Chapter 1

Preliminaries and Basic Concepts

Basic concepts defined in this chapter will be helpful in succeeding chapters.

1.1 Order Statistics

Perhaps the earliest model for ordered random variables is order statistics. If sample observations are ascending in order according to their eminence then we call them ordered values. Let the random variables $\{Z_u\},\ \ u = 1, 2, \ldots, n$ are written in ascending order of eminence like

$$Z_{1:n} \ \leq\ Z_{2:n} \ \leq\ \ldots\ \leq\ Z_{n:n}\ ,$$

then, we represents uth-order statistics $Z_{u:n}$. Usually in sampling theory, we assume that $\{Z_u\}$ are identically distributed and statistically independent. But in case of order statics $Z_{u:n}$ are necessarily dependent. Some commonly used order statistics are the extremes $Z_{1:n}$ and $Z_{n:n}$, the range $W = Z_{n:n} - Z_{1:n}$, deviation from sample mean of extremes, $Z_{n:n} - \bar{Z}$, and studentized range,W/S_v from a random sample of $N(\mu, \sigma^2)$, where S_v is estimate of σ with v degrees of freedom. The extremes occures in study of droughts, floods, fatigue failure and fracture toughness. Range is a common tool in quality control to estimate standard deviation. Extreme deviation is an essential tool in outliers detection process. While the studentized range is the basis of many quick tests in small samples, and important in the analysis of variance to rank treatment means.

1.2 Distribution of Order Statistics

Let $Z_1, Z_2, \ldots, Z_n$, a sample of size n, randomly selected from a continuous population with cumulative distribution function (cdf) $H(z)$ and probability density function (pdf) $h(z)$. Then uth order statistics' pdf is

$$h_{Z_{u:n}}(z) = C_{u:n}\, H^{u-1}(z)\ [1 - H(z)]^{n-u}\ h(z); \quad -\infty < z < \infty, \tag{1.1}$$

where

$$C_{u:n} = \frac{n!}{(u-1)!(n-u)!}.$$

Special Cases

The pdf of first order statistics is

$$h_{Z_{1:n}}(z) = n\ [1 - H(z)]^{n-1}\ h(z). \tag{1.2}$$

The pdf of nth order statistics is

$$h_{Z_{n:n}}(z) = n\, H^{n-1}(z) h(z). \tag{1.3}$$

and $Z_{u:n}$ follows the distribution function

$$\begin{aligned}
H_{u:n}(z) &= Pr\left[Z_{u:n} \leq z\right] \\
&= Pr\left[\text{at least } u \text{ of } Z_1, Z_2, \ldots, Z_n \leq z\right] \\
&= \sum_{r=u}^{n} Pr\left[\text{exactly } r \text{ of } Z_1, Z_2, \ldots, Z_n \leq z\right] \\
&= \sum_{r=u}^{n} \binom{n}{r} \; [H(z)]^r \;\; [1-H(z)]^{n-r} && (1.4) \\
&= C_{u:n} \int_o^{H(z)} t^{u-1}(1-t)^{n-u} dt && (1.5) \\
&= I_{H(z)}\,(u, n-u+1)\,, && (1.6)
\end{aligned}$$

where

$$\begin{aligned}
I_z(\alpha,\beta) &= \frac{1}{B(\alpha,\beta)} \int_0^z t^{\alpha-1}\,(1-t)^{\beta-1} dt, \\
&\text{and} \\
B(\alpha,\beta) &= \int_0^1 t^{\alpha-1}\,(1-t)^{\beta-1} dt.
\end{aligned}$$

The result in RHS of (1.6) is obtained with the help of incomplete beta function and binomial sums. Khan (1991) used negative binomial sums to obtain this result.

$$\begin{aligned}
H_{u:n}(z) &= \sum_{r=0}^{n-u} \binom{r+u-1}{u-1} [H(z)]^u \,[1-H(z)]^r \\
&= \sum_{r=u}^{n} \binom{r-1}{u-1} [H(z)]^u \;\; [1-H(z)]^{r-u}\,. && (1.7)
\end{aligned}$$

The pdf of $Z_{u:n}$ can be derived by differentiating (1.6) with respect to z for continuous case.

The pth moment of $Z_{u:n}$ can be calculated by

$$E\left(Z_{u:n}^p\right) = \int_{-\infty}^{\infty} z^p h_{u:n}(z) dz. \quad (1.8)$$

