Chapter 2

Inferences for Type-II Exponentiated Log-logistic Distribution based on Order Statistics with Application*

2.1 Introduction

This chapter follows the following structure: In Sections 2.2 and 2.3, we derive the exact expressions for the single and product moments of order statistics from TIIELLD. In Section 2.4, we obtain BLUEs for δ and φ by using these moments. These BLUEs are then used in Section 2.5 to obtain $(1-\alpha)100\%$ confidence intervals (CIs) for the location and scale parameters of the BLUEs based on the pivotal quantities. Besides, lower and upper percentage points of pivotal quantities through Edgeworth approximations are obtained and compare the results with simulated percentage points. A real data application is provided in Section 2.6. Finally, in Section 2.7 , we draw a conclusion for the chapter.

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In recent past several authors have tabulated the moments of order statistics quite extensively for several distributions and also obtained MLEs and BLUEs for the scale and location parameters of the distributions based on complete and type-II censored samples. Further, they developed point prediction and goodness-of-fit tests. In this regard, readers may refer to the works of Balakrishnan and Cohen (1991), Balakrishnan and Sultan (1998), Saran and Pushkarna (2000), Childs et al. (2000), Sultan et al. (2000), Jabeen et al. (2013), Balakrishnan et al. (2015), Sultan and Al-Thubyani (2016), Kumar et al. (2017), Kumar and Dey (2017a,b), Ahsanullah and Alzaatreh (2018), Kumar and Goyal (2019a,b) Kumar et al. (2020b) and many others.

Rao et al. (2012) suggested a generalization of the log-logistic distribution called Type-II exponentiated log-logistic (TIIELL) distribution with pdf

$$h(z)=\frac{\tau\eta\left(\frac{z}{\varphi}\right)^{\eta-1}}{\varphi\left[1+\left(\frac{z}{\varphi}\right)^{\eta}\right]^{\tau+1}},\quad z>0,\ \ (\tau,\varphi)>0,\eta>1. \tag{2.1}$$

The associated cdf and quantile function are, respectively given by

$$H(z)=1-\left[1+\left(\frac{z}{\varphi}\right)^{\eta}\right]^{-\tau},\quad z>0,\ \ (\tau,\varphi)>0,\eta>1. \tag{2.2}$$

and

$$H^{-1}(z)=\varphi\left(\left(\frac{1}{1-z}\right)^{\frac{1}{\tau}}-1\right)^{\frac{1}{\eta}}. \tag{2.3}$$

where, φ is the scale parameter, and η and τ are the shape parameters of the distribution. If $\tau=1$, then Eq. (2.1) becomes log-logistic distribution, and if $\eta=1$, then TIIELL distribution becomes Pareto type-II distribution. The pth moments of the TIIELL distribution in (2.1) can be easily computed as

$$E(Z^p)=\varphi^p\tau B\left(\tau-\frac{p}{\eta},1+\frac{p}{\eta}\right). \tag{2.4}$$

where $B(.,.)$ is the beta function. Note that the pth moment exists iff $\eta>\max\{1,p/\tau\}$. A more compact form of (2.4) can be derived using the fact that $\Gamma(z)\Gamma(1-z)=\pi csc(\pi z)$ [see

Abramowitz and Stegun (1964)] as follows

$$E(Z^p) = \frac{\varphi^p \frac{p}{\eta}\pi csc\frac{p\pi}{\eta}}{\Gamma\tau}\prod_{i=1}^{\tau-1}\left(\tau - \frac{p}{\eta} - i\right). \tag{2.5}$$

Therefore,

$$E(Z) = \frac{\varphi \frac{\pi}{\eta} csc\frac{\pi}{\eta}}{\Gamma\tau}\prod_{i=1}^{\tau-1}\left(\tau - \frac{1}{\eta} - i\right),$$

$$E(Z^2) = \frac{\varphi^2 \frac{2\pi}{\eta} csc\frac{2\pi}{\eta}}{\Gamma\tau}\prod_{i=1}^{\tau-1}\left(\tau - \frac{2}{\eta} - i\right)$$

and

$$Var(Z) = \left[\frac{\varphi^2 \frac{2\pi}{\eta} csc\frac{2\pi}{\eta}}{\Gamma\tau}\prod_{i=1}^{\tau-1}\left(\tau - \frac{2}{\eta} - i\right) - \left(\frac{\varphi \frac{\pi}{\eta} csc\frac{\pi}{\eta}}{\Gamma\tau}\prod_{i=1}^{\tau-1}\left(\tau - \frac{1}{\eta} - i\right)\right)^2\right].$$

2.2 Relations for Single Moments of Order Statistics

Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from TIIELL distribution, the pdf of uth order statistic is obtained by using equation (2.1) and (2.2) in (1.1) as follows:

$$h_{Z_{u:n}}(z) = \frac{\tau\eta}{\varphi} C_{u:n}\sum_{i=0}^{u-1}\binom{u-1}{i}(-1)^i \frac{\left(\frac{z}{\varphi}\right)^{\eta-1}}{\left[1+\left(\frac{z}{\varphi}\right)^{\eta}\right]^{\tau i+\tau+1+\tau(n-u)}}, \quad z \geq 0. \tag{2.6}$$

The pth moments of $Z_{u:n}$, $\mu_{u:n}^{(p)} = E(Z_{u:n}^p)$ can be derived from (2.6) as

$$\mu_{u:n}^{(p)} = \varphi^p \tau C_{u:n}\sum_{i=0}^{u-1}\binom{u-1}{i}(-1)^i B\left(\tau(i+n-u+1) - \frac{p}{\eta}, 1+\frac{p}{\eta}\right). \tag{2.7}$$

Similarly as in (2.5), one can show that

$$\mu_{u:n}^{(p)} = \frac{\varphi^p p\tau n!\pi csc\frac{p\pi}{\eta}}{\eta(u-1)!(n-u)!}\sum_{i=0}^{u-1}\binom{u-1}{i}(-1)^i \prod_{j=1}^{\tau(i+n-u+1)-1}\frac{\tau(i+n-u+1) - j - \frac{p}{\eta}}{\Gamma(\tau(i+n-u+1)-1)}. \tag{2.8}$$

Note that from (2.8), the first and second moments of $Z_{u:n}$ are, respectively, given by

$$\mu_{u:n}^{(1)} = \frac{\varphi\tau n!\pi csc\frac{\pi}{\eta}}{\eta(u-1)!(n-u)!}\sum_{i=0}^{u-1}\binom{u-1}{i}(-1)^i\prod_{j=1}^{\tau(i+n-u+1)-1}\frac{\tau(i+n-u+1)-j-\frac{1}{\eta}}{\Gamma(\tau(i+n-u+1)-1)},$$

and

$$\mu_{u:n}^{(2)} = \frac{\varphi^2\tau n!\pi csc\frac{2\pi}{\eta}}{\eta(u-1)!(n-u)!}\sum_{i=0}^{u-1}\binom{u-1}{i}(-1)^i\prod_{j=1}^{\tau(i+n-u+1)-1}\frac{\tau(i+n-u+1)-j-\frac{2}{\eta}}{\Gamma(\tau(i+n-u+1)-1)},$$

Some special cases from equation (2.8) are

1. For $\tau = 1$, in (2.8), we get the explicit expression for order statistic of log-logistic distribution

$$\mu_{u:n}^{(p)} = \frac{\varphi^p pn!\pi csc\frac{p\pi}{\eta}}{\eta(u-1)!(n-u)!}(-1)^u\prod_{j=1}^{n}\frac{n-u+1-j-\frac{p}{\eta}}{\Gamma((n-u+1)-1)}.$$

2. If $u = n = 1$, we get

$$\mu_{1:1}^{(p)} = \varphi^p\tau B\left(\tau-\frac{p}{\eta},1+\frac{p}{\eta}\right)$$

which agrees with (2.4).

