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Research Article

Inference on Exponentiated Power Lindley Distribution Based on Order Statistics with Application

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Exponentiated power Lindley distribution is proposed as a generalization of some widely well-known distributions such as Lindley, power Lindley, and generalized Lindley distributions. In this paper, the exact explicit expressions for moments of order statistics from the exponentiated power Lindley distribution are derived. By using these relations, the best linear unbiased estimates of the location and scale parameters, based on type-II right-censored sample, are obtained. Next, the mean, variance, and coefficients of skewness and kurtosis of some certain linear functions of order statistics are calculated and then used to derive the approximate confidence interval for the location and scale parameters using the Edgeworth approximation. Finally, some numerical illustrations and two real data applications are presented.

1. Introduction

Ashour and Eltehiwy [1] introduced the exponentiated power Lindley (EPL) distribution as a generalization of two-parameter power Lindley distribution and they studied some mathematical properties of the EPL distribution. They also showed that this distribution provides more flexibility to analyze a complex real data set. Due to its practicality, the EPL distribution can be used for many applications, including accelerated life testing, survival analysis, reliability, biology, and others. The EPL distribution has attractive feature that its hazard rate function could be decreasing, increasing, and decreasing-increasing-decreasing but not constant, depending on the shape parameter. The EPL distribution has a unimodal and right skewed probability density function (PDF) for $\delta > 1$. Many authors have developed generalization of Lindley distribution. Prominent among these generalizations which we are aware of are generalized Lindley distribution by Nadarajah et al. [2]; power Lindley distribution by Ghitany et al. [3]; generalized inverse Lindley distribution by Sharma et al. [4]; pseudo-Lindley distribution by Nedjar and Zeghdoudi [5]; size-biased gamma Lindley distribution by Beghriche and Zeghdoud [6]. Ashour and Eltehiwy [1] observed that in numerous situations, the EPL distribution provided a better fit than Lindley, power Lindley, generalized Lindley, exponentiated exponential, modified Weibull, and Weibull distributions.

Over the past six decades or so, several authors have shown keen interest in order statistics. These order statistics and their moments are of great significance in many real life applications involving data relating to flood, drought, reliability, engineering, and life testing situation. In recent times, several papers and books have been published on order statistics and their distributional properties. Among them are David and Nagaraja [7]; Arnold et al. [8]; Balakrishnan and Cohen [9]; Balakrishnan and Ahsanullah [10]; Balakrishnan and Sultan [11]; Malik et al. [12];

Mahmoud et al. [13]; Genc [14]; MirMostafaee [15]; Balakrishnan et al. [16]; Kumar and Dey [17]; Sultan et al. [18]; Sultan and AL-Thubyani [19]; Kumar et al. [20]; and so on. Due to the wide applications of order statistics, several authors have carried out extensive studies on these kinds of ordered data. Numerous papers dealing with moments and estimation of parameters of different lifetime distributions based on order statistics can be found in literature. For example, Balakrishnan and Cohen [9] discussed the moments and estimation of parameters. Sultan et al. [18] obtained the moments of power function distribution. Sultan and AL-Thubyani [19] developed the higher-order moments and inferential procedure for estimating parameters of Lindley distribution. Ahsanullah and Alzaatreh [21] studied the moments and inferential procedure for log-logistic distribution. Recently, Kumar and Goyal [22, 23] and Kumar et al. [24, 25] developed inferential procedure for power Lindley and generalized Lindley, modified power function, and generalized inverse Lindley distribution.

Let $Y_1, Y_2, \ldots, Y_m$ be an independent random variable from the EPL distribution with cumulative distribution function (CDF) and PDF is given, respectively, by

$$F(y) = \left[1 - \left(\frac{1 + v + vy^{\gamma}}{1 + v}\right)\exp\left(-vy^{\gamma}\right)\right]^{\delta}, \quad y > 0,\ \delta > 0,\ v > 0,\ \gamma > 0, \tag{1}$$

$$f(y) = \frac{\delta v^2 \gamma\, y^{\gamma-1}}{1 + v}\left(1 + y^{\gamma}\right)\exp\left(-vy^{\gamma}\right) \cdot \left[1 - \left(\frac{1 + v + vy^{\gamma}}{1 + v}\right)\exp\left(-vy^{\gamma}\right)\right]^{\delta-1}, \quad y > 0. \tag{2}$$

The hazard function of the EPL distribution is given by

$$h(y) = \frac{\delta v^2 \gamma\, y^{\gamma-1}}{1 + v}\left(1 + y^{\gamma}\right)\left[V(y)\right]^{\delta-1} \cdot \left[1 - V^{\delta}(y)\right]^{-1}\exp\left(-vy^{\gamma}\right), \quad y > 0, \delta, v, \gamma > 0, \tag{3}$$

where

$$V(y) = \left[1 - \left(\frac{1 + v + vy}{1 + v}\right)\exp\left(-vy^{\gamma}\right)\right]. \tag{4}$$

Some widely well-known distributions can be obtained as special cases from the EPL distribution. By setting $\gamma = 1$ and $\delta = 1$ in both (1) and (2), the CDF and PDF of the generalized Lindley distribution and power Lindley distribution are obtained, respectively, while setting $\delta = \gamma = 1$ gives the CDF and PDF of Lindley distributions.

To the best of our knowledge, there are no reports on moments and estimation for the EPL distribution using order statistics. In this paper, first is to obtain the exact explicit expressions for moments of order statistics from the EPL distribution, and second is to derive the best linear unbiased estimates (BLUEs) of the location and scale parameters of the EPL distribution based on order statistics.