The uth and vth order statistics' joint pdf is

$$\begin{aligned} h_{Z_{u:n},Z_{v:n}}(z,y) &= C_{u,v:n}\, H^{u-1}(z)\,[H(y)-H(z)]^{v-1-u} \\ &\times\ [1-H(y)]^{n-v}\, h(z)h(y); \quad -\infty < z < y < \infty, \end{aligned} \tag{1.9}$$

for $z < y$, $u, v = 1, 2, \ldots$, $u < v$ and

$$C_{u,v:n} = \frac{n!}{(u-1)!\,(v-u-1)!\,(n-v)!}.$$

The cdf of $Z_{u:n}$, & $Z_{v:n}$, $1 \le u < v \le n$ is

$$\begin{aligned} H_{u,v:n}(z,y) &= Pr\left(Z_{u:n} \le z, Z_{v:n} \le y\right) \\ &= Pr\Big(\text{ at leaset } u \text{ of } Z_1, Z_2, \ldots, Z_n \text{ are at most } z \ \& \\ &\quad \text{at least } v \text{ of } Z_1, Z_2, \ldots, Z_n \text{ are at most } y\Big) \\ &= \sum_{s=v}^{n}\sum_{r=u}^{s} Pr\Big(\text{ exactly } r \text{ of } Z_1, Z_2, \ldots, Z_n \text{ are at most } z \ \& \\ &\quad \text{exactly } s \text{ of } Z_1, Z_2, \ldots, Z_n \text{ are at most } y\Big) \\ &= \sum_{s=v}^{n}\sum_{r=u}^{s} \frac{n!}{r!\,(s-r)!\,(n-s)!}\ [H(z)]^{r}\ [H(y)-H(z)]^{s-r}\ [1-H(y)]^{n-s} \end{aligned} \tag{1.10}$$

The cdf of $Z_{u:n}$ and $Z_{v:n}$ can be written as

$$\begin{aligned} H_{u,v:n}(z,\ y) &= C_{u,v:n} \int_0^{H(z)} \int_x^{H(y)} x^{u-1}(t-x)^{v-u-1}(1-t)^{n-v}\,dx\,dt \\ &= I_{H(z),H(y)}(u, v-u, n-v+1); \quad -\infty < z < y < \infty. \end{aligned} \tag{1.11}$$

which is incomplete bivariate beta function.

For $z \le y$

$$H_{u,v:n}(z,y) = H_{v:n}(y). \tag{1.12}$$

The pth and qth order product moment of $Z_{u:n}$ and $Z_{v:n}$ is given by

$$E\left(Z_{u:n}^{p} Z_{v:n}^{q}\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} z^{p} y^{q} h_{u,v:n}(z,y) dz dy. \tag{1.13}$$

In general the pdf of $Z_{i_1:n}, Z_{i_2:n}, \ldots, Z_{i_k:n}$, for $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ and $-\infty < Z_{i_1} < Z_{i_2} < \ldots < Z_{i_k} < \infty$, is given by

$$h_{i_1,i_2,\ldots,i_k:n}(z_{i_1}, z_{i_2}, \ldots, z_{i_k}) = n! \left(\prod_{j=1}^{k} h(z_{i_j})\right) \prod_{j=0}^{k} \left[\frac{\left(H(z_{i_{j+1}}) - H(z_{i_j})\right)^{i_{j+1}-i_j-1}}{(i_{j+1}-i_j-1)!}\right], \tag{1.14}$$

where, $z_0 = -\infty, z_{k+1} = \infty, i_0 = 0, i_{k+1} = n+1$.

Also the accuracy of calculation can be checked by the following relation David and Nagaraja (2003),

$$\sum_{u=1}^{n} E\left(Z_{u:n}^{p}\right) = nE\left(Z^{p}\right); \quad p = 1,2,\ldots \tag{1.15}$$

$$\sum_{u=1}^{n} \sum_{v=1}^{n} E\left(Z_{u:n}^{p} Z_{v:n}^{q}\right) = nE\left(Z^{p+q}\right) + n(n-1)E\left(Z^{p}\right)E\left(Z^{q}\right); \quad p,q = 1,2,\ldots \tag{1.16}$$

and

$$\sum_{u=1}^{n} \sum_{v=1}^{n} Cov\left(Z_{u:n}, Z_{v:n}\right) = nVar(Z), \tag{1.17}$$

where, $E(Z^{p}) = E(Z_{1:1}^{p})$.

1.3 Moments and Recurrence Relations

In the last six decades or so, we see a spur in the efforts of order statistics, which are applied successfully to almost every possible sphere of human activity. Also, the order statistics' moments are fruitful in broad practical and theoretical situations like best linear unbiased estimators (BLUEs) of scale and location parameters for instance of censored or complete samples, entropy estimation, quality control, goodness of fit tests, characterization of probability distributions, reliability etc. It's miles visible that reliability of an object will be high if duration of failure of gadgets is high, which increases the cost of product in terms of time and money. In this situation experimenter is not able to predict failure of products by analyzing them for a short period. So he requires few early failures for prediction and this can be achieved through order statistics' moments. The early applications order statistics were concerned with empirical economic studies and coordination among various projects and efficient utilization of future emergencies.

Since the turn of this century, lot of attention paid to order statistics and their moments. Pearson (1902) and Galton (1902) explored the distribution of the difference of successive order statistics. For more information, see Arnold et al. (2008), Arnold and Balakrishnan (2012), Sarhān and Greenberg (1962), and David and Nagaraja (2003).