3. If $p = u = 1$ in (2.8), we get

$$\mu_{1:n}^{(1)} = \frac{\varphi\tau n\pi csc\frac{\pi}{\eta}}{\eta}\prod_{j=1}^{\tau n-1}\frac{\left[\tau n-j-\frac{1}{\eta}\right]}{\Gamma(\tau n-1)},$$

4. If $p = 1$, $u = n$ in (2.8), we get

$$\mu_{n:n}^{(1)} = \frac{\varphi\tau n\pi csc\frac{\pi}{\eta}}{\eta}\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i\prod_{j=1}^{\tau(i+1)-1}\frac{\left[\tau(i+1)-j-\frac{1}{\eta}\right]}{\Gamma(\tau(i+1)-1)},$$

5. If $p = u = n = 1$ in (2.8), we get

$$\mu_{1:1}^{(1)} = \frac{\varphi\tau\pi csc\frac{\pi}{\eta}}{\eta}\prod_{j=1}^{\tau-1}\frac{\left[\tau-j-\frac{1}{\eta}\right]}{\Gamma(\tau-1)}, \tag{2.9}$$

which agree with equation (2.5) for $p=1$.

It is interesting to note that (2.8) can be used easily to derive several recurrence relations for the moments of order statistics. Some of these recurrence relations already exist in the literature. Below, we provide some of these recurrence relations.

I. From equation (2.8) we can write

$$\mu_{u:n}^{(p)} = \frac{\varphi^p \tau p n(n-1)! \pi csc\frac{p\pi}{\eta}}{\eta(u-1)(u-2)!(n-u)!} \sum_{i=0}^{u-1} \binom{u-1}{i} (-1)^{u-1+i} \prod_{j=1}^{\tau(i+n-u+1)-1} \frac{\tau(i+n-u+1)-j-\frac{p}{\eta}}{\Gamma(\tau(i+n-u+1)-1)}.$$

Let us define $\Delta(\tau,\eta) = \frac{\varphi^p \tau p (n-1)! \pi csc\frac{p\pi}{\eta}}{\eta(u-2)!(n-u)!}$, we have

$$\begin{aligned}
\mu_{u-1:n-1}^{(p)} &= \Delta(\tau,\eta) \sum_{i=0}^{u-2} \binom{u-2}{i} (-1)^{i} \prod_{j=1}^{\tau(i+n-u+1)-1} \frac{\tau(i+n-u+1)-j-\frac{p}{\eta}}{\Gamma(\tau(i+n-u+1)-1)} \\
&= \Delta(\tau,\eta) \binom{u-2}{0} (-1)^{0} \prod_{j=1}^{\tau(n-u+1)-1} \frac{\tau(n-u+1)-j-\frac{p}{\eta}}{\Gamma(\tau(i+n-u+1)-1)} \\
&+ \Delta(\tau,\eta) \binom{u-2}{1} (-1)^{1} \prod_{j=1}^{\tau(n-u+2)-1} \frac{\tau(n-u+2)-j-\frac{p}{\eta}}{\Gamma(\tau(n-u+2)-1)} \\
&\vdots \\
&+ \Delta(\tau,\eta) \binom{u-2}{u-2} (-1)^{u-2} \prod_{j=1}^{\tau(n-1)-1} \frac{\tau(n-1)-j-\frac{p}{\eta}}{\Gamma(\tau(n-1)-1)},
\end{aligned}$$

which can be written in vector form as

$$\mu_{u-1:n-1}^{(p)} = \mathbf{1}' \, {}_i\mu_{u-1:n-1}^{(p)},$$

where $\mathbf{1}' = (1,1,\ldots,1)$ and ${}_i\mu_{u-1:n-1}^{(p)}$ denotes a vector of order $(1\times u-2)$ and $((u-2)\times 1)$, respectively, where

$${}_i\mu_{u-1:n-1}^{(p)} = \begin{pmatrix} \Delta(\tau,\eta)\binom{u-2}{0}(-1)^0 \prod_{j=1}^{\tau(n-u+1)-1} \frac{\tau(n-u+1)-j-\frac{p}{\eta}}{\Gamma(\tau(n-u+1)-1)} \\ \vdots \\ \vdots \\ \Delta(\tau,\eta)\binom{u-2}{u-2}(-1)^{u-2} \prod_{j=1}^{\tau(n-1)-1} \frac{\tau(n-1)-j-\frac{p}{\eta}}{\Gamma(\tau(n-1)-1)}. \end{pmatrix}$$

Therefore, we can write $\mu_{u:n}^{(p)}$ as

$$\begin{aligned}\mu_{u:n}^{(p)} &= \frac{n}{u-1}\Delta(\tau,\eta)\left[\binom{u-1}{u-1}(-1)^{u-1}\prod_{j=1}^{\tau n-1}\frac{\tau n-j-\frac{p}{\eta}}{\Gamma(\tau n-1)}\right. \\ &+ \left.\sum_{i=0}^{u-2}\frac{u-1}{u-1-i}\binom{u-2}{i}(-1)^{i}\prod_{j=1}^{\tau(i+n-u+1)-1}\frac{\tau(i+n-u+1)-j-\frac{p}{\eta}}{\Gamma(\tau(i+n-u+1)-1)}\right]\end{aligned}$$

$$\begin{aligned}&= \frac{n}{u-1}\Delta(\tau,\eta)(-1)^{u-1}\prod_{j=1}^{\tau n-1}\frac{\tau n-j-\frac{p}{\eta}}{\Gamma(\tau n-1)} \\ &+ n\Delta(\tau,\eta)\sum_{i=0}^{u-2}\frac{1}{(u-1-i)}\binom{u-2}{i}(-1)^{i}\prod_{j=1}^{\tau(i+n-u+1)-1}\frac{\tau(i+n-u+1)-j-\frac{p}{\eta}}{\Gamma(\tau(i+n-u+1)-1)} \\ &= \frac{n}{u-1}\Delta(\tau,\eta)(-1)^{u-1}\prod_{j=1}^{\tau n-1}\frac{\tau n-j-\frac{p}{\eta}}{\Gamma(\tau n-1)}+nv'_{i}\mu_{u-1:n-1}^{(p)},\end{aligned}$$

where $v' = \left(\frac{1}{u-1},\frac{1}{u-2},\frac{1}{u-3},\ldots,1\right)$ is vector of order $(1\times(u-2))$

II If $u=1$ in equation (2.8), we get

$$\mu_{1:n}^{(p)} = \prod_{j=1}^{\tau}\frac{n\left(j-\frac{p}{\eta}\right)}{\eta(n-1)\prod_{h=1}^{\tau+1}[\tau(n)-h]}\mu_{1:n-1}^{(p)}.$$

2.3 Relations for Product Moments of Order Statistics

Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from TIIELL distribution, the joint pdf of uth and vth order statistic is obtained by using equation (2.1) and (2.2) in (1.9) as follows:

$$\begin{aligned}h_{u,v:n}(z,y) &= \frac{\tau^2\eta^2}{\varphi^2}C_{u,v:n}\sum_{i=0}^{u-1}\sum_{j=0}^{v-u-1}\binom{u-1}{i}\binom{v-u-1}{j}(-1)^{i+j} \\ &\times \frac{\left(\frac{z}{\varphi}\right)^{\eta-1}\left(\frac{y}{\varphi}\right)^{\eta-1}}{\left(1+\left(\frac{z}{\varphi}\right)^{\eta}\right)^{\tau(i+v-u-j)+1}\left(1+\left(\frac{y}{\varphi}\right)^{\eta}\right)^{\tau(n-v+1+j)+1}}\end{aligned}$$