The rest of this paper is organized as follows: in Section 2, the moments of EPL distribution are derived based on order statistics. The BLUEs of the location and scale parameters are provided in Section 3. Next, these BLUEs are used in Section 4 to obtain the coefficient of skewness and kurtosis of some pivotal quantities. Also, we obtain Edgeworth approximations for the distributions of these pivotal quantities. Simulation study and real data analysis are presented in Sections 5 and 6. Finally, the paper ends with a conclusion in Section 7.

2. Moments of Order Statistics

In this section, the moments of order statistics are established for a given random sample $Y_1, Y_2, \ldots, Y_m$ from the EPL distribution.

2.1. Single Moments. In this subsection, the explicit expressions for single moments of order statistics are derived. Let $Y_{1:m} \le Y_{2:m} \le \cdots \le Y_{m:m}$ be the order statistics for a given random sample $Y_1, Y_2, \ldots, Y_m$ from the EPL distribution, and $Y_{r:m}$, $r = 1, 2, \ldots, m$, denotes the rth order statistics with PDF which is given by (see [7])

$$\% \ f_{Y_{r:m}}(x) = C_{r:m}F^{r-1}(y)\left[1 - F(y)\right]^{m-r}f(y), \quad -\infty < y < \infty, \tag{5}$$

where $F(.)$ and $f(.)$ are given by (1) and (2), respectively, and

$$C_{r:m} = \frac{m!}{(r-1)!\,(m-r)!}. \tag{6}$$

Theorem 1. *For the EPL distribution given in (1), the p_1th moment of the rth order statistics $\mu_{r:m}^{(p_1)}$, $p_1 = 0, 1, \ldots$ and $1 \le r \le m$, is as follows:*

$$\mu_{r:m}^{(p_1)} = C_{r:m}\frac{\delta v^2}{(1 + v)}\sum_{j_1=0}^{m-r}\sum_{j_2=0}^{\delta(r+j_1)-1}\sum_{j_3=0}^{j_2}\sum_{j_4=0}^{j_3+1}(-1)^{j_1+j_2}\frac{v^{j_3}}{(1 + v)^{j_2}}\binom{m-r}{j_1} \times \binom{\delta(r + j_1) - 1}{j_2}\binom{j_2}{j_3}\binom{j_3 + 1}{j_4}\frac{\Gamma\left(\left(p_1 + \gamma\left(j_4 + 1\right)\right)/\gamma\right)}{\left[v\left(j_2 + 1\right)\right]^{\left(p_1+\gamma\left(j_4+1\right)\right)/\gamma}}. \tag{7}$$

Proof. From (5), we have

$$\mu_{r:m}^{(p_1)} = E\left(Y_{r:m}^{(p_1)}\right) = C_{r:m}\int_0^\infty y^{p_1}F^{r-1}(y)[1-F(y)]^{m-r}f(y) = C_{r:m}\sum_{j_1=0}^{m-r}(-1)^{j_1}\binom{m-r}{j_1}\int_0^\infty y^{p_1}[F(y)]^{j_1+r-1}f(y)\mathrm{d}y. \quad (8)$$

By using (1) and (2), one can write

$$\begin{aligned}\mu_{r:m}^{(p_1)} &= C_{r:m}\frac{\delta\gamma v^2}{(1+v)}\sum_{j_1=0}^{m-r}(-1)^{j_1}\binom{m-r}{j_1} \cdot \int_0^\infty y^{p_1+\gamma-1}(1+y^\gamma)e^{-vy^\gamma} \times\left[1-\left(\frac{1+v+vy^\gamma}{1+v}\right)e^{-vy^\gamma}\right]^{\delta(r+j_1)-1}\mathrm{d}y \\ &= C_{r:m}\frac{\delta\gamma v^2}{(1+v)}\sum_{j_1=0}^{m-r}\sum_{j_2=0}^{\delta(r+j_1)-1}\sum_{j_3=0}^{j_2}\sum_{j_4=0}^{j_3+1}(-1)^{j_1+j_2} \cdot \frac{v^{j_3}}{(1+v)^{j_2}}\binom{m-r}{j_1}\binom{\delta(r+j_1)-1}{j_2} \\ &\quad\times\binom{j_2}{j_3}\binom{j_3+1}{j_4}\int_0^\infty y^{p_1+\gamma(j_4+1)-1}e^{-v(j_2+1)y^\gamma}\mathrm{d}y,\end{aligned} \quad (9)$$

hence the result.

Some special cases from equation (7):

(1) If $\delta = 1$, we get

$$\mu_{r:m}^{(p_1)} = C_{r:m}\frac{v^2}{(1+v)}\sum_{j_1=0}^{m-r}\sum_{j_2=0}^{r+j_1-1}\sum_{j_3=0}^{j_2}\sum_{j_4=0}^{j_3+1}(-1)^{j_1+j_2}\frac{v^{j_3}}{(1+v)^{j_2}}\binom{m-r}{j_1} \times\binom{r+j_1-1}{j_2}\binom{j_2}{j_3}\binom{j_3+1}{j_4}\frac{\Gamma((p_1+\gamma(j_4+1))/\gamma)}{[v(j_2+1)]^{(p_1+\gamma(j_4+1))/\gamma}}. \quad (10)$$

(2) If $\gamma = 1$, we get

$$\mu_{r:m}^{(p_1)} = C_{r:m}\frac{\delta v^2}{(1+v)}\sum_{j_1=0}^{m-r}\sum_{j_2=0}^{\delta(r+j_1)-1}\sum_{j_3=0}^{j_2}\sum_{j_4=0}^{j_3+1}(-1)^{j_1+j_2}\frac{v^{j_3}}{(1+v)^{j_2}}\binom{m-r}{j_1} \times\binom{\delta(r+j_1)-1}{j_2}\binom{j_2}{j_3}\binom{j_3+1}{j_4}\frac{\Gamma(p_1+j_4+1)}{[v(j_2+1)]^{p_1+j_4+1}}, \quad (11)$$

as obtained by Kumar and Goyal [23].

