0.008567
	10	0	0.000733	0.063908	-0.001967
		1	0.000747	0.094983	-0.002616
		2	0.000761	0.124390	-0.003258
		3	0.000778	0.157157	-0.004003
		4	0.000800	0.197936	-0.004964
2.5	7	0	0.001048	0.118123	-0.003776
		1	0.001087	0.178267	-0.005298
		2	0.001136	0.246417	-0.007119
	10	0	0.000478	0.076564	-0.001690
		1	0.000487	0.107209	-0.002208
		2	0.000496	0.136061	-0.002718
		3	0.000507	0.168343	-0.003311
		4	0.000522	0.208741	-0.004080

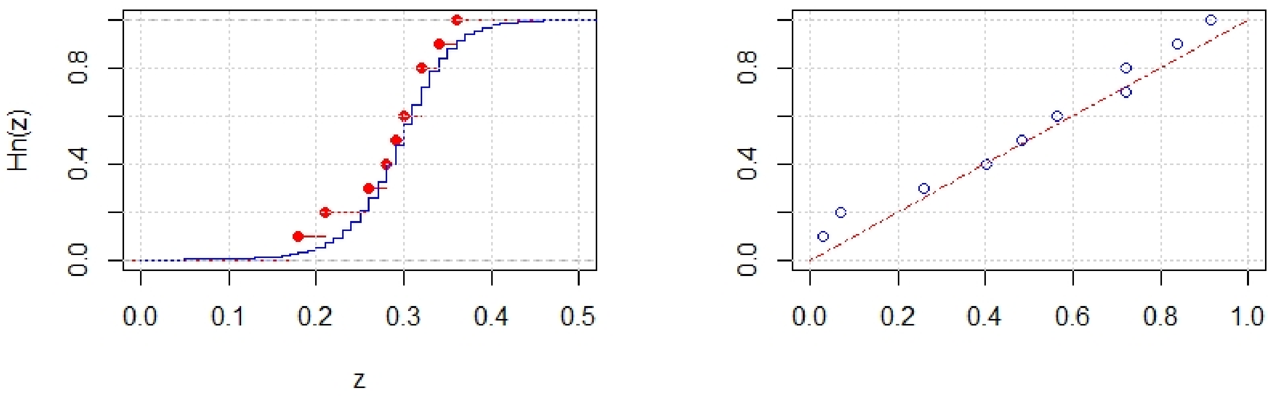

FIGURE 3.1: ECDF-plot and QQ-plot

have obtained means, variance and covariances for the order statistics. In simulation study we observed that means variance and covariances do decrease when a high number of *n* are taken into account however, the behaviour is found opposite when a large value of shape parameter is considered correspond to a fixed value of scale parameter. In practice the reported values can also be used to obtain the BLUEs of the location and scale parameters.

Chapter 4

Inference for the Extended Power Lindley Distribution based on Order Statistics with Application

4.1 Introduction

This chapter follows the following structure: In section 4.2, we introduce some lemmas on EPL distribution. We discuss single and product moments of order statistics in Sections 4.3 and 4.4. Section 4.5 describes how these moments can be used to calculate BLUEs for δ and φ. Section 4.6 demonstrate a real data application. This chapter is concluded in Section 4.7.

The Lindley distribution can be viewed as a mixture of $exp(\xi)$ and $gamma(2,\xi)$ distributions, which was introduced by Lindley (1958) in the context of fiducial and Bayesian statistics. Later, Ghitany et al. (2008) investigated the mathematical and statistical properties of this distribution and also demonstrate that this distribution makes better than exponential distribution in a number of ways. The Lindley distribution has a single scale parameter and can model the data with increasing monotonous failure rates and by virtue of this Lindley distribution offers insufficient flexibility to analyze different kinds of lifetime data. In order to increase flexibility for modelling, it will be helpful to consider other alternatives to this distribution. Also, Bouchahed and Zeghdoudi and Zeghdoudi et al. (2018) generalized the the Lindley distribution. Recently the three parameter extended power Lindley distribution was proposed by Alkarni (2015) for the flexibility of purpose.

The extended power Lindley (EPL) distribution is specified by the following pdf

$$h(z;\tau,\xi,\kappa)=\frac{\tau\xi^2}{\xi+\kappa}(1+\kappa z^{\tau})\,z^{\tau-1}\,e^{-\xi z^{\tau}},\quad z>0;\quad \tau>0,\xi>0,\kappa>0 \tag{4.1}$$

and the associated cdf is

$$H(z;\tau,\xi,\kappa)=1-\left(1+\frac{\kappa\xi}{\xi+\kappa}z^{\tau}\right)e^{-\xi z^{\tau}},\quad z>0;\quad \tau>0,\xi>0,\kappa>0. \tag{4.2}$$

For $\kappa=1$ and $\kappa=1,\ \ \tau=1$, the EPL distribution reduces to power Lindley (PL) and Lindley distributions respectively.

4.2 Technical Lemmas

In this section, we provide and prove some lemmas.

Lemma 1 Let $h(z)$ and $H(z)$ be given by (4.1) and (4.2), respectively. For $a>0$, $b>0$ and $p>0$, let

$$K(a,b,p)=\int_0^{\infty} z^p H^a(z)[1-H(z)]^b h(z)dz.$$

Then,

$$
\begin{aligned}
K(a,b,p) &= \frac{\xi^2}{(\xi+\kappa)^{b+1}} \sum_{i_1=0}^{a} \sum_{i_2=0}^{i_1+b} \sum_{i_3=0}^{i_2+1} (-1)^{i_1} \frac{\xi^{i_2}\kappa^{b+i_1-i_2+i_3}}{(\xi+\kappa)^{i_1}} \binom{a}{i_1} \binom{i_1+b}{i_2} \binom{i_2+1}{i_3} \\
&\times \frac{\Gamma\left(\frac{p+\tau(i_3+1)}{\tau}\right)}{[\xi(i_1+b+1)]^{\frac{p+\tau(i_3+1)}{\tau}}}.
\end{aligned}
$$

Proof. By using (4.1) and (4.2), we get

$$
\begin{aligned}
K(a,b,p) &= \sum_{i_1=0}^{a} (-1)^{i_1} \binom{a}{i_1} \int_0^\infty z^p [1-H(z)]^{i_1+b} h(z) dz. \\
&= \frac{\tau\xi^2}{(\xi+\kappa)} \sum_{i_1=0}^{a} (-1)^{i_1} \binom{a}{i_1} \int_0^\infty z^{p+\tau-1} (1+\kappa z^\tau) e^{-\xi(i_1+b+1)z^\tau} \\
&\times \left[\left(1+\frac{\kappa\xi}{\xi+\kappa} z^\tau\right)\right]^{i_1+b} dz \\
&= \frac{\tau\xi^2}{(\xi+\kappa)} \sum_{i_1=0}^{a} (-1)^{i_1} \binom{a}{i_1} \int_0^\infty z^{p+\tau-1} (1+\kappa z^\tau) e^{-\xi(i_1+b+1)z^\tau} \\
&\times \left[\left(\frac{\xi(1+\kappa z^\tau)+\kappa}{\xi+\kappa}\right)\right]^{i_1+b} dz \\
&= \frac{\tau\xi^2}{(\xi+\kappa)^{b+1}} \sum_{i_1=0}^{a} \sum_{i_2=0}^{i_1+b} \sum_{i_3=0}^{i_2+1} (-1)^{i_1} \frac{\xi^{i_2}\kappa^{b+i_1-i_2+i_3}}{(\xi+\kappa)^{i_1}} \binom{a}{i_1} \binom{i_1+b}{i_2} \binom{i_2+1}{i_3} \\
&\times \int_0^\infty z^{p+\tau(i_3+1)-1} \, e^{-\xi(i_1+b+1)z^\tau} dz \\
&= \frac{\xi^2}{(\xi+\kappa)^{b+1}} \sum_{i_1=0}^{a} \sum_{i_2=0}^{i_1+b} \sum_{i_3=0}^{i_2+1} (-1)^{i_1} \frac{\xi^{i_2}\kappa^{b+i_1-i_2+i_3}}{(\xi+\kappa)^{i_1}} \binom{a}{i_1} \binom{i_1+b}{i_2} \binom{i_2+1}{i_3} \\
&\times \frac{1}{[\xi(i_1+b+1)]^{\frac{p+\tau(i_3+1)}{\tau}}} \int_0^\infty y^{\frac{p+\tau i_3}{\tau}} \, e^{-y} dy,
\end{aligned}
$$

where $y = \xi(i_1+b+1)z^\tau$.

Lemma 2 Let $h(z)$ and $H(z)$ be given by (4.1) and (4.2), respectively. For $a > 0$, $b > 0$, c, $p > 0$ and $q > 0$, let

$$
L(a,b,c,p,q) = \int_0^\infty \int_z^\infty z^p y^q [H(z)]^a [H(y)-H(z)]^b [1-H(y)]^c h(z)h(y) dy dz.
$$

Then,

$$\begin{aligned} L(a,b,c,p,q) &= \frac{\xi^4}{(\xi+\kappa)^{b+c+2}} \sum_{i_1=0}^{a} \sum_{i_2=0}^{b} \sum_{i_3=0}^{i_1+b-i_2} \sum_{i_4=0}^{i_2+c} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} \sum_{i_7=0}^{\frac{q+\tau i_6}{\tau}} (-1)^{i_1+i_2} \binom{a}{i_1} \binom{b}{i_2} \\ &\times \binom{i_1+b-i_2}{i_3} \binom{i_2+c}{i_4} \binom{i_3+1}{i_5} \binom{i_4+1}{i_6} \frac{\xi^{i_3+i_4} \kappa^{i_1+b-i_3-i_4+i_5+i_6}}{(\xi+\kappa)^{i_1}} \\ &\times \frac{\left(\frac{q+\tau i_6}{\tau}\right)!}{[\xi(i_2+c+1)]^{\frac{q+\tau(i_6+1)}{\tau}}} \frac{[\xi(i_2+c+1)]^{i_7}}{i_7!} \frac{\Gamma\left(\frac{p+\tau(i_5+i_7+1)}{\tau}\right)}{[\xi(i_1+b+c+2)]^{\frac{p+\tau(i_5+i_7+1)}{\tau}}}. \end{aligned}$$

Proof. We have

$$\begin{aligned} L(a,b,c,p,q) &= \sum_{i_1=0}^{a} \sum_{i_2=0}^{b} (-1)^{i_1+i_2} \binom{a}{i_1} \binom{b}{i_2} \\ &\times \int_0^\infty \int_z^\infty z^p y^q [1-H(z)]^{i_1+b-i_2} [1-H(y)]^{i_2+c} h(z) h(y) dy dz \\ &= \frac{\tau^2 \xi^4}{(\xi+\kappa)^2} \sum_{i_1=0}^{a} \sum_{i_2=0}^{b} (-1)^{i_1+i_2} \binom{a}{i_1} \binom{b}{i_2} \int_0^\infty \int_z^\infty z^{p+\tau-1} y^{q+\tau-1} \\ &\times (1+\kappa z^\tau)(1+\kappa y^\tau) e^{-\xi(i_1+b-i_2+1)z^\tau} e^{-\xi(i_2+c+1)y^\tau} \\ &\times \left[\left(1+\frac{\xi\kappa}{\xi+\kappa} z^\tau\right)\right]^{i_1+b-i_2} \left[\left(1+\frac{\xi\kappa}{\xi+\kappa} y^\tau\right)\right]^{i_2+c} dz dy \\ &= \frac{\tau^2 \kappa^4}{(\xi+\kappa)^{b+c+2}} \sum_{i_1=0}^{a} \sum_{i_2=0}^{b} \sum_{i_3=0}^{i_1+b-i_2} \sum_{i_4=0}^{i_2+c} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} (-1)^{i_1+i_2} \frac{\kappa^{i_1+b-i_3-i_4+i_5+i_6}}{(\xi+\kappa)^{i_1}} \\ &\times \binom{a}{i_1} \binom{b}{i_2} \binom{i_1+b-i_2}{i_3} \binom{i_2+c}{i_4} \binom{i_3+1}{i_5} \binom{i_4+1}{i_6} \\ &\times \int_0^\infty \int_z^\infty z^{p+\tau(i_5+1)-1} y^{q+\tau(i_6+1)-1} e^{-\xi(i_1+b-i_2+1)z^\tau} e^{-\xi(i_2+c+1)y^\tau} dz dy \\ &= \frac{\tau^2 \xi^4}{(\xi+\kappa)^{b+c+2}} \sum_{i_1=0}^{a} \sum_{i_2=0}^{b} \sum_{i_3=0}^{i_1+b-i_2} \sum_{i_4=0}^{i_2+c} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} (-1)^{i_1+i_2} \\ &\times \frac{\xi^{i_3+i_4} \kappa^{i_1+b-i_3-i_4+i_5+i_6}}{(\xi+\kappa)^{i_1}} \binom{a}{i_1} \binom{b}{i_2} \binom{i_1+b-i_2}{i_3} \binom{i_2+c}{i_4} \binom{i_3+1}{i_5} \\ &\times \binom{i_4+1}{i_6} \frac{1}{[\xi(i_2+c+1)]^{\frac{q+\tau(i_6+1)}{\tau}}} \int_0^\infty z^{p+\tau(i_5+1)-1} e^{-\xi(i_1+b-i_2+1)z^\tau} \\ &\times \Gamma\left(\frac{q+\tau(i_6+1)}{\tau},\ \xi(i_2+c+1)z^\tau\right) dz, \end{aligned}$$

By using the relation

$$\Gamma(p,\ y) = (p-1)!\ e^{-y}\sum_{l=0}^{p-1}\frac{y^l}{l!}.$$

$$\begin{aligned}
L(a,b,c,p,q) &= \frac{\tau^2\xi^4}{(\xi+\kappa)^{b+c+2}}\sum_{i_1=0}^{a}\sum_{i_2=0}^{b}\sum_{i_3=0}^{i_1+b-i_2}\sum_{i_4=0}^{i_2+c}\sum_{i_5=0}^{i_3+1}\sum_{i_6=0}^{i_4+1}\sum_{i_7=0}^{\frac{q+\tau i_6}{\tau}}(-1)^{i_1+i_2}\binom{a}{i_1}\binom{b}{i_2}\\
&\times \frac{\xi^{i_3+i_4}\kappa^{i_1+b-i_3-i_4+i_5+6}}{(\xi+\kappa)^{i_1}}\binom{i_1+b-i_2}{i_3}\binom{i_2+c}{i_4}\binom{i_3+1}{i_5}\binom{i_4+1}{i_6}\\
&\times \frac{\left(\frac{q+\tau i_6}{\tau}\right)!}{[\xi(i_2+c+1)]^{\frac{q+\tau(i_6+1)}{\tau}}}\frac{[\xi(i_2+c+1)]^{i_7}}{i_7!}\\
&\times \int_0^\infty z^{p+\tau(i_5+i_7+1)-1}e^{-\xi(i_1+b+c+2)z^\tau}dz\\
&= \frac{\xi^4}{(\xi+\kappa)^{b+c+2}}\sum_{i_1=0}^{a}\sum_{i_2=0}^{b}\sum_{i_3=0}^{i_1+b-i_2}\sum_{i_4=0}^{i_2+c}\sum_{i_5=0}^{i_3+1}\sum_{i_6=0}^{i_4+1}\sum_{i_7=0}^{\frac{q+\tau i_6}{\tau}}(-1)^{i_1+i_2}\binom{a}{i_1}\binom{b}{i_2}\\
&\times \frac{\xi^{i_3+i_4}\kappa^{i_1+b-i_3-i_4+i_5+i_6}}{(\xi+\kappa)^{i_1}}\binom{i_1+b-i_2}{i_3}\binom{i_2+c}{i_4}\binom{i_3+1}{i_5}\binom{i_4+1}{i_6}\\
&\times \frac{\left(\frac{q+\tau i_6}{\tau}\right)!}{[\xi(i_2+c+1)]^{\frac{q+\tau(i_6+1)}{\tau}}}\frac{[\xi(i_2+c+1)]^{i_7}}{i_7!}\\
&\times \frac{1}{[\xi(i_1+b+c+2)]^{\frac{p+\tau(i_5+i_7+1)}{\tau}}}\int_0^\infty t^{\frac{p+\tau(i_5+i_7)}{\tau}}e^{-t}dt,
\end{aligned}$$

where $t=\xi(i_1+b+c+2)z^\tau$.

4.3 Relations for Single Moments of Order Statistics

Here, we present some new expressions for single moments of uth order statistics, $E\left(Z_{u:n}^{(p)}\right)=\mu_{u:n}^{(p)}$ for the given random sample $Z_1,Z_2,\ldots,Z_n$ from the EPL distribution.

Theorem 1. For the EPL distribution given in (4.1) and for, $1 \leq u \leq n$

$$\mu_{u:n}^{(p)} = \frac{\xi^2 C_{u:n}}{(\xi+\kappa)^{n-u+1}} \sum_{i_1=0}^{u-1} \sum_{i_2=0}^{i_1+n-u} \sum_{i_3=0}^{i_2+1} (-1)^{i_1} \binom{u-1}{i_1} \binom{i_1+n-u}{i_2} \binom{i_2+1}{i_3} \times \frac{\xi^{i_2} \kappa^{i_1+n-u-i_2+i_3}}{(\xi+\kappa)^{i_1}} \frac{\Gamma(\frac{p+\tau(i_3+1)}{\tau})}{[\xi(i_1+n-u+1)]^{\frac{p+\tau(i_3+1)}{\tau}}}. \quad (4.3)$$

Proof. By using (1.1), we get

$$\mu_{u:n}^{(p)} = C_{u:n} \int_0^\infty z^p H^{u-1}(z) [1-H(z)]^{n-u} h(z) dz. \quad (4.4)$$

By lemma 1, we get the result.

Special Cases

(i) For $\tau = 1$ and $\kappa = 1$ in (4.3), we obtain relation for order statistic of Lindley distribution

$$\mu_{u:n}^{(p)} = \frac{\xi^2 C_{u:n}}{(\xi+1)^{n-u+1}} \sum_{i_1=0}^{u-1} \sum_{i_2=0}^{i_1+n-u} \sum_{i_3=0}^{i_2+1} (-1)^{i_1} \binom{u-1}{i_1} \binom{i_1+n-u}{i_2} \binom{i_2+1}{i_3} \times \frac{\xi^{i_2}}{(\xi+1)^{i_1}} \frac{\Gamma(p+i_3+1)}{[\xi(i_1+n-u+1)]^{p+i_3+1}},$$

as obtained by Sultan and Al-Thubyani (2016).

(ii) For $\kappa = 1$ in (4.3), we obtain relation for order statistic of power Lindley distribution

$$\mu_{u:n}^{(p)} = \frac{\xi^2 C_{u:n}}{(\xi+1)^{n-u+1}} \sum_{i_1=0}^{u-1} \sum_{i_2=0}^{i_1+n-u} \sum_{i_3=0}^{i_2+1} (-1)^{i_1} \binom{u-1}{i_1} \binom{i_1+n-u}{i_2} \binom{i_2+1}{i_3} \times \frac{\xi^{i_2}}{(\xi+1)^{i_1}} \frac{\Gamma\frac{(p+\tau(i_3+1))}{\tau}}{[\xi(i_1+n-u+1)]^{\frac{p+\tau(i_3+1)}{\tau}}},$$

as obtained by Kumar and Goyal (2019b).

(iii) If $p=u=1$ in (4.3), we obtain

$$\mu_{1:n}^{(1)} = \frac{n!}{(n-1)!}\frac{\xi^2}{(\xi+\kappa)^n}\sum_{i_2=0}^{n-1}\sum_{i_3=0}^{i_2+1}\binom{n-1}{i_2}\binom{i_2+1}{i_3}\xi^{i_2}\kappa^{n-1-i_2+i_3} \times \frac{\Gamma(\frac{\tau(i_3+1)+1}{\tau})}{(\xi n)^{\frac{\tau(i_3+1)+1}{\tau}}}.$$

(iv) If $p=1$, $u=n$ in (4.3), we obtain

$$\mu_{n:n}^{(1)} = \frac{n!}{(n-1)!}\frac{\xi^2}{(\xi+\kappa)}\sum_{i_1=0}^{n-1}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_2+1}(-1)^{i_1}\binom{n-1}{i_1}\binom{i_1}{i_2}\binom{i_2+1}{i_3} \times \frac{\xi^{i_2}\kappa^{i_1-i_2+i_3}}{(\xi+\kappa)^{i_1}}\frac{\Gamma(\frac{\tau(i_3+1)+1}{\tau})}{[\xi(i_1+1)]^{\frac{\tau(i_3+1)+1}{\tau}}}.$$

(v) If $p=n=u=1$ in (4.3), we obtain unordered mean of the extended power Lindley distribution

$$\mu_{1:1}^{(1)} = \frac{\xi^2}{(\xi+\kappa)}\sum_{i_3=0}^{1}\frac{\kappa^{i_3}\,\Gamma(\frac{\tau(i_3+1)+1}{\tau})}{\xi^{\frac{\tau(i_3+1)+1}{\tau}}} = E(Z),$$

as obtained by Alkarni (2015).

Specially, the first moment (mean) of the uth order statistic is

$$\mu_{u:n} = \mu_{u:n}^{(1)} = \vartheta(\tau,\xi,\kappa,u,n,1),$$

where

$$\vartheta(\tau,\xi,\kappa,u,n,p) = \frac{\xi^2\,C_{u:n}}{(\xi+\kappa)^{n-u+1}}\sum_{i_1=0}^{u-1}\sum_{i_2=0}^{i_1+n-u}\sum_{i_3=0}^{i_2+1}(-1)^{i_1}\binom{u-1}{i_1}\binom{i_1+n-u}{i_2}\binom{i_2+1}{i_3} \times \frac{\xi^{i_2}\kappa^{i_1+n-u-i_2+i_3}}{(\xi+\kappa)^{i_1}}\frac{\Gamma(\frac{p+\tau(i_3+1)}{\tau})}{[\xi(i_1+n-u+1)]^{\frac{p+\tau(i_3+1)}{\tau}}}.$$

In addition, the variance of $Z_{u:n}$ is found to be

$$\sigma^2_{u:n} \;=\; \mu^{(2)}_{u:n} - [\mu^{(1)}_{u:n}]^2 = \vartheta(\tau,\xi,\kappa,u,n,2) - [\vartheta(\tau,\xi,\kappa,u,n,1)]^2.$$

4.4 Relations for Product Moments of Order Statistics

Here, we present some new expressions for product moments of uth and vth order statistics, $E\left(Z^{(p,q)}_{u,v:n}\right) = \mu^{(p,q)}_{u,v:n}$ for the given random sample $Z_1, Z_2, \ldots, Z_n$ from the EPL distribution.