Therefore the product moments, $\mu_{u,v:n} = E(Z_{u:n}Z_{v:n})$, can be written as

$$\begin{aligned}
\mu_{u,v:n} &= \tau^2\eta^2 C_{u,v:n} \sum_{i=0}^{u-1}\sum_{j=0}^{v-u-1} \binom{u-1}{i}\binom{v-u-1}{j}(-1)^{i+j} \\
&\times \int_0^\infty \int_0^y \frac{\left(\frac{z}{\varphi}\right)^\eta \left(\frac{y}{\varphi}\right)^\eta}{\left(1+\left(\frac{z}{\varphi}\right)^\eta\right)^{\tau(i+v-u-j)+1}\left(1+\left(\frac{y}{\varphi}\right)^\eta\right)^{\tau(n-v+1+j)+1}} dzdy \\
&= \varphi\tau^2\eta^2 C_{u,v:n} \sum_{i=0}^{u-1}\sum_{j=0}^{v-u-1} \binom{u-1}{i}\binom{v-u-1}{j}(-1)^{i+j} \\
&\times \int_0^\infty \frac{\left(\frac{y}{\varphi}\right)^\eta}{\left(1+\left(\frac{y}{\varphi}\right)^\eta\right)^{\tau(n-v+1+j)+1}} \left(\frac{1}{\eta}\int_0^{\left(\frac{y}{\varphi}\right)^\eta} \frac{x^{\frac{1}{\eta}}}{(1+x)^{\tau(i+v-u-j)+1}} dx\right) dy \\
&= \varphi\tau^2\eta^2 C_{u,v:n} \sum_{i=0}^{u-1}\sum_{j=0}^{v-u-1} \binom{u-1}{i}\binom{v-u-1}{j}(-1)^{i+j} \\
&\times \int_0^\infty \frac{\left(\frac{y}{\varphi}\right)^\eta}{\left(1+\left(\frac{y}{\varphi}\right)^\eta\right)^{\tau(n-v+1+j)+1}} \underbrace{\left(\frac{1}{\eta}\int_{\frac{1}{1+\left(\frac{y}{\varphi}\right)^\eta}}^1 (1-t)^{\frac{1}{\eta}} t^{\tau(i+v-u-j)-\frac{1}{\eta}-1} dt\right)}_{I} dy,
\end{aligned} \tag{2.10}$$

where $z^\eta = x$ and $t = \frac{1}{x+1}$, it is not difficult to show that I can be simplified

$$\begin{aligned}
I &= B\left(\tau(i+v-u-j)-\frac{1}{\eta}, \frac{1}{\eta}+1\right) - \frac{\left(\frac{1}{1+\left(\frac{y}{\varphi}\right)^\eta}\right)^{\tau(i+v-u-j)+1-\frac{1}{\eta}}}{\tau(i+v-u-j)+1-\frac{1}{\eta}} \\
&\times {}_2F_1\left[\tau(i+v-u-j)-\frac{1}{\eta}, \frac{1}{\eta}, \tau(i+v-u-j)-\frac{1}{\eta}+1, \frac{1}{1+\left(\frac{y}{\varphi}\right)^\eta}\right],
\end{aligned} \tag{2.11}$$

where ${}_pF_q$ is the generalized hypergeometric function defined as

$$ {}_pF_q(r_1,\ldots,s_1,\ldots,s_q;z) = \sum_{k=0}^\infty \frac{(r_1)_k \ldots (r_p)_k}{(s_1)_k \ldots (s_q)_k} \frac{z^k}{n!} $$

Using (2.11) and (2.10), we get the

$$\begin{aligned}
\mu_{u,v:n} &= \varphi^2\tau^2 C_{u,v:n}\sum_{i=0}^{u-1}\sum_{j=0}^{v-u-1}\binom{u-1}{i}\binom{v-u-1}{j}(-1)^{i+j}\Bigg[B\left(\tau(i+v-u-j)-\frac{1}{\eta},\frac{1}{\eta}+1\right)\\
&\times\ B\left(\frac{1}{\eta}+1,\tau(n-v+1+j)-\frac{1}{\eta}\right)-\psi(\tau,\eta)\\
&\times\ \sum_{k=0}^{\infty}B\left(\frac{1}{\eta}+1,\tau(i+n-u+1)-\frac{2}{\eta}+k+1\right)\Bigg],
\end{aligned} \tag{2.12}$$

where

$$\psi(\tau,\eta) = \frac{\left[\tau(i+v-u-j)-\frac{1}{\eta}\right]_k\left[\frac{1}{\eta}\right]_k}{\left[\tau(i+v-u-j)-\frac{1}{\eta}+1\right]_k\left[\tau(i+v-u-j)-\frac{1}{\eta}+1\right]k!}.$$

2.4 BLUEs of the Location and Scale Parameters

Here, we study parameter estimation for the TIIELL distribution based on order statistics. Let $Z_1 \leq Z_2 \leq \cdots \leq Z_n$ be a random sample of size n from TIIELL distribution, the pdf of the scale-parameter TIIELL distribution is

$$h(z)=\frac{\tau\eta\left(\frac{z}{\varphi}\right)^{\eta-1}}{\varphi\left(1+\left(\frac{z}{\varphi}\right)^{\eta}\right)^{\tau+1}},\ \ z>0,\ \ (\tau,\varphi)>0,\ \ \eta>1 \tag{2.13}$$

and the pdf of the location-scale parameter is

$$h(z)=\frac{\tau\eta\left(\frac{z-\delta}{\varphi}\right)^{\eta-1}}{\varphi\left(1+\left(\frac{z-\delta}{\varphi}\right)^{\eta}\right)^{\tau+1}},\ \ z>\delta,\ \ (\tau,\varphi)>0,\ \ \eta>1. \tag{2.14}$$

The expression for the BLUEs of location and scale parameter are given in (1.18) and also variances and covariance for these parameters are given in eqn (1.21), (1.22) and (1.23).

Tables 2.3 and 2.4 display the coefficient of the BLUEs for type-II right censored sample of various values of $n=7,\ 10$ and censoring cases $c=0(1)([n/2]-1)$. Also, Table 2.5 shows variances and covariances of the BLUEs.

2.5 Approximate Inference

Here, we derive the $(1-\alpha)100\%$ confidence intervals for the location and scale parameters of the BLUEs δ^* and φ^* based on the pivotal quantities

$$U_1 = \frac{\delta^* - \delta}{\varphi\sqrt{W_1}}, \quad U_2 = \frac{\varphi^* - \varphi}{\varphi\sqrt{W_2}}, \quad U_3 = \frac{\delta^* - \delta}{\varphi^*\sqrt{W_1}}, \tag{2.15}$$

where δ^* and φ^* are the BLUEs of δ and φ with variances $\varphi^2 W_1$ and $\varphi^2 W_2$, respectively. U_1 used to draw inference for δ when φ is known, while U_3 can be used to draw inference for δ when φ is unknown. Similarly, U_2 can be used to draw inference for φ.

To derive the CIs of the location and scale parameters based on the pivotal quantities in Eqn. (2.15), the moments presented in Section 2.2, are used.