(3) If $\gamma = 1$ and $\delta = 1$, we get

$$\mu_{r:m}^{(p_1)} = C_{r:m}\frac{v^2}{(1+v)}\sum_{j_1=0}^{m-r}\sum_{j_2=0}^{r+j_1-1}\sum_{j_3=0}^{j_2}\sum_{j_4=0}^{j_3+1}(-1)^{j_1+j_2}\frac{v^{j_3}}{(1+v)^{j_2}}\binom{m-r}{j_1} \times\binom{r+j_1-1}{j_2}\binom{j_2}{j_3}\binom{j_3+1}{j_4}\frac{\Gamma(p_1+j_4+1)}{[v(j_2+1)]^{p_1+j_4+1}}. \quad (12)$$

(4) If $p_1 = 1$ and $r = 1$, we get

$$\mu_{1:m}^{(1)} = \frac{m\delta v^2}{(1+v)}\sum_{j_1=0}^{m-1}\sum_{j_2=0}^{\delta(r+j_1)-1}\sum_{j_3=0}^{j_2}\sum_{j_4=0}^{j_3+1}(-1)^{j_1+j_2}\frac{v^{j_3}}{(1+v)^{j_2}}\binom{m-1}{j_1} \times\binom{\delta(j_1+1)-1}{j_2}\binom{j_2}{j_3}\binom{j_3+1}{j_4}\frac{\Gamma((1+\gamma(j_4+1))/\gamma)}{[v(j_2+1)]^{(1+\gamma(j_4+1))/\gamma}}. \quad (13)$$

(5) If $p_1 = 1$ and $r = m$, we get

$$\mu_{m:m}^{(1)} = \frac{m\delta v^2}{(1+v)}\sum_{j_2=0}^{\delta m-1}\sum_{j_3=0}^{j_2}\sum_{j_4=0}^{j_3+1}(-1)^{j_2}\frac{v^{j_3}}{(1+v)^{j_2}} \times\binom{\delta m-1}{j_2}\binom{j_2}{j_3}\binom{j_3+1}{j_4}\frac{\Gamma((1+\gamma(j_4+1))/\gamma)}{[v(j_2+1)]^{(1+\gamma(j_4+1))/\gamma}}. \quad (14)$$

(6) If $p_1 = 1$ and $m = r = 1$, we get

$$\mu_{1:1}^{(1)} = \frac{\delta v^2}{(1+v)} \sum_{j_2=0}^{\delta-1} \sum_{j_3=0}^{j_2} \sum_{j_4=0}^{j_3+1} (-1)^{j_2} \frac{v^{j_3}}{(1+v)^{j_2}} \cdot \binom{\delta-1}{j_2} \binom{j_2}{j_3} \binom{j_3+1}{j_4} \frac{\Gamma((\gamma(j_4+1)+1)/\gamma)}{[v(j_2+1)]^{(\gamma(j_4+1)+1)/\gamma}} = E(Y), \tag{15}$$

as obtained by Ashour and Eltehiwy [1]. □

2.2. Double Moments. Let $Y_{1:m} \le Y_{2:m} \le \cdots \le Y_{n:m}$ be the order statistics from the EPL distribution. Then, the joint PDF of the rth and sth order statistics is given as

$$f_{Y_{(r:m)},Y_{(s:m)}}(y,z) = C_{r,s:m} F^{r-1}(y)[F(z)-F(y)]^{s-1-r} \cdot [1-F(z)]^{m-s} f(y) f(z), \quad 0<y<z, \tag{16}$$

where $1 \le r < s \le m$ and $C_{r,s:m} = (m!/((r-1)!(s-r-1)!(m-s)!))$.

Theorem 2. *For the EPL distribution given in (1), the double moment of rth and sth order statistics,* $\mu_{r,s:m}^{(p_1,p_2)}$, $p_1, p_2 = 1, 2, \ldots$, *is as follows:*

$$\begin{aligned}\mu_{r,s:m}^{(p_1,p_2)} = {} & C_{r,s:m} \frac{\delta^2 v^4}{(1+v)} \sum_{j_1=0}^{s-r-1} \sum_{j_2=0}^{m-s} \sum_{j_3=0}^{\delta(r+j_1)-1} \sum_{i_4=0}^{\delta(s-r+j_2-j_1)-1} \sum_{j_5=0}^{j_3} \sum_{i_6=0}^{j_4} \sum_{i_7=0}^{j_5+1} \sum_{j_8=0}^{j_6+1} \sum_{j_9=0}^{((p_1+\gamma(j_8+1))/\gamma)-1} \\ & \times (-1)^{j_1+j_2+j_3+j_4} \frac{v^{j_5+j_6}}{(1+v)^{j_1+j_3+j_4}} \binom{s-r-1}{j_1} \binom{m-s}{j_2} \binom{\delta(r+j_1)-1}{j_3} \\ & \times \binom{\delta(s-r+j_2-j_1)-1}{j_4} \binom{j_3}{j_5} \binom{j_4}{j_6} \binom{j_5+1}{j_7} \binom{j_6+1}{j_8} \\ & \times \frac{[v(j_4+1)]^{j_9-((p_1+\gamma(j_8+1))/\gamma)} (((p_1+\gamma(j_8+1))/\gamma)-1)!}{j_9!} \frac{\Gamma((p_1+\gamma(j_8+1)/\gamma)+1)}{[v(j_3+j_4+2)]^{((p_1+\gamma(j_8+1))/\gamma)+1}}.\end{aligned} \tag{17}$$

Proof. Using (16), we have

$$\mu_{r,s:m}^{(p_1,p_2)} = E\left(Y_{r,s:m}^{(p_1,p_2)}\right) = C_{r,s:m} \int_0^\infty \int_y^\infty y^{p_1} z^{p_2} [F(y)]^{r-1} [F(z)-F(y)]^{s-r-1} [1-F(z)]^{n-s} f(y) f(z) \mathrm{d}z \, \mathrm{d}y. \tag{18}$$