Theorem 2. For the EPL distribution given in (4.1) and for $1 \le u < v \le n$

$$\begin{aligned}
\mu^{(p,q)}_{u,v:n} \;=\;& C_{u,v:n}\frac{\xi^4}{(\xi+\kappa)^{n-u+1}} \sum_{i_1=0}^{u-1}\sum_{i_2=0}^{v-u-1}\sum_{i_3=0}^{i_1+v-u-i_2-1}\sum_{i_4=0}^{i_2+n-v}\sum_{i_5=0}^{i_3+1}\sum_{i_6=0}^{i_4+1}\sum_{i_7=0}^{\frac{q+\tau i_6}{\tau}}(-1)^{i_1+i_2} \\
\times\;& \frac{\xi^{i_3+i_4}\kappa^{i_1+v-u-1-i_3-i_4+i_5+i_6}}{(\xi+\kappa)^{i_1}}\binom{u-1}{i_1}\binom{v-u-1}{i_2}\binom{i_1+v-u-i_2-1}{i_3} \\
\times\;& \binom{i_2+n-v}{i_4}\binom{i_3+1}{i_5}\binom{i_4+1}{i_6}\frac{\left(\frac{q+\tau i_6}{\tau}\right)!}{[\xi(i_2+n-v+1)]^{\frac{q+\tau(i_6+1)}{\tau}}} \\
\times\;& \frac{[\xi(i_2+n-v+1)]^{i_7}}{i_7!}\frac{\Gamma\left(\frac{p+\tau(i_5+i_7+1)}{\tau}\right)}{[\xi(i_1+n-u+1)]^{\frac{p+\tau(i_5+i_7+1)}{\tau}}}.
\end{aligned}$$

Proof. By using (1.9), we obtain

$$\mu^{(p,q)}_{u,v:n} = C_{u,v:n}\int_0^\infty\int_z^\infty z^p\, y^q\, H^{u-1}(z)\,[H(y)-H(z)]^{v-1-u}\,[1-H(y)]^{n-v}\,h(z)h(y)dzdy \qquad (4.5)$$

The result follow by using lemma 2.

Special Cases

(i) For $\tau = 1$ and $\kappa = 1$ in Therom 2, we obtain the relation for order statistic of Lindley distribution

$$\begin{aligned}
\mu_{u,v:n}^{(p,q)} &= \frac{\xi^4 C_{u,v:n}}{(\xi+1)^{n-u+1}} \sum_{i_1=0}^{u-1} \sum_{i_2=0}^{v-u-1} \sum_{i_3=0}^{i_1+v-u-i_2-1} \sum_{i_4=0}^{i_2+n-v} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} \sum_{i_7=0}^{q+i_6} (-1)^{i_1+i_2} \frac{\xi^{i_3+i_4}}{(\xi+1)^{i_1}} \\
&\times \binom{u-1}{i_1} \binom{v-u-1}{i_2} \binom{i_1+v-u-i_2-1}{i_3} \binom{i_2+n-v}{i_4} \binom{i_3+1}{i_5} \binom{i_4+1}{i_6} \\
&\times \frac{(q+i_6)!}{[\xi(i_2+n-v+1)]^{q+i_6+1}} \frac{[\xi(i_2+n-v+1)]^{i_7}}{i_7!} \frac{\Gamma(p+i_5+i_7+1)}{[\xi(i_1+n-u+1)]^{p+i_5+i_7+1}}.
\end{aligned}$$

as obtained by Sultan and Al-Thubyani (2016).

(ii) For $\kappa = 1$ in Therom 2, we obtain the relation for order statistic of power Lindley distribution

$$\begin{aligned}
\mu_{u,v:n}^{(p,q)} &= \frac{\xi^4 C_{u,v:n}}{(\xi+1)^{n-u+1}} \sum_{i_1=0}^{u-1} \sum_{i_2=0}^{v-u-1} \sum_{i_3=0}^{i_1+v-u-i_2-1} \sum_{i_4=0}^{i_2+n-v} \sum_{i_5=0}^{i_3+1} \sum_{i_6=0}^{i_4+1} \sum_{i_7=0}^{\frac{q+\tau i_6}{\tau}} (-1)^{i_1+i_2} \frac{\xi^{i_3+i_4}}{(\xi+1)^{i_1}} \\
&\times \binom{u-1}{i_1} \binom{v-u-1}{i_2} \binom{i_1+v-u-i_2-1}{i_3} \binom{i_2+n-v}{i_4} \binom{i_3+1}{i_5} \binom{i_4+1}{i_6} \\
&\times \frac{\left(\frac{q+\tau i_6}{\tau}\right)!}{[\xi(i_2+n-v+1)]^{\frac{q+\tau(i_6+1)}{\tau}}} \frac{[\xi(i_2+n-v+1)]^{i_7}}{i_7!} \frac{\Gamma\left(\frac{p+\tau(i_5+i_7+1)}{\tau}\right)}{[\xi(i_1+n-u+1)]^{\frac{p+\tau(i_5+i_7+1)}{\tau}}}.
\end{aligned}$$

as obtained by Kumar and Goyal (2019b).

Let $\mu_{u,v:n}$ represents (1, 1)-th moment of $Z_{u:n}$ and $Z_{v:n}$ order statistics. So, covariance term can be written as

$$\sigma_{u,v:n} = Cov(Z_{u:n}, Z_{v:n}) = \mu_{u,v:n} - \mu_{u:n}\mu_{v:n}.$$

Tables 4.1 and 4.2 contains first order moments, second order moments and variances for various values of parameters and sample size n, of uth order statistic from EPL disribution. Table 4.3, shows product moments and covariance of uth and vth order statistic from EPL distribution.

TABLE 4.1: Moments, variances skewness and kurtosis of order statistic from EPL distribution $\tau = 2$, $\kappa = 0.5$ and $\xi = 5$

u	n	$E(Z)$	$E(Z^2)$	$V(Z)$	δ_1	δ_2	γ_1	γ_2
1	1	0.414348	0.218182	0.046498	0.379571	3.207079	0.616093	0.207079
	2	0.293422	0.109504	0.023408	0.387238	3.219388	0.622284	0.219388
	3	0.239709	0.073106	0.015646	0.389946	3.229529	0.624457	0.229529
	4	0.207652	0.054870	0.011751	0.391543	3.234472	0.625734	0.234472
	5	0.185762	0.043916	0.009408	0.392307	3.236857	0.626344	0.236857
	6	0.169597	0.036608	0.007845	0.394587	3.229235	0.628161	0.229235
	7	0.157029	0.031385	0.006727	0.395586	3.221588	0.628956	0.221588
	8	0.146897	0.027467	0.005888	0.395741	3.229525	0.629079	0.229525
	9	0.138503	0.024418	0.005235	0.394610	3.235330	0.628180	0.235330
	10	0.131400	0.021978	0.004712	0.396862	3.226521	0.629970	0.226521
2	2	0.535273	0.326860	0.040343	0.241593	3.215795	0.491521	0.215795
	3	0.400850	0.182301	0.021620	0.189444	3.115636	0.435251	0.115636
	4	0.335877	0.127814	0.015001	0.174957	3.088871	0.418279	0.088871
	5	0.295213	0.098686	0.011535	0.170039	3.070969	0.412358	0.070969
	6	0.266589	0.080456	0.009386	0.167111	3.066828	0.408792	0.066828
	7	0.245001	0.067944	0.007919	0.165394	3.065728	0.406687	0.065728
	8	0.227956	0.058815	0.006851	0.164916	3.061962	0.406099	0.061962
	9	0.214051	0.051856	0.006038	0.164358	3.069852	0.405411	0.069852
	10	0.202423	0.046373	0.005398	0.164208	3.048848	0.405226	0.048848
3	3	0.602485	0.399139	0.036151	0.225634	3.273408	0.475009	0.273408
	4	0.465823	0.236788	0.019797	0.145361	3.125628	0.381262	0.125628
	5	0.396873	0.171505	0.013997	0.123096	3.072745	0.350850	0.072745
	6	0.352462	0.135146	0.010917	0.112742	3.054180	0.335770	0.054180
	7	0.320559	0.111736	0.008978	0.108337	3.051939	0.329145	0.051939
	8	0.296135	0.095332	0.007636	0.105760	3.035578	0.325207	0.035578
	9	0.276626	0.083171	0.006649	0.105379	3.008423	0.324621	0.008423
	10	0.260564	0.073785	0.005891	0.104265	3.025598	0.322900	0.025598
4	4	0.648039	0.453256	0.033301	0.229320	3.321976	0.478874	0.321976
	5	0.511790	0.280311	0.018382	0.130464	3.147984	0.361198	0.147984
	6	0.441284	0.207865	0.013133	0.102281	3.090069	0.319815	0.090069
	7	0.394999	0.166359	0.010335	0.090404	3.056551	0.300673	0.056551
	8	0.361267	0.139076	0.008562	0.084264	3.040724	0.290282	0.040724
	9	0.335151	0.119652	0.007326	0.081505	3.010495	0.285491	0.010495
	10	0.314105	0.105073	0.006411	0.078213	3.023921	0.279667	0.023921
5	5	0.682101	0.496492	0.031230	0.237912	3.362265	0.487762	0.362265
	6	0.547043	0.316534	0.017278	0.124926	3.165975	0.353448	0.165975
	7	0.475998	0.238994	0.012420	0.092794	3.093716	0.304621	0.093716
	8	0.428730	0.193642	0.009833	0.078295	3.075215	0.279812	0.075215
	9	0.393912	0.163355	0.008188	0.071074	3.053238	0.266597	0.053238
	10	0.366721	0.141522	0.007038	0.067297	3.019226	0.259417	0.019226
6	6	0.709112	0.532483	0.029643	0.247927	3.393989	0.497922	0.393989
	7	0.575461	0.347549	0.016394	0.123722	3.179939	0.351742	0.179939
	8	0.504359	0.266206	0.011828	0.087950	3.098604	0.296564	0.098604
	9	0.456584	0.217871	0.009402	0.072183	3.067347	0.268669	0.067347
	10	0.421103	0.185187	0.007859	0.063867	3.048346	0.252719	0.048346
7	7	0.731387	0.563306	0.028379	0.257507	3.423281	0.507451	0.423281
	8	0.599162	0.374664	0.015668	0.123781	3.197735	0.351825	0.197735
	9	0.528246	0.290374	0.011330	0.081133	3.116019	0.284838	0.116019
	10	0.480238	0.239660	0.009031	0.067515	3.077528	0.259836	0.077528
8	8	0.750277	0.590255	0.027339	0.267491	3.444852	0.517195	0.444852
	9	0.619424	0.398746	0.015060	0.125692	3.209343	0.354531	0.209343
	10	0.548821	0.312108	0.010904	0.083717	3.116898	0.289339	0.116898
9	9	0.766633	0.614193	0.026467	0.276388	3.468726	0.525726	0.468726
	10	0.637074	0.420406	0.014543	0.127287	3.217234	0.356773	0.217234
10	10	0.781029	0.635725	0.025719	0.285348	3.482843	0.534180	0.482843

TABLE 4.2: Moments, variances skewness and kurtosis of order statistic from EPL distribution for $\tau = 2$, $\kappa = 0.5$ and $\xi = 10$

u	n	$E(Z)$	$E(Z^2)$	$V(Z)$	δ_1	δ_2	γ_1	γ_2
1	1	0.286922	0.104762	0.022438	0.392513	3.233671	0.626509	0.233671
	2	0.202969	0.052438	0.011242	0.395237	3.235008	0.628679	0.235008
	3	0.165748	0.034972	0.007500	0.395367	3.241712	0.628782	0.241712
	4	0.143552	0.026234	0.005627	0.395894	3.257377	0.629201	0.257377
	5	0.128403	0.020990	0.004503	0.397574	3.250389	0.630535	0.250389
	6	0.117219	0.017493	0.003753	0.396251	3.264481	0.629485	0.264481
	7	0.108526	0.014995	0.003217	0.399772	3.270879	0.632275	0.270879
	8	0.101518	0.013121	0.002815	0.399592	3.202086	0.632133	0.202086
	9	0.095714	0.011663	0.002502	0.396203	3.258955	0.629447	0.258955
	10	0.090803	0.010497	0.002252	0.403750	3.146694	0.635413	0.146694
2	2	0.370875	0.157086	0.019538	0.252992	3.233830	0.502984	0.233830
	3	0.277412	0.087369	0.010412	0.195887	3.130170	0.442592	0.130170
	4	0.232333	0.061185	0.007206	0.180837	3.092581	0.425249	0.092581
	5	0.204150	0.047211	0.005534	0.173430	3.076871	0.416449	0.076871
	6	0.184323	0.038474	0.004499	0.171049	3.090499	0.413580	0.090499
	7	0.169376	0.032482	0.003794	0.165917	3.090481	0.407329	0.090481
	8	0.157579	0.028111	0.003280	0.166400	3.110573	0.407921	0.110573
	9	0.147956	0.024781	0.002890	0.166052	3.079916	0.407494	0.079916
	10	0.139911	0.022158	0.002583	0.163607	3.108243	0.404484	0.108243
3	3	0.417607	0.191945	0.017549	0.236218	3.294003	0.486023	0.294003
	4	0.322490	0.113553	0.009553	0.151694	3.141847	0.389479	0.141847
	5	0.274609	0.082146	0.006736	0.127620	3.077971	0.357239	0.077971
	6	0.243803	0.064685	0.005245	0.117231	3.069729	0.342389	0.069729
	7	0.221690	0.053456	0.004310	0.111750	3.067425	0.334290	0.067425
	8	0.204769	0.045593	0.003663	0.108367	3.027919	0.329192	0.027919
	9	0.191258	0.039767	0.003187	0.107573	2.974637	0.327983	0.025360
	10	0.180137	0.035272	0.002823	0.107969	3.008948	0.328587	0.008948
4	4	0.449313	0.218075	0.016193	0.239672	3.345870	0.489563	0.345870
	5	0.354410	0.134491	0.008885	0.136586	3.148446	0.369575	0.148446
	6	0.305414	0.099607	0.006329	0.106752	3.095144	0.326730	0.095144
	7	0.273287	0.079658	0.004972	0.094570	3.035524	0.307522	0.035524
	8	0.249893	0.066561	0.004114	0.086248	3.032981	0.293680	0.032981
	9	0.231790	0.057244	0.003517	0.082227	3.081452	0.286752	0.081452
	10	0.217207	0.050255	0.003076	0.078999	3.035468	0.281068	0.035468
5	5	0.473039	0.238971	0.015205	0.248785	3.378142	0.498784	0.378142
	6	0.378909	0.151933	0.008361	0.131396	3.170584	0.362486	0.170584
	7	0.329508	0.114569	0.005993	0.096047	3.099872	0.309914	0.099872
	8	0.296682	0.092756	0.004736	0.081285	3.081199	0.285105	0.081199
	9	0.272522	0.078207	0.003939	0.073627	3.027151	0.271344	0.027151
	10	0.253665	0.067728	0.003382	0.069769	2.985835	0.264139	0.014160
6	6	0.491865	0.256379	0.014448	0.258173	3.415604	0.508108	0.415604
	7	0.398669	0.166879	0.007942	0.128823	3.206365	0.358919	0.206365
	8	0.349204	0.127657	0.005714	0.090977	3.107751	0.301623	0.107751
	9	0.316010	0.104395	0.004533	0.074676	3.078239	0.273269	0.078239
	10	0.291379	0.088686	0.003784	0.065202	3.049056	0.255346	0.049056
7	7	0.507398	0.271295	0.013842	0.268815	3.439121	0.518474	0.439121
	8	0.415157	0.179953	0.007598	0.130030	3.189014	0.360597	0.189014
	9	0.365801	0.139288	0.005478	0.087421	3.153352	0.295671	0.153352
	10	0.332431	0.114868	0.004358	0.071826	3.031533	0.268003	0.031533
8	8	0.520575	0.284344	0.013346	0.277922	3.467195	0.527183	0.467195
	9	0.429259	0.191572	0.007309	0.131214	3.202192	0.362235	0.202192
	10	0.380103	0.149753	0.005275	0.087888	3.122100	0.296459	0.122100
9	9	0.531990	0.295940	0.012927	0.287613	3.487318	0.536295	0.487318
	10	0.441548	0.202027	0.007062	0.132007	3.237307	0.363328	0.237307
10	10	0.542039	0.306375	0.012569	0.296542	3.497892	0.544557	0.497892

TABLE 4.3: Covariances of order statistics.

			$\tau=2,\xi=5,\kappa=0.5$				$\tau=2,\xi=10,\kappa=0.5$			
v	u	n	$\mu_{u,v:n}$	$E(Z_{u:n})$	$E(Z_{v:n})$	$\sigma_{u,v:n}$	$\mu_{u,v:n}$	$E(Z_{u:n})$	$E(Z_{v:n})$	$\sigma_{u,v:n}$
2	1	2	0.125100	0.293422	0.535273	-0.03196	0.059192	0.202969	0.370875	-0.01608
		3	0.073220	0.239709	0.400850	-0.02287	0.034402	0.165748	0.277412	-0.01158
		4	0.052133	0.207652	0.335877	-0.01761	0.024407	0.143552	0.232333	-0.00895
		5	0.040552	0.185762	0.295213	-0.01429	0.018943	0.128403	0.204150	-0.00727
		6	0.033203	0.169597	0.266589	-0.01201	0.015488	0.117219	0.184323	-0.00612
		7	0.028119	0.157029	0.245001	-0.01035	0.013102	0.108526	0.169376	-0.00528
		8	0.024389	0.146897	0.227956	-0.00910	0.011355	0.101518	0.157579	-0.00464
		9	0.021534	0.138503	0.214051	-0.00811	0.010020	0.095714	0.147956	-0.00414
		10	0.019279	0.131400	0.202423	-0.00732	0.008966	0.090803	0.139911	-0.00374
3	1	3	0.146966	0.239709	0.602485	0.002545	0.066242	0.165748	0.417607	-0.00298
		4	0.100791	0.207652	0.465823	0.004062	0.044251	0.143552	0.322490	-0.00204
		5	0.079630	0.185762	0.396873	0.005906	0.034146	0.128403	0.274609	-0.00112
		6	0.067164	0.169597	0.352462	0.007387	0.028186	0.117219	0.243803	-0.00039
		7	0.058882	0.157029	0.320559	0.008545	0.024224	0.108526	0.221690	0.000165
		8	0.052962	0.146897	0.296135	0.009461	0.021392	0.101518	0.204769	0.000604
		9	0.048512	0.138503	0.276626	0.010199	0.019262	0.095714	0.191258	0.000956
		10	0.045043	0.131400	0.260564	0.010805	0.017601	0.090803	0.180137	0.001244
3	2	3	0.155112	0.400850	0.602485	-0.08639	0.076932	0.277412	0.417607	-0.03892
		4	0.087823	0.335877	0.465823	-0.06864	0.044543	0.232333	0.322490	-0.03038
		5	0.059381	0.295213	0.396873	-0.05778	0.031058	0.204150	0.274609	-0.02500
		6	0.043331	0.266589	0.352462	-0.05063	0.023522	0.184323	0.243803	-0.02142
		7	0.032948	0.245001	0.320559	-0.04559	0.018679	0.169376	0.221690	-0.01887
		8	0.025656	0.227956	0.296135	-0.04185	0.015295	0.157579	0.204769	-0.01697
		9	0.020245	0.214051	0.276626	-0.03897	0.012794	0.147956	0.191258	-0.01550
		10	0.016067	0.202423	0.260564	-0.03668	0.010868	0.139911	0.180137	-0.01434
4	1	4	0.157123	0.207652	0.648039	0.022556	0.067875	0.143552	0.449313	0.003375
		5	0.116281	0.185762	0.511790	0.021210	0.048633	0.128403	0.354410	0.003126
		6	0.096719	0.169597	0.441284	0.021879	0.039214	0.117219	0.305414	0.003413
		7	0.084955	0.157029	0.394999	0.022929	0.033425	0.108526	0.273287	0.003766
		8	0.077118	0.146897	0.361267	0.024049	0.029472	0.101518	0.249893	0.004104
		9	0.071578	0.138503	0.335151	0.025159	0.026598	0.095714	0.231790	0.004413
		10	0.067511	0.131400	0.314105	0.026238	0.024416	0.090803	0.217207	0.004693
	2	4	0.172829	0.335877	0.648039	-0.04483	0.084966	0.232333	0.449313	-0.01942
		5	0.112133	0.295213	0.511790	-0.03895	0.055696	0.204150	0.354410	-0.01666
		6	0.082851	0.266589	0.441284	-0.03479	0.041888	0.184323	0.305414	-0.01441
		7	0.064709	0.245001	0.394999	-0.03207	0.033554	0.169376	0.273287	-0.01273
		8	0.052012	0.227956	0.361267	-0.03034	0.027892	0.157579	0.249893	-0.01149
		9	0.042428	0.214051	0.335151	-0.02931	0.023756	0.147956	0.231790	-0.01054
		10	0.034802	0.202423	0.314105	-0.02878	0.020579	0.139911	0.217207	-0.00981
	3	4	0.179898	0.465823	0.648039	-0.12197	0.089110	0.322490	0.449313	-0.05579
		5	0.104111	0.396873	0.511790	-0.09900	0.052452	0.274609	0.354410	-0.04487
		6	0.071720	0.352462	0.441284	-0.08382	0.036947	0.243803	0.305414	-0.03751
		7	0.053410	0.320559	0.394999	-0.07321	0.028192	0.221690	0.273287	-0.03239
		8	0.041644	0.296135	0.361267	-0.06534	0.022532	0.204769	0.249893	-0.02864
		9	0.033502	0.276626	0.335151	-0.05921	0.018569	0.191258	0.231790	-0.02576
		10	0.027592	0.260564	0.314105	-0.05425	0.015641	0.180137	0.217207	-0.02349
5	1	5	0.165311	0.185762	0.682101	0.038602	0.068527	0.128403	0.473039	0.007787
		6	0.128166	0.169597	0.547043	0.035389	0.051311	0.117219	0.378909	0.006896
		7	0.110052	0.157029	0.475998	0.035306	0.042581	0.108526	0.329508	0.006820
		8	0.099086	0.146897	0.428730	0.036107	0.037070	0.101518	0.296682	0.006952
		9	0.091832	0.138503	0.393912	0.037274	0.033237	0.095714	0.272522	0.007153
		10	0.086817	0.131400	0.366721	0.038630	0.030414	0.090803	0.253665	0.007380
	2	5	0.173400	0.295213	0.682101	-0.02796	0.085162	0.204150	0.473039	-0.01141
		6	0.119750	0.266589	0.547043	-0.02609	0.059439	0.184323	0.378909	-0.01040
		7	0.092013	0.245001	0.475998	-0.02461	0.046480	0.169376	0.329508	-0.00933
		8	0.073911	0.227956	0.428730	-0.02382	0.038296	0.157579	0.296682	-0.00846

TABLE 4.3: Continued.