Hence U_1 and U_2 can be rewritten as

$$U_1 = \frac{1}{\sqrt{W_1}}\left(\sum_{u=1}^{n-c} p_u Z_{u:n}\right) = \frac{U_1^*}{\sqrt{W_1}}, \qquad U_2 = \frac{1}{\sqrt{W_2}}\left(\sum_{u=1}^{n-c} q_u Z_{u:n} - 1\right) = \frac{U_2^* - 1}{\sqrt{W_2}}, \tag{2.16}$$

where $Z_{u:n} = (Y_{u:n} - \delta)/\varphi$, $u = 1, 2, \dots, n-c$, is the standardized form of the available Type-II right-censored sample $Y_{u:n}$, $u = 1, 2, \dots, n-c$. Now, we consider to find the approximate distribution by using Edgeworth approximation for a statistic S (with mean 0 and variance 1) as

$$H(s) \approx \Phi(s) - \phi(s)\left[\frac{\sqrt{\tau_1}}{6}(s^2 - 1) + \frac{\tau_2 - 3}{24}(s^3 - 3s) + \frac{\tau_1}{72}(s^5 - 10s^2 + 15s)\right], \tag{2.17}$$

where $\sqrt{\tau_1}$ and τ_2 are the coefficients of skewness and kurtosis of S, respectively and $\Phi(s)$, $\phi(s)$ are the cdf and pdf of the standard normal distribution, respectively.

To obtain the the coefficients of skewness and kurtosis of linear functions of order statistics, single moments $E(Z^{p}_{u:n})$ denoted by $\mu^{p}_{u:n}$, the double moments $Z^{p}_{u:n}Z^{q}_{v:n}$, denoted by $\mu^{(p;q)}_{u,v:n}$ of the TIIELL distribution for $1 \leq u < v \leq (n-c)$ are required.

Table 2.9 displays the values of the mean, variance, coefficients of skewness and kurtosis ($\sqrt{\tau_1}$ and τ_2) of U_1^* and U_2^*. From Table 2.9 it is observed that the distributions of U_1^* and U_2^* and hence of U_1 and U_2 are positively skewed and heavier tailed than normal. Also we can see that $\sqrt{\tau_1}$ of U_1^* and U_2^* increases as η increases and decreases as n increases and decreases as c increases. τ_2 of U_1^* decreases as n increases and increases as η increases and decreases as c increases, while τ_2 of U_2^* increases as n and η increases and decreases as c increases.

We also obtained the lower and upper 1%, 2.5%, 5%, *and* 10% points of U_1 and U_2 through Edgeworth approximation (see Tables 2.6 and 2.7). From Tables 2.6 and 2.7, we can observe that the percentage points of U_1 and U_2 increases as η increases for $n = 7$ and decreases for $n = 10$ in most of the cases and increases as n increases in most of the cases for $\eta = 2$, while decreases as n increases for $\eta = 3$ and decreases as c increases. Similarly the percentage points of U_3 increases as n increases.

The performance of the developed inference can be shown from the simulated average width of confidence intervals in Table 2.10. We observe that the Edgeworth approximations of the distributions of U_1 and U_2 both work quite satisfactory; this is also clear from the average width of the confidence intervals based on U_1 and U_2 which are presented in Table 2.10. In addition we can see that average width decreases as η increases for most of the cases.

TABLE 2.1: Expected values, second moments, variances, skewness and kurtosis of the *u*th order statistic from TIIELL distribution for $n = 1, 2, ..., 10$, $\tau = 2.5$, $\eta = 2$ and $\varphi = 0.25$

u	n	$E(Z)$	$E(Z^2)$	$V(Z)$	τ_1	τ_2	γ_1	γ_2
1	1	0.166667	0.041667	0.013889	6.124244	24.00119	2.474721	21.00119
	2	0.107379	0.015625	0.004095	1.480527	5.834856	1.216769	2.834856
	3	0.085248	0.009615	0.002348	0.970243	4.645654	0.985009	1.645654
	4	0.072834	0.006944	0.001639	0.775930	4.337018	0.880869	1.337018
	5	0.064627	0.005435	0.001258	0.690985	4.060399	0.831255	1.060399
	6	0.058687	0.004464	0.001020	0.653052	3.734868	0.808116	0.734868
	7	0.054132	0.003788	0.000858	0.577341	4.350134	0.759829	1.350134
	8	0.050495	0.003289	0.000739	0.576750	3.853975	0.759440	0.853975
	9	0.047505	0.002907	0.000650	0.534243	4.254706	0.730919	1.254706
	10	0.044990	0.002604	0.000580	0.479173	3.332768	0.692223	0.332768
2	2	0.225955	0.067708	0.016652	6.669762	26.89585	2.582588	23.89585
	3	0.151640	0.027644	0.004649	1.311287	5.810189	1.145115	2.810189
	4	0.122490	0.017628	0.002624	0.742933	4.483691	0.861936	1.483691
	5	0.105661	0.012983	0.001819	0.537407	4.080492	0.733081	1.080492
	6	0.094328	0.010287	0.001389	0.445519	3.949799	0.667472	0.949799
	7	0.086020	0.008523	0.001124	0.391159	3.375693	0.625427	0.375693
	8	0.079588	0.007277	0.000943	0.335838	3.046048	0.579516	0.046048
	9	0.074417	0.006349	0.000811	0.307304	3.904693	0.554350	0.904693
	10	0.070140	0.005632	0.000712	0.278758	3.848412	0.527975	0.848412
3	3	0.263112	0.087740	0.018512	7.184192	28.97034	2.680334	25.97034
	4	0.180790	0.037660	0.004975	1.322886	5.987013	1.150168	2.987013
	5	0.147734	0.024596	0.002771	0.707746	4.471739	0.841277	1.471739
	6	0.128326	0.018375	0.001907	0.492489	3.942891	0.701776	0.942891
	7	0.115099	0.014699	0.001451	0.380277	4.011283	0.616666	1.011283
	8	0.105315	0.012261	0.001170	0.312036	3.453741	0.558602	0.453741
	9	0.097688	0.010522	0.000979	0.302289	3.434712	0.549808	0.434712
	10	0.091521	0.009218	0.000842	0.251679	2.882750	0.501676	0.399884
4	4	0.290553	0.104434	0.020013	7.578064	30.47057	2.752828	27.47057
	5	0.202827	0.046370	0.005231	1.363788	6.098496	1.167813	3.098496
	6	0.167141	0.030817	0.002881	0.708859	4.536456	0.841938	1.536456
	7	0.145963	0.023276	0.001971	0.489885	3.966303	0.699918	0.966303
	8	0.131407	0.018762	0.001494	0.363963	3.895946	0.603293	0.895946
	9	0.120569	0.015738	0.001201	0.301321	3.393636	0.548927	0.393636
	10	0.112075	0.013565	0.001004	0.272530	3.072031	0.522044	0.072031
5	5	0.312484	0.118950	0.021304	7.887611	31.61464	2.808489	28.61464
	6	0.220670	0.054146	0.005451	1.404782	6.200086	1.185235	3.200086
	7	0.183024	0.036473	0.002975	0.717814	4.548271	0.847239	1.548271
	8	0.160518	0.027791	0.002025	0.477320	4.057296	0.690884	1.057296
	9	0.144955	0.022541	0.001529	0.368251	3.717102	0.606837	0.717102
	10	0.133308	0.018998	0.001227	0.293470	3.286510	0.541729	0.286501
6	6	0.330847	0.131910	0.022450	8.139275	32.53265	2.852941	29.53265
	7	0.235729	0.061216	0.005648	1.443160	6.280985	1.201316	3.280985
	8	0.196528	0.041682	0.003059	0.735319	4.529456	0.857508	1.529456
	9	0.172968	0.031990	0.002072	0.488891	4.037825	0.699208	1.037825
	10	0.156602	0.026085	0.001561	0.363625	3.532627	0.603013	0.532627
7	7	0.346701	0.143693	0.023492	8.346127	33.28143	2.888966	30.28143
	8	0.248795	0.067727	0.005828	1.477872	6.381171	1.215677	3.381171
	9	0.208308	0.046527	0.003135	0.750557	4.606563	0.866347	1.606563
	10	0.183879	0.035927	0.002116	0.499695	4.026693	0.706891	1.026693
8	8	0.360687	0.154545	0.024450	8.522843	33.91907	2.919391	30.91907
	9	0.260363	0.073784	0.005995	1.505565	6.457144	1.227015	3.457144
	10	0.218777	0.051071	0.003207	0.757008	4.664468	0.870062	1.664468
9	9	0.373227	0.164640	0.025342	8.674506	34.46545	2.945251	31.46545
	10	0.270760	0.079463	0.006152	1.531512	6.518105	1.237543	3.518105
10	10	0.384613	0.174104	0.026177	8.808048	34.94389	2.967835	31.94389