By using the same argument as in Theorem 1, we get the result given in (17). □

2.3. Triple Moments. Let $Y_{1:m} \le Y_{2:m} \le \cdots \le Y_{n:m}$ denote the order statistics drawn from the EPL distribution. The joint PDF of the rth, sth, and tth order statistics is given by

$$f_{Y_{(r:m)},Y_{(s:m)},Y_{(t:m)}}(y,z,u) = C_{r,s,t:m} F^{r-1}(y)[F(z)-F(y)]^{s-1-r} [F(u)-F(z)]^{t-1-s} \times [1-F(u)]^{m-t} f(y) f(z) f(u), \quad 0<y<z<u, \tag{19}$$

where $1 \le r < s < t \le m$ and

$$C_{r,s,t:m} = \frac{m!}{(r-1)!(s-r-1)!(t-s-1)!(m-t)!}. \tag{20}$$

Theorem 3. *For the EPL distribution given in (1), The triple moment of rth, sth, and tth order statistics* $\mu_{r,s,t:m}^{(p_1,p_2,p_3)}$, $p_1, p_2, p_3 = 1, 2, \ldots$, *is as follows:*

$$
\begin{aligned}
\mu_{r,s,t:m}^{(p_1,p_2,p_3)} = {} & \delta^3 C_{r,s,t:m} \sum_{\ell_1=0}^{s-1-r} \sum_{\ell_2=0}^{t-1-s} \sum_{\ell_3=0}^{m-t} \binom{s-1-r}{\ell_1} \binom{t-1-s}{\ell_2} \binom{m-t}{\ell_3} (-1)^{\ell_1+\ell_2+\ell_3} \\
& \times \sum_{j_1=0}^{\delta(t-s-\ell_2+\ell_3)-1} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2+1} \sum_{j_4=0}^{(p_3/\gamma)+j_3} \sum_{j_5=0}^{\delta(s-r-\ell_1+\ell_2)-1} \sum_{j_6=0}^{j_5} \sum_{j_7=0}^{j_6+1} \sum_{j_8=0}^{(p_2/\gamma)+j_4+j_7} \sum_{j_9=0}^{\delta(r+\ell_1)-1} \sum_{j_{10}=0}^{j_9} \sum_{j_{11}=0}^{j_{10}+1} \\
& \times \binom{\delta(t-s-\ell_2+\ell_3)-1}{j_1} \binom{j_1}{j_2} \binom{j_2+1}{j_3} \binom{\delta(s-r-\ell_1+\ell_2)-1}{j_5} \binom{j_5}{j_6} \binom{j_6+1}{j_7} \\
& \times \binom{\delta(r+\ell_1)-1}{j_9} \binom{j_9}{j_{10}} \binom{j_{10}+1}{j_{11}} \frac{(-1)^{j_1+j_5+j_9}\, \upsilon^{j_2-j_3+j_6-j_7+j_{10}-j_{11}-((p_1+p_2+p_3)/\gamma)+3}}{j_4!\, j_8!\, (1+\upsilon)^{j_1+j_5+j_9+3} (j_1+1)^{(l/\beta)+j_3-j_4+1}} \\
& \times \frac{((p_3/\gamma)+j_3)!\, ((p_2/\gamma)+j_4+j_7)!\, \Gamma((p_1/\gamma)+j_8+j_{11}+1)}{(j_5+j_1+2)^{(p_2/\gamma)+j_4+j_7-j_8+1} (j_9+j_5+j_1+3)^{(p_1/\gamma)+j_8+j_{11}+1}}.
\end{aligned}
\tag{21}
$$

Proof. From equation (19), we have

$$
\begin{aligned}
\mu_{r,s,t:m}^{(p_1,p_2,p_3)} = E\left(Y_{r:m}^{(p_1)} Y_{s:m}^{(p_2)} Y_{t:m}^{(p_3)}\right) = {} & C_{r,s,t:m} \int_0^\infty \int_y^\infty \int_z^\infty y^{p_1} z^{p_2} u^{p_3} F^{r-1}(y) [F(z)-F(y)]^{s-r-1} \\
& \times [F(u)-F(z)]^{t-s-1} [1-F(u)]^{m-t} f(y) f(z) f(u) \mathrm{d}u\mathrm{d}z\mathrm{d}y.
\end{aligned}
\tag{22}
$$

By using the same argument as in Theorems 1 and 2, we obtain the result given in (21). □

2.4. Quadruple Moments

Let $Y_{1:m} \le Y_{2:m} \le \cdots \le Y_{n:m}$ denote the order statistics drawn from the EPL distribution. The joint PDF of the rth, sth, tth, and wth order statistics is given as

$$
\begin{aligned}
f_{Y_{(r:m)},Y_{(s:m)},Y_{(t:m)},Y_{(w:m)}}(y,z,u,v) = {} & C_{r,s,t,w:m} F^{r-1}(y) [F(z)-F(y)]^{s-1-r} [F(u)-F(z)]^{t-1-s} \\
& \times [F(v)-F(u)]^{w-1-t} [1-F(v)]^{m-w} f(y) f(z) f(u) f(v),
\end{aligned}
\tag{23}
$$

where $1 \le r < s < t < u \le m$ and

$$
C_{r,s,t,w:m} = \frac{m!}{(r-1)!(s-r-1)!(t-s-1)!(w-t-1)(m-w)!}.
\tag{24}
$$

Theorem 4. *For the EPL distribution given in (1), the quadruple moment of rth, sth, tth, and wth order statistics* $\mu_{r,s,t:m}^{(p_1,p_2,p_3,p_4)}$ *is as follows:*