			$\tau=2,\xi=5,\kappa=0.5$				$\tau=2,\xi=10,\kappa=0.5$			
v	u	n	$\mu_{u,v:n}$	$E(Z_{u:n})$	$E(Z_{v:n})$	$\sigma_{u,v:n}$	$\mu_{u,v:n}$	$E(Z_{u:n})$	$E(Z_{v:n})$	$\sigma_{u,v:n}$
		9	0.060674	0.214051	0.393912	-0.02364	0.032550	0.147956	0.272522	-0.00777
		10	0.050264	0.202423	0.366721	-0.02397	0.028244	0.139911	0.253665	-0.00725
	3	5	0.202606	0.396873	0.682101	-0.06810	0.099405	0.274609	0.473039	-0.03050
		6	0.133799	0.352462	0.547043	-0.05901	0.065762	0.243803	0.378909	-0.02662
		7	0.100903	0.320559	0.475998	-0.05168	0.049799	0.221690	0.329508	-0.02325
		8	0.080899	0.296135	0.428730	-0.04606	0.040135	0.204769	0.296682	-0.02062
		9	0.067311	0.276626	0.393912	-0.04166	0.033574	0.191258	0.272522	-0.01855
		10	0.057471	0.260564	0.366721	-0.03808	0.028798	0.180137	0.253665	-0.01690
	4	5	0.197590	0.511790	0.682101	-0.15150	0.097898	0.354410	0.473039	-0.06975
		6	0.115810	0.441284	0.547043	-0.12559	0.058351	0.305414	0.378909	-0.05737
		7	0.080302	0.394999	0.475998	-0.10772	0.041419	0.273287	0.329508	-0.04863
		8	0.059934	0.361267	0.428730	-0.09495	0.031757	0.249893	0.296682	-0.04238
		9	0.046659	0.335151	0.393912	-0.08536	0.025457	0.231790	0.272522	-0.03771
		10	0.037334	0.314105	0.366721	-0.07786	0.021014	0.217207	0.253665	-0.03408
6	1	6	0.173540	0.169597	0.709112	0.053277	0.069056	0.117219	0.491865	0.011400
		7	0.138937	0.157029	0.575461	0.048573	0.053325	0.108526	0.398669	0.010059
		8	0.122004	0.146897	0.504359	0.047915	0.045185	0.101518	0.349204	0.009735
		9	0.111775	0.138503	0.456584	0.048537	0.039958	0.095714	0.316010	0.009712
		10	0.105094	0.131400	0.421103	0.049761	0.036274	0.090803	0.291379	0.009816
	2	6	0.169539	0.266589	0.709112	-0.01950	0.083624	0.184323	0.491865	-0.00704
		7	0.121311	0.245001	0.575461	-0.01968	0.060716	0.169376	0.398669	-0.00681
		8	0.095139	0.227956	0.504359	-0.01983	0.048693	0.157579	0.349204	-0.00633
		9	0.077345	0.214051	0.456584	-0.02039	0.040853	0.147956	0.316010	-0.00590
		10	0.063859	0.202423	0.421103	-0.02138	0.035208	0.139911	0.291379	-0.00556
	3	6	0.206018	0.352462	0.709112	-0.04392	0.100332	0.243803	0.491865	-0.01959
		7	0.145146	0.320559	0.575461	-0.03932	0.070483	0.221690	0.398669	-0.01790
		8	0.114479	0.296135	0.504359	-0.03488	0.055437	0.204769	0.349204	-0.01607
		9	0.095232	0.276626	0.456584	-0.03107	0.045952	0.191258	0.316010	-0.01449
		10	0.081919	0.260564	0.421103	-0.02781	0.039327	0.180137	0.291379	-0.01316
	4	6	0.223267	0.441284	0.709112	-0.08965	0.110002	0.305414	0.491865	-0.04022
		7	0.148164	0.394999	0.575461	-0.07914	0.073256	0.273287	0.398669	-0.03569
		8	0.111855	0.361267	0.504359	-0.07035	0.055703	0.249893	0.349204	-0.03156
		9	0.089461	0.335151	0.456584	-0.06356	0.045006	0.231790	0.316010	-0.02824
		10	0.073972	0.314105	0.421103	-0.05830	0.037697	0.217207	0.291379	-0.02559
	5	6	0.211616	0.547043	0.709112	-0.17630	0.104759	0.378909	0.491865	-0.08161
		7	0.125475	0.475998	0.575461	-0.14844	0.063092	0.329508	0.398669	-0.06827
		8	0.087690	0.428730	0.504359	-0.12854	0.045094	0.296682	0.349204	-0.05851
		9	0.065827	0.393912	0.456584	-0.11403	0.034744	0.272522	0.316010	-0.05138
		10	0.051487	0.366721	0.421103	-0.10294	0.027951	0.253665	0.291379	-0.04596
7	1	7	0.182405	0.157029	0.731387	0.067556	0.069687	0.108526	0.507398	0.014621
		8	0.149550	0.146897	0.599162	0.061535	0.055063	0.101518	0.415157	0.012917
		9	0.133567	0.138503	0.528246	0.060403	0.047410	0.095714	0.365801	0.012398
		10	0.124002	0.131400	0.480238	0.060899	0.042442	0.090803	0.332431	0.012256
	2	7	0.163819	0.245001	0.731387	-0.01537	0.081635	0.169376	0.507398	-0.00431
		8	0.119737	0.227956	0.599162	-0.01685	0.060918	0.157579	0.415157	-0.00450
		9	0.094876	0.214051	0.528246	-0.01820	0.049741	0.147956	0.365801	-0.00438
		10	0.077366	0.202423	0.480238	-0.01984	0.042276	0.139911	0.332431	-0.00423
	3	7	0.205092	0.320559	0.731387	-0.02936	0.099057	0.221690	0.507398	-0.01343
		8	0.150629	0.296135	0.599162	-0.02680	0.072334	0.204769	0.415157	-0.01268
		9	0.122355	0.276626	0.528246	-0.02377	0.058356	0.191258	0.365801	-0.01161
		10	0.104276	0.260564	0.480238	-0.02086	0.049294	0.180137	0.332431	-0.01059
	4	7	0.227234	0.320559	0.731387	-0.00722	0.111556	0.273287	0.507398	-0.02711
		8	0.160109	0.296135	0.599162	-0.01732	0.078663	0.249893	0.415157	-0.02508
		9	0.125850	0.276626	0.528246	-0.02028	0.062003	0.231790	0.365801	-0.02279
		10	0.103939	0.260564	0.480238	-0.02119	0.051445	0.217207	0.332431	-0.02076

TABLE 4.3: Continued.

			$\tau=2,\xi=5,\ \kappa=0.5$				$\tau=2,\xi=10,\ \kappa=0.5$			
v	u	n	$\mu_{u,v:n}$	$E(Z_{u:n})$	$E(Z_{v:n})$	$\sigma_{u,v:n}$	$\mu_{u,v:n}$	$E(Z_{u:n})$	$E(Z_{v:n})$	$\sigma_{u,v:n}$
	5	7	0.240059	0.475998	0.731387	-0.10808	0.118411	0.329508	0.507398	-0.04878
		8	0.160346	0.428730	0.599162	-0.09653	0.079330	0.296682	0.415157	-0.04384
		9	0.121653	0.393912	0.528246	-0.08643	0.060576	0.272522	0.365801	-0.03911
		10	0.097711	0.366721	0.480238	-0.07840	0.049095	0.253665	0.332431	-0.03523
	6	7	0.223156	0.575461	0.731387	-0.19773	0.110362	0.398669	0.507398	-0.09192
		8	0.133614	0.504359	0.599162	-0.16858	0.067043	0.349204	0.415157	-0.07793
		9	0.094015	0.456584	0.528246	-0.14717	0.048208	0.316010	0.365801	-0.06739
		10	0.070923	0.421103	0.480238	-0.13131	0.037307	0.291379	0.332431	-0.05956
8	1	8	0.192134	0.146897	0.750277	0.081920	0.070480	0.101518	0.520575	0.017632
		9	0.160463	0.138503	0.619424	0.074671	0.056695	0.095714	0.429259	0.015609
		10	0.145243	0.131400	0.548821	0.073128	0.049442	0.090803	0.380103	0.014928
	2	8	0.156885	0.227956	0.750277	-0.01415	0.079554	0.157579	0.520575	-0.00248
		9	0.116038	0.214051	0.619424	-0.01655	0.060572	0.147956	0.429259	-0.00294
		10	0.092204	0.202423	0.548821	-0.01889	0.050124	0.139911	0.380103	-0.00306
	3	8	0.203197	0.296135	0.750277	-0.01899	0.097193	0.204769	0.520575	-0.00940
		9	0.153848	0.276626	0.619424	-0.01750	0.072961	0.191258	0.429259	-0.00914
		10	0.127820	0.260564	0.548821	-0.01518	0.059979	0.180137	0.380103	-0.00849
	4	8	0.225773	0.361267	0.750277	-0.04528	0.110449	0.249893	0.520575	-0.01964
		9	0.165186	0.335151	0.619424	-0.04241	0.080817	0.231790	0.429259	-0.01868
		10	0.133147	0.314105	0.548821	-0.03924	0.065255	0.217207	0.380103	-0.01731
	5	8	0.245110	0.428730	0.750277	-0.07656	0.120610	0.296682	0.520575	-0.03384
		9	0.173431	0.393912	0.619424	-0.07057	0.085377	0.272522	0.429259	-0.03161
		10	0.136810	0.366721	0.548821	-0.06445	0.067482	0.253665	0.380103	-0.02894
	6	8	0.253982	0.504359	0.750277	-0.12443	0.125347	0.349204	0.520575	-0.05644
		9	0.170589	0.456584	0.619424	-0.11223	0.084412	0.316010	0.429259	-0.05124
		10	0.129956	0.421103	0.548821	-0.10115	0.064702	0.291379	0.380103	-0.04605
	7	8	0.232942	0.504359	0.750277	-0.14547	0.115084	0.415157	0.520575	-0.10104
		9	0.140652	0.456584	0.619424	-0.14217	0.070427	0.365801	0.429259	-0.08660
		10	0.099571	0.421103	0.548821	-0.13154	0.050909	0.332431	0.380103	-0.07545
9	1	9	0.202844	0.138503	0.766633	0.096663	0.071447	0.095714	0.531990	0.020528
		10	0.171939	0.131400	0.637074	0.088228	0.058309	0.090803	0.441548	0.018215
	2	9	0.148835	0.214051	0.766633	-0.01526	0.077490	0.147956	0.531990	-0.00122
		10	0.110573	0.202423	0.637074	-0.01839	0.059909	0.139911	0.441548	-0.00187
	3	9	0.201458	0.276626	0.766633	-0.01061	0.095239	0.191258	0.531990	-0.00651
		10	0.156264	0.260564	0.637074	-0.00973	0.073008	0.180137	0.441548	-0.00653
	4	9	0.222547	0.335151	0.766633	-0.03439	0.108528	0.231790	0.531990	-0.01478
		10	0.167173	0.314105	0.637074	-0.03294	0.081537	0.217207	0.441548	-0.01437
	5	9	0.244147	0.393912	0.766633	-0.05784	0.119779	0.272522	0.531990	-0.02520
		10	0.179218	0.366721	0.637074	-0.05441	0.087879	0.253665	0.441548	-0.02413
	6	9	0.260024	0.456584	0.766633	-0.09001	0.128154	0.316010	0.531990	-0.03996
		10	0.184607	0.421103	0.637074	-0.08367	0.091033	0.291379	0.441548	-0.03762
	7	9	0.265867	0.528246	0.766633	-0.13910	0.131234	0.365801	0.531990	-0.06337
		10	0.179461	0.480238	0.637074	-0.12649	0.088775	0.332431	0.441548	-0.05801
	8	9	0.241423	0.619424	0.766633	-0.23345	0.119156	0.429259	0.531990	-0.10921
		10	0.146845	0.548821	0.637074	-0.20279	0.073382	0.380103	0.441548	-0.09445
10	1	10	0.214615	0.131400	0.781029	0.111988	0.072587	0.090803	0.542039	0.023369
	2	10	0.139577	0.202423	0.781029	-0.01852	0.075465	0.139911	0.542039	-0.00037
	3	10	0.200369	0.260564	0.781029	-0.00314	0.093369	0.180137	0.542039	-0.00427
	4	10	0.218699	0.314105	0.781029	-0.02663	0.106389	0.217207	0.542039	-0.01135
	5	10	0.241199	0.366721	0.781029	-0.04522	0.117950	0.253665	0.542039	-0.01955
	6	10	0.259491	0.421103	0.781029	-0.06940	0.127622	0.291379	0.542039	-0.03032
	7	10	0.272860	0.480238	0.781029	-0.10222	0.134607	0.332431	0.542039	-0.04558
	8	10	0.276212	0.548821	0.781029	-0.15243	0.136336	0.380103	0.542039	-0.06969
	9	10	0.248896	0.637074	0.781029	-0.24868	0.122731	0.441548	0.542039	-0.11661

4.5 BLUEs of the Location and Scale Parameters

Let $Y_{1:n} \leq Y_{2:n} \leq \cdots \leq Y_{n-c:n}$, $c = 0(1)([n/2]-1)$ denote Type-II right-censored sample from the location-scale parameter extended power Lindley distribution in Equation (4.1).

Let us consider scale-parameter pdf of EPL distribution as

$$h(z;\tau,\kappa,\xi,\varphi) = \frac{\tau\xi^2}{\varphi(\xi+\kappa)}\left[1+\kappa\left(\frac{z}{\varphi}\right)^{\tau}\right]\left(\frac{z}{\varphi}\right)^{\tau-1} e^{-\xi\left(\frac{z}{\varphi}\right)^{\tau}},\ z>0;\ \tau>0,\kappa>0,\xi>0,\varphi>0, \tag{4.6}$$

and location-scale parameter pdf of EPL distribution is

$$\begin{aligned} h(z;\tau,\kappa,\xi,\delta,\varphi) &= \frac{\tau\xi^2}{\varphi(\xi+\kappa)}\left[1+\kappa\left(\frac{z-\delta}{\varphi}\right)^{\tau}\right]\left(\frac{z-\delta}{\varphi}\right)^{\tau-1} \\ &\times\ e^{-\xi\left(\frac{z-\delta}{\varphi}\right)^{\tau}},\ z>\delta;\ \tau>0,\kappa>0,\xi>0,\varphi>0,\delta>0, \end{aligned} \tag{4.7}$$

The expression for the BLUEs of location and scale parameter are given in (1.18) and also variances and covariance for these parameters are given in eqn (1.21), (1.22) and (1.23).

Tables 4.4 and 4.5 display the coefficient of the BLUEs for type-II right censored sample of various values of $n = 4,\ 8,\ 10,\ \tau = 2,3$ and censoring cases $c = 0(1)([n/2]-1)$. Also, Table 4.6 shows variances and covariances of the BLUEs.

4.6 Real Data Application

The data set refers to a study on the vinyl chloride from clean upgradient monitoring wells in mg/L which studied by Bhaumik et al. (2009). The data are:

```
5.1 1.2 1.3 0.6 0.5 2.4 0.5 1.1 8.0 0.8 0.4 0.6 0.9 0.4 2.0 0.5
5.3 3.2 2.7 2.9 2.5 2.3 1.0 0.2 0.1 0.1 1.8 0.9 2.0 4.0 6.8 1.2
0.4 0.2
```

TABLE 4.4: Coefficients of the BLUEs of the location parameter for $\tau = 0.5$, $\kappa = 2$.

ξ	n	c	$a_u, u = 1,2,3,\ldots,(n-c)$						
0.5	4	0	0.60056	0.52054	-0.0249	-0.0962			
		1	1.14951	-0.0022	-0.1473				
		2	1.46179	-0.4618					
	8	0	0.94978	-0.0079	-0.0355	0.11756	0.04068	-0.0357	-0.0211
			-0.0079						
		1	0.99500	-0.0542	-0.0168	0.13125	0.02120	-0.0520	-0.0245
		2	1.10008	-0.1932	0.02847	0.16721	-0.0256	-0.0769	
		3	-0.1442	3.22795	-1.6960	-1.0836	0.69588		
		4	1.22957	-5E-05	-0.2371	0.00762			
		5	1.20093	0.03644	-0.2374				
		6	1.55875	-0.5587					
	10	0	0.84633	0.46665	-0.1943	0.02331	0.04968	-0.1587	-0.0944
			0.02940	0.02612	0.00600				
		1	0.87109	0.40252	-0.1787	0.02622	0.03714	-0.1427	-0.0747
			0.03345	0.02564					
		2	0.92580	0.23990	-0.1220	0.04079	0.00905	-0.0958	-0.0266
			0.02881						
		3	1.14784	-0.0594	-0.0458	0.05472	-0.0716	-0.0521	0.02645
		4	0.97357	0.11809	-0.0719	0.05459	-0.0127	-0.0617	
		5	1.28460	-0.3778	0.12364	0.10493	-0.1354		
		6	1.19440	0.06579	-0.2574	-0.0028			
1	4	0	3.49685	-2.3573	-0.4864	0.34688			
		1	1.11260	0.02270	-0.1353				
		2	1.41595	-0.4159					
	8	0	0.95517	0.00129	-0.0557	0.09979	0.04894	-0.0232	-0.0185
			-0.0078						
		1	1.00829	-0.0616	-0.0298	0.11856	0.02769	-0.0411	-0.0220
		2	1.12022	-0.2225	0.02545	0.16109	-0.0176	-0.0667	
		3	0.33441	1.88369	-0.9937	-0.5809	0.35650		
		4	1.21720	-0.0211	-0.2102	0.01419			
		5	1.15723	0.05438	-0.2116				
		6	1.49807	-0.4981					
	10	0	0.88677	0.32334	-0.1486	0.03122	0.03628	-0.1013	-0.0611
			0.01480	0.01489	0.00371				
		1	0.91353	0.27429	-0.1386	0.03356	0.02542	-0.0940	-0.0482
			0.01886	0.01508					
		2	0.97087	0.15491	-0.1026	0.04442	0.00171	-0.0720	-0.0174
			0.02004						
		3	1.27311	-0.2207	-0.0148	0.06681	-0.1006	-0.0450	0.04122
		4	0.98785	0.08119	-0.0667	0.05692	-0.0081	-0.0512	
		5	1.32640	-0.4955	0.17390	0.13470	-0.1395		
		6	1.17694	0.04512	-0.2260	0.00389			
		7	1.15952	0.06823	-0.2278				
		8	1.51118	-0.5112					

TABLE 4.5: Coefficients of the BLUEs of the scale parameter for $\tau = 0.5$, $\kappa = 2$.

ξ	n	c	$b_u,\ u = 1,2,3,\ldots,(n-c)$						
0.5	4	0	-0.19072	0.15597	0.04744	-0.01269			
		1	-0.11835	0.08705	0.03130				
		2	-0.18472	0.18472					
	8	0	-0.11319	0.09364	0.00620	-0.02865	0.01231	0.02046	0.00619
			0.00305						
		1	-0.13061	0.11148	-0.00102	-0.03392	0.01981	0.02676	0.00750
		2	-0.16282	0.15410	-0.01488	-0.04495	0.03417	0.03438	
		3	0.39367	-1.37593	0.75633	0.51444	-0.28851		
		4	-0.17592	-0.03759	0.15149	0.06202			
		5	-0.40913	0.25959	0.14954				
		6	-0.63456	0.63456					
	10	0	-0.05656	-0.13237	0.10417	0.01245	-0.02274	0.06273	0.04769
			-0.00559	-0.00908	-0.00070				
		1	-0.05946	-0.12487	0.10235	0.01211	-0.02127	0.06086	0.04538
			-0.00606	-0.00902					
		2	-0.07871	-0.06765	0.08238	0.00698	-0.01139	0.04436	0.02846
			-0.00443						
		3	-0.11285	-0.02163	0.07068	0.00484	0.00101	0.03764	0.02030
		4	-0.24665	0.11466	0.05071	0.00474	0.04621	0.03032	
		5	-0.39958	0.35850	-0.04542	-0.02001	0.10651		
		6	-0.32861	0.00947	0.25437	0.06478			
1	4	0	2.94543	-3.06133	-0.39787	0.51376			
		1	-0.58588	0.46372	0.12215				
		2	-0.85975	0.85975					
	8	0	-0.57040	0.50408	0.01908	-0.14321	0.05548	0.09304	0.02875
			0.01319						
		1	-0.66076	0.61109	-0.02508	-0.17515	0.09163	0.12358	0.03469
		2	-0.83696	0.86427	-0.11200	-0.24208	0.16300	0.16378	
		3	1.09377	-4.31054	2.39211	1.58093	-0.75627		
		4	-0.77897	-0.26961	0.73003	0.31855			
		5	-2.12522	1.42604	0.69918				
		6	-3.25139	3.25139					
	10	0	-0.24289	-0.51009	0.44639	0.05160	-0.11371	0.20398	0.18654
			0.00347	-0.02489	-0.00040				
		1	-0.24574	-0.50487	0.44532	0.05135	-0.11255	0.20320	0.18517
			0.00304	-0.02491					
		2	-0.34047	-0.30765	0.38585	0.03342	-0.07339	0.16686	0.13428
			0.00109						
		3	-0.32406	-0.32804	0.39062	0.03463	-0.07894	0.16833	0.13747
		4	-1.27553	0.67903	0.21763	0.00163	0.22956	0.14769	
		5	-2.25242	2.34293	-0.47649	-0.22280	0.60877		
		6	-1.60032	-0.01563	1.26804	0.34791			
		7	-3.15724	2.05023	1.10701				
		8	-4.86649	4.86649					

TABLE 4.6: Variances and covariance of the BLUEs when $\tau = 0.5$, $\kappa = 2$ and $\delta = 0$ and $\varphi = 1$.