Recurrence relations and identities have achieved prominence for three primary reasons:

i. Shorten the time and labour and also lessen the number of direct computation.

ii. They give relationship between higher and lower order moments and hence higher order moments can easily assessed.

iii. Dispense some easy checks to check exactness of order statistics' moments.

For logistic distribution, Tarter (1966) and Shah (1966, 1970) found order statistics' moments. For the gamma distribution, Joshi (1979b) and Krishnaiah and Rizvi (1967) derived recurrence relations for order statistics' moments. Joshi (1982) also discovered several mixed aspects of order statistics recurrence relations for exponential and truncated exponential distributions. For the power function distribution, Malik (1967) developed recurrence relations for order statistics' moments.

Some recurrence relations of generalized Lindley, power Lindley, power gereralize Weibull, Extended exponential, Lindley and complementary exponential-geometric distributions for single and product order statistics' moments are established by Kumar and Goyal (2019a,b), Kumar and Dey (2017a,b),Kumar et al. (2017), Sultan and Al-Thubyani (2016), Balakrishnan et al. (2015), respectively. For extended exponential distribution, Kumar et al. (2017) entrenched order statistics' single and product moments and BLUEs of scale parameter besed on type-II right censored and complete samples. For Log-logistic distribution, Ahsanullah and Alzaatreh (2018) obtained order statistics' moments and estimate of parameters.

Several researchers have worked in the field of order statistics have appeared in the literature, see Kamps (1991), and Mohie El-Din et al. (1991), Childs et al. (2000), Sultan et al. (2000), Mahmoud et al. (2005), Sultan and Al-Thubyani (2016), Genç (2012), Kumar (2015), Balakrishnan et al. (2015), Kumar and Dey (2017b), Kumar and Goyal (2019a,b), Kumar et al. (2020b), Balakrishnan and Cohen (1991), Sanmel and Thomas (1997), Balakrishnan et al. (1996), Sultan et al. (2000), Mahmoud et al. (2005), Jabeen et al. (2013), Sultan and Al-Thubyani (2016), Kumar et al. (2017), Kumar and Dey (2017a,b), Ahsanullah and Alzaatreh (2018), Kumar and Goyal (2019a,b), Kumar et al. (2020a,b), Lieblein (1955), Balakrishnan and Joshi (1981), Saleh et al. (1975), Joshi (1978, 1979a) and many others.

1.4 BLUEs of the Location and Scale Parameters

Let $Z_{1:n} \leq Z_{2:n} \leq \cdots \leq Z_{n:n}$ be the order statistics from the continuous population with pdf of the location-scale parameter be $h(z)$. Let δ and φ are the location and scale parameters, respectively. To compute the BLUEs of the location and scale parameters δ and φ, we utilize the single and product moments. There are many applications of the scale-parameter and location-scale parameter distributions, see Arnold et al. (2008), Meyer (1987) and Wasserman (2003). Let $Z_{1:n} \leq Z_{2:n} \leq \cdots \leq Z_{n-c:n}$, $c = 0(1)([n/2]-1)$, denote Type-II right censored sample of $h(z)$. Let us denote $Y_{u:n} = (Z_{u:n} - \delta)/\varphi$, $E(Y_{u:n}) = \delta^{(1)}_{u;n}$, $1 \leq u \leq (n-c)$, and $Cov(Y_{u:n}, Y_{v:n}) = \varphi_{u,v:n} = \delta^{(1,1)}_{u,v:n} - \delta^{(1)}_{u:n}\ \delta^{(1)}_{v:n}$, $1 \leq u < v \leq (n-c)$. We shall use the following notations

$$\begin{aligned}\mathbf{Z} &= (Z_{1:n}, Z_{2:n}, \ldots, Z_{(n-c):n})^T,\\ \delta &= (\delta_{1:n}, \delta_{2:n}, \ldots, \delta_{(n-c):n})^T,\\ \mathbf{1} &= \underbrace{(1,\ 1,\ \ldots,\ 1)^T}_{n-c},\end{aligned}$$

and

$$\Psi = ((\varphi_{u,v}));\ \ 1 \le u,\ \ v \le n-c,$$

where, $\delta_{u:n} = E(Y_{u:n})$, $\varphi_{uu} = Var(Y_{u:n})$ and $\varphi_{uv} = Cov(Y_{u:n}, Y_{v:n})$; $u, v = 1, 2, \ldots (n-c)$. Then the BLUEs of δ and φ are given by Arnold et al. (2008)

$$\delta^* = \sum_{u=1}^{n-c} a_u Z_{u:n} \quad \text{and} \qquad \varphi^* = \sum_{u=1}^{n-c} \varphi_u Z_{u:n}, \tag{1.18}$$

where,

$$a_u = \left\{\frac{\boldsymbol{\delta^T\Psi^{-1}\delta 1^T\Psi^{-1} - \delta^T\Psi^{-1}1\delta^T\Psi^{-1}}}{\boldsymbol{(\delta^T\Psi^{-1}\delta)(1^T\Psi^{-1}1) - (\delta^T\Psi^{-1}1)^2}}\right\}, \tag{1.19}$$

$$b_u = \left\{\frac{\boldsymbol{1^T\Psi^{-1}1\delta^T\Psi^{-1} - 1^T\Psi^{-1}\delta 1^T\Psi^{-1}}}{\boldsymbol{(\delta^T\Psi^{-1}\delta)(1^T\Psi^{-1}1) - (\delta^T\Psi^{-1}1)^2}}\right\}. \tag{1.20}$$