$$\mu_{r,s,t:m}^{(p_1,p_2,p_3,p_4)} = \delta^4 C_{r,s,t,u:m} \sum_{\ell_1=0}^{s-1-r} \sum_{\ell_2=0}^{t-1-s} \sum_{\ell_3=0}^{u-1-t} \sum_{\ell_4=0}^{m-u} \binom{s-1-r}{\ell_1}\binom{t-1-s}{\ell_2}\binom{u-1-t}{\ell_3}\binom{m-u}{\ell_4}$$
$$\times (-1)^{\ell_1+\ell_2+\ell_3+\ell_4} \sum_{j_1=0}^{\delta(w-t-\ell_3+\ell_4)-1} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2+1} \sum_{j_4=0}^{(p_4/\gamma)+j_3} \sum_{j_5=0}^{\delta(t-s-\ell_2+\ell_3)-1} \sum_{j_6=0}^{j_5} \sum_{j_7=0}^{j_6+1} \sum_{j_8=0}^{(p_3/\gamma)+j_4+j_7}$$
$$\times \sum_{j_9=0}^{\delta(s-r-\ell_1+\ell_2)-1} \sum_{j_{10}=0}^{j_9} \sum_{j_{11}=0}^{j_{10}+1} \sum_{i_{12=0}}^{(p_2/\gamma)+j_8+j_{11}} \sum_{j_{13}=0}^{\delta(r+\ell_1)-1} \sum_{j_{14}=0}^{j_{13}} \sum_{j_{15}=0}^{i_{14+1}} \binom{\delta(w-t-\ell_3+\ell_4)-1}{j_1}$$
$$\times \binom{j_1}{j_2}\binom{j_2+1}{j_3}\binom{\delta(t-s-\ell_2+\ell_3)-1}{j_5}\binom{j_5}{j_6}\binom{j_6+1}{j_7}\binom{\delta(s-r-\ell_1+\ell_2)-1}{j_9} \tag{25}$$
$$\times \binom{j_9}{j_{10}}\binom{j_{10}+1}{j_{11}}\binom{\delta(r+\ell_1)-1}{j_{13}}\binom{j_{13}}{j_{14}}\binom{j_{14}+1}{j_{15}}$$
$$\times \frac{(-1)^{j_1+j_5+j_9+j_{13}}\, v^{j_2-j_3+j_6-j_7+j_{10}-j_{11}+j_{14}-j_{15}-((p_1+p_2+p_3+p_4)/\gamma)+4}}{j_4!\, j_8!\, j_{12}!\, (1+v)^{j_1+j_5+j_9+j_{13}+4}\, (j_1+1)^{(p_4/\gamma)+j_3-j_4+1}}$$
$$\times \frac{((p_4/\gamma)+j_3)!\, ((p_3/\gamma)+j_4+j_7)!\, ((p_2/\gamma)+j_8+j_{11})!}{(j_5+j_1+2)^{(p_4/\gamma)+j_4+j_7-j_8+1}\, (j_9+j_5+j_1+3)^{(p_2/\gamma)+j_8+j_{11}-j_{12}+1}}$$
$$\times \frac{\Gamma((p_1/\gamma)+j_{12}+j_{15}+1)}{(j_{13}+j_9+j_5+j_1+4)^{(p_1/\gamma)+j_{12}+j_{15}+1}}.$$

Proof. From equation (23), we have

$$\begin{aligned}\mu_{r,s,t:m}^{(p_1,p_2,p_3,p_4)} &= E\left(Y_{r:m}^{(p_1)} Y_{s:m}^{(p_2)} Y_{t:m}^{(p_3)} Y_{w:m}^{(p_4)}\right)\\ &= C_{r,s,t,w:m} \int_0^\infty \int_y^\infty \int_z^\infty \int_u^\infty\\ &\quad \cdot y^{p_1} z^{p_2} u^{p_3} v^{p_4} [F(z)-F(y)]^{s-1-r}\\ &\quad \times F^{r-1}(y)[F(u)-F(z)]^{t-1-s}\\ &\quad \cdot [F(v)-F(u)]^{w-1-t}[1-F(v)]^{m-w}\\ &\quad \cdot f(y)f(v)f(u)f(v)\,\mathrm{d}v\mathrm{d}u\mathrm{d}z\mathrm{d}y.\end{aligned} \tag{26}$$

By using the same argument as in Theorem 3, we get the result given in (25). □

3. Estimation of Parameters

Here, we study parameter estimation for the EPL distribution based on order statistics.

3.1. BLUEs of Parameters. The PDF of the scale-parameter EPL distribution is

$$\begin{aligned}f(y) &= \frac{\delta\gamma v^2}{\sigma(1+v)}\left(\frac{y}{\sigma}\right)^{\gamma-1}\left[1+\left(\frac{y}{\sigma}\right)^{\gamma}\right]\\ &\quad \cdot \left[1-\frac{1+v+v(y/\sigma)^{\gamma}}{1+v}e^{-v(y/\sigma)^{\gamma}}\right]^{\delta-1}\\ &\quad \cdot e^{-v(y/\sigma)^{\gamma}}, \quad y>0,\ (\delta,\gamma,v,\sigma)>0,\end{aligned} \tag{27}$$

and the PDF of the location-scale parameter EPL distribution is

$$\begin{aligned}f(y) &= \frac{\delta\gamma v^2}{\sigma(1+v)}\left(\frac{y-\mu}{\sigma}\right)^{\gamma-1}\left[1+\left(\frac{y-\mu}{\sigma}\right)^{\gamma}\right]e^{-v((y-\mu)/\sigma)^{\gamma}}\\ &\quad \times\left[1-\frac{1+v+v((y-\mu)/\sigma)^{\gamma}}{1+v}e^{-v((y-\mu)/\sigma)^{\gamma}}\right]^{\delta-1},\\ &\quad y>0,\ (\delta,\gamma,v,\sigma,\mu)>0.\end{aligned} \tag{28}$$