ξ	n	c	$Var(\delta^*)$	$Var(\varphi^*)$	$Cov(\delta^*,\ \varphi^*)$
0.5	4	0	17.65257	1.410936	-0.566364
		1	14.02154	1.347831	-1.045046
		2	4.033713	0.896757	1.077511
	8	0	2.337863	0.846369	-0.520900
		1	2.167946	0.821161	-0.455453
		2	1.696970	0.776917	-0.311100
		3	20.75401	4.588457	-8.833814
		4	4.587047	1.809407	-2.130921
		5	4.594606	2.310717	-2.069361
		6	0.286717	0.600882	0.644635
	10	0	2.750427	1.010938	-1.077970
		1	2.628375	1.009270	-1.063700
		2	2.206944	0.957081	-0.915396
		3	1.727925	0.945757	-0.841744
		4	1.848615	1.016895	-0.749085
		5	0.544039	0.701471	-0.107607
		6	3.260254	2.383097	-2.244815
		7	3.261429	3.000961	-2.271760
		8	0.140384	0.562046	0.487219
1	4	0	0.053382	0.191120	-1.066744
		1	0.641031	1.480219	-0.196377
		2	0.306267	1.207343	0.105863
	8	0	0.086850	0.871200	-0.080502
		1	0.078748	0.847759	-0.066722
		2	0.059150	0.799190	-0.035870
		3	0.444527	3.125664	-0.982743
		4	0.146996	1.786670	-0.351560
		5	0.148080	2.333096	-0.327222
		6	0.027926	1.021373	0.069779
	10	0	0.076264	0.962850	-0.152530
		1	0.073788	0.962822	-0.152266
		2	0.065425	0.939994	-0.138449
		3	0.048548	0.939945	-0.139365
		4	0.056977	1.033724	-0.111248
		5	0.005818	0.607766	0.036372
		6	0.094559	2.297031	-0.350807
		7	0.094639	2.932194	-0.343704
		8	0.013512	1.015521	0.050624

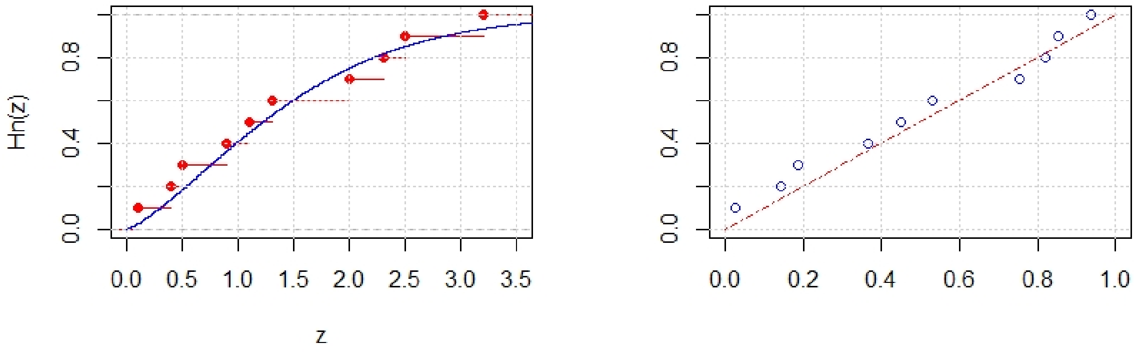

FIGURE 4.1: ECDF-plot and QQ-plot

In this data set, we select a random sample of size 10 and data are: 0.1, 1.1, 0.9, 2.3, 1.3, 2.5, 0.4, 2.0, 0.5, 3.2. $\tau_{ML} = 1.2945$, $\kappa_{ML} = 0.5514$ and $\xi_{ML} = 0.8099$ are the MLEs of EPL distribution for the given sample. The K-S statistic and corresponding p-value are 0.1525 and 0.9476, respectively, which conforms the suitability of the EPL distribution. Figure 4.1 shows ecdf and Q-Q plot of the sample.

By using Tables 4.4 and 4.5, we have

$$\delta^* = \sum_{u=1}^{n} a_u Z_{u:n} = 0.20081 \quad \text{and} \quad \varphi^* = \sum_{u=1}^{n} b_u Z_{u:n} = 0.36501.$$

4.7 Conclusion

Here, we come up with new expressions for moments of order statistics. Also, these results are reduced for special cases of power Lindley and Lindley distribution which is obtained by Kumar and Anju (2019a) and Sultan and AL-Thubyani (2016) respectively. The BLUEs for the corresponding parameters have been derived using moments of the order statistics and simulation is used for the case of Type-II right-censoring. Finally, for illustration, a real data set is analysed

Chapter 5

Inferences for Generalized Topp-Leone Distribution under Order Statistics with Application to Polyester Fibers Data

5.1 Introduction

This chapter follows the following structure: In GTL distribution case, we provide expressions for the single moments of order statistics in Section 5.2. A discussion of the lifetimes of coherent systems is provided in Section 5.3 in application of the theoretical results presented in Section 5.2. Section 5.4 is devoted to the product moments of order statistics. We use these moments in Section 5.5 to obtain BLUEs for the location and scale parameters. A real data application is provided in Section 5.6. Finally, in Section 5.7 , we draw a conclusion for the chapter.

Shekhawat and Sharma (2020) proposed two parameter GTL distribution by using the transformation $X = Z^{\alpha}$. The hazard function of this model has increasing and bathtub shape such as Weibull and exponentiated exponential distributions; and so on and also show that GTL model is superior model tissue damage proportions data as compare to the unit-gamma distributions, Beta, Kumaraswamy and unit-Weibull. They also study properties and MLEs of unknown parameters and significance of concentration level of drugs on tissue damage proportion by developing a parameteric regression model.

Shekhawat and Sharma (2020) proposed two parameter GTL distribution with pdf

$$h(z;\kappa,\xi) \;=\; 2\kappa\xi z^{\kappa\xi-1}(1-z^{\kappa})(2-z^{\kappa})^{\xi-1},\; 0<z<1,\; \kappa,\; \xi>0. \tag{5.1}$$

The associated cdf and quantile function are, respectively, given by

$$H(z;\kappa,\xi)=(z^{\kappa}(2-z^{\kappa}))^{\xi},\; 0<z<1,\; \kappa,\; \xi>0. \tag{5.2}$$

and let $H(z_p;\kappa,\xi)=p$, $p\in\{0,1\}$, then quantile function is

$$z_p=\left\{1-\sqrt{1-p^{\frac{1}{\xi}}}\right\}^{\frac{1}{\kappa}},$$

The pth moment can be easily computed as

$$\mu_p^{'}=\xi 2^{2\xi+\frac{p}{\kappa}}\left\{B_{1/2}\left(\xi+\frac{p}{\kappa},\,\xi\right)-2B_{1/2}\left(\xi+1+\frac{p}{\kappa},\,\xi\right)\right\}, \tag{5.3}$$

where $B_z(.,.)$ is defined as $B_z(\kappa,\xi)=\int_0^z t^{\kappa-1}(1-t)^{\xi-1}dt$.

5.2 Relations for Single Moments of Order Statistics

Let $Z_1,Z_2,\ldots,Z_n$ be a random sample from GTL distribution, the pdf of uth order statistic is obtained by using equation (5.1) and (5.2) in (1.1) as follows:

$$h_{Z_{u:n}}(z)=\frac{2n!\kappa\xi\, z^{\kappa\xi-1}}{(u-1)!(n-u)!}\left[z^{\kappa}(2-z^{\kappa})\right]^{\xi(u-1)}\left[1-\{z^{\kappa}(2-z^{\kappa})\}^{\xi}\right]^{n-u}(1-z^{\kappa})(2-z^{\kappa})^{\xi-1} \tag{5.4}$$

The pth moments of uth order statistics can be obtain from (5.4) as

$$\mu_{u:n}^{(p)} = \frac{\xi\, n!}{(u-1)!(n-u)!}\sum_{j=0}^{n-u}\binom{n-u}{j}(-1)^j\, 2^{\frac{p+2\kappa\xi(u+j)}{\kappa}} \times \left\{B_{1/2}\left(\frac{p+\kappa\xi(u+j)}{\kappa},\xi(u+j)\right)-2B_{1/2}\left(\frac{p+\kappa\xi(u+j)+\kappa}{\kappa},\xi(u+j)\right)\right\} \quad (5.5)$$

Note that when $u=n=1$, $\mu_{1:1}^{(p)}=\xi 2^{2\xi+\frac{p}{\kappa}}\left\{B_{1/2}\left(\xi+\frac{p}{\kappa},\,\xi\right)-2B_{1/2}\left(\xi+1+\frac{p}{\kappa},\,\xi\right)\right\}$, which agrees with (5.3). In addition from (5.5), the first and second moments of order statistics are, respectively, given by

$$\mu_{u:n}^{(1)} = \frac{\xi\, n!}{(u-1)!(n-u)!}\sum_{j=0}^{n-u}\binom{n-u}{j}(-1)^j\, 2^{\frac{1+2\kappa\xi(u+j)}{\kappa}} \times \left\{B_{1/2}\left(\frac{1+\kappa\xi(u+j)}{\kappa},\xi(u+j)\right)-2B_{1/2}\left(\frac{1+\kappa\xi(u+j)+\kappa}{\kappa},\xi(u+j)\right)\right\} \quad (5.6)$$

and

$$\mu_{u:n}^{(2)} = \frac{\xi\, n!}{(u-1)!(n-u)!}\sum_{j=0}^{n-u}\binom{n-u}{j}(-1)^j\, 2^{\frac{2+2\kappa\xi(u+j)}{\kappa}} \times \left\{B_{1/2}\left(\frac{2+\kappa\xi(u+j)}{\kappa},\xi(u+j)\right)-2B_{1/2}\left(\frac{2+\kappa\xi(u+j)+\kappa}{\kappa},\xi(u+j)\right)\right\} \quad (5.7)$$

The variance of uth order statistics of GTL distribution can be calculated by using the formula $V_{u:n}=\mu_{u:n}^{(2)}-(\mu_{u:n}^{(1)})^2$.

Theorem 1. For $\frac{p}{\kappa}=m, m\in Z^{+}$, the pth moment of uth order statistics can be expressed as,

$$\begin{aligned}\mu_{u:n}^{(p)} &= \frac{n!}{(u-1)!(n-u)!}\sum_{j=0}^{n-u}\binom{n-u}{j}\frac{(-1)^j}{(u+j)}\left[\frac{(-1)^{m+1}(-m-\xi(u+j)-1)_{m+1}}{(\xi(u+j))_{m+1}}\right.\\ &- 2^{m+2\xi(u+j)-1}(m+2)Be(\xi(u+j),m+\xi(u+j)+1)\\ &- \left.\sum_{k=1}^{m}(-1)^k(k-1)\frac{(-m-\xi(u+j))_k}{(\xi(u+j))_k}\right].\end{aligned} \quad (5.8)$$

Proof. By using some relations given below, we will prove the result.

$$\begin{aligned}
(i) &\quad F(\kappa,\xi;1+\xi;u) = \xi u^{-\xi} Be(\xi,1-\kappa;u),\\
(ii) &\quad Be(m+v,v,\frac{1}{2}) = \frac{2^{-(m+v)}}{v} F(1-v,m+v;1+m+v;\frac{1}{2}),\\
(iii) &\quad F(\kappa,\xi;y;u) = (1-u)^{-\kappa} F(\kappa,y-\xi;y;\frac{-u}{1-u}),\\
(iv) &\quad F(\kappa,\xi;y;u) = \sum_{k=0}^{\infty} \frac{(\kappa)_k(\xi)_k u^k}{(\gamma)_k k!}, (a)_k = a(a+1)\ldots(a+k-1),\\
(v) &\quad F(a,1,l-a,-1) = 2^{l-2a-2}\frac{\Gamma(1-a)\Gamma(l-a)}{\Gamma(l-2a)} - \frac{1}{2}\sum_{k=1}^{l-2}(-1)^k \frac{(1-l+a)_k}{(1-a)_k}.
\end{aligned}$$

From (i)-(iii) in (5.5) and simplifying, we get

$$\begin{aligned}
\mu_{u:n}^{(p)} &= \frac{n!}{(u-1)!(n-u)!}\sum_{j=0}^{n-u}\binom{n-u}{j}\frac{2(-1)^j}{u+j}\big[F(1-\xi(u+j),1,1+m+\xi(u+j),-1)say(I_1)\\
&- F(1-\xi(u+j),1,2+m+\xi(u+j),-1)say(I_2)\big] \qquad (5.9)
\end{aligned}$$

Using relations (iv) and (v), the terms I_1 and I_2 in (5.9) can be modified as

$$\begin{aligned}
I_1 &= 2^{m+2\xi(u+j)-2}\frac{\Gamma(\xi(u+j))\Gamma(1+m+\xi(u+j))}{\Gamma(m+2\xi(u+j))}\\
&- \frac{1}{2}\sum_{k=1}^{m}(-1)^k\frac{(-m-\xi(u+j))_k}{(\xi(u+j))_k}\\
I_2 &= 2^{m+2\xi(u+j)-1}\frac{\Gamma(\xi(u+j))\Gamma(2+m+\xi(u+j))}{\Gamma(1+m+2\xi(u+j))}\\
&- \frac{1}{2}\sum_{k=1}^{m+1}(-1)^k\frac{(-m-\xi(u+j)-1)_k}{(\xi(u+j))_k}.
\end{aligned}$$

Substituting the above expressions of I_1 and I_2 in (5.9) we get (5.8).

In this theorem, relation for single moments are derived under the condition that $0<\xi<1$ and $m=1/\xi$ is a positive integer. If $m+1\le u\le n-1$ for all positive integer $n\ge 3$.

Theorem 2. *Let $0 < \xi < 1$ so that $m = 1/\xi$ is a positive integer. If $m+1 \leq u \leq n-1$ for all positive integer $n \geq 3$, then we have the following moment relation for $p = 1, 2, \ldots$.*

$$\begin{aligned} \mu_{u:n}^{(p-1)} &= C\Bigg\{\left(1 - \frac{\kappa - 1}{p + \kappa - 1}\right)\left[\mu_{u-m+1:n-m}^{(p+\kappa-1)} - \mu_{u-m:n-m}^{(p+\kappa-1)}\right] \\ &+ \left(1 - \frac{2\kappa - 1}{p + 2\kappa - 1}\right)\left[\mu_{u-m:n-m}^{(p+2\kappa-1)} - \mu_{u-m+1:n-m}^{(p+2\kappa-1)}\right]\Bigg\}, \end{aligned}$$

where $p \in N$ and

$$C = \frac{2\kappa n!(u-m)!}{pm(u-1)!(n-m)!}.$$

Proof. We have

$$\mu_{u:n}^{(p-1)} = \frac{n!}{(u-1)!(n-u)!}\int_0^1 z^{p-1} H^{u-1}(z)\left[1 - H(z)\right]^{n-u} h(z) dz.$$

Integrating by parts and noting that

$$h^2(z) = 2\kappa\xi(1 - z^\kappa) z^{\kappa-1} H^{1-m}(z) h(z)$$

and

$$h^{'}(z) = 2\kappa(\xi - 1)(1 - z^\kappa) z^{\kappa-1} H^{-m}(z) h(z) + 2\kappa\xi\left[(\kappa - 1) z^{\kappa-2} - (2\kappa - 1) z^{2\kappa-2}\right] H^{1-m}(z)$$

we have

$$\begin{aligned} \mu_{u:n}^{(p-1)} &= C_{u:n}\frac{z^p}{p} H^{u-1}(z)\left[1 - H(z)\right]^{n-u} h(z)|_0^1 - C_{u:n}\int_0^1 \frac{z^p}{p}\Big\{\Big[(u-1) H^{u-2}(z)\left[1 - H(z)\right]^{n-u} \\ &- (n-u) H^{u-1}(z)\left[1 - H(z)\right]^{n-u-1}\Big] h^2(z) + H^{u-1}(z)\left[1 - H(z)\right]^{n-u} h^{'}(z)\Big\} dz \end{aligned}$$

$$\begin{aligned}
&= \frac{2\kappa(1-\xi u)C_{u:n}}{p}\int_0^1 z^{p+\kappa-1}(1-z^{\kappa})H^{u-m-1}(z)\left[1-H(z)\right]^{n-u}h(z)dz \\
&+ \frac{2\kappa\xi(n-u)C_{u:n}}{p}\int_0^1 z^{p+\kappa-1}(1-z^{\kappa})H^{u-m}(z)\left[1-H(z)\right]^{n-u-1}h(z)dz \\
&+ \frac{2\kappa\xi C_{u:n}}{p}\int_0^1 z^{p+\kappa-2}\left[(2\kappa-1)z^{\kappa}-(\kappa-1)\right]H^{u-m}(z)\left[1-H(z)\right]^{n-u}dz. \qquad (5.10)
\end{aligned}$$

After some manipulation the first two integrals above will be

$$\frac{2\kappa(1-\xi u)n!(u-m-1)!}{p(u-1)!(n-m)!}\left\{\mu_{u-m:n-m}^{(p+\kappa-1)}-\mu_{u-m:n-m}^{(p+2\kappa-1)}\right\}$$

and

$$\frac{2\kappa\xi n!(u-m)!}{p(u-1)!(n-m)!}\left\{\mu_{u-m+1:n-m}^{(p+\kappa-1)}-\mu_{u-m+1:n-m}^{(p+2\kappa-1)}\right\},$$

respectively. The third integral in (5.10) will be

$$\begin{aligned}
&\frac{2\kappa\xi(1-2\kappa)n!(u-m)!}{p(p+2\kappa-1)(u-1)!(n-m)!}\left\{\mu_{u-m:n-m}^{(p+2\kappa-1)}-\mu_{u-m+1:n-m}^{(p+2\kappa-1)}\right\} \\
&+\frac{2\kappa\xi(\kappa-1)n!(u-m)!}{p(p+\kappa-1)(u-1)!(n-m)!}\left\{\mu_{u-m:n-m}^{(p+\kappa-1)}-\mu_{u-m+1:n-m}^{(p+\kappa-1)}\right\}.
\end{aligned}$$

after some manipulation and substituting these results into (5.10) completes the proof. □

The recurrence relation without any restriction is presented by the following theorem.

Theorem 3.. *Let $Z_1,\ldots,Z_n$ be a random sample of size n from the GTL distribution, and let $Z_{1:n}\le Z_{2:n}\le\cdots\le Z_{n:n}$ be the corresponding order statistics, then for $1\le u<n$ and $p\in N$, we have*

$$\mu_{u+1:n}^{(p+\kappa)} = \left(1+\frac{\kappa}{p}\right)\mu_{u+1:n}^{(p)}+\left(\frac{p+\kappa}{2\kappa\xi u}+1\right)\mu_{u:n}^{(p+\kappa)}-\left(\frac{p+\kappa}{\kappa\xi u}+\frac{\kappa}{p}+1\right)\mu_{u:n}^{(p)}. \qquad (5.11)$$

Proof. We have

$$\begin{aligned}
2\mu_{u:n}^{(p)} - \mu_{u:n}^{(p+\kappa)} &= \frac{n!}{(u-1)!(n-u)!}\int_0^1 z^{p-1}H^{u-1}(z)\left[1-H(z)\right]^{n-u}(2z - z^{\kappa+1})h(z)dz \\
&= \frac{2\kappa\xi\, n!}{(u-1)!(n-u)!}\int_0^1 z^{p-1}(1-z^{\kappa})H^{u}(z)\left[1-H(z)\right]^{n-u}dz,
\end{aligned}$$

where the last equality is obtained from (5.1) and (5.2) note that

$$(2z - z^{\kappa+1})h(z) = 2\kappa\xi(1-z^{\kappa})H(z). \tag{5.12}$$

Integrating by parts by treating $z^{p-1}(1-z^{\kappa})$ for integration and $H^{u}(z)\left[1-H(z)\right]^{n-u}$ for differentiation, we have

$$\begin{aligned}
2\mu_{u:n}^{(p)} - \mu_{u:n}^{(p+\kappa)} &= 2\frac{\kappa\xi\, n!}{(u-1)!(n-u)!}\Big\{\left(\frac{1}{p} - \frac{z^{\kappa}}{p+\kappa}\right)z^{p}H^{u}(z)\left[1-H(z)\right]^{n-u}|_0^1 \\
&- u\int_0^1\left(\frac{1}{p} - \frac{z^{\kappa}}{p+\kappa}\right)z^{p}H^{u-1}(z)\left[1-H(z)\right]^{n-u}h(z)dz \\
&+ (n-u)\int_0^1\left(\frac{1}{p} - \frac{z^{\kappa}}{p+\kappa}\right)z^{p}H^{u}(z)\left[1-H(z)\right]^{n-u-1}h(z)dz\Big\}.
\end{aligned} \tag{5.13}$$

If $u < n$, right hand side of (5.13) vanishes. Therefore, we have

$$\begin{aligned}
\frac{1}{2\kappa\xi u}\left(2\mu_{u:n}^{(p)} - \mu_{u:n}^{(p+\kappa)}\right) &= \int_0^1\left(\frac{1}{p} - \frac{z^{\kappa}}{p+\kappa}\right)z^{p}h_{Z_{u+1:n}}(z;\kappa,\xi)dz \\
&- \int_0^1\left(\frac{1}{p} - \frac{z^{\kappa}}{p+\kappa}\right)z^{p}h_{Z_{u:n}}(z;\kappa,\xi)dz \\
&= \frac{\mu_{u+1:n}^{(p)} - \mu_{u:n}^{(p)}}{p} - \frac{\mu_{u+1:n}^{(p+\kappa)} - \mu_{u:n}^{(p+\kappa)}}{p+\kappa}
\end{aligned}$$

we can get result by above equation.

Remark 5.1. Under the assumptions of Theorem 1 and for $p, n \in N$, we have

$$\mu_{n:n}^{(p+\kappa)} = \left(\frac{1}{p+\kappa} + \frac{1}{2n\kappa\xi}\right)^{-1}\left\{\left(\frac{1}{p} + \frac{1}{n\kappa\xi}\right)\mu_{n:n}^{(p)} - \frac{\kappa}{p(p+\kappa)}\right\}. \tag{5.14}$$

The proof is similar to that of Theorem 1.

Next, by using the techniques which is introduced by Thomas and Samuel (2008), if we assume that ξ is a positive integer value to obtain the several recurrence relations for single moments of lower sample sizes for GTL distribution.