Furthermore, the variances and covariance of these BLUEs are given by Arnold et al. (2008)

$$Var(\delta^*) = \varphi^2\left\{\frac{\boldsymbol{\delta^T\Psi^{-1}\delta}}{\boldsymbol{(\delta^T\Psi^{-1}\delta)(1^T\Psi^{-1}1) - (\delta^T\Psi^{-1}1)^2}}\right\}, \tag{1.21}$$

$$Var(\varphi^*) = \varphi^2\left\{\frac{\boldsymbol{1^T\Psi^{-1}1}}{\boldsymbol{(\delta^T\Psi^{-1}\delta)(1^T\Psi^{-1}1) - (\delta^T\Psi^{-1}1)^2}}\right\}, \tag{1.22}$$

and

$$Cov(\delta^*,\ \varphi^*) = \varphi^2\left\{\frac{\boldsymbol{-\delta^T\Psi^{-1}1}}{\boldsymbol{(\delta^T\Psi^{-1}\delta)(1^T\Psi^{-1}1) - (\delta^T\Psi^{-1}1)^2}}\right\}. \tag{1.23}$$

The values of a_u and b_u can be obtained for different values of sample sizes for example $n = 7,\ 10$, and different censoring cases $c = 0(1)([n/2]-1)$, and for some selected values for parameters. The coefficient of the BLUEs a_u and b_u given by (1.19) and (1.20) respectively, the conditions,

$$\sum_{i=1}^{n-c} a_i = 1$$

and

$$\sum_{i=1}^{n-c} b_i = 0,$$

which are used to check the computations accuracy.

1.5 Method of Maximum Likelihood Estimation

Methods name clearly indicates the way of obtaining estimator at which likelihood function attains its maximum. Let Z be a random variable with density $h(z;\Delta)$, where $\Delta = (\delta_1, \delta_2, \ldots, \delta_k)$ is a k-dimensional parameter vector. Therefore the M.L.E. of Δ usually denoted by $\hat{\Delta}_{mle} = (\hat{\delta}_{mle1}, \hat{\delta}_{mle2}, \ldots, \hat{\delta}_{mlek})$ is obtained by solving the following system of equations

$$\frac{\partial log\,[\ell(\Delta|\underline{z})]}{\partial \delta_i} = 0 \xrightarrow{s.t.} \frac{\partial^2 log\,[\ell(\Delta|\underline{z})]}{\partial \delta_i^2} < 0;\ \ i = 1,2,\ldots,k, \tag{1.24}$$

where, $log\,[\ell(\Delta|\underline{z})] = \sum_{u=1}^{n} log\,[h(z_u,\Delta)]$ and $\underline{z} = (z_1, z_2, \ldots, z_n)$, denotes respectively the log-likelihood function and a random sample of size n. In general, M.L.E.s are not unbiased but consistent estimators. M.L.E.s are also satisfies invariance property i.e. if $\hat{\Delta}_{mle}$ is the M.L.E. of Δ, then the M.L.E. of one-to-one transformation $g(\Delta)$ is $g(\hat{\Delta}_{mle})$.

1.6 Method of Moments

Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from a population with pdf or pmf $H(z, \Delta)$, where $\Delta = (\delta_1, \delta_2, \ldots, \delta_k)$. Then the moment estimator of δ i.e. $\hat{\Delta}_{mm} = (\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_k)$ is obtained by solving the following system of k equation.

$$\sum_{u=1}^{n} \frac{z_u^r}{n} = \int_{-\infty}^{\infty} z^r h(z) dz \quad \text{for continuous}$$
$$\sum_{u=1}^{n} \frac{z_u^r}{n} = \sum_{x} z^r h(z), \quad r = 1, 2, \ldots, k, \quad \text{for discrete} \tag{1.25}$$

where, $\sum_{u=1}^{n} \frac{z_u^r}{n}$ is the rth sample moment and $\int_{-\infty}^{\infty} z^r h(z) dz$ or $\sum_x z^r h(z)$ is the rth population moment.

1.7 Methods of Generating Distribution

The amount of data available for analysis is growing increasingly faster, requiring new probabilistic distributions to better describe each phenomenon or experiment studied. Distributions with more complexity and greater parameters can be with the help of computer softwares.