Let $Y_{1:m} \le Y_{2:m} \le \cdots \le Y_{n-c:m}$, $c = 0, 1, \ldots, n-1$ denote type-II right-censored sample from the location-scale parameter EPL distribution in equation (1). Let us denote $X_{r:m} = (Y_{r:m}-\mu)/\sigma$, $E(X_{r:m}) = \mu_{r;m}^{(1)}$, $1 \le r \le (m-c)$, and $\mathrm{Cov}(X_{r:m}, X_{s:m}) = \sigma_{r,s:m} = \mu_{r,s:m}^{(1,1)} - \mu_{r:m}^{(1)}\mu_{s:m}^{(1)}$, $1 \le r < s \le (m-c)$. Therefore,

$$\mathbf{Y} = \left(Y_{1:m}, Y_{2:m}, \ldots, Y_{(m-c):m}\right)^T,$$
$$\boldsymbol{\mu} = \left(\mu_{1:m}, \mu_{2:m}, \ldots, \mu_{(m-c):m}\right)^T,$$
$$1 = \underbrace{(1, 1, \ldots, 1)^T}_{m-c},$$
$$\sum = \left(\left(\sigma_{r,s}\right)\right), \quad 1 \le r,\ s \le m-c, \tag{29}$$

where $\mu_{i:m} = E(Y_{i:m})$, $\sigma_{ii} = \mathrm{Var}(Y_{i:m})$ and $\sigma_{ij} = \mathrm{Cov}(Y_{i:m}, Y_{j:m})$, and $i = 1, 2, \ldots (m-c)$. Then, the BLUEs of μ and σ can be computed as follows (see Arnold et al. [8]):

$$\mu^* = \sum_{r=1}^{m-c} p_r Y_{r:m},$$
$$\sigma^* = \sum_{r=1}^{m-c} q_r Y_{r:m}, \tag{30}$$

where

$$p_r = \left\{ \frac{\mu^T \sum^{-1} \mu 1^T \sum^{-1} - \mu^T \sum^{-1} 1 \mu^T \sum^{-1}}{\left(\mu^T \sum^{-1} \mu\right)\left(1^T \sum^{-1} 1\right) - \left(\mu^T \sum^{-1} 1\right)^2} \right\} \tag{31}$$

$$q_r = \left\{ \frac{1^T \sum^{-1} 1 \mu^T \sum^{-1} - 1^T \sum^{-1} \mu 1^T \sum^{-1}}{\left(\mu^T \sum^{-1} \mu\right)\left(1^T \sum^{-1} 1\right) - \left(\mu^T \sum^{-1} 1\right)^2} \right\}. \tag{32}$$

The variances and covariance of these BLUEs can be computed as follows (see Arnold et al. [8]):

$$\mathrm{Var}(\mu^*) = \sigma^2 \left\{ \frac{\mu^T \sum^{-1} \mu}{\left(\mu^T \sum^{-1} \mu\right)\left(1^T \sum^{-1} 1\right) - \left(\mu^T \sum^{-1} 1\right)^2} \right\} = \sigma^2 W_1,$$
$$\mathrm{Var}(\sigma^*) = \sigma^2 \left\{ \frac{1^T \sum^{-1} 1}{\left(\mu^T \sum^{-1} \mu\right)\left(1^T \sum^{-1} 1\right) - \left(\mu^T \sum^{-1} 1\right)^2} \right\} = \sigma^2 W_2,$$
$$\mathrm{Cov}(\mu^*, \sigma^*) = \sigma^2 \left\{ \frac{-\mu^T \sum^{-1} 1}{\left(\mu^T \sum^{-1} \mu\right)\left(1^T \sum^{-1} 1\right) - \left(\mu^T \sum^{-1} 1\right)^2} \right\} = \sigma^2 W_3. \tag{33}$$

The values of p_r and q_r are displayed in Tables 1 and 2 for different values of sample sizes $n = 7, 10$ and different censoring cases $c = 0(1)([n/2] - 1)$ and for some selected values for $v = 1, 2$. The coefficient of the BLUEs p_r and q_r are given by (17) and (18), respectively, with conditions

$$\sum_{i=1}^{n-c} p_i = 1,$$
$$\sum_{i=1}^{n-c} q_i = 0, \tag{34}$$

which are used to check the computations accuracy.

4. Approximate Inference

Here, we derive the $(1-\alpha)100\%$ confidence intervals for the parameters μ and σ based on the following pivotal quantities:

$$U_1 = \frac{\mu^* - \mu}{\sigma\sqrt{W_1}},$$
$$U_2 = \frac{\sigma^* - \sigma}{\sigma\sqrt{W_2}},$$
$$U_3 = \frac{\mu^* - \mu}{\sigma^*\sqrt{W_1}}, \tag{35}$$

where μ^* and σ^* are the BLUEs of μ and σ with variances $\sigma^2 W_1$ and $\sigma^2 W_2$, respectively. U_1 is used to draw inference for μ when σ is known, while U_3 can be used to draw inference for μ when σ is unknown. Similarly, U_2 can be used to draw inference for σ when μ is unknown.

The moments presented in Section 2 are used to derive the confidence intervals of the location and scale parameters based on the pivotal quantities in equation (35).