Theorem 4. Suppose that $p, \xi \in N$, then we have

(i) If $2 \leq u \leq n$, then

$$\mu_{u:n}^{(p)} = \frac{n}{u-1} \sum_{r=0}^{\xi} \binom{\xi}{r} (-1)^r 2^{\xi-r} \mu_{u-1:n-1}^{(p+\kappa(\xi+r))}. \tag{5.15}$$

(ii) If $1 \leq u < n$, then

$$\mu_{u:n}^{(p)} = \frac{n}{n-u} \left\{ \mu_{u:n-1}^{(p)} - \sum_{r=0}^{\xi} \binom{\xi}{r} (-1)^r 2^{\xi-r} \mu_{u:n-1}^{(p+\kappa(\xi+r))} \right\}. \tag{5.16}$$

Proof. Proof of (i) is similar to (ii), so we are proving (ii) only. Since $\xi \in Z^+$, the cdf of the GTL distribution can be expressed as

$$H(z) = \sum_{r=0}^{\xi} \binom{\xi}{r} (-1)^r 2^{\xi-r} z^{\kappa(\xi+r)}, \quad 0 < z < 1. \tag{5.17}$$

Therefore if $1 \leq u \leq n-1$, then we may write

$$\mu_{u:n}^{(p)} = \frac{n!}{(u-1)!(n-u)!} \int_0^1 z^p \left\{ 1 - \sum_{r=0}^{\xi} \binom{\xi}{r} (-1)^r 2^{\xi-r} z^{\kappa(\xi+r)} \right\} H^{u-1}(z) \left[1 - H(z)\right]^{n-u-1} h(z) dz,$$

By simplifying above expression, we get the result. □

5.3 Application to the Lifetime of Coherent Systems

Here, we will derive the expression for coherent systems' expected lifetimes by using relations of previous sections. Coherent systems are beneficial in reliability. Samaniego (1985)

TABLE 5.1: First moments, second moments and variances of the uth order statistics from GTL distribution.

κ	ξ	u	n	$E(Z)$	$E(Z^2)$	$V(Z)$	κ	ξ	u	n	$E(Z)$	$E(Z^2)$	$V(Z)$
0.5	1	1	1	0.166667	0.066667	0.038889	0.5	2	1	1	0.266667	0.119048	0.047936
			2	0.066667	0.014286	0.009841				2	0.146032	0.040115	0.018790
			3	0.035714	0.004762	0.003486				3	0.098968	0.019297	0.009503
			4	0.022222	0.002020	0.001526				4	0.074313	0.011134	0.005612
			5	0.015152	0.000999	0.000769				5	0.059259	0.007173	0.003661
			6	0.010989	0.000550	0.000429				6	0.049158	0.004975	0.002559
			7	0.008333	0.000327	0.000257				7	0.041931	0.003638	0.001880
			8	0.006536	0.000206	0.000164				8	0.036516	0.002768	0.001435
			9	0.005263	0.000137	0.000109				9	0.032312	0.002173	0.001128
			10	0.004329	0.000094	0.000075				10	0.028957	0.001748	0.000909
		2	2	0.266667	0.119048	0.047936			2	2	0.387302	0.197980	0.047977
			3	0.128571	0.033333	0.016803				3	0.240160	0.081752	0.024075
			4	0.076191	0.012987	0.007182				4	0.172933	0.043787	0.013882
			5	0.050505	0.006105	0.003554				5	0.134527	0.026979	0.008881
			6	0.035964	0.003247	0.001953				6	0.109765	0.018161	0.006113
			7	0.026923	0.001885	0.001161				7	0.092518	0.012996	0.004437
			8	0.020915	0.001170	0.000732				8	0.079841	0.009728	0.003353
			9	0.016718	0.000764	0.000484				9	0.070146	0.007535	0.002615
			10	0.013671	0.000520	0.000333				10	0.062500	0.005998	0.002091
		3	3	0.335714	0.161905	0.049201			3	3	0.460873	0.256094	0.043690
			4	0.180952	0.053680	0.020936				4	0.307387	0.119716	0.025229
			5	0.114719	0.023310	0.010150				5	0.230541	0.069000	0.015851
			6	0.079587	0.011822	0.005487				6	0.184053	0.044615	0.010740
			7	0.058566	0.006650	0.003220				7	0.152882	0.031072	0.007699
			8	0.044947	0.004033	0.002012				8	0.130547	0.022803	0.005760
			9	0.035604	0.002589	0.001322				9	0.113775	0.017401	0.004456
			10	0.028909	0.001739	0.000904				10	0.100731	0.013686	0.003540
		4	4	0.387302	0.197980	0.047977			4	4	0.512034	0.301553	0.039374
			5	0.225108	0.073926	0.023252				5	0.358617	0.153526	0.024920
			6	0.149850	0.034799	0.012343				6	0.277030	0.093385	0.016639
			7	0.107615	0.018717	0.007136				7	0.225614	0.062673	0.011771
			8	0.081265	0.011013	0.004409				8	0.190107	0.044854	0.008713
			9	0.063634	0.006919	0.002870				9	0.164091	0.033608	0.006682
			10	0.051224	0.004573	0.001949				10	0.144212	0.026068	0.005270
		5	5	0.427850	0.228993	0.045938			5	5	0.550389	0.338560	0.035632
			6	0.262737	0.093490	0.024459				6	0.399411	0.183597	0.024068
			7	0.181527	0.046860	0.013908				7	0.315592	0.116419	0.016820
			8	0.133964	0.026420	0.008474				8	0.261122	0.080492	0.012307
			9	0.103304	0.016129	0.005458				9	0.222626	0.058911	0.009349
			10	0.082249	0.010440	0.003675				10	0.193910	0.044918	0.007317
		6	6	0.460873	0.256094	0.043690			6	6	0.580584	0.369553	0.032474
			7	0.295221	0.112142	0.024986				7	0.432938	0.210469	0.023033
			8	0.210064	0.059124	0.014997				8	0.348274	0.137975	0.016680
			9	0.158492	0.034653	0.009533				9	0.291918	0.097756	0.012540
			10	0.124359	0.021819	0.006354				10	0.251343	0.072904	0.009731
		7	7	0.488481	0.280086	0.041472			7	7	0.605192	0.396067	0.029809
			8	0.323607	0.129815	0.025093				8	0.461159	0.234633	0.021965
			9	0.235851	0.071359	0.015734				9	0.376452	0.158084	0.016368
			10	0.181248	0.043209	0.010359				10	0.318968	0.114324	0.012583
		8	8	0.512034	0.301553	0.039374			8	8	0.625768	0.419129	0.027543
			9	0.348680	0.146516	0.024938				9	0.485361	0.256504	0.020929
			10	0.259252	0.083423	0.016212				10	0.401088	0.176839	0.015967
		9	9	0.532454	0.320933	0.037426			9	9	0.643319	0.439457	0.025597
			10	0.371038	0.162289	0.024620				10	0.506430	0.276421	0.019950
		10	10	0.550389	0.338560	0.035632			10	10	0.658529	0.457572	0.023911

derived system reliability and coherent systems' expected lifetimes using the concept of signature vector. Let a system having n components with lifetimes represented by independently and identically distributed $Z_1, Z_2, \ldots, Z_n$, random variables. Moreover, let lifetime of whole system be Y and $s_u = Pr(Y = Z_{u:n})$ for $u = 1, \ldots, n$. Then for $p \in N$, we have Samaniego (1985)

$$E(Y^p) \quad = \quad \sum_{u=1}^{n} s_u \, \mu_{u:u}^{(p)}, \tag{5.18}$$

and the vector $s = (s_1, \ldots, s_n)$ is called the signature vector.

Navarro et al. (2007) gave concept of exchangeable components for a coherent system and can be written as

$$E(Y^p) \quad = \quad \sum_{u=1}^{n} \lambda_u \, \mu_{1:u}^{(p)} = \sum_{u=1}^{n} \theta_u \, \mu_{u:u}^{(p)}, \tag{5.19}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is minimal and $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$is maximal signature, respectively.

Suppose, independently and identically distributed $Z_1, Z_2, \ldots, Z_n$ are lifetimes of components of a coherent system and follows generalized Topp-Leone distribution then using relations of previous sections, we find expected system lifetime. For instance, from (5.14) and (5.19) and for $p \in \mathbb{N}$, we have

$$E(Y^{p+\kappa}) \quad = \quad \sum_{u=1}^{n} \theta_u \left(\frac{1}{p+\kappa} + \frac{1}{2u\kappa\xi} \right)^{-1} \left\{ \left(\frac{1}{p} + \frac{1}{u\kappa\xi} \right) \mu_{u:u}^{(p)} - \frac{\kappa}{p(p+\kappa)} \right\},$$

or from (5.16) and (5.18), for $\xi, p \in \mathbb{N}$, we can state

$$E(Y^p) \quad = \quad \sum_{u=1}^{n} \frac{n s_u}{n-u} \left\{ \mu_{u:n-1}^{(p)} - \sum_{r=0}^{\xi} \binom{\xi}{r} (-1)^r 2^{\xi - r} \mu_{u:n-1}^{(p+\kappa(\xi+r))} \right\}.$$

5.4 Relations for Product Moments of Order Statistics

Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from GTL distribution, then under the condition $\xi \in \mathbb{N}$, we can express product moment of uth and vth order statistic of GTL distribution as follows:

Theorem 5. *For* $1 \leq u < v \leq n$, $n \in N$ *and* $\xi \in N$ *we have,*

$$\begin{aligned}
\mu_{u,v:n} &= C_{u,v:n} \sum_{r=0}^{n-v} \sum_{s=0}^{v-u-1} \binom{n-v}{r} \binom{v-u-1}{s} (-1)^{r+s} \frac{1}{(s+u)\Delta_{r,s}} \mu_{s+u:s+u} \mu_{\Delta_{r,s}:\Delta_{r,s}} \\
&+ C_{u,v:n} \sum_{r=0}^{n-v} \sum_{s=0}^{v-u-1} \sum_{t=0}^{\xi\Delta_{r,s}-1} \binom{n-v}{r} \binom{v-u-1}{s} \binom{\xi\Delta_{r,s}-1}{t} (-1)^{r+s+t} \frac{2^{\xi\Delta_{r,s}-t} \kappa\xi}{(s+u)} \\
&\times \left\{ \frac{1}{1+\kappa+\kappa t+\kappa\xi\Delta_{r,s}} \mu_{s+u:s+u}^{2+\kappa+\kappa t+\kappa\xi\Delta_{r,s}} - \frac{1}{1+\kappa t+\kappa\xi\Delta_{r,s}} \mu_{s+u:s+u}^{2+\kappa t+\kappa\xi\Delta_{r,s}} \right\}, \qquad (5.20)
\end{aligned}$$

where $\Delta_{r,s} = v - u - s + r$.

Proof. From joint pdf of uth and vth order statistics, we get

$$\begin{aligned}
\mu_{u,v:n} &= C_{u,v:n} \int_0^1 \int_z^1 zyH^{u-1}(z) \left[H(y) - H(z)\right]^{v-1-u} \left[1 - H(y)\right]^{n-v} h(z)h(y)dzdy \\
&= C_{u,v:n} \sum_{r=0}^{n-v} \sum_{s=0}^{v-u-1} \binom{n-v}{r} \binom{v-u-1}{s} (-1)^{r+s} \int_0^1 zH^{s+u-1}(z)h(z)I(z)dz. \qquad (5.21)
\end{aligned}$$

where

$$\begin{aligned}
I(z) &= \int_z^1 y[H(y)]^{v-u-s+r-1} h(y)dy \\
&= \frac{1}{v-u-s+r} \mu_{v-u-s+r:v-u-s+r} - \int_0^z y[H(y)]^{v-u-s+r-1} h(y)dy \\
&= \frac{1}{\Delta_{r,s}} \mu_{\Delta_{r,s}:\Delta_{r,s}} + \kappa\xi 2^{\xi\Delta_{r,s}} \left[\int_0^1 z^{\kappa\xi\Delta_{r,s}+\kappa+1} u^{\kappa\xi\Delta_{r,s}+\kappa} \left[1 - \frac{1}{2}(zu)^{\kappa}\right]^{\xi\Delta_{r,s}-1} dz \right. \\
&- \left. \int_0^1 z^{\kappa\xi\Delta_{r,s}+1} u^{\kappa\xi\Delta_{r,s}} \left[1 - \frac{1}{2}(zu)^{\kappa}\right]^{\xi\Delta_{r,s}-1} dz \right] \qquad (5.22)
\end{aligned}$$

Now substituting the values of $I(z)$ with some manipulations we get the result. □

Next, we elucidate the (p,q)th product moment for unconditional ξ which comprise the simple product moments as well, the result.

Theorem 6. For $1 \leq u < v \leq n$ and $n,p,q \in \mathbb{N}$, we have

$$\begin{aligned}
\mu_{u,v:n}^{(p,q)} &= \frac{\xi^2 n!}{(u-1)!(v-u-1)!(n-v)!} \sum_{r=0}^{n-v} \sum_{s=0}^{v-u-1} \sum_{l=0}^{\infty} \binom{n-v}{r} \binom{v-u-1}{s} \binom{\xi(s+u)-1}{l} \\
&\times (-1)^{r+s+l} 2^{\frac{q+p+2\kappa\xi(v+r)}{\kappa}} \left\{ \frac{1}{\eta_{u,p,\kappa,\xi}(s,l)} B_{1/2}(\eta^*_{v,p,q,\kappa,\xi}(r,l), \xi\Delta_{r,s}) \right. \\
&- \frac{2(2\eta_{u,p,\kappa,\xi}(s,l)+1)}{\eta_{u,p,\kappa,\xi}(s,l)(\eta_{u,p,\kappa,\xi}(s,l)+1)} B_{1/2}(\eta^*_{v,p,q,\kappa,\xi}(r,l)+1, \xi\Delta_{r,s}) \\
&+ \left. \frac{4}{\eta_{u,p,\kappa,\xi}(s,l)+1} B_{1/2}(\eta^*_{v,p,q,\kappa,\xi}(r,l)+2, \xi\Delta_{r,s}) \right\}. \qquad (5.23)
\end{aligned}$$

where, $\eta_{u,p,\kappa,\xi}(s,l) = \frac{p}{\kappa} + \xi(s+u) + l$ and $\eta^*_{v,p,q,\kappa,\xi}(r,l) = \frac{q+p}{\kappa} + \xi(v+r) + l$. If $\xi \in \mathbb{N}$, then the third summation stops at $l = \xi(s+u) - 1$.

Proof. Using (1.9), the (p,q)-th product moment of (u,v)th order statistics can be written as

$$\begin{aligned}
\mu_{u,v:n}^{(p,q)} &= C_{u,v:n} \int_0^1 \int_0^y z^p y^q H^{u-1}(z) [H(y) - H(z)]^{v-1-u} [1 - H(y)]^{n-v} h(z)h(y) dz dy \\
&= C_{u,v:n} \sum_{r=0}^{n-v} \sum_{s=0}^{v-u-1} \binom{n-v}{r} \binom{v-u-1}{s} (-1)^{r+s} \int_0^1 y^q h(y) [H(y)]^{r,s-1} I(y) dy \qquad (5.24)
\end{aligned}$$

where

$$\begin{aligned}
I(y) &= \int_0^y z^p [H(z)]^{s+u-1} h(z) dz \\
&= 2\kappa\xi \int_0^y z^{p+\kappa\xi(s+u)-1} (2 - z^\kappa)^{\xi(s+u)-1} (1 - z^\kappa) dz \\
&= \xi 2^{\frac{p}{\kappa}+2\xi(s+u)} \int_0^{y^\kappa/2} t^{\frac{p}{\kappa}+\xi(s+u)-1} (1-t)^{\xi(s+u)-1} (1-2t) dt \\
&= \xi 2^{\frac{p}{\kappa}+2\xi(s+u)} \sum_{l=0}^{\infty} \binom{\xi(s+u)-1}{l} (-1)^l \int_0^{y^\kappa/2} t^{\eta_{u,p,\kappa,\xi}(s,l)-1} (1-2t) dt \\
&= \xi \sum_{l=0}^{\infty} 2^{\xi(s+u)-l} \binom{\xi(s+u)-1}{l} (-1)^l \left(\frac{1}{\eta_{u,p,\kappa,\xi}(s,l)} - \frac{y^\kappa}{\eta_{u,p,\kappa,\xi}(s,l)+1} \right) y^{\kappa\eta_{u,p,\kappa,\xi}(s,l)}
\end{aligned}$$

TABLE 5.2: Covariances of order statistics.

κ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$	κ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$
0.5	1	2	1	2	0.027778	0.010001	0.5	2	2	1	2	0.071111	0.014553
				3	0.008730	0.004138					3	0.032354	0.008586
				4	0.003608	0.001914					4	0.018197	0.005346
				5	0.001758	0.000992					5	0.011550	0.003578
				6	0.000957	0.000562					6	0.007933	0.002537
				7	0.000566	0.000341					7	0.005761	0.001882
				8	0.000356	0.000219					8	0.004361	0.001445
				9	0.000235	0.000147					9	0.003408	0.001141
				10	0.000161	0.000102					10	0.002732	0.000922
		3	1	3	0.015873	0.003883			3	1	3	0.052117	0.006505
				4	0.006118	0.002097					4	0.027582	0.004739
				5	0.002886	0.001148					5	0.017013	0.003351
				6	0.001543	0.000668					6	0.011489	0.002441
				7	0.000901	0.000412					7	0.008250	0.001839
				8	0.000561	0.000267					8	0.006194	0.001427
				9	0.000368	0.000180					9	0.004811	0.001135
				10	0.000251	0.000126					10	0.003838	0.000921
			2	3	0.058730	0.015567				2	3	0.128862	0.018179
				4	0.021587	0.007801					4	0.065441	0.012284
				5	0.009879	0.004085					5	0.039322	0.008307
				6	0.005173	0.002310					6	0.026078	0.005876
				7	0.002973	0.001396					7	0.018478	0.004333
				8	0.001831	0.000891					8	0.013731	0.003308
				9	0.001190	0.000594					9	0.010579	0.002598
				10	0.000807	0.000411					10	0.008384	0.002088
		4	1	4	0.010476	0.001870			4	1	4	0.041590	0.003539
				5	0.004607	0.001196					5	0.024186	0.002935
				6	0.002382	0.000735					6	0.015880	0.002262
				7	0.001363	0.000466					7	0.011213	0.001752
				8	0.000838	0.000307					8	0.008324	0.001382
				9	0.000545	0.000210					9	0.006413	0.001111
				10	0.000369	0.000148					10	0.005084	0.000908
			2	4	0.036421	0.006913				2	4	0.097705	0.009158
				5	0.015607	0.004237					5	0.055513	0.007269
				6	0.007923	0.002534					6	0.035848	0.005440
				7	0.004472	0.001575					7	0.025001	0.004127
				8	0.002722	0.001022					8	0.018381	0.003203
				9	0.001753	0.000689					9	0.014053	0.002542
				10	0.001181	0.000481					10	0.011072	0.002059
			3	4	0.088456	0.018373				3	4	0.176151	0.018759
				5	0.036286	0.010462					5	0.096526	0.013850
				6	0.017917	0.005990					6	0.060923	0.009934
				7	0.009922	0.003619					7	0.041819	0.007327
				8	0.005954	0.002301					8	0.030392	0.005574
				9	0.003795	0.001529					9	0.023028	0.004358
				10	0.002535	0.001054					10	0.018015	0.003488
		5	1	5	0.007511	0.001028			5	1	5	0.034785	0.002169
				6	0.003627	0.000740					6	0.021599	0.001964
				7	0.002009	0.000496					7	0.014844	0.001611
				8	0.001212	0.000336					8	0.010843	0.001308
				9	0.000777	0.000233					9	0.008262	0.001069
				10	0.000522	0.000166					10	0.006499	0.000884

TABLE 5.2: Continued.

κ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$	κ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$
			2	5	0.025241	0.003633				2	5	0.079409	0.005367
				6	0.011993	0.002544					6	0.048563	0.004721
				7	0.006559	0.001672					7	0.032991	0.003793
				8	0.003918	0.001116					8	0.023878	0.003030
				9	0.002492	0.000765					9	0.018062	0.002446
				10	0.001664	0.000540					10	0.014122	0.002003
			3	5	0.058009	0.008926				3	5	0.137098	0.010211
				6	0.026904	0.005994					6	0.082128	0.008615
				7	0.014463	0.003832					7	0.054979	0.006731
				8	0.008528	0.002507					8	0.039359	0.005271
				9	0.005372	0.001694					9	0.029521	0.004192
				10	0.003559	0.001181					10	0.022925	0.003392
			4	5	0.115995	0.019683				4	5	0.215710	0.018332
				6	0.051660	0.012288					6	0.125061	0.014412
				7	0.027063	0.007528					7	0.082007	0.010805
				8	0.015676	0.004789					8	0.057876	0.008235
				9	0.009747	0.003173					9	0.042955	0.006424
				10	0.006393	0.002180					10	0.033089	0.005124
		6	1	6	0.005684	0.000620			6	1	6	0.029980	0.001440
				7	0.002947	0.000487					7	0.019544	0.001390
				8	0.001722	0.000349					8	0.013913	0.001196
				9	0.001083	0.000249					9	0.010439	0.001006
				10	0.000718	0.000180					10	0.008125	0.000847
			2	6	0.018701	0.002126				2	6	0.067187	0.003459
				7	0.009586	0.001638					7	0.043326	0.003271
				8	0.005550	0.001156					8	0.030576	0.002769
				9	0.003465	0.000816					9	0.022779	0.002302
				10	0.002285	0.000585					10	0.017627	0.001918
			3	6	0.041676	0.004997				3	6	0.113165	0.006307
				7	0.021036	0.003746					7	0.071990	0.005802
				8	0.012036	0.002594					8	0.050282	0.004816
				9	0.007447	0.001804					9	0.037158	0.003945
				10	0.004874	0.001279					10	0.028566	0.003248
			4	6	0.079265	0.010203				4	6	0.171378	0.010538
				7	0.039110	0.007340					7	0.106985	0.009308
				8	0.022016	0.004945					8	0.073730	0.007521
				9	0.013458	0.003372					9	0.053944	0.006043
				10	0.008727	0.002357					10	0.041152	0.004905
			5	6	0.141262	0.020173				5	6	0.249456	0.017565
				7	0.067095	0.013505					7	0.151106	0.014474
				8	0.036865	0.008724					8	0.102175	0.011233
				9	0.022160	0.005787					9	0.073779	0.008790
				10	0.014195	0.003966					10	0.055740	0.007002

TABLE 5.2: Continued.