The literature in the field describes several generalizations and extensions of symmetric, asymmetric, discrete and continuous distributions. The relevance of these new models is that, according to situation, each one of them can better fit the mass of data. We presents several classes of distributions described in literature, their nomenclature and the title of the work where they have been presented.

1. **Exponentiated Generalized:** For constant $\alpha > 0$ Mudholkar et al. (1995) defined exponentiated generalized as

$$G(z) = H^{\alpha}(z)$$

2. **Beta1 Generalized:** Eugene et al. (2002) presents beta1 generalized model as

$$G(z) = \frac{1}{B(\alpha,\beta)} \int_0^{H(z)} t^{\alpha-1}(1-t)^{\beta-1}dt; \;\; \alpha,\beta > 0 \; and \; 0 < t < 1.$$

3. **Beta2 Generalized:** Tahir and Nadarajah (2013) presents beta2 generalized model as

$$G(z) = \frac{1}{B(\alpha,\beta)} \int_0^{H(z)} t^{\alpha-1}(1+t)^{-(\alpha+\beta)}dt; \;\; \alpha,\beta > 0 \; and \; t > 0.$$

4. **Mc1 Generalized:** McDonald (1984) presents Mc1 generalized model as

$$G(z) = \frac{1}{B(\alpha,\beta)} \int_0^{H^{\gamma}(z)} t^{\alpha-1}(1-t)^{\beta-1}dt; \;\; \alpha,\beta,\gamma > 0 \; and \; 0 < t < 1.$$

5. **Mc2 Generalized:** Tahir and Nadarajah (2013) presents Mc2 generalized model as

$$G(z) = \frac{1}{B(\alpha,\beta)} \int_0^{H^{\gamma}(z)} t^{\alpha-1}(1+t)^{-(\alpha+\beta)}dt; \;\; \alpha,\beta,\gamma > 0 \; and \; t > 0.$$

6. **Kumaraswamy G_1:** Cordeiro and de Castro (2011) defined Kumaraswamy G_1 model as

$$G(z) = 1-(1-H^{\alpha}(z))^{\beta}$$

7. **Kumaraswamy Type 2:** For $\alpha > 0$ and $\beta > 0$ Tahir and Nadarajah (2013) defined Kumaraswamy type 2 model as

$$G(z) = 1-\left[1-(1-H(z))^{\alpha}\right]^{\beta}$$

8. **Marshall-Olkin:** Marshall and Olkin (1997) presented Masrshall-Olkin model as

$$G(z) = \frac{H(z)}{H(z)+\alpha(1-H(z))}; \;\; \alpha > 0.$$

9. **Marshall-Olkin G_1:** Jayakumar and Mathew (2008) presented Masrshall-Olkin G_1 model as

$$G(z) = 1 - \left[\frac{\alpha(1-H(z))}{H(z)+\alpha(1-H(z))}\right]^{\beta}; \;\; \alpha,\beta > 0.$$

10. **Marshall-Olkin G_1:** Tahir and Nadarajah (2013) presented a different type of Masrshall-Olkin G_1 model as

$$G(z) = \left[\frac{H(z)}{H(z)+\alpha(1-H(z))}\right]^{\theta}; \;\; \alpha,\theta > 0.$$

11. **Gamma-Generated:** Zografos and Balakrishnan (2009) defined Gamma-Generated model as

$$G(z) = \frac{\theta^{\gamma}}{\Gamma(\gamma)}\int_0^{-ln(1-H(z))} t^{\gamma-1}e^{-\theta t}dt.$$

12. **Gamma-Generated:** Cordeiro et al. (2017b) also defined a different form of Gamma-Generated model as

$$G(z) = 1 - \frac{\theta^{\gamma}}{\Gamma(\gamma)}\int_0^{-ln(H(z))} t^{\gamma-1}e^{-\theta t}dt.$$

13. **Extended Weibull Distribution:** Silva et al. (2013) defined extended Weibull distribution as

$$G(z) = 1 - \frac{C(\gamma e^{-\beta H(z)})}{C(\gamma)},$$

where $z > 0, \gamma > 0$ and $C(\gamma) = \sum_{n=1}^{\infty} a_n \gamma^n$.