Hence, U_1 and U_2 can be rewritten as

$$U_1 = \frac{1}{\sqrt{W_1}}\left(\sum_{i=1}^{m-c} p_i X_{i:m}\right) = \frac{U_1^*}{\sqrt{W_1}},$$
$$U_2 = \frac{1}{\sqrt{W_2}}\left(\sum_{i=1}^{m-c} q_i X_{i:m} - 1\right) = \frac{U_2^* - 1}{\sqrt{W_2}}, \tag{36}$$

where $X_{i:m} = (Y_{i:m} - \mu)/\sigma$, $i = 1, 2, \ldots, m-c$, is the standardized form of the available type-II right-censored sample $Y_{i:m}$, $i = 1, 2, \ldots, m-c$.

we consider finding the approximate distribution by using Edgeworth approximation for a statistic S (with mean 0 and variance 1) as

$$H(s) \approx \Phi(s) - \phi(s)\left[\frac{\sqrt{\alpha_1}}{6}\left(s^2 - 1\right) + \frac{\alpha_2 - 3}{24}\left(s^3 - 3s\right) + \frac{\alpha_1}{72}\left(s^5 - 10s^2 + 15s\right)\right], \tag{37}$$

where $\sqrt{\alpha_1}$ and α_2 are the coefficients of skewness and kurtosis of S, respectively, and $\Phi(s)$ and $\phi(s)$ are the CDF and PDF of the standard normal distribution, respectively.

5. Numerical Aspects

The relations obtained in the preceding sections allow us to evaluate coefficients of the BLUEs. The coefficients of the BLUEs are presented in Tables 1 and 2 while the variances and covariances of the BLUEs are presented in Table 3. From the results presented in Table 3, we observe that the variance of the BLUEs increases as the censoring level increases while the variance of the BLUEs decreases when the sample size increases and increases as v increases. In addition, we observe that the covariances of the BLUEs decrease as the censoring level increases while the covariances of the BLUEs increase when the sample size increases and decrease as v increases.

Tables 4–6 display the simulated percentage points at $v = 1(1)3$ and sample size $n = 7, 10$. Table 7 presents the values of the mean, variance, and coefficient of skewness and

Table 1: Coefficients of the BLUEs of the location parameter.

v	n	c	p_i, $i = 1, 2, \ldots, (n-c)$									
1	7	0	1.136733	0.005506	−0.023661	−0.026954	−0.024014	−0.032131	−0.035482			
		1	1.161942	−0.000791	−0.040361	−0.029379	−0.039803	−0.051603				
		2	1.215986	−0.044508	−0.036811	−0.061522	−0.073146					
	10	0	1.092681	−0.014652	0.027777	−0.023314	−0.001847	−0.019939	−0.014359	−0.014698	−0.014324	−0.017325
		1	1.105425	−0.005128	0.000896	−0.017679	0.001242	−0.025909	−0.023216	−0.012611	−0.023022	
		2	1.129876	−0.005171	−0.017773	−0.019798	−0.008271	−0.031966	−0.019298	−0.027601		
		3	1.136733	0.005506	−0.023616	−0.026954	−0.024014	−0.032131	−0.035482			
		4	1.161942	−0.000796	−0.040361	−0.029379	−0.039803	−0.051603				
2	7	0	1.131801	0.004217	−0.030454	−0.006347	−0.030267	−0.030084	−0.038867			
		1	1.164716	−0.015401	−0.017184	−0.042061	−0.038315	−0.051755				
		2	1.204871	−0.030066	−0.038861	−0.061012	−0.074933					
	10	0	1.078095	0.018066	−0.012711	0.010341	−0.018799	−0.013232	−0.014746	−0.014122	−0.014181	−0.018722
		1	1.088818	0.021605	−0.014117	−0.007688	−0.019094	−0.010235	−0.016854	−0.018684	−0.023699	
		2	1.107122	0.015291	−0.019433	−0.017916	−0.008332	−0.019141	−0.029435	−0.028157		
		3	1.131801	0.004217	−0.030454	−0.006347	−0.030267	−0.030084	−0.038867			
		4	1.164716	−0.015401	−0.017184	−0.04206	−0.038315	−0.051755				
3	7	0	1.125991	0.003349	−0.003324	−0.026256	−0.035498	−0.02695	−0.037311			
		1	1.158534	−0.006501	−0.023706	−0.036411	−0.039839	−0.052078				
		2	1.203149	−0.025636	−0.048175	−0.053915	−0.075387					
	10	0	1.084513	−0.001082	0.012616	−0.006894	−0.025682	0.004272	−0.023087	−0.010631	−0.015161	−0.018865
		1	1.089608	0.005737	0.014226	−0.023515	−0.010205	−0.013899	−0.022786	−0.016716	−0.022451	
		2	1.113338	−0.006922	0.010884	−0.026283	−0.014407	−0.026124	−0.022595	−0.027892		
		3	1.125991	0.003349	−0.003324	−0.026256	−0.035498	−0.026951	−0.037311			
		4	1.158534	−0.006501	−0.023706	−0.036411	−0.039839	−0.052078				

Table 2: Coefficients of the BLUEs of the scale parameter.