κ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$	κ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$
		7	1	7	0.004470	0.000399			7	1	7	0.026389	0.001013
				8	0.002451	0.000335					8	0.017866	0.001026
				9	0.001495	0.000253					9	0.013080	0.000916
				10	0.000973	0.000188					10	0.010030	0.000794
			2	7	0.014494	0.001342				2	7	0.058373	0.002382
				8	0.007880	0.001111					8	0.039196	0.002376
				9	0.004773	0.000830					9	0.028502	0.002095
				10	0.003090	0.000612					10	0.021734	0.001798
			3	7	0.031674	0.003065				3	7	0.096746	0.004223
				8	0.017036	0.002490					8	0.064334	0.004131
				9	0.010230	0.001833					9	0.046420	0.003589
				10	0.006577	0.001337					10	0.035174	0.003044
			4	7	0.058559	0.005991				4	7	0.143309	0.006769
				8	0.031038	0.004740					8	0.094118	0.006449
				9	0.018431	0.003422					9	0.067269	0.005497
				10	0.011746	0.002462					10	0.050596	0.004597
			5	7	0.099657	0.010985				5	7	0.201510	0.010516
				8	0.051693	0.008341					8	0.130045	0.009626
				9	0.030227	0.005863					9	0.091801	0.007992
				10	0.019045	0.004137					10	0.068410	0.006559
			6	7	0.164416	0.020206				6	7	0.278715	0.016705
				8	0.082268	0.014289					8	0.174891	0.014281
				9	0.047015	0.009634					9	0.121288	0.011395
				10	0.029155	0.006616					10	0.089283	0.009113
		8	1	8	0.003617	0.000271			8	1	8	0.023594	0.000744
				9	0.002075	0.000240					9	0.016466	0.000783
				10	0.001311	0.000189					10	0.012335	0.000720
			2	8	0.011606	0.000896				2	8	0.051684	0.001722
				9	0.006615	0.000786					9	0.035837	0.001790
				10	0.004158	0.000614					10	0.026698	0.001630
			3	8	0.025021	0.002006				3	8	0.084684	0.002992
				9	0.014148	0.001734					9	0.058289	0.003067
				10	0.008834	0.001340					10	0.043164	0.002762
			4	8	0.045423	0.003813				4	8	0.123631	0.004668
				9	0.025422	0.003234					9	0.084338	0.004694
				10	0.015744	0.002464					10	0.062007	0.004166
			5	8	0.075291	0.006697				5	8	0.170366	0.006964
				9	0.041552	0.005532					9	0.114878	0.006824
				10	0.025459	0.004136					10	0.083720	0.005945
			6	8	0.118997	0.011436				6	8	0.228260	0.010322
				9	0.064336	0.009073					9	0.151410	0.009724
				10	0.038844	0.006604					10	0.109066	0.008256

TABLE 5.2: Continued.

κ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$	κ	ξ	v	u	n	$\mu_{u,v:n}$	$\sigma_{u,v:n}$
			7	8	0.185677	0.019979				7	8	0.304429	0.015850
				9	0.097007	0.014771					9	0.196672	0.013956
				10	0.057307	0.010318					10	0.139328	0.011394
		9	1	9	0.002994	0.000191			9	1	9	0.021352	0.000565
				10	0.001783	0.000177					10	0.015278	0.000613
			2	9	0.009527	0.000625				2	9	0.046418	0.001292
				10	0.005646	0.000574					10	0.033042	0.001391
			3	9	0.020337	0.001379				3	9	0.075405	0.002211
				10	0.011979	0.001252					10	0.053358	0.002345
			4	9	0.036452	0.002570				4	9	0.108949	0.003386
				10	0.021307	0.002301					10	0.076590	0.003557
			5	9	0.059395	0.004390				5	9	0.148136	0.004917
				10	0.034376	0.003859					10	0.103251	0.005050
			6	9	0.091576	0.007186				6	9	0.194801	0.007005
				10	0.052295	0.006154					10	0.134318	0.007030
			7	9	0.137247	0.011667				7	9	0.252222	0.010043
				10	0.076847	0.009597					10	0.171222	0.009687
			8	9	0.205262	0.019606				8	9	0.327281	0.015038
				10	0.111231	0.015038					10	0.216691	0.013568
		10	1	10	0.002522	0.000139			10	1	10	0.019510	0.000441
			2	10	0.007976	0.000452				2	10	0.042156	0.000998
			3	10	0.016897	0.000985				3	10	0.068024	0.001690
			4	10	0.030003	0.001809				4	10	0.097514	0.002546
			5	10	0.048301	0.003031				5	10	0.131332	0.003637
			6	10	0.073274	0.004828				6	10	0.170556	0.005040
			7	10	0.107274	0.007517				7	10	0.217004	0.006954
			8	10	0.154439	0.011750				8	10	0.273856	0.009728
			9	10	0.223369	0.019154				9	10	0.347783	0.014284

If $\xi \in \mathbb{N}$, summation stops at $\xi(s+u)-1$. Substituting the value of $I(y)$ in (5.24) and after simplification, we get the result given in (5.23). $\square$

Next, we present a recurrence relation for the product moments of order statistics for GTL distribution.

Theorem 7. For the GTL distribution $1 \leq u \leq v-2, v \leq n$ and $p,q \in \mathbb{N}$, we have

$$\mu_{u+1,v:n}^{(p+\kappa,q)} = \left(1+\frac{\kappa}{p}\right)\mu_{u+1,v:n}^{(p,q)} + \left(\frac{p+\kappa}{2\kappa\xi u}+1\right)\mu_{u,v:n}^{(p+\kappa,q)} - \left(\frac{p+\kappa}{\kappa\xi u}+\frac{\kappa}{p}+1\right)\mu_{u,v:n}^{(p,q)} \quad (5.25)$$

Proof. Form (1.9) and (5.12), we get

$$2\mu_{u,v:n}^{(p,q)} - \mu_{u,v:n}^{(p+\kappa,q)} = \frac{n!}{(u-1)!(v-u-1)!(n-v)!}\int_0^1 y^q h(y)[1-H(y)]^{n-v}G(y)dy$$

where

$$\begin{aligned} G(y) &= \int_0^y z^{p-1}[H(z)]^{u-1}[H(y)-H(z)]^{v-u-1}z(2-z^\kappa)h(z)dz \\ &= 2\kappa\xi\int_0^y z^{p-1}(1-z^\kappa)[H(z)]^u[H(y)-H(z)]^{v-u-1}dz \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} G(y) &= 2\kappa\xi\left\{\left(\frac{1}{p}-\frac{z^\kappa}{p+\kappa}\right)z^p H^u(z)\,[H(y)-H(z)]^{v-u-1}|_0^y\right. \\ &- u\int_0^y\left(\frac{1}{p}-\frac{z^\kappa}{p+\kappa}\right)z^p H^{u-1}(z)\,[H(y)-H(z)]^{v-u-1}h(z)dz \\ &+ \left.(v-u-1)\int_0^y\left(\frac{1}{p}-\frac{z^\kappa}{p+\kappa}\right)z^p H^u(z)\,[H(y)-H(z)]^{v-u-2}h(z)dz\right\}. \end{aligned} \tag{5.26}$$

If $v > u-1$, then the right hand side of (5.26) is vanishes and as a effect we get

$$\begin{aligned} \frac{1}{2\kappa\xi u}\left(2\mu_{u,v:n}^{(p,q)}-\mu_{u,v:n}^{(p+\kappa,q)}\right) &= \int_0^1\int_0^y\left(\frac{1}{p}-\frac{z^\kappa}{p+\kappa}\right)z^p y^q h_{Z_{u+1,v:n}}(z,y;\kappa,\xi)dzdy \\ &- \int_0^1\int_0^y\left(\frac{1}{p}-\frac{z^\kappa}{p+\kappa}\right)z^p y^q h_{Z_{u,v:n}}(z,y;\kappa,\xi)dzdy \\ &= \frac{\mu_{u+1,v:n}^{(p,q)}-\mu_{u,v:n}^{(p,q)}}{p}-\frac{\mu_{u+1,v:n}^{(p+\kappa,q)}-\mu_{u,v:n}^{(p+\kappa,q)}}{p+\kappa} \end{aligned}$$

and hence the result follows. □

Remark 5.2. For $1 \le u \le n-1$ and $p,q \in \mathbb{N}$, we have

$$\mu_{u,u+1:n}^{(p+\kappa,q)} = -\left(\frac{p+\kappa}{2\kappa\xi u}+1\right)^{-1}\left\{\left(1+\frac{\kappa}{p}\right)\mu_{u+1:n}^{(p+q)}-\mu_{u+1:n}^{(p+\kappa+q)}-\left(\frac{p+\kappa}{\kappa\xi u}+\frac{\kappa}{p}+1\right)\mu_{u,u+1:n}^{(p,q)}\right\}.$$

Next, by using the techniques which is introduced by Thomas and Samuel (2008), if we assume that ξ is a positive integer value to prove the several recurrence relations for product moments of lower sample sizes for GTL distribution.

Theorem 8. Let $\xi \in \mathbb{N}$, then we have

(i) If $2 \leq u < v \leq n$, then

$$\mu_{u,v:n}^{(p,q)} = \frac{n}{u-1}\sum_{r=0}^{\xi}\binom{\xi}{r}(-1)^r 2^{\xi-r}\mu_{u-1,v-1:n-1}^{(p+\kappa(\xi+r),q)} \tag{5.27}$$

(ii) If $1 \leq u < v-1 \leq n-1$, then

$$\mu_{u,v:n}^{(p,q)} = \frac{n}{v-u-1}\sum_{r=0}^{\xi}\binom{\xi}{r}(-1)^r 2^{\xi-r}\left\{\mu_{u,v-1:n-1}^{(p,q+\kappa(\xi+r))} - \mu_{u,v-1:n-1}^{(p+\kappa(\xi+r),q)}\right\} \tag{5.28}$$

(iii) If $1 \leq u < v \leq n-1$, then

$$\mu_{u,v:n}^{(p,q)} = \frac{n}{n-v}\left\{\mu_{u,v:n-1}^{(p,q)} - \sum_{r=0}^{\xi}\binom{\xi}{r}(-1)^r 2^{\xi-r}\mu_{u,v:n-1}^{(p,q+\kappa(\xi+r))}\right\} \tag{5.29}$$

Proof. Proof of (i) and (iii) are similar to (ii), so we proved only (ii). If $1 \leq u < v-1 \leq n-1$, then using the equation (5.11) and expanding the term $H(y)-H(z)$, we may write

$$\begin{aligned}
\mu_{u,v:n}^{(p,q)} &= C_{u,v:n}\int_0^1\int_0^y z^p y^q H^{u-1}(z)\left[H(y)-H(z)\right]^{v-1-u}\left[1-H(y)\right]^{n-v}h(z)h(y)dzdy \\
&= C_{u,v:n}\sum_{r=0}^{\xi}\binom{\xi}{r}(-1)^r 2^{\xi-r}\int_0^1\int_0^y z^p y^{q+\kappa(\xi+r)} \\
&\times H^{u-1}(z)\left[H(y)-H(z)\right]^{v-u-2}\left[1-H(y)\right]^{n-v}h(z)h(y)dzdy \\
&- C_{u,v:n}\sum_{r=0}^{\xi}\binom{\xi}{r}(-1)^r 2^{\xi-r}\int_0^1\int_0^y z^{p+\kappa(\xi+r)} y^q \\
&\times H^{u-1}(z)\left[H(y)-H(z)\right]^{v-u-2}\left[1-H(y)\right]^{n-v}h(z)h(y)dzdy
\end{aligned}$$

After some manipulations of above expression, we get the result. □

5.5 BLUEs of the Location and Scale Parameters

Let $Z_1 \leq Z_2 \leq \cdots \leq Z_n$ be a random sample of size n from GTL distribution with the pdf of scale-parameter GTL distribution is

$$h(z;\kappa,\xi,\varphi) = \frac{2\kappa\xi}{\varphi}\left(\frac{z}{\varphi}\right)^{\kappa\xi-1}\left[1-\left(\frac{z}{\varphi}\right)^{\kappa}\right]\left[2-\left(\frac{z}{\varphi}\right)^{\kappa}\right]^{\xi-1}, \tag{5.30}$$

where, $0 < z < 1$, κ, $\xi, \varphi > 0$. The pdf of the location-scale parameter is

$$h(z;\kappa,\xi,\delta,\varphi) = \frac{2\kappa\xi}{\varphi}\left(\frac{z-\delta}{\varphi}\right)^{\kappa\xi-1}\left[1-\left(\frac{z-\delta}{\varphi}\right)^{\kappa}\right]\left[2-\left(\frac{z-\delta}{\varphi}\right)^{\kappa}\right]^{\xi-1}, \tag{5.31}$$

where $0 < z < 1$, κ, $\xi, \delta, \varphi > 0$.

The expression for the BLUEs of location and scale parameter are given in (1.18) and also variances and covariance for these parameters are given in eqn (1.21), (1.22) and (1.23).

Tables 5.3 and 5.4 display the coefficient of the BLUEs for type-II right censored sample of various values of $n = 7,\ 10$ and censoring cases $c = 0(1)([n/2]-1)$. Also, Table 5.5 shows variances and covariances of the BLUEs.

5.6 Real Data Application

To see the practical utility of the model a real data set of 30 observations from Quesenberry and Hales (1980) is taken. The data are:

```
0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148,
0.169, 0.188, 0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395,
0.432, 0.463, 0.481, 0.519, 0.529, 0.567, 0.642, 0.674, 0.752,
0.823, 0.887, 0.926
```

TABLE 5.3: Coefficient of the BLUEs of the a_u

κ	ξ	n	c	$a_u,\ u=1,2,\ldots,(n-c)$						
0.5	1	7	0	1.122123	-0.075374	-0.023245	-0.009345	-0.004594	-0.002852	-0.006712
			1	1.129701	-0.076824	-0.024211	-0.010194	-0.005477	-0.012996	
			2	1.141753	-0.07901	-0.025653	-0.011471	-0.025621		
		10	0	1.120081	-0.075032	-0.023505	-0.009452	-0.004447	-0.002353	-0.001384
				-0.000921	-0.000737	-0.002249				
			1	1.122372	-0.075425	-0.023748	-0.009646	-0.004623	-0.002525	-0.001566
				-0.001129	-0.003709					
			2	1.125214	-0.075894	-0.024035	-0.009874	-0.00483	-0.002731	-0.001783
				-0.006068						
			3	1.129322	-0.076545	-0.024429	-0.010186	-0.005113	-0.003011	-0.010038
			4	1.135917	-0.077546	-0.025026	-0.010658	-0.005544	-0.017143	
	2	7	0	1.129977	-0.025254	-0.016827	-0.013254	-0.012016	-0.012837	-0.049791
			1	1.162885	-0.029873	-0.021133	-0.017699	-0.017052	-0.077129	
			2	1.205879	-0.035752	-0.026621	-0.023376	-0.120132		
		10	0	1.095827	-0.018305	-0.011527	-0.008401	-0.006652	-0.005841	-0.005352
				-0.005636	-0.006409	-0.027705				
			1	1.111367	-0.019731	-0.013508	-0.009195	-0.008641	-0.007048	-0.007175
				-0.007635	-0.038435					
			2	1.128081	-0.021833	-0.014442	-0.011367	-0.009619	-0.008913	-0.008861
				-0.053046						
			3	1.148688	-0.023804	-0.016537	-0.013022	-0.011512	-0.010857	-0.072956
			4	1.176809	-0.026692	-0.018993	-0.015504	-0.013931	-0.101689	

In this data set, we select a random sample of size 10 and data are: 0.105, 0.432, 0.642, 0.529, 0.069, 0.361, 0.887, 0.674, 0.216, 0.081. By using the GTL in Eq. (5.1) for the given sample, we have the maximumlikelihood estimate of $\kappa_{ML} = 2.9967$ and $\xi_{ML} = 0.32063$. The K-S statistic and corresponding p-value are 0.15679 and 0.9357, respectively, which conforms the suitability of GTL distribution. Figure 5.1 shows ecdf and Q–Q plot of the sample.

Then by using the BLUEs coefficients in Tables 5.3 and 5.4, we have

$$\delta^* = \sum_{u=1}^{n} a_u Z_{u:n} = .030836 \quad \text{and} \quad \varphi^* = \sum_{u=1}^{n} b_u Z_{u:n} = 1.329179$$

TABLE 5.4: Coefficient of the BLUEs of the b_u

κ	ξ	n	c	$b_u, u = 1,2,\ldots,(n-c)$						
0.5	1	7	0	-3.46878	0.549522	0.341474	0.287862	0.289566	0.344431	1.655924
			1	-5.33835	0.907121	0.579507	0.497268	0.507453	2.846996	
			2	-7.97863	1.386104	0.895646	0.776871	4.920006		
		10	0	-3.40004	0.483125	0.278286	0.213224	0.188383	0.181781	0.188651
				0.212461	0.271219	1.382916				
			1	-4.80901	0.725179	0.427867	0.332547	0.296316	0.287677	0.300279
				0.340695	2.098449					
			2	-6.41672	0.990501	0.590117	0.461615	0.413331	0.403162	0.423261
				3.134731						
			3	-8.53949	1.326667	0.793374	0.622812	0.559831	0.548687	4.688121
			4	-11.6192	1.793964	1.072399	0.843379	0.760811	7.148644	
	2	7	0	-2.46947	0.220001	0.198251	0.199872	0.222026	0.280561	1.348755
			1	-3.36093	0.345122	0.314918	0.320282	0.358451	2.022157	
			2	-4.48814	0.499256	0.458759	0.469136	3.060991		
		10	0	-2.3293	0.155376	0.131581	0.126299	0.121686	0.131588	0.141512
				0.168971	0.222404	1.129882				
			1	-2.96309	0.213516	0.212391	0.158725	0.202792	0.180793	0.215848
				0.250522	1.528504					
			2	-3.62776	0.297134	0.249521	0.245132	0.241661	0.254961	0.282914
				2.056436						
			3	-4.42664	0.373541	0.330768	0.309272	0.315052	0.330328	2.767681
			4	-5.49345	0.483106	0.423914	0.403443	0.406811	3.776179	

TABLE 5.5: Variances and covariance of the BLUEs

κ	ξ	n	c	$var(\delta^*)$	$var(\varphi^*)$	$cov(\delta^*, \varphi^*)$
0.5	1	7	0	0.000251	0.167441	-0.000853
			1	0.000252	0.284615	-0.001328
			2	0.000255	0.430011	-0.001992
		10	0	0.000071	0.109877	-0.000236
			1	0.000072	0.171915	-0.000337
			2	0.000072	0.236574	-0.000451
			3	0.000072	0.313967	-0.000601
			4	0.000073	0.414729	-0.000817
	2	7	0	0.002146	0.086711	-0.005036
			1	0.002217	0.138541	-0.006951
			2	0.002309	0.201735	-0.009361
		10	0	0.000991	0.054401	-0.002226
			1	0.001006	0.080332	-0.002861
			2	0.001022	0.106462	-0.003519
			3	0.001043	0.137329	-0.004315
			4	0.001071	0.177438	-0.005372

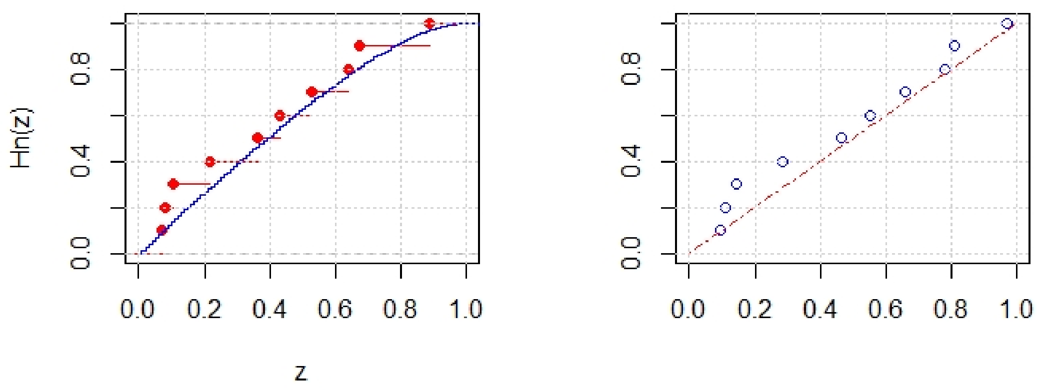

FIGURE 5.1: ECDF-plot and QQ-plot

5.7 Conclusion

Here, we come up with expressions of single and product moments of order statistics from GTL distribution. The BLUEs for the corresponding parameters have been derived using the moments of the order statistics. Also, the performance of the BLUEs is checked by variances and covariances. Finally, for illustration, a real data set is analysed. Based on our results we can claim for well behaviour of moments of order statistics.

Chapter 6

Weibull Marshall-Olkin Lomax Distribution with Applications to Bladder and Head Cancer Data*

6.1 Introduction

This chapter is sketched into the following sections: In section 6.2, we introduce the WMOL distribution and some special cases are presented. We derive two linear representations for the WMOL density which hold for $0 < \alpha < 1$ and $\alpha > 1$ in Sections 6.3. Some mathematical and statistical properties of the WMOL distribution are presented in Section 6.4. Section 6.5 describes the method of maximum likelihood for estimation of the model parameters. A simulation study is investigated in Section 6.6. In Section 6.7, we analyze two real data sets. Finally, in Section 6.8, we offer some concluding remarks.

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The proposal of new families has been worked out by many authors over recent years. Many ways to generate new families have been developed as the methods of addition, linear combination, composition and, one of the newer, the T-X family of distributions. Using this latter method, Korkmaz et al. (2019) proposed a new class called Weibull Marshall-Olkin-G (WMO-G) family. Here, we come up with a distribution, based on the WMO-G family, using the Lomax distribution as baseline, called Weibull Marshall-Olkin Lomax (WMOL) distribution. This distribution can have different shape of hazard rate function, like unimodal, decreasing, increasing, decreasing-increasing-decreasing and bathtub-shaped. Some properties of proposed model are developed. We also find the maximum likelihood estimates of unknown parameters of the WMOL distribution. For the confirmation of asymptotic behaviour of maximum likelihood estimates we provide simulation study and also used two real data sets to check the applicability of model in real life.

Abdul-Moniem and Abdel-Hameed (2012) proposed exponentiated Lomax distribution by generalizing Lomax distributionto analyze failure time data. Also, they proved it may provide better fits than exponential distribution and gave some mathematical properties of the exponentiated Lomax distribution. The statistical literary works contains many extended structures of the lomax distribution. For example, the Exponentiated Weibull-Lomax distribution (Hassan and Abd-Allah (2018)), Kumaraswamy exponentiated Lomax distribution (Elbatal and Kareem (2014)), exponentiated Lomax geometric distribution (Hassan and Abdelghafar (2017)), Weibull-Lomax distribution (Tahir et al. (2015)), Kumaraswamy-generalized Lomax distribution (Shams (2013)), power Lomax distribution (Rady et al. (2016)),transmuted Weibull Lomax distribution (Afify et al. (2015)), Marshall-Olkin power generalized Weibull distribution (Afify et al. (2020a)), Weibull Marshall–Olkin Lindley distribution (Afify et al. (2020b)).