14. **Kumaraswamy-G Poisson:** Ramos (2014) defined Kumaraswamy-G Poisson model as

$$G(z) = \frac{1-exp(-\theta H(z))}{1-exp(-\theta)}.$$

15. **Kumaraswamy-G Exponentiated:** Ramos (2014) defined Kumaraswamy-G exponentiated model as

$$G(z) = \left[1-(1-H^{\gamma}(z))^{\alpha}\right]^{\beta};\ \ \alpha,\beta,\gamma > 0.$$

16. **Beta Weibull Poisson Family:** Paixao (2014) defined Beta Weibull Poisson family model as

$$G(z) = \frac{exp(\theta exp(-\gamma H^{\beta}(z))) - exp(\theta)}{1-exp(\theta)}.$$

17. **Beta Kummer Generalized:** Pescim et al. (2012) defined Beta Kummer generalized model as

$$G(z) = \int_0^{H(z)} Kt^{\alpha-1}(1-t)^{\beta-1}e^{-\gamma t}dt;\ \ \alpha,\beta > 0, -\infty < \gamma < \infty.$$

18. **Weibull Gneralized Poisson Distribution:** Paixao (2014) defined Beta Weibull generalized Poisson distribution model as

$$G(z) = \frac{exp\left(-\frac{\theta}{\beta}R(-\beta e^{-\beta})\right) - exp\left(-\frac{\theta}{\beta}R(\xi(z))\right)}{exp\left(-\frac{\theta}{\beta}R(-\beta e^{-\beta})\right) - 1},$$

where, $R(z) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}m^{m-2}}{(m-1)!}z^m$ and $\xi(z) = -\beta exp(-\beta - \gamma z^{\alpha})$.

19. **G-Negative Binomial Family:** Paixao (2014) defined G-Negative Binomial family model as

$$G(z) = \frac{(1-\theta)^{-m} - [1-\theta(1-H(z))]^{-m}}{(1-\theta)^{-m} - 1}.$$

20. **Zeta-G:** Paixao (2014) defined Zeta-G model as

$$G(z) = \frac{\xi(t) - Li_t[1-H(z)]}{\xi(t)},$$

where, $Li_t(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^t}$ and $\xi(t) = \sum_{m=1}^{\infty} \frac{1}{m^t}$.

21. **Power Series Distributions Family:** Consul and Famoye (2006) defined Power Series distributions family model as

$$G(z) = \sum_{m=0}^{z} \frac{B^(m)(\alpha)}{m!B(\gamma)}(\gamma - \alpha)^m.$$

22. **Basic Lagrangian:** Consul and Famoye (2006) defined Basic Lagrangian model as

$$G(z) = \sum_{m=1}^{z} \frac{1}{m!}\left[(B(0))^m\right]^{m-1}.$$

23. **Lagrangian Delta:** Consul and Famoye (2006) defined Lagrangian delta model as

$$G(z) = \sum_{m=n}^{z} \frac{n}{(m-n)!m}\left[(B(0))^m\right]^{m-n}.$$

24. **Weibull Marshall-Olkin-G (WMO-G) Family:** Korkmaz et al. (2019) proposed the *Weibull Marshall-Olkin-G* (WMO-G) family as

$$H(z;\alpha,\beta,\eta) = 1 - \exp\left(-\left\{-\log\left[\frac{\alpha\bar{G}(z;\eta)}{1-\bar{\alpha}\bar{G}(z;\eta)}\right]\right\}^{\beta}\right).$$

1.8 Some Continuous Distributions

1. **Type-II Exponentiated Log-logistic Distribution**

   Recently, Rao et al. (2012) proposed Type-II exponentiated log-logistic (TIIELL) distribution with pdf

$$h(z;\tau,\varphi,\eta) = \frac{\tau\eta\left(\frac{z}{\varphi}\right)^{\eta-1}}{\varphi\left[1+\left(\frac{z}{\varphi}\right)^{\eta}\right]^{\tau+1}}, \quad z>0, \quad (\tau,\varphi)>0, \eta>1$$

and associated cdf is

$$H(z;\tau,\varphi,\eta) = 1 - \left[1 + \left(\frac{z}{\varphi}\right)^{\eta}\right]^{-\tau}, \quad z > 0, \quad (\tau,\varphi) > 0, \eta > 1$$

where φ is the scale parameter, and η and τ are the shape parameters of the distribution. If $\tau = 1$, then TIIELL distribution becomes log-logistic distribution, and if $\eta = 1$, then TIIELL distribution becomes Pareto type-II distribution.

2. **Log-logistic Distribution**

Let Z, a random variable, is said to follow log-logistic distribution with parameters φ, η denoted by $Z \sim LL(\varphi, \eta)$ if its pdf is

$$h(z;\varphi,\eta) = \frac{\eta\left(\frac{z}{\varphi}\right)^{\eta-1}}{\varphi\left[1 + \left(\frac{z}{\varphi}\right)^{\eta}\right]^{2}}, \quad z > 0, \quad \varphi > 0, \eta > 1. \tag{1.26}$$

and associated cdf is

$$H(z;\varphi,\eta) = 1 - \left[1 + \left(\frac{z}{\varphi}\right)^{\eta}\right]^{-1}, \quad z > 0, \quad \varphi > 0, \eta > 1$$

Log-logistic distribution is closed under scaling, i.e., if $Z \sim LL(\varphi, \eta)$, then for some $p > 0, pZ \sim (p\varphi; \eta)$. If $Z \sim LL(\varphi, \eta)$ then the transformation
$Y = log(Z) \sim logistic\ distribution[L(log(\varphi), 1/\eta)]$.