TABLE 2.2: Expected values, second moments, variances, skewness and kurtosis of the uth order statistic from TIIELL distribution for $n = 1, 2, ..., 10$, $\tau = 5$, $\eta = 2$ and $\varphi = 0.5$

u	n	$E(Z)$	$E(Z^2)$	$V(Z)$	τ_1	τ_2	γ_1	γ_2
1	1	0.214757	0.062501	0.016379	1.482629	5.832059	1.217633	2.832059
	2	0.145668	0.027778	0.006559	0.782284	4.113691	0.884468	1.113691
	3	0.117375	0.017857	0.004080	0.626158	3.790735	0.791301	0.790735
	4	0.100991	0.013158	0.002959	0.560859	3.566212	0.748905	0.566212
	5	0.089980	0.010417	0.002321	0.525832	3.442966	0.725143	0.442966
	6	0.081930	0.008621	0.001908	0.500077	3.502186	0.707161	0.502186
	7	0.075714	0.007353	0.001620	0.495238	3.237636	0.703732	0.237636
	8	0.070728	0.006410	0.001408	0.478453	3.227514	0.691703	0.227514
	9	0.066612	0.005682	0.001245	0.456279	3.397797	0.675484	0.397797
	10	0.063141	0.005102	0.001115	0.451449	3.303742	0.671901	0.303742
2	2	0.283846	0.097222	0.016653	1.400052	6.014629	1.183238	3.014629
	3	0.202256	0.047619	0.006712	0.563894	4.001324	0.750929	1.001324
	4	0.166526	0.031955	0.004224	0.391693	3.627183	0.625854	0.627183
	5	0.145032	0.024123	0.003089	0.322388	3.480211	0.567792	0.480211
	6	0.130231	0.019397	0.002437	0.284753	3.357070	0.533623	0.357070
	7	0.119225	0.016227	0.002012	0.265080	3.246232	0.514859	0.246232
	8	0.110621	0.013952	0.001715	0.245458	3.320959	0.495437	0.320959
	9	0.103652	0.012238	0.001494	0.230641	3.469610	0.480251	0.469610
	10	0.097858	0.010901	0.001324	0.233139	2.927740	0.482845	0.218501
3	3	0.324642	0.122024	0.016632	1.466340	6.256670	1.210925	3.256670
	4	0.237985	0.063283	0.006646	0.526732	4.039598	0.725763	1.039598
	5	0.198768	0.043703	0.004194	0.342041	3.552107	0.584842	0.552107
	6	0.174635	0.033575	0.003078	0.263913	3.429646	0.513725	0.429646
	7	0.157746	0.027320	0.002436	0.222956	3.410097	0.472182	0.410097
	8	0.145037	0.023054	0.002018	0.193353	3.346856	0.439719	0.346856
	9	0.135010	0.019951	0.001723	0.176914	3.377494	0.420612	0.377494
	10	0.126829	0.017590	0.001504	0.175914	2.934875	0.419421	0.192158
4	4	0.353527	0.141604	0.016623	1.540551	6.459855	1.241189	3.459855
	5	0.264129	0.076337	0.006573	0.522017	4.094437	0.722507	1.094437
	6	0.222901	0.053831	0.004146	0.322694	3.617912	0.568062	0.617912
	7	0.197152	0.041916	0.003047	0.241729	3.378629	0.491659	0.378629
	8	0.178928	0.034431	0.002416	0.197982	3.283768	0.444952	0.283768
	9	0.165092	0.029259	0.002004	0.179578	3.085929	0.423766	0.085929
	10	0.154098	0.025460	0.001714	0.161099	3.196057	0.401371	0.196057
5	5	0.375877	0.157921	0.016637	1.608648	6.618557	1.268325	3.618557
	6	0.284744	0.087590	0.006511	0.529866	4.110246	0.727919	1.110246
	7	0.242213	0.062767	0.004101	0.317573	3.609708	0.563536	0.609708
	8	0.215376	0.049401	0.003014	0.232813	3.364292	0.482507	0.364292
	9	0.196223	0.040895	0.002392	0.189503	3.326950	0.435320	0.326950
	10	0.181583	0.034958	0.001986	0.166596	3.238532	0.408162	0.238532
6	6	0.394104	0.171987	0.016669	1.668102	6.757334	1.291550	3.757334
	7	0.301756	0.097519	0.006462	0.540625	4.142795	0.735272	1.142795
	8	0.258315	0.070787	0.004060	0.313873	3.650085	0.560244	0.650085
	9	0.230698	0.056205	0.002983	0.229549	3.451351	0.479112	0.451351
	10	0.210864	0.046832	0.002368	0.180792	3.353638	0.425196	0.353638
7	7	0.409495	0.184398	0.016712	1.719701	6.877223	1.311373	3.877223
	8	0.316236	0.106430	0.006425	0.551682	4.167832	0.742753	1.167832
	9	0.272123	0.078077	0.004026	0.318976	3.677382	0.564780	0.677382
	10	0.243921	0.062454	0.002957	0.228410	3.391861	0.477923	0.391861
8	8	0.422818	0.195537	0.016762	1.764809	6.979383	1.328461	3.979383
	9	0.328840	0.114530	0.006394	0.562789	4.217233	0.750193	1.217233
	10	0.284209	0.084773	0.003998	0.320279	3.693316	0.565932	0.693316
9	9	0.434565	0.205663	0.016816	1.805957	7.064301	1.343859	4.064301
	10	0.339998	0.121970	0.006371	0.572928	4.244680	0.756920	1.244680
10	10	0.445072	0.214962	0.016873	1.842618	7.141398	1.357431	4.141398

TABLE 2.3: Cofficient of the BLUEs of the a_u for $\tau = 1.5$

η	n	c										
2	7	0	0.013963	0.101597	0.558978	0.263992	0.051795	0.009676	-5.27E-07			
		1	-0.34963	0.833878	0.401217	0.108408	0.006114	1.22E-05				
		2	-0.11492	0.812707	0.290103	0.012084	2.69E-05					
	10	0	0.026562	0.067891	0.131564	0.228105	0.328165	0.132287	0.060663	0.023448	0.001311	3.53E-06
		1	0.039625	0.100778	0.188777	0.284606	0.265189	0.082232	0.036252	0.002535	4.15E-06	
		2	0.047991	0.127099	0.250121	0.374234	0.143044	0.053643	0.003868	1.68E-06		
		3	0.029534	0.131456	0.511993	0.235001	0.087684	0.004320	1.37E-05			
		4	-0.25036	0.761078	0.334894	0.145439	0.008938	9.13E-06				
3	7	0	-0.46467	-0.74389	1.449078	0.545318	0.204818	0.009332	1.17E-05			
		1	-0.29246	-0.60828	1.444277	0.453126	0.003324	1.28E-05				
		2	-1.58802	1.838166	0.728422	0.021388	4.29E-05					
	10	0	-0.01692	-0.03763	-0.06087	-0.04818	0.118261	0.764299	0.199978	0.078079	0.002988	1.05E-06
		1	0.002353	0.007739	0.019326	0.051598	0.170786	0.611624	0.131185	0.005383	5.88E-06	
		2	-0.01192	-0.03172	-0.03954	0.060255	0.846969	0.166772	0.009176	7.58E-06		
		3	-0.06968	-0.16761	-0.15093	1.136136	0.240728	0.011335	2.05E-05			
		4	-0.29882	-0.61856	1.540259	0.349781	0.027336	8.17E-06				