For a baseline G distribution with parameter vector η, Korkmaz et al. (2019) proposed a wider class of continuous distributions called the *Weibull Marshall-Olkin-G* (WMO-G) family. They defined this family based on the T-X generator by choosing $r(t) = \beta t^{\beta-1} e^{-t^{\beta}}$, $t > 0$, where $\beta > 0$ is a shape parameter and $W[G(z;\eta)] = -\log\left[\frac{\alpha\bar{G}(z;\eta)}{G(z;\eta)+\alpha\bar{G}(z;\eta)}\right]$.

The cdf of the WMO-G family is

$$H(z;\alpha,\beta,\eta) = 1-\exp\left(-\left\{-\log\left[\frac{\alpha\bar{G}(z;\eta)}{1-\bar{\alpha}\bar{G}(z;\eta)}\right]\right\}^{\beta}\right). \tag{6.1}$$

The pdf corresponding to (6.1) is

$$\begin{aligned} h(z;\alpha,\beta,\eta) &= \frac{\beta g(z;\eta)}{\bar{G}(z;\eta)\left[1-\bar{\alpha}\bar{G}(z;\eta)\right]}\left\{-\log\left[\frac{\alpha\bar{G}(z;\eta)}{1-\bar{\alpha}\bar{G}(z;\eta)}\right]\right\}^{\beta-1} \\ &\quad \times\exp\left(-\left\{-\log\left[\frac{\alpha\bar{G}(z;\eta)}{1-\bar{\alpha}\bar{G}(z;\eta)}\right]\right\}^{\beta}\right), \end{aligned} \tag{6.2}$$

where $g(z;\eta)$ is the baseline PDF, $\bar{\alpha} = 1-\alpha$, and α and β are two extra positive shape parameters.

The hazard rate function (HRF) of the WMO-G family takes the form

$$\tau(z;\alpha,\beta,\eta) = \frac{\beta\, w(z;\eta)}{\left[1-\bar{\alpha}\bar{G}(z;\eta)\right]}\left\{-\log\left[\frac{\alpha\bar{G}(z;\eta)}{1-\bar{\alpha}\bar{G}(z;\eta)}\right]\right\}^{\beta-1},$$

where $w(z;\eta) = g(z;\eta)/\bar{G}(z;\eta)$ is the baseline HRF.

For $\alpha = 1$, we obtain the Weibull-X family (Alzaatreh et al. (2013); Cordeiro et al. (2015)) as a special case of the WMO-G family. For $\beta = 1$, we obtain the MO-G family (Marshall and Olkin (1997)). For $\alpha = \beta = 1$, we have the baseline distribution. Further details on the WMO-G family can be explored in Korkmaz et al. (2019).

6.2 Proposed Model

We propose *Weibull Marshall-Olkin Lomax* (WMOL) distribution with four parameters by setting the Lomax cdf $G(z;\lambda,\theta) = 1-(1+\lambda z)^{-\theta}$ in (6.1), we obtain

$$H(z) = 1-\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}}\right]\right)^{\beta}\right\}. \tag{6.3}$$

The associated pdf to (6.3) is

$$
\begin{aligned}
h(z) &= \frac{\beta\theta\lambda}{(1+\lambda z)[1-\bar{\alpha}(1+\lambda z)^{-\theta}]}\left(-log\left[\frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}}\right]\right)^{\beta-1} \\
&\times \exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}}\right]\right)^{\beta}\right\}, \qquad (6.4)
\end{aligned}
$$

where $\bar{\alpha} = 1-\alpha$, $\theta > 0$, $\beta > 0$ are shape and $\alpha > 0$, $\lambda > 0$ are scale parameters.

The HRF of Z is

$$
\tau(z) = \frac{\beta\theta\lambda}{(1+\lambda z)[1-\bar{\alpha}(1+\lambda z)^{-\theta}]}\left(-log\left[\frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}}\right]\right)^{\beta-1}.
$$

where $w(z;\eta) = g(z;\eta)/\bar{G}(z;\eta)$ is the baseline HRF.

It is observed that the density function of the new model provides a wide range of shapes based on its additional shape parameter, for example a monotonically decreasing density of exponentiated Lomax (EL) will become monotonically decreasing, decreasing, symmetric, reversed J, right-skewed and left-skewed. The WMOL distribution have decreasing, increasing-decreasing, increasing, constant and upside down bathtub shaped hazard function based on its additional parameter and can be used to provide a good fit for the real data than well-known distributions (see Figure 6.1).

Some mathematical properties of WMOL model can directly obtained from Lehmann type II (LTII) exponentiated Lomax model properties because it can be expressed as the linear combination of LTIIEL and EL densities. Additionally, the new model contains some distributions as special cases, these sub-models being listed in Table 6.1.

TABLE 6.1: Special cases of the WMOL distribution

Parametric values in Eq. (4)	Sub-models
$\beta = 1$	Marshall-Olkin Lomax distribution(α,θ,λ)
$\alpha = 1$	Weibull Lomax distribution(β,θ,λ)
$\alpha = \beta = 1$	Lomax distribution (θ)

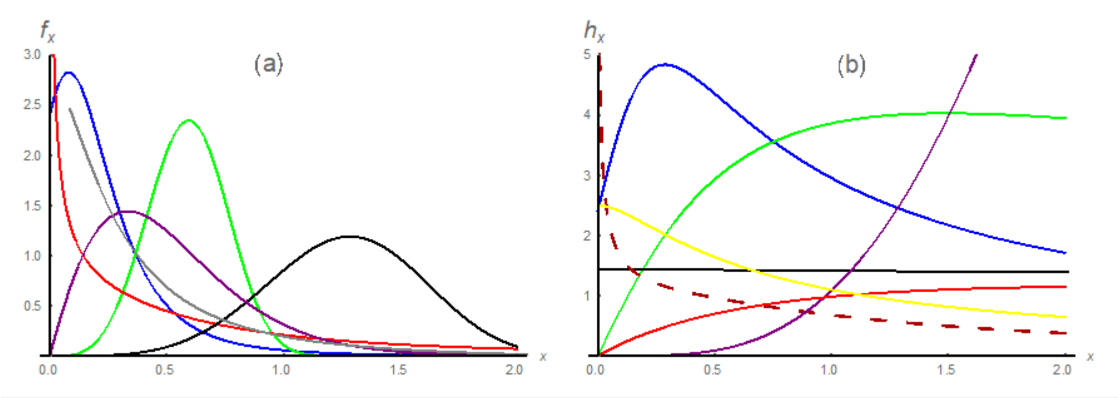

FIGURE 6.1: Plots of the WMOL density and hazard functions. (a) ($\theta = 1.5, \beta = 1, \alpha = 3, \lambda = 5$) (gray), ($\theta = 2, \beta = 5, \alpha = 1, \lambda = 1$) (green), ($\theta = 2, \beta = 5, \alpha = 2, \lambda = 0.8$) (black), ($\theta = 3, \beta = 2, \alpha = 1, \lambda = 0.7$) (purple), ($\theta = 4, \beta = 0.5, \alpha = 5, \lambda = 1$) (red), ($\theta = 4, \beta = 1, \alpha = 5, \lambda = 3$) (blue) (b) ($\theta = 1, \beta = 1, \alpha = 5, \lambda = 1$) (black), ($\theta = 1.5, \beta = 1, \alpha = 3, \lambda = 5$) (yellow), ($\theta = 1.5, \beta = 2, \alpha = 1, \lambda = 0.7$) (red), ($\theta = 2, \beta = 2, \alpha = 2, \lambda = 2$) (green), ($\theta = 2, \beta = 5, \alpha = 2, \lambda = 0.8$) (purple), ($\theta = 4, \beta = 0.5, \alpha = 5, \lambda = 1$) (dashes-red), ($\theta = 4, \beta = 1, \alpha = 5, \lambda = 3$) (blue).

6.3 Linear Representation

In this section, we provide two linear representations for the WMOL density depending on α. By using the power series

$$e^z = \sum_{i=0}^{\infty} \frac{z^i}{i!},$$

the cdf in (6.3) can be expressed as

$$H(z) = \sum_{h=1}^{\infty} \frac{(-1)^h}{\Gamma(h+1)} \left(-log \left[1 - \left(1 - \frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}} \right) \right] \right)^{h\beta}. \tag{6.5}$$

For a real number d and $z \in (0,1)$, we have

$$[-\log(1-z)]^d = z^d + \sum_{i=0}^{\infty} \psi_i(d)\, z^{i+d+1}, \tag{6.6}$$

where

$$\begin{aligned}\psi_0(d) &= \frac{1}{2}d,\ \ \psi_1(d)=\frac{1}{24}[d(3d+5)],\ \ \psi_2(d)=\frac{1}{48}[d(d^2+5d+6)],\\ \psi_3(d) &= \frac{1}{5760}[d(15d^3+150d^2+485d+502)],\ldots,\end{aligned}$$

are Stirling polynomials. The proof is given in Theorem 3A of Flajolet and Odlyzko (1990) and in Theorem VI.2 of Flajolet and Sedgewick (2009). The previous results have been used by Cordeiro et al. (2017a). We can write

$$[-\log(1-z)]^{h\beta}=\sum_{i=0}^{\infty}\psi_{i-1}(h\beta)\,z^{i+h\beta}, \tag{6.7}$$

where $\psi_{-1}(h\beta)=0$ by convention and $\psi_i(h\beta)$ for $i\geq 0$ can be obtained from (6.6). Then, the cdf (6.5) can be expressed using (6.7) as

$$H(z)=\sum_{h=1}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^h}{\Gamma(h+1)}\psi_{i-1}(h\beta)\left[1-\frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}}\right]^{i+h\beta}.$$

For a real non-integer d and $|z|<1$, we have

$$(1-z)^d=\sum_{k=0}^{\infty}(-1)^k\binom{d}{k}z^k.$$

Hence, we can write

$$\begin{aligned}H(z) &= \sum_{h=1}^{\infty}\sum_{i,k=0}^{\infty}\frac{(-1)^{h+k}\alpha^k}{\Gamma(h+1)}\psi_{i-1}(h\beta)\binom{i+h\beta}{k}(1+\lambda z)^{-\theta k}\\ &\times\left[1-\bar{\alpha}(1+\lambda z)^{-\theta}\right]^{-k}.\end{aligned} \tag{6.8}$$

For a positive integer ϑ and $|z|<1$, a convergent power series can be expressed as

$$(1-z)^{-\vartheta}=\sum_{l=0}^{\infty}(-1)^l\binom{-\vartheta}{l}z^l,$$

For $\alpha \in (0,1)$, $H(z)$ can be written as

$$H(z) \;=\; \sum_{k,l=0}^{\infty} \upsilon_{k,l}\, \bar{G}(z;\theta,\lambda)^{k+l}, \tag{6.9}$$

where

$$\upsilon_{k,l} = \sum_{h=1}^{\infty}\sum_{i,k=0}^{\infty} \frac{(-1)^{i+k+l}\,\alpha^{k}\;\phi_{i-1}(h\beta)}{\Gamma(h+1)\;(1-\alpha)^{-l}} \binom{h\beta+i}{k}\binom{-k}{l}$$

and $\bar{G}(z;\theta,\lambda) = 1 - G(z;\theta,\lambda)$, is the exponentiated Lomax survival function.

For a baseline $G(z)$ and power parameter e, $\Pi_e(z) = 1 - \{1 - G(z)\}^e$ Lehmann (1953) is known as LTII cdf. Thus, the LTII density is given by $\pi_e(z) = e\,\bar{G}(z)^{e-1}\,g(z)$, where $g(z) = dG(z)/dz$.

Let $J = \{(k,l); k,l = 0,1,2,\ldots; k+l \geq 1\}$ be a set of non-negative integers. By differentiating the last equation for $H(z)$, the pdf of Z is

$$f(z) = \sum_{(k,l)\in J} \upsilon_{k,l}\,\pi_{k+l}(z;\theta,\lambda), \tag{6.10}$$

where $\pi_{k+l}(z) = (k+l)\,\bar{G}(z;\theta,\lambda)^{k+l-1}\,g(z;a)$ is known as LTII exponentiated Lomax density function with power parameter $k+l$.

If $\alpha > 1$, (6.8) can be written as

$$\begin{aligned} H(z) \;&=\; \sum_{h=1}^{\infty}\sum_{i,k=0}^{\infty} \frac{(-1)^{h+k}\alpha^{k}}{\Gamma(h+1)}\phi_{i-1}(h\beta)\binom{i+h\beta}{k}[(1+\lambda z)^{-\theta}]^{k} \\ &\times\; \alpha^{-k}\left[1-(1-\alpha^{-1})(1+\lambda z)^{-\theta}\right]^{-k}. \end{aligned}$$

Using series expansion, we have

$$H(z) \;=\; \sum_{k,l=0}^{\infty} \nu_{k,l}\,\bar{G}(z;\theta,\lambda)^{k+l},$$

where,

$$\nu_{k,l} = \sum_{h=1}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{i+k+l} \quad \phi_{i-1}(h\beta)}{\Gamma(h+1)\,(1-\alpha)^{-l}} \binom{h\beta+i}{k}\binom{-k}{l}$$

So, pdf of Z is

$$h(z) = \sum_{(k,l)\in J} \nu_{k,l}\, \pi_{k+l}(z;\theta,\lambda), \tag{6.11}$$

From (6.10) and (6.11), we find that WMOL density function can be expressed as linear combination of EL density function and LTII exponentiated Lomax densities for both cases.

Every LTII Lindley can be a linear combination of EL densities. By expanding $\Pi_e(z) = 1 - \{1-G(z)\}^e$ (for e real), the power series converges everywhere

$$\Pi_e(z) = \sum_{r=1}^{\infty}(-1)^{r+1}\binom{e}{r} G(z)^r.$$

Differentiating last equation, we get

$$\pi_e(z) = \sum_{r=0}^{\infty}(-1)^{r}\binom{e}{r+1}\rho_{r+1}(z), \tag{6.12}$$

where $\rho_{r+1}(z) = (r+1)\,G(z)^r\,g(z)$ represents EL density with $r+1$ as power parameter. Sum lasts at e, if e is a positive integer.

6.4 Properties of the WMOL Distribution

This section deals with statistical properties of WMOL distribution. We will use weights $\upsilon_{k,l}$ and $\nu_{k,l}$ according to $0<\alpha<1$ and $\alpha>1$, respectively, to derive properties.

6.4.1 Quantiles Function

Quantiles are fruitful in estimation and simulation. The root of the equation given below will give the pth quantile for WMOL distribution.

$$\xi_p = \frac{1}{\lambda}\left[\left(\frac{\alpha-\bar{\alpha}\, e^{-(-log(1-p))^{\frac{1}{\beta}}}}{e^{-(-log(1-p))^{\frac{1}{\beta}}}}\right)^{\frac{1}{\theta}}-1\right],\ 0<p<1,\ \lambda,\ \theta>0. \tag{6.13}$$

A random sample of size n can be generated with the help of uniform distribution and equation (6.13) for WMOL distribution as follows

$$\xi_i = \frac{1}{\lambda}\left[\left(\frac{\alpha-\bar{\alpha}\, e^{-(-log(1-u_i))^{\frac{1}{\beta}}}}{e^{-(-log(1-u_i))^{\frac{1}{\beta}}}}\right)^{\frac{1}{\theta}}-1\right].$$

In particular, the first three quantiles, Q_1, Q_2 and Q_3, can be derived for specific values of p.

6.4.2 Moments and Generating Functions

Moments tell us about important features and characteristics of a distribution. Here, we derive raw moments and moment generating function (MGF) of WMOL distribution.

Now, the nth raw moment of the WMOL can be written as

$$\begin{aligned}
\mu_n^{'} &= E[Z^n]=\int_0^\infty z^n h(z)dz=\sum_{(k,l)\in J}\sum_{r=0}^{k+l}(-1)^r\binom{k+l}{r+1}\upsilon_{k,l}\lambda\theta(r+1)\\
&\times \int_0^\infty z^n[1-(1+\lambda z)^{-\theta}]^{(r+1)-1}(1+\lambda z)^{-\theta-1}dz\\
&= \frac{1}{\lambda^n}\sum_{(k,l)\in J}\sum_{r=0}^{k+l}\sum_{m=0}^{n}(-1)^{r+m}\binom{k+l}{r+1}\binom{n}{m}\upsilon_{k,l}(r+1)\\
&\times B\left(1-\frac{1}{\theta}(n-m),r+1\right),
\end{aligned} \tag{6.14}$$

where, $B(a,\ b)=\int_0^1 z^{a-1}(1-z)^{b-1}dz$.

The nth central moments μ_n and cumulants k_n of Z can be determined from (6.14) as

$$\mu_n = E(Z-\mu)^n = \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} \mu_1'^n \mu_{n-k}',$$

and

$$k_n = \mu_n' - \sum_{k=1}^{n-1} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} k_n \mu_{n-k}',$$

where $k_1 = \mu_1'$. Cumulants are useful to calculate moments, skewness and kurtosis. The MGF of Z easily follows from (6.10) as

$$\begin{aligned} M(t) &= \frac{1}{\lambda^p} \sum_{(k,l)\in J} \sum_{r=0}^{k+l} \sum_{m=0}^{n} \sum_{p=0}^{\infty} \frac{t^p (-1)^{r+m}}{p!} \binom{k+l}{r+1} \binom{p}{m} \upsilon_{k,l}(r+1) \\ &\times B\left(1-\frac{1}{\theta}(p-m), r+1\right). \end{aligned}$$

6.4.3 Conditional Moments, Mean Residual Life and Mean Deviations

Conditional moments, $E(Z^n|Z>z)$, of WMOL distribution can be derived as

$$E(Z^n|Z>z) = \frac{1}{S(z)} J_n(z)$$

where,

$$\begin{aligned} J_n(z) &= \int_z^{\infty} y^n f(y)dy = \frac{1}{\lambda^n} \sum_{(k,l)\in J} \sum_{r=0}^{k+l} \sum_{m=0}^{n} \sum_{p=0}^{\infty} (-1)^{r+m} \binom{k+l}{r+1} \binom{n}{m} \\ &\times \upsilon_{k,l}(r+1) \frac{(1-\{r+1\})_p \, (1+\lambda z)^{n-m-\theta(p+1)}}{p! \left[\frac{1}{\theta}(m-n)+p+1\right]}, \end{aligned} \quad (6.15)$$

where, $S(z) = 1 - H(z)$, defined in (6.3).

Conditional moments are helpful in deriving mean residual life (MRL). MRL is expected residual life of an item with the condition that it has survived for time z. Using the conditional moment, the MRL function can be expressed as

$$m_Z(z) = E(Z - z|Z > z) = \frac{1}{S(z)} J_1(z) - z.$$

where, $J_1(z)$ can be derived from (6.15) where $n = 1$.

Also, conditional moments can be used to derive the mean deviation about mean and median. Let M and μ represents median and mean, then mean deviations can be expressed as

$$\begin{aligned} \delta_\mu &= \int_0^\infty |z-\mu|\, h(z)dz = 2\mu H(\mu) - 2\mu + 2J_1(\mu) \\ \delta_M &= \int_0^\infty |z-M|\, h(z)dz = 2J_1(M) - \mu \end{aligned}$$

respectively. Where $J_1(\mu)$ and $J_1(M)$ are derived from (6.15). Also, $H(\mu)$ and $H(M)$ are calculated from (6.3).

6.4.4 Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves evaluate disparity of the distribution of a random variable and they are applicable in economics, reliability, medical and demography, among other areas. For a probability p, these curves are given by

$$B(p) = \frac{1}{p\mu_1'} \int_0^q z h(z)dz \quad \text{and} \quad L(p) = pB(p),$$

respectively, where $\mu_1' = E(Z)$ and $q = F^{-1}(p)$.

Bonferroni and Lorenz curves for the WMOL distribution can be expressed as

$$\begin{aligned} B(p) &= \frac{1}{p} - \frac{1}{\lambda p \mu_1'} \sum_{(k,l)\in J} \sum_{r=0}^{k+l} \sum_{m=0}^{n} \sum_{s=0}^{\infty} (-1)^{r+m} \binom{k+l}{r+1} \binom{n}{m} \upsilon_{k,l}(r+1) \\ &\times \frac{(1-\{r+1\})_s \, (1+\lambda q)^{1-m-\theta(s+1)}}{s!\left[\frac{1}{\theta}(m-1)+s+1\right]} \end{aligned}$$

and $L(p) = pB(p)$, respectively.