3. **Modified Power Function Distribution**

Recently, Okorie et al. (2017) introduced a two parameter modified power function (MPF) distribution with pdf

$$h(z;\alpha,\beta) \; = \; \frac{\alpha\, \beta\, (1-z)^{\beta-1}}{[1-(1-\alpha)\,(1-z)^{\beta}]^{2}}, \; 0 < z < 1, \; \alpha, \; \beta > 0,$$

and associated cdf is

$$H(z;\alpha,\beta) \ = \ 1-\frac{\alpha\ (1-z)^{\beta}}{[1-(1-\alpha)\ (1-z)^{\beta}]}\ 0<z<1,\ \alpha,\ \beta>0.$$

If $\alpha=1$, then MPF distribution becomes power function distribution.

4. **Power Function Distribution**

Let Z, a random variable, is said to follow power function distribution with shape parameter β denoted by $Z\sim power(\beta)$ if its pdf is

$$h(z;\beta) \ = \ \beta\ (1-z)^{\beta-1},\ \ z\in(0,1),\ \beta>0. \tag{1.27}$$

and associated cdf is

$$H(z;\beta) \ = \ 1-(1-z)^{\beta},\ \ z\in(0,1),\ \beta>0. \tag{1.28}$$

The density of power function is monotone increasing in nature with global maximum occurring at $z=\lambda$. Power distribution is closed under scaling, i.e., if $Z\sim power(\lambda,\beta)$, then for some $p<0, pZ\sim power(\lambda/p;\beta)$. It is also closed under maximum, i.e., if $Z\sim power(\lambda,\beta)$ and $Z_1\sim power(\lambda,\beta_1)$, then $max(Z,Z_1)\sim power(\lambda,\beta+\beta_1)$. Power function distribution with $\beta=1$ reduces to $U(0,\lambda)$ distribution. $power(1;\beta)$ is a special case of Kumaraswamy distribution whose density is given by $\theta\beta z^{\beta-1}(1-z^{\beta})^{\theta-1},\ \ 0<z<1$. If $Y\sim exp(\theta)$, then the transformation $Z=(\lambda e^{Y})^{-1}\sim power(\lambda,\beta)$. Inverse of power function random variable follows Pareto distribution.

5. **Extended Power Lindley Distribution**

Recently the three parameter extended power Lindley distribution was proposed by Alkarni (2015) for the flexibility of purpose. A random variable Z said to follow extended power Lindley (EPL) distribution if it has following pdf

$$h(z;\tau,\xi,\kappa)=\frac{\tau\xi^{2}}{\xi+\kappa}(1+\kappa z^{\tau})\ z^{\tau-1}\ e^{-\xi z^{\tau}},\quad z>0;\quad \tau>0,\xi>0,\kappa>0$$

and associated cdf is

$$H(z;\tau,\xi,\kappa) = 1 - \left(1 + \frac{\kappa\xi}{\xi+\kappa} z^{\tau}\right) e^{-\xi z^{\tau}}, \quad z > 0; \quad \tau > 0, \xi > 0, \kappa > 0.$$

For $\kappa = 1$ and $\kappa = 1, \quad \tau = 1$, the EPL distribution reduces to power Lindley (PL) and Lindley distributions respectively.

6. **Power Lindley Distribution**

Let Z, a random variable, is said to follow power Lindley distribution with parameters ξ, τ if its pdf is

$$h(z;\xi,\tau) = \frac{\tau\xi^2}{(1+\xi)}(1+z^{\tau})z^{\tau-1}exp(-\xi z^{\tau}), \;\; z > 0, \xi, \tau > 0. \tag{1.29}$$

and associated cdf is

$$H(z;\tau,\xi) = 1 - \left(1 + \frac{\xi}{\xi+1} z^{\tau}\right) e^{-\xi z^{\tau}}, \quad z > 0; \quad \xi, \tau > 0.$$

7. **Lindley Distribution**

Let Z, a random variable is said to follow Lindley distribution with parameter ξ if its pdf is

$$h(z;\xi) = \frac{\xi^2}{(1+\xi)}(1+z)exp(-\xi z), \;\; z > 0, \xi > 0. \tag{1.30}$$

and associated cdf is

$$H(z;\xi) = 1 - \left(1 + \frac{\xi}{\xi+1} z\right) e^{-\xi z}, \quad z > 0; \quad \xi > 0.$$

It is also useful in medicine, engineering and biology. Ghitany et al. (2008) used it for modeling in mortality studies. The parameter, $\xi > 0$ can result in either a unimodal or monotone decreasing distribution.