TABLE 2.4: Cofficient of the BLUEs of the b_u for $\tau = 1.5$

η	n	c										
2	7	0	-1.42624	-2.69332	2.763219	1.092811	0.224505	0.039025	-3.63E-06			
		1	-5.05483	3.111726	1.515761	0.403758	0.023545	4.36E-05				
		2	-4.45187	3.226687	1.177263	0.047808	0.000117					
	10	0	-0.40703	-0.92138	-1.29380	-0.64137	2.361500	0.514100	0.274523	0.107245	0.006194	1.47E-05
		1	-0.28225	-0.68770	-1.08679	-0.84100	2.287784	0.409765	0.185349	0.014828	1.58E-05	
		2	-0.56606	-1.31300	-1.47408	2.431089	0.650346	0.253884	0.017819	7.43E-06		
		3	-1.34466	-2.59115	2.584963	0.962910	0.369273	0.018615	5.26E-05			
		4	-4.68322	2.837779	1.270177	0.540726	0.034506	3.06E-05				
3	7	0	-2.17816	-3.85459	4.049251	1.422057	0.537002	0.024409	3.09E-05			
		1	-1.65382	-3.79265	4.181823	1.255093	0.009517	3.47E-05				
		2	-6.66135	4.733542	1.872912	0.054788	0.000108					
	10	0	-0.21816	-0.51608	-0.92486	-1.24654	-0.76193	2.846231	0.570023	0.242188	0.009133	2.23E-06
		1	-0.12748	-0.32472	-0.62717	-1.01443	-0.95517	2.569284	0.460068	0.019596	1.77E-05	
		2	-0.25649	-0.69056	-1.27040	-1.44404	3.097968	0.533783	0.029706	2.53E-05		
		3	-0.60690	-1.57047	-2.26163	3.697287	0.707580	0.034080	5.81E-05			
		4	-1.67530	-3.83228	4.469613	0.962392	0.075552	2.27E-05				

TABLE 2.5: Variances and covariance of the BLUEs when $\tau = 1.5$ $\delta = 0$ and $\varphi = 1$

η	n	c	$Var(\delta)$	$Var(\varphi)$	$Cov(\delta, \varphi)$
2	7	0	0.074821	0.970035	0.253528
		1	0.093972	1.102865	0.310651
		2	0.125613	1.597531	0.426752
	10	0	0.044879	0.616469	0.152306
		1	0.052941	0.848929	0.185043
		2	0.060478	0.878651	0.209287
		3	0.073563	0.955970	0.248770
		4	0.087637	1.020583	0.287920
3	7	0	0.153237	0.952101	0.377268
		1	0.214149	1.459957	0.549955
		2	0.248820	1.525382	0.610006
	10	0	0.106367	0.810697	0.284275
		1	0.132891	1.206524	0.379329
		2	0.150195	1.238222	0.414675
		3	0.182625	1.375398	0.488490
		4	0.211345	1.424032	0.539068

TABLE 2.6: Edgeworth approximate and the simulated (*) values of the distribution of U_1 when $\tau = 1.5$, $\delta = 0$ and $\varphi = 1$

η	n	c	1%	2.50%	5%	10%	90%	95%	97.50%	99%
2	7	0	-0.77841	-0.76196	-0.73504	-0.68271	1.083601	1.925702	1.854568	4.255494
			-1.03231*	-0.97438*	-0.90792*	-0.81937*	1.083648*	1.768060*	2.560536*	3.758601*
		1	-2.27628	-2.32369	-0.67909	-0.62752	0.208921	1.889575	1.822893	4.241948
			-1.47652*	-1.41945*	-1.35722*	-1.26558*	0.682143*	1.351194*	2.104918*	3.327379*
		2	-2.34755	-2.41664	-2.56902	-2.86364	0.324397	0.432937	1.791339	3.749634
			-2.06661*	-2.01487*	-1.95338*	-1.86370*	0.126588*	0.818807*	1.556711*	2.749675*
	10	0	-0.63703	-0.62091	-0.59465	-0.54415	1.968333	1.825218	3.499767	4.188481
			-1.19869*	-1.11596*	-1.03013*	-0.90966*	1.176293*	1.816772*	2.499808*	3.605541*
		1	-2.56341	-1.08063	-1.02533	-0.92635	0.773829	1.106087	1.429446	3.896156
			-1.68973*	-1.60117*	-1.51202*	-1.38262*	0.766383*	1.428123*	2.120552*	3.137602*
		2	-2.51757	-2.77431	-2.97307	-2.89647	0.704014	0.993738	1.354706	3.585678
			-2.19815*	-2.10884*	-2.02115*	-1.89652*	0.271168*	0.924273*	1.627683*	2.585719*
		3	-2.53477	-3.16427	-2.98272	-2.98269	0.732411	1.042713	1.382714	1.382703
			-2.72401*	-2.63791*	-2.55189*	-2.42756*	-0.25733*	0.390673*	1.068124*	2.103971*
		4	-3.83866	-3.30443	-2.96218	-2.96219	0.181199	0.835414	1.163553	1.314588
			-3.25567*	-3.17872*	-3.09859*	-2.98289*	-0.81876*	-0.16451*	0.504467*	1.485206*
3	7	0	-0.38025	-0.37311	-0.36123	-0.33769	1.936298	1.888275	1.865233	4.491297
			-0.95660*	-0.90678*	-0.85151*	-0.77025*	1.032663*	1.716643*	2.505057*	3.743581*
		1	-2.28575	-2.32319	-0.66079	-0.61545	0.161453	1.927139	1.872661	4.263209
			-1.45259*	-1.40138*	-1.34743*	-1.26801*	0.586968*	1.278905*	2.080461*	3.263277*
		2	-2.30645	-2.35543	-2.45118	-2.81325	0.226085	0.306988	1.837418	3.764807
			-1.99110*	-1.94309*	-1.89224*	-1.81329*	0.111492*	0.792065*	1.608480*	2.764848*
	10	0	-0.83021	-0.81163	-0.78146	-0.72326	1.296581	1.916373	1.828839	4.202645
			-1.08730*	-1.01830*	-0.94733*	-0.84676*	1.096046*	1.772315*	2.542991*	3.727137*
		1	-2.18033	-2.21922	-2.29146	-2.39668	0.156439	1.849359	1.797111	4.137826
			-1.65246*	-1.57954*	-1.50311*	-1.39675*	0.607943*	1.262664*	2.027025*	3.137897*
		2	-2.41444	-2.52619	-2.92543	-2.92543	0.480104	0.644207	1.703217	3.567014
			-2.28619*	-2.21093*	-2.12713*	-2.01619*	0.051319*	0.725112*	1.482041*	2.567054*
		3	-2.44641	-2.58122	-2.94305	-2.94304	0.424358	0.740532	0.873089	1.518757
			-2.94442*	-2.86884*	-2.78695*	-2.67421*	-0.57559*	0.095727*	0.809759*	1.841936*
		4	-3.94473	-3.48833	-2.95145	-2.95145	-0.14498	0.510946	0.891869	1.527782
			-3.48544*	-3.41859*	-3.34497*	-3.23734*	-1.14494*	-0.48899*	0.243865*	1.308830*

TABLE 2.7: Edgeworth approximate and the simulated (*) values of the distribution of U_2 when $\tau = 1.5$, $\delta = 0$ and $\varphi = 1$