6.4.5 Residuals Life Function

Let Z follows the pdf $h(z)$ given by (6.4). The conditional random variable $R_{(t)} = Z - t|Z > t$, $t \geq 0$ describes the residual life. Using (6.3), the survival function of residual lifetime $R_{(t)}$ is given by

$$S_{R_{(t)}}(z) = \frac{S(z+t)}{S(t)} = \frac{\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda(z+t))^{-\theta}}{1-\bar{\alpha}(1+\lambda(z+t))^{-\theta}}\right]\right)^{\beta}\right\}}{\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda t)^{-\theta}}{1-\bar{\alpha}(1+\lambda t)^{-\theta}}\right]\right)^{\beta}\right\}}, \quad z > 0.$$

The associated cdf is given by

$$H_{R_{(t)}}(z) = \frac{\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda t)^{-\theta}}{1-\bar{\alpha}(1+\lambda t)^{-\theta}}\right]\right)^{\beta}\right\} - \exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda(z+t))^{-\theta}}{1-\bar{\alpha}(1+\lambda(z+t))^{-\theta}}\right]\right)^{\beta}\right\}}{\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda t)^{-\theta}}{1-\bar{\alpha}(1+\lambda t)^{-\theta}}\right]\right)^{\beta}\right\}}.$$

Then, the associated pdf is given by

$$\begin{aligned} h_{R_{(t)}}(z) &= \beta\,\theta\,\lambda \frac{\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda(z+t))^{-\theta}}{1-\bar{\alpha}(1+\lambda(z+t))^{-\theta}}\right]\right)^{\beta}\right\}}{\exp\left\{-\left(-\log\left[\frac{\alpha(1+\lambda t)^{-\theta}}{1-\bar{\alpha}(1+\lambda t)^{-\theta}}\right]\right)^{\beta}\right\}} \\ &\times \frac{\left(-\log\left[\frac{\alpha(1+\lambda(z+t))^{-\theta}}{1-\bar{\alpha}(1+\lambda(z+t))^{-\theta}}\right]\right)^{\beta-1}}{[1-\bar{\alpha}(1+\lambda(z+t))^{-\theta}][1+\lambda(z+t)]}, \quad z > 0. \end{aligned}$$

The associated hazard rate function is given by

$$\tau_{R_{(t)}}(z) \quad = \quad \frac{\beta\theta\lambda\left(-\log\left[\frac{\alpha(1+\lambda(z+t))^{-\theta}}{1-\bar{\alpha}(1+\lambda(z+t))^{-\theta}}\right]\right)^{\beta-1}}{[1-\bar{\alpha}(1+\lambda(z+t))^{-\theta}][1+\lambda(z+t)]},\ z>0.$$

The nth moments of residual life of Z, $m_n(t) = E[(Z-t)|Z>t]$ for $n = 1,2,\ldots,$ uniquely determines $H(z)$, we have

$$\begin{aligned} m_n(t) &= E(R_{(t)}) = E[(Z-t)^n|Z>t] = \frac{1}{S(t)}\int_t^\infty z^n dH(z) - t \\ &= \frac{1}{S(t)}\left(E(Z^n) - \int_0^t z^n dH(z)\right) - t. \end{aligned} \tag{6.16}$$

On the other hand, the variance residual life is given by

$$\begin{aligned} V(t) &= Var(R_{(t)}) = Var[Z-t|Z>t] = \frac{2}{S(t)}\int_t^\infty zS(z)dz - 2tm_1(t) - [m_1(t)]^2 \\ &= \frac{1}{S(t)}\left(E(Z^2) - \int_0^t z^2 h(z)dz\right) - t^2 - 2tm_1(t) - [m_1(t)]^2. \end{aligned}$$

The conditional random variable $\bar{R}_{(t)} = t - Z|Z \le t$, $t \ge 0$ describes reverse residual life. Using the cdf (6.3), the survival function of the reverse residual lifetime $\bar{R}_{(t)}$ is given by

$$S_{\bar{R}_{(t)}}(z) = \frac{1-\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda(t-z))^{-\theta}}{1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}}\right]\right)^{\beta}\right\}}{1-\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda t)^{-\theta}}{1-\bar{\alpha}(1+\lambda t)^{-\theta}}\right]\right)^{\beta}\right\}},\ \ 0 \le z \le t.$$

The associated cdf is given by

$$H_{\bar{R}_{(t)}}(z) = \frac{\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda(t-z))^{-\theta}}{1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}}\right]\right)^{\beta}\right\} - \exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda t)^{-\theta}}{1-\bar{\alpha}(1+\lambda t)^{-\theta}}\right]\right)^{\beta}\right\}}{1-\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda t)^{-\theta}}{1-\bar{\alpha}(1+\lambda t)^{-\theta}}\right]\right)^{\beta}\right\}}.$$

Therefore, the associated pdf is given by

$$\begin{aligned} h_{\bar{R}_{(t)}}(z) &= \beta\,\theta\,\lambda\,\frac{\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda(t-z))^{-\theta}}{1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}}\right]\right)^{\beta}\right\}}{1-\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda t)^{-\theta}}{1-\bar{\alpha}(1+\lambda t)^{-\theta}}\right]\right)^{\beta}\right\}} \\ &\times\ \frac{\left(-log\left[\frac{\alpha(1+\lambda(t-z))^{-\theta}}{1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}}\right]\right)^{\beta-1}}{[1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}][1+\lambda(t-z)]}. \end{aligned}$$

The associated hazard rate is given by

$$\begin{aligned} \tau_{\bar{R}_{(t)}}(z) &= \beta\,\theta\,\lambda\,\frac{\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda(t-z))^{-\theta}}{1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}}\right]\right)^{\beta}\right\}}{1-\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda(t-z))^{-\theta}}{1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}}\right]\right)^{\beta}\right\}} \\ &\times\ \frac{\left(-log\left[\frac{\alpha(1+\lambda(t-z))^{-\theta}}{1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}}\right]\right)^{\beta-1}}{[1-\bar{\alpha}(1+\lambda(t-z))^{-\theta}][1+\lambda(t-z)]}. \end{aligned}$$

In the similar manner, Navarro et al. (1998) prove that the nth moment of the reversed residual life, say $M_n(t) = E[(t-Z)^n|Z \le t]$ for $t > 0$ and $n = 1, 2, \ldots$, uniquely determines $H(z)$. We obtain

$$M_n(t) = \frac{1}{H(t)}\int_0^t (t-z)^n dH(z).$$

The mean reversed residual life is define as

$$M(t) = E(\bar{R}_{(t)}) = E(t-z|z \le t) = t - \frac{1}{H(t)}\int_0^t zh(z)dz,$$

The variance reversed residual life can be derived as

$$\begin{aligned} w(t) &= Var(\bar{R}_{(t)}) = Var(t-z|z \le t) = 2tM(t) - (M(t))^2 - \frac{2}{H(t)}\int_0^t zH(z)dz \\ &= 2tM(t) - (M(t))^2 - t^2 + \frac{1}{H(t)}\int_0^t z^2 h(z)dz. \end{aligned}$$

6.5 Maximum Likelihood Estimation

Let a random sample $z_1,\ldots,z_n$ of size n be selected from WMOL distribution. Using (6.4), log-likelihood function can be written as

$$\begin{aligned}\ell \quad &\propto \quad n\log(\beta)+n\log(\theta)+n\log(\lambda)-\sum_{i=1}^{n}\log(1+\lambda z_i)-\sum_{i=1}^{n}\log[1-\bar{\alpha}(1+\lambda z_i)^{-\theta}]\\ &- \quad \sum_{i=1}^{n}\left\{-\log\left[\frac{\alpha(1+\lambda z_i)^{-\theta}}{1-\bar{\alpha}(1+\lambda z_i)^{-\theta}}\right]\right\}^{\beta}+(\beta-1)\sum_{i=1}^{n}\log\left\{-\log\left[\frac{\alpha(1+\lambda z_i)^{-\theta}}{1-\bar{\alpha}(1+\lambda z_i)^{-\theta}}\right]\right\}.\end{aligned}$$

The MLEs of α, β, λ and θ, denoted by $\hat{\alpha}_{MLE}$, $\hat{\beta}_{MLE}$, $\hat{\lambda}_{MLE}$ and $\hat{\theta}_{MLE}$, can be obtained numerically by maximizing the log-likelihood function ℓ or by solving the nonlinear equations:

$$\begin{aligned}\frac{\partial\ell}{\partial\alpha} \quad &= \quad -\sum_{i=1}^{n}\frac{(1+\lambda z_i)^{-\theta}}{1-\bar{\alpha}(1+\lambda z_i)^{-\theta}}+\sum_{i=1}^{n}\left[\beta(-\log(\xi_i))^{\beta-1}+\frac{(\beta-1)}{\log(\xi_i)}\right]\\ &\times \quad \left(\frac{1}{\alpha}-\frac{(1+\lambda z_i)^{-\theta}}{1-\bar{\alpha}(1+\lambda z_i)^{-\theta}}\right)=0,\\ \frac{\partial\ell}{\partial\beta} \quad &= \quad \frac{n}{\beta}+\sum_{i=1}^{n}\log(-\log(\xi_i))[1-(-\log(\xi_i))^{\beta}]=0,\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\lambda} \quad &= \quad \frac{n}{\lambda}-\sum_{i=1}^{n}\left[\beta(-\log(\xi_i))^{\beta-1}+\frac{(\beta-1)}{\log(\xi_i)}\right]\frac{\theta z_i}{(1+\lambda z_i)[1-\bar{\alpha}(1+\lambda z_i)^{-\theta}]}\\ &- \quad \sum_{i=1}^{n}\frac{z_i}{(1+\lambda z_i)}-\sum_{i=1}^{n}\frac{\xi_i\theta z_i}{(1+\lambda z_i)}=0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\ell}{\partial\theta} \quad &= \quad \frac{n}{\theta}-\sum_{i=1}^{n}\left[\beta(-\log(\xi_i))^{\beta-1}+\frac{(\beta-1)}{\log(\xi_i)}\right]\frac{\log(1+\lambda z_i)}{[1-\bar{\alpha}(1+\lambda z_i)^{-\theta}]}\\ &- \quad \sum_{i=1}^{n}\xi_i\log(1+\lambda z_i)=0,\end{aligned}$$

where,

$$\xi_i=\xi(\alpha,a;z_i)=\frac{\alpha(1+\lambda z_i)^{-\theta}}{1-\bar{\alpha}(1+\lambda z_i)^{-\theta}}. \tag{6.17}$$

6.6 Simulation Study

This section deals with simulation study to verify asymptotic properties of MLE's. We perform a Monte Carlo simulation, with 1000 replications and using the R software. To do this, we choose some scenarios:

- 1. $(\alpha, \beta, \lambda, \theta) = (1, 2, 1, 2)$, $n = 50, 100, 150$ and uncensored;
- 2. Some parameter value, $n = 50, 100, 150$ with 10% censorship;
- 3. Some parameter value, $n = 50, 100, 150$ with 20% censorship;

and we calculate the average estimates (AEs) of the MLEs and the mean squared errors (MSEs), for each parameter point.

The results are present in Table 6.2 and indicates that the AEs become closer to the true parameter values and MSEs of MLEs of the model parameters approach to zero when n increases.

TABLE 6.2: Simulation study

		0% censured		10% censured		20% censured	
n	Parameter	AE	MSE	AE	MSE	AE	MSE
	α	1.0019	0.0143	1.1523	0.0429	1.1706	0.0503
50	β	2.0346	0.0500	1.9525	0.0562	1.9421	0.0541
	λ	1.0103	0.0093	0.9013	0.0171	0.8904	0.0198
	θ	2.0173	0.0209	1.8489	0.0400	1.8315	0.0467
	α	1.0000	0.0065	1.1210	0.0231	1.1757	0.0413
100	β	2.0129	0.0226	1.9591	0.0278	1.9223	0.0308
	λ	1.0050	0.0041	0.9164	0.0105	0.8818	0.0176
	θ	2.0079	0.0093	0.9164	0.0105	1.8174	0.0419
	α	1.0003	0.0048	1.0881	0.0141	1.2100	0.0513
150	β	2.0153	0.0155	1.9625	0.0161	1.9054	0.0265
	λ	1.0039	0.0030	0.9374	0.0068	0.8597	0.0219
	θ	2.0068	0.0068	1.9049	0.0156	1.7830	0.0524

6.7 Real Data Application

By making use of two practical data sets, we illustrate the applicability of the WMOL distribution among a set of classical and recent models containing beta exponentiated Lomax, transmuted Weibull Lomax, exponentiated Weibull Lomax, beta Marshall-Olkin Lomax, Gompertz Lomax and Kumaraswamy generalized Lomax , based on a set of goodness-of-fit statistics. ML method is used to estimate model parameters and compared with the help of K-S statistic, p-value, Cramer-von Mises (W^*) and Anderson Darling (A^*). Generally larger p-value and smaller values of these statistics indicates a better fit to data.

TABLE 6.3: Some competitive models to the WMOL distribution.

Distribution	Author(s)
Beta exponentiated Lomax (BEL)	Mead (2016)
Transmuted Weibull Lomax (TWL)	Afify et al. (2015)
exponentiated Weibull-Lomax (EWL)	Hassan and Abd-Allah (2018)
Beta Marshall-Olkin Lomax (BMOL)	Tablada and Cordeiro (2019)
Gompertz-Lomax (GL)	Oguntunde et al. (2017)
Kumaraswamy generalized Lomax (KGL)	Shams (2013)

Description of the data is as follows: The bladder cancer patient's data: The first data set is remission time (in months) of a group of 128 bladder cancer patients taken from Lee and Wang (2003).

```
0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23,
3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09,
9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24,
25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81,
2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32,
7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66,
15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01,
1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33,
5.49, 7.66, 11.25, 2.07, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87,
11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46,
```

4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 3.36, 6.93, 8.65, 12.63, 22.69.

Survival times of patients treated using RT: The second real data represents the survival time of head cancer patients, who treated using radiotherapy (RT). The data were initially reported by Efron (1988). These data consists of 58 observations:

6.53, 7, 10.42, 14.48, 16.1, 22.7, 34, 41.55, 42, 45.28, 49.4, 53.62, 63, 64, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 146, 149, 154, 157, 160, 160, 165, 173, 176, 218, 225, 241, 248, 273, 277, 297, 405, 417, 420, 440, 523, 583, 594, 1101, 1146, 1417.

Table 6.4 gives some descriptive statistics for both data sets and using it we note that the two data sets have positive skewness and kurtosis.

On the other side, comparing the WMOL distribution with other classical and recent distributions is done as follows. For the two data sets, ML method is used to estimate the parameters of models and by these estimates, we provide the statistics K-S, p-value, W^* and A^*. The obtained results are reported in Tables 6.5-6.8. From these tables, the smallest values of the K-S, W^*,A^*, and the largest p-value is obtained for the WMOL distribution. Hence, we infer that WMOL distribution provides the best fit among the compared distributions.

TABLE 6.4: Descriptive statistics of both data sets (MD:= Mean deviation, Kr:= kurtosis, SK:= skewness, SE:= Shannon entropy).

Data	Mean	Median	SD	SK	Kr	MD-mean	MD-median	SE
First data	9.36561	6.395	10.5081	3.2737	15.338	6.72060	6.12812	2.083
Second data	226.174	151.5	273.943	2.6999	7.5399	172.048	145.068	1.649

Figure 6.2 shows the TTT plot (see Aarset (1987)) for both data sets. Note that the TTT plot for the first data set indicates a bathtub hazard rate function, while the second one indicates increasing-decreasing-increasing-decreasing hazard.

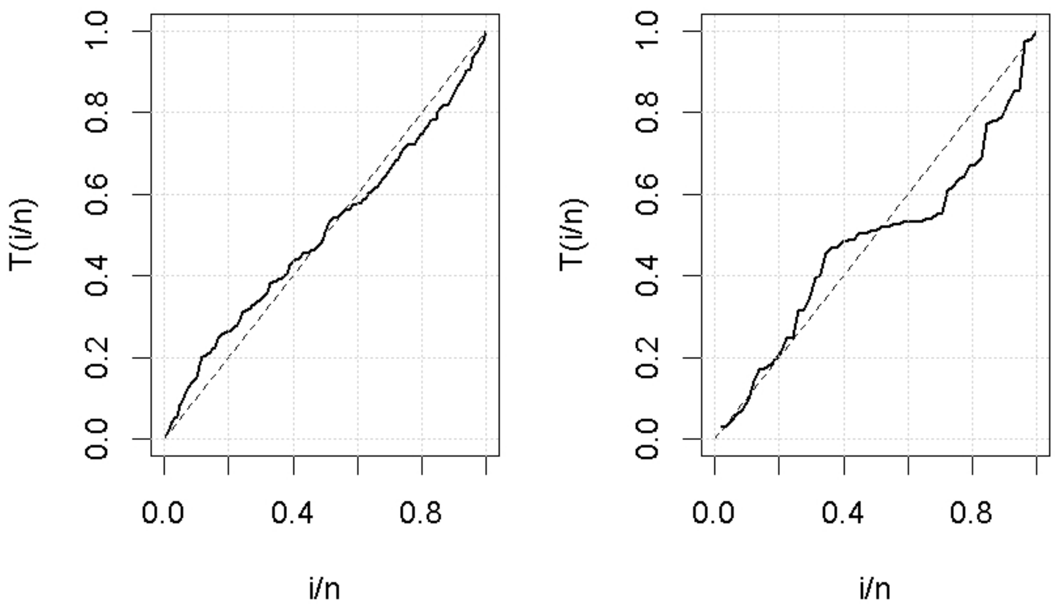

FIGURE 6.2: TTT plot for both data sets.

TABLE 6.5: The MLEs of the parameters of some models fitted to the bladder cancer patient's data.

Distribution		Estimates			
BEL(a,b,λ,θ,β)	0.71556	5.18863	0.08855	0.80847	2.10461
TWL(α,β,λ,a,b)	0.19165	9.60751	0.68149	12.7477	1.48715
EWL(a,α,β,θ,λ)	57.9147	1.20253	1.27849	0.07811	11.0033
BMOL(a,b,c,α)	1.06537	1.43179	46.2985	1.74832	-
GL(θ,γ,α,β)	1.22647	1.04043	0.87032	0.10565	-
KGL(a,b,α,λ)	1.51371	23.9726	0.22322	11.1227	-
WMOL(α,β,λ,θ)	15.6523	1.13049	0.55624	1.90448	-

The empirical and fitted densities are demonstrated in figure 6.3 for this data set. We are comparing only two models WMOL and BEL because of the smallest values of the statistics and goodness of fit measures and according to figure WMOL distribution fits better.

The empirical and fitted densities are demonstrated in figure 6.4 for this data set. We are comparing only two models WMOL and BEL because of the smallest values of the statistics and goodness of fit measures and according to figure WMOL distribution fits better.

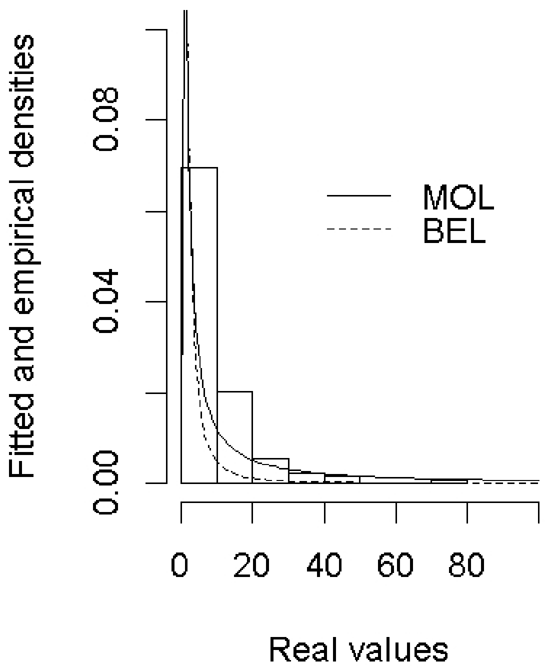

FIGURE 6.3: Fitted and empirical densities for the first data set

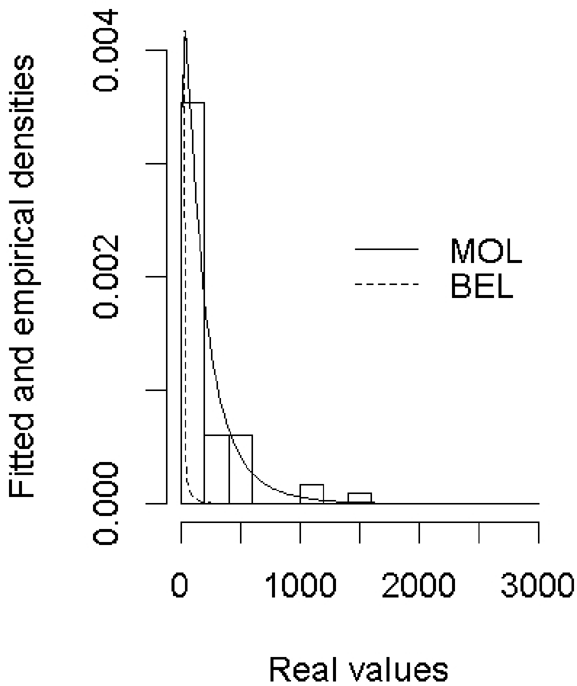

FIGURE 6.4: Fitted and empirical densities for the second data set

TABLE 6.6: The values of K-S, p- value, W^* and A^* statistics for some models fitted to the bladder cancer patient's data.

Distribution	K-S	p-value	(W^*)	(A^*)
BEL(a,b,λ,θ,β)	0.03911	0.88959	0.02330	0.1594
TWL(α,β,λ,a,b)	0.03827	0.87194	0.02160	0.1494
EWL(a,α,β,θ,λ)	0.03990	0.88694	0.02566	0.1755
BMOL(a,b,c,α)	0.02965	0.86980	0.01456	0.0921
GL(θ,γ,α,β)	0.09264	0.22180	0.20280	1.3288
KGL(a,b,α,λ)	0.16266	0.14182	0.02380	0.1635
WMOL(α,β,λ,θ)	0.02922	0.99990	0.01410	0.0903

TABLE 6.7: The MLEs of the parameters of some models fitted to the survival times of patients treated using RT data.

Distribution		Estimates			
BEL(a,b,λ,θ,β)	0.73642	15.2639	0.00504	0.29914	1.86758
TWL(α,β,λ,a,b)	0.64257	2.57857	-1.0000	0.09276	1.02731
EWL(a,α,β,θ,λ)	3.40069	0.90215	3.07405	0.13213	4.63291
BMOL(a,b,c,α)	4.25699	10.5517	23.9284	0.47569	-
GL(θ,γ,α,β)	14.7599	1.009×10^{-6}	0.45312	0.00078	-
WMOL(α,β,λ,θ)	74.9829	0.61133	0.00545	6.88208	-

TABLE 6.8: The values of K-S, p- value,(W^*) and (A^*) statistics for some models fitted to survival times of patients treated using RT data.

Distribution	K-S	p-value	(W^*)	(A^*)
BEL(a,b,λ,θ,β)	0.13473	0.24303	0.18937	0.9141
TWL(α,β,λ,a,b)	0.13581	0.23501	0.21332	1.0387
EWL(a,α,β,θ,λ)	0.14371	0.18208	0.22394	1.0982
BMOL(a,b,c,α)	0.16727	0.07786	0.25599	1.2571
GL(θ,γ,α,β)	0.14522	0.17306	0.25671	1.2410
WMOL(α,β,λ,θ)	0.11319	0.44717	0.12648	0.6496

6.8 Conclusion

Here, we come up with a new lifetime model christended the Weibull Marshall-Olkin Lomax (WMOL) distribution, which has two shape and two scale parameters. It can be reduced to Weibull-Lomax, Marshall-Olkin Lomax and Lomax distributions. The failure rate function of WMOL model can have decreasing, increasing, upside down bathtub and bathtub curve according to its shape parameters. Therefore, WMOL model can be used quite effectively as an alternative to some extended form of Lomax and Weibull distributions and works better than the cited models. Maximum likelihood method is used to estimate the parameters of model and to check the efficiency of estimators we did a simulation study. We hope that the new distribution can be widely used in many different fields.

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