8. **Generalized Topp-Leone Distribution**

   Recently, Shekhawat and Sharma (2020) proposed a generalization of the Topp-Leone distribution called generalized Topp-Leone (GTL) distribution. A random variable Z said to follow generalized Topp-Leone (GTL) distribution if it has following pdf

$$h(z;\kappa,\xi) \;=\; 2\kappa\xi z^{\kappa\xi-1}(1-z^{\kappa})(2-z^{\kappa})^{\xi-1},\; 0<z<1,\; \kappa,\; \xi>0.$$

   and associated cdf is

$$H(z;\kappa,\xi)=(z^{\kappa}(2-z^{\kappa}))^{\xi},\; 0<z<1,\; \kappa,\; \xi>0.$$

9. **Topp-Leone Distribution**

   The single parameter (ξ) Topp-Leone distribution is defined by the pdf

$$h(z;\xi) \;=\; 2\xi z^{\xi-1}(1-z)(2-z)^{\xi-1},\; 0<z<1,\; \xi>0.$$

   and associated cdf is

$$H(z;\xi)=(z(2-z))^{\xi},\; 0<z<1,\; \xi>0.$$

   This distribution has J-shaped frequency curve for $\xi<1$. Topp-Leone distribution is also effective for the generation of new flexible families of distributions.

10. **Weibull Marshall-Olkin Lomax (WMOL) Distribution**

   Let Z, a random variable is said to follow *Weibull Marshall-Olkin Lomax* (WMOL) distribution if its pdf is

$$\begin{aligned} h(z;\beta,\theta,\lambda,\alpha) \;&=\; \frac{\beta\theta\lambda}{(1+\lambda z)[1-\bar{\alpha}(1+\lambda z)^{-\theta}]}\left(-log\left[\frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}}\right]\right)^{\beta-1} \\ &\times\; \exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}}\right]\right)^{\beta}\right\},\; z\geq 0,\; \theta,\lambda,\beta,\alpha>0. \end{aligned}$$

The associated cdf is

$$H(z;\beta,\theta,\lambda,\alpha)=1-\exp\left\{-\left(-log\left[\frac{\alpha(1+\lambda z)^{-\theta}}{1-\bar{\alpha}(1+\lambda z)^{-\theta}}\right]\right)^{\beta}\right\},\ z\geq 0,\ \theta,\lambda,\beta,\alpha>0.$$

where, $\theta > 0$ and $\beta > 0$ are two shape parameters and $\alpha > 0$, $\lambda > 0$ are the scale parameters.

Additionally, the new model contains some distributions as special cases, these sub-models being listed in Table 1.

TABLE 1.1: Special cases of the WMOL distribution

Parametric values in WMOL distribution	Sub-models
$\beta=1$	Marshall-Olkin Lomax distribution(α,θ,λ)
$\alpha=1$	Weibull Lomax distribution(β,θ,λ)
$\alpha=\beta=1$	Lomax distribution (θ)

11. **Marshall-Olkin Lomax Distribution**

The pdf corresponding to MOL distribution is

$$\begin{aligned} h(z;\theta,\lambda,\alpha) &= \frac{\theta\lambda}{(1+\lambda z)[1-\bar{\alpha}(1+\lambda z)^{-\theta}]} \\ &\times \exp\left\{-\left(-log\left[\frac{\alpha\ (1+\lambda z)^{-\theta}}{1-\bar{\alpha}\ (1+\lambda z)^{-\theta}}\right]\right)\right\},\ z\geq 0,\ \theta,\lambda,\alpha>0. \end{aligned}$$

and associated cdf is

$$H(z;\theta,\lambda,\alpha)=1-\exp\left\{-\left(-log\left[\frac{\alpha\ (1+\lambda z)^{-\theta}}{1-\bar{\alpha}\ (1+\lambda z)^{-\theta}}\right]\right)\right\},\ z\geq 0,\ \theta,\lambda,\alpha>0.$$

12. **Weibull Lomax Distribution**

The pdf corresponding to WL distribution is

$$\begin{aligned} h(z;\theta,\lambda,\beta) &= \frac{\beta\theta\lambda}{(1+\lambda z)}\left(\theta log(1+\lambda z)\right)^{\beta-1} \\ &\times \exp\left\{-\left(\theta log(1+\lambda z)\right)^{\beta}\right\},\ z\geq 0,\ \theta,\lambda,\beta>0. \end{aligned} \tag{1.31}$$

and associated cdf is

$$H(z;\theta,\lambda,\beta) = 1-\exp\left\{-\left(-log\left[(1+\lambda z)^{-\theta}\right]\right)^{\beta}\right\},\ \ z\geq 0,\ \theta,\lambda,\beta>0.$$

13. **Lomax Distribution**

The pdf corresponding to Lomax distribution is

$$h(z;\theta,\lambda) \ \ = \ \ \frac{\theta\lambda}{(1+\lambda z)^{\theta+1}},\ \ z\geq 0,\ \theta,\lambda>0. \tag{1.32}$$

and associated cdf is

$$H(z;\theta,\lambda) = 1-\exp\left\{\ -\left(\ -log\left[\ (1+\lambda z)^{-\theta}\ \right]\ \right)\right\},\ \ z\geq 0,\ \theta,\lambda>0.$$