η	n	c	1%	2.50%	5%	10%	90%	95%	97.50%	99%
2	7	0	-0.68625	-0.67387	-0.65342	-0.61301	1.120145	1.963351	1.917835	4.386462
			-0.89515*	-0.85571*	-0.81026*	-0.74149*	1.016280*	1.700985*	2.516495*	3.864885*
		1	-2.23387	-2.26766	-0.58962	-0.54756	0.133227	1.900113	1.852216	4.326887
			-1.37336*	-1.33204*	-1.28475*	-1.21241*	0.661287*	1.334092*	2.137424*	3.366250*
		2	-2.28594	-2.32889	-2.41048	-2.68876	0.190544	0.261264	1.847944	3.857855
			-1.90759*	-1.87523*	-1.83573*	-1.77180*	0.101144*	0.809872*	1.640061*	2.857924*
	10	0	-0.43146	-0.42262	-0.40801	-0.37914	1.928908	1.865193	1.835121	4.403044
			-1.03058*	-0.97355*	-0.91151*	-0.82102*	1.084735*	1.761624*	2.523320*	3.710513*
		1	-2.39482	-0.86625	-0.83388	-0.77177	0.366606	1.928228	1.824014	4.173098
			-1.50745*	-1.44968*	-1.38733*	-1.29935*	0.641782*	1.332129*	2.118734*	3.318632*
		2	-2.25389	-2.31073	-2.42521	-2.81980	0.242052	0.315944	1.774996	3.702989
			-2.09084*	-2.03112*	-1.96779*	-1.87474*	0.139231*	0.812459*	1.597538*	2.703031*
		3	-2.39193	-2.48068	-2.72223	-2.91628	0.413258	0.553318	0.636243	1.679891
			-2.62983*	-2.57057*	-2.50472*	-2.40972*	-0.38170*	0.297366*	1.054344*	2.169122*
		4	-3.95033	-3.49826	-2.94629	-2.94627	0.159258	0.734727	0.863918	0.968469
			-3.14328*	-3.08945*	-3.02804*	-2.93252*	-0.84070*	-0.15407*	0.568380*	1.601999*
3	7	0	-0.37383	-0.36704	-0.35582	-0.33353	1.947225	1.903340	1.882234	4.526831
			-0.89584*	-0.8538*	-0.80822*	-0.73897*	1.003720*	1.702315*	2.507245*	3.772574*
		1	-2.25553	-0.60699	-0.58794	-0.55025	0.094725	1.941317	1.901147	4.364091
			-1.33671*	-1.30106*	-1.26096*	-1.19843*	0.562082*	1.280316*	2.118576*	3.364137*
		2	-2.28995	-2.33181	-2.41096	-2.66945	0.186123	0.257073	1.855683	3.870927
			-1.88758*	-1.85427*	-1.81481*	-1.74995*	0.119499*	0.832150*	1.645907*	2.871001*
	10	0	-0.72809	-0.71409	-0.69103	-0.64574	1.167925	1.949229	1.893921	4.330301
			-0.95512*	-0.90828*	-0.85568*	-0.77823*	1.042522*	1.747703*	2.533224*	3.812363*
		1	-2.12219	-2.13917	-2.16865	-2.23302	0.037221	1.888342	1.864152	4.247133
			-1.43782*	-1.39501*	-1.34838*	-1.27871*	0.511022*	1.219516*	2.011160*	3.247213*
		2	-2.33214	-2.38482	-2.48992	-2.88514	0.254338	0.346561	1.841463	3.647901
			-2.11302*	-2.06494*	-2.01162*	-1.93031*	-0.02184*	0.681566*	1.499523*	2.647942*
		3	-2.42043	-2.50059	-3.24692	-2.94216	0.379866	0.576944	0.665016	1.735106
			-2.76610*	-2.71757*	-2.66356*	-2.58041*	-0.62009*	0.089863*	0.870971*	2.015355*
		4	-4.07898	-3.68032	-2.72291	-2.94198	-0.13232	0.561205	0.679897	1.720768
			-3.29861*	-3.25072*	-3.19745*	-3.11502*	-1.13227*	-0.43873*	0.318227*	1.532725*

TABLE 2.8: Simulated values of the distribution of U_3 when $\tau = 1.5$, $\delta = 0$ and $\varphi = 1$

η	n	c	1%	2.50%	5%	10%	90%	95%	97.50%	99%
2	7	0	-8.58565	-6.00463	-4.36124	-2.92522	0.577351	0.740279	0.891971	1.127982
		1	-22.0015	-14.7982	-10.3426	-7.00606	0.490966	0.754123	0.961019	1.207395
		2	-64.3401	-40.6360	-27.6904	-17.7657	0.128992	0.633082	1.003714	1.452917
	10	0	-6.71847	-4.88497	-3.67234	-2.55740	0.671541	0.860460	1.024841	1.245431
		1	-14.4160	-10.4421	-7.84722	-5.55788	0.624766	0.977244	1.314201	1.778569
		2	-25.1951	-18.3949	-14.0381	-10.1702	0.274881	0.778888	1.175830	1.693537
		3	-43.5958	-31.0430	-23.3486	-16.8049	-0.31343	0.390987	0.902069	1.484084
		4	-80.3106	-55.2902	-40.0114	-27.9420	-1.15339	-0.18127	0.463265	1.118295
3	7	0	-8.28370	-5.77654	-4.18322	-2.83497	0.526657	0.656970	0.744066	0.828474
		1	-25.6916	-17.0079	-12.0546	-8.10941	0.431894	0.703867	0.882413	1.061739
		2	-68.5725	-41.7708	-28.1831	-17.9228	0.104877	0.574845	0.890182	1.167284
	10	0	-7.14072	-5.19687	-3.88199	-2.69139	0.589233	0.735267	0.842694	0.964767
		1	-18.5664	-13.4642	-10.1707	-7.19092	0.482042	0.788716	1.012966	1.290519
		2	-34.5303	-25.0795	-19.0127	-13.4866	0.055358	0.591351	0.969128	1.348443
		3	-60.0887	-43.3305	-32.6716	-23.4520	-0.75955	0.100226	0.669017	1.197819
		4	-104.838	-72.0204	-52.8649	-36.8763	-1.75691	-0.58387	0.226488	0.958274

TABLE 2.9: Mean, Variance and coefficients of skewness and kurtosis of U_1^* and U_2^* when $\tau = 1.5$, $\delta = 0$ and $\varphi = 1$

			U_1				U_2			
η	n	c	Mean	V_1	$\sqrt{\tau_1}$	τ_2	Mean	V_1	$\sqrt{\tau_1}$	τ_2
2	7	0	0.000963	0.074656	1.904050	32.19869	1.005801	0.982068	2.114422	46.51046
		1	-0.08758	0.041760	1.864872	32.90590	0.714386	0.465816	1.986304	42.78576
		2	-0.15359	0.023106	1.763638	25.11762	0.498366	0.258819	1.922601	35.45901
	10	0	0.000786	0.046427	1.746194	31.89380	1.001825	0.652555	2.067376	60.53579
		1	-0.07138	0.026898	1.489195	12.70246	0.722236	0.351057	1.801017	24.78170
		2	-0.12518	0.017567	1.485461	13.29467	0.554844	0.192955	1.758955	28.75834
		3	-0.16467	0.012416	1.478552	12.92351	0.448749	0.127834	1.699910	21.34141
		4	-0.19641	0.009247	1.512846	14.19956	0.374720	0.093584	1.610215	17.27234
3	7	0	0.002024	0.157938	2.234621	78.61409	1.004534	0.982242	2.312447	86.37732
		1	-0.13231	0.077857	1.991401	39.44096	0.676460	0.499386	2.126234	50.97791
		2	-0.21177	0.044925	1.879843	32.08649	0.503454	0.265125	1.939696	36.16978
	10	0	-0.00042	0.105689	1.831395	27.65306	0.999542	0.796568	2.021134	39.48200
		1	-0.11958	0.052235	1.851664	37.89021	0.669847	0.405057	2.214512	75.30156
		2	-0.19250	0.029811	1.632681	18.67683	0.502932	0.196356	1.870876	30.37217
		3	-0.24453	0.019317	1.595066	16.93230	0.398287	0.118517	1.764992	22.59934
		4	-0.28135	0.014526	1.606979	17.01038	0.331043	0.086028	1.748858	22.01017

TABLE 2.10: Average width of the Edgeworth and simulated(*) C.I.'s

		U_1				U_2				U_3	
n	c	90%		95%		90%		95%		90%	95%
7	0	2.660745	2.675980*	2.616533	3.534921*	2.616774	2.511241*	2.591710	3.372206*	5.101516*	6.896599*
	1	2.568667	2.708410*	4.146580	3.524364*	2.489716	2.618837*	4.119881	3.469468*	11.09670*	15.75922*
	2	3.001957	2.772187*	4.207975	3.571577*	2.671747	2.645604*	4.176834	3.515287*	28.32346*	41.63974*
10	0	2.419866	2.846901*	4.120680	3.615770*	2.273196	2.673121*	2.257739	3.496869*	4.532798*	5.909810*
	1	2.131419	2.940145*	2.510081	3.721726*	2.762113	2.719455*	2.690262	3.568417*	8.824464*	11.75634*
	2	3.966812	2.945421*	4.129015	3.736519*	2.741151	2.780245*	4.085728	3.628634*	14.81699*	19.57076*
	3	4.025414	2.942563*	4.546983	3.706037*	3.275550	2.802063*	3.116921	3.624910*	23.73962*	31.94510*
	4	3.797594	2.934083*	4.467987	3.683186*	3.681019	2.873962*	4.362180	3.657831*	39.83013*	55.75350*
7	0	2.249506	2.568142*	2.238341	3.411832*	2.259165	2.510530*	2.249274	3.361044*	4.840185*	6.520606*
	1	2.587927	2.626336*	4.195851	3.481836*	2.529257	2.541276*	2.508136	3.419633*	12.75846*	17.89027*
	2	2.758169	2.684303*	4.192849	3.551572*	2.668035	2.646963*	4.187487	3.500173*	28.75793*	42.66099*
10	0	2.697834	2.719643*	2.640473	3.561290*	2.640264	2.603388*	2.608009	3.441503*	4.617257*	6.039560*
	1	4.140821	2.765767*	4.016333	3.606564*	4.056995	2.567898*	4.003325	3.406160*	10.95945*	14.47721*
	2	3.569634	2.852243*	4.229410	3.692967*	2.836481	2.693163*	4.226280	3.564463*	19.60404*	26.04864*
	3	3.683578	2.882681*	3.454311	3.678602*	3.823864	2.753427*	3.165611	3.588544*	32.77182*	43.99951*
	4	3.462393	2.855982*	4.380202	3.662456*	3.284114	2.758720*	4.360213	3.568945*	52.28107*	72.24690*

2.6 Real Data Application

To demonstrate how the proposed methods can be used in practice, we consider the following real-life data set (see Bhaumik et al. (2009)). The data set represents vinyl chloride from clean upgradient monitoring wells in mg/L. The data are:

```
5.1 1.2 1.3 0.6 0.5 2.4 0.5 1.1 8.0 0.8 0.4 0.6 0.9 0.4 2.0 0.5
5.3 3.2 2.7 2.9 2.5 2.3 1.0 0.2 0.1 0.1 1.8 0.9 2.0 4.0 6.8 1.2
0.4 0.2
```

Now a random sample of size 10 is selected from the given data set and data are: 0.1, 1.1, 0.9, 2.3, 1.3, 2.5, 0.4, 2.0, 0.5, 3.2. Figure 2.1 shows ecdf and QQ-plot of the sample. The Kolmogorov-Smirnov (K-S) statistic is 0.17491 and the corresponding p-value is 0.8697. This shows the suitability of the TIIELL distribution for this data set.

Then by using the BLUEs coefficients in Tables 2.3 and 2.4, we have

$$\delta^* = \sum_{u=1}^{n} a_u Z_{u:n} = 1.012389 \quad \text{and} \quad \varphi^* = \sum_{u=1}^{n} b_u Z_{u:n} = 2.443834$$

By using Table 2.8, the CIs for the location parameter δ as

	90% CI	95% CI
Edgeworth	(0.6257, 1.13836)	(0.27096, 1.14393)
Simulated	(0.6275, 1.23062)	(0.48281, 1.24881)

By using Table 2.9, the CIs for the scale parameter φ as

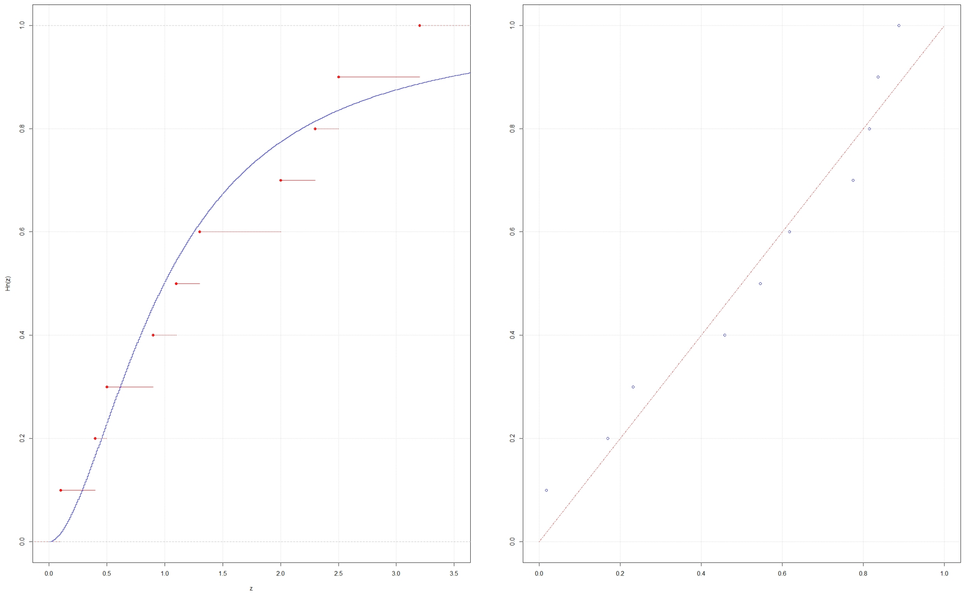

FIGURE 2.1: ECDF-plot and QQ-plot

	90% CI	95% CI
Edgeworth	(0.99163, 3.59568)	(1.00122, 3.65746)
simulated	(1.02546, 3.59603)	(0.91974, 3.67228)

By using Table 2.10, the CIs for the location parameter δ when φ is unknown as

	90% CI	95% CI
simulated	(0.56691, 2.91362)	(0.48181, 3.54143)

We note that the average width of the CIs increase as the level of significant increases.

2.7 Conclusion

In this chapter, the single and product moments of the order statistics from the type II exponentiated log-logistic distribution are derived in explicit forms. The single and product moments are used to obtain the BLUEs of the location and scale parameters of TIIELL distribution. The variances and covariances are calculated to show the performance of the BLUEs. Next, we calculate mean, variance, coefficient of skewness and kurtosis for some linear pivotal quantities. The distributions of the pivotal quantities are calculated in terms of Edgeworth approximation based on BLUEs which in turn can be used to develop confidence intervals. Hence, the distributions of the pivotal quantities are used to construct the interval estimation for the location and scale parameters. The accuracy of the estimated confidence intervals is investigated in terms of the average width. Finally, one real data set has been used to obtain the MLEs of the model parameters, BLUEs of δ and φ and CIs of δ and φ.