v	n	c	q_i, $i = 1, 2, \ldots, (n-c)$									
1	7	0	−0.607191	0.049265	0.107357	0.098563	0.106076	0.115131	0.130798			
		1	−0.619096	0.065174	0.148212	0.116684	0.133273	0.155753				
		2	−0.650729	0.142376	0.160171	0.154025	0.194157					
	10	0	−0.615392	0.081514	0.028793	0.038189	0.074711	0.069523	0.077051	0.087084	0.071246	0.087283
		1	−0.623159	0.089041	0.027265	0.079845	0.064478	0.087226	0.094728	0.082462	0.098115	
		2	−0.610879	0.067192	0.042666	0.108053	0.081235	0.102664	0.097707	0.111363		
		3	−0.607191	0.049265	0.107357	0.098563	0.106076	0.115131	0.130798			
		4	−0.619096	0.065174	0.148212	0.116684	0.133273	0.155753				
2	7	0	−0.603985	0.056041	0.086529	0.098091	0.120122	0.117091	0.126112			
		1	−0.624402	0.081701	0.122939	0.128854	0.140442	0.150465				
		2	−0.647063	0.145811	0.136075	0.176638	0.188541					
	10	0	−0.575495	0.011332	0.070903	0.025755	0.098121	0.040347	0.079868	0.082063	0.081904	0.085201
		1	−0.580427	0.010583	0.078087	0.062771	0.081424	0.065365	0.098493	0.087668	0.096037	
		2	−0.598187	0.043684	0.069971	0.095037	0.077136	0.097313	0.106356	0.108691		
		3	−0.603985	0.056041	0.086529	0.098091	0.120122	0.117091	0.126112			
		4	−0.624402	0.081701	0.122939	0.128854	0.140442	0.150465				
3	7	0	−0.610973	0.071403	0.064731	0.127295	0.117376	0.097999	0.132169			
		1	−0.635591	0.101063	0.121287	0.132771	0.123643	0.156826				
		2	−0.635278	0.123788	0.159474	0.157116	0.194901					
	10	0	−0.608131	0.067851	0.040462	0.026115	0.087299	0.069354	0.083118	0.079318	0.063311	0.091304
		1	−0.601453	0.054542	0.050085	0.045373	0.094679	0.099195	0.082039	0.075534	0.100006	
		2	−0.605026	0.060417	0.060951	0.063863	0.126491	0.094747	0.085135	0.113422		
		3	−0.610973	0.071403	0.064731	0.127295	0.117376	0.097999	0.132169			
		4	−0.635591	0.101063	0.121287	0.132771	0.123643	0.156826				

kurtosis U_1^* and U_2^*. The simulated average widths of confidence intervals are presented in Table 8. We observe that the Edgeworth approximations of the distributions of U_1 and U_2 both work quite satisfactorily; this is also clear from the average width of the confidence intervals based on U_1 and U_2 which are presented in Table 8.

Table 3: Variance and covariance of the BLUEs when $\mu = 0$ and $\sigma = 1$.

v	n	c	Var (μ)	Var (σ)	Cov (μ, σ)
1	7	0	0.068821	0.135051	−0.041267
		1	0.093333	0.160655	−0.055683
		2	0.141613	0.201843	−0.083785
	10	0	0.033913	0.088141	−0.020113
		1	0.042059	0.100032	−0.024623
		2	0.053405	0.116162	−0.031974
		3	0.068821	0.135051	−0.041267
		4	0.093333	0.160655	−0.055683
2	7	0	0.070562	0.135301	−0.041106
		1	0.098388	0.164369	−0.057362
		2	0.146701	0.208599	−0.085694
	10	0	0.034945	0.089757	−0.020554
		1	0.042393	0.101301	−0.025082
		2	0.054041	0.114633	−0.030894
		3	0.070562	0.135301	−0.041106
		4	0.098388	0.164369	−0.057362
3	7	0	0.072062	0.137961	−0.041643
		1	0.100875	0.167103	−0.058229
		2	0.142707	0.211536	−0.084917
	10	0	0.033831	0.091732	−0.020271
		1	0.042178	0.103774	−0.025141
		2	0.053543	0.119151	−0.031737
		3	0.072062	0.137961	−0.041643
		4	0.100875	0.167103	−0.058229

6. Real Data Analysis

In this section, we analyze two real data sets to show the importance of the proposed estimators. One data set from environmental monitoring and another data is from maximum flood level for the Susquehanna River at Harrisburg.

Example 1. Analysis of clean upgradient ground-water monitoring wells subjected to mg/L.In the first data set, we consider vinyl chloride data obtained from clean upgradient monitoring wells which was studied by many authors such as Bhaumik and Gibbons [26]; Krishnamoorthy et al. [27]; Bhaumik et al. [28]; and Kumar and Goyal [22, 23]. The data are as follows:

$$\begin{array}{ccccccccccccccccc} 5.1 & 1.2 & 1.3 & 0.6 & 0.5 & 2.4 & 0.5 & 1.1 & 8.0 & 0.8 & 0.4 & 0.6 & 0.9 & 0.4 & 2.0 & 0.5 & 5.3 \\ 3.2 & 2.7 & 2.9 & 2.5 & 2.3 & 1.0 & 0.2 & 0.1 & 0.1 & 1.8 & 0.9 & 2.0 & 4.0 & 6.8 & 1.2 & 0.4 & 0.2\,. \end{array} \tag{38}$$

Now, a random sample of size 10 is selected from the given data set 1.0, 1.2, 3.2, 2.4, 0.8, 2.0, 0.4, 0.2, 2.9, 1.2. By the EPL distribution for the given sample, we have obtained the maximum likelihood estimate of $\delta_{ML} = 0.59023$, $\gamma_{ML} = 1.808770$, and $v_{ML} = 0.434371$. By computation, the Kolmogorov–Smirnov (K–S) distance and the corresponding p values are 0.163772 and 0.9513, respectively, which implies that the EPL distribution provides a reasonable model for this data. Moveover, the empirical cumulative distribution function (ECDF) plot and the Quantile-Quantile (Q-Q) plots are also presented under the vinyl chloride data as shown in Figure 1, which also imply that the EPL distribution can be used as a proper model to fit this data.

Then, by using the BLUE coefficients in Tables 1 and 2, we have

$$\begin{aligned} \mu^* &= \sum_{j=1}^{n} p_j X_{j:\,n} = 0.024467, \\ \sigma^* &= \sum_{j=1}^{n} q_j X_{j:\,n} = 0.992854. \end{aligned} \tag{39}$$

Example 2. Analysis of the maximum flood level for the Susquehanna River at Harrisburg, Pennsylvania.The second data set presents the maximum flood level for the Susquehanna River at Harrisburg, Pennsylvania, and it was studied by Dumonceaux and Antle [29]. The data are

$$\begin{array}{cccccccccc} 0.654 & 0.613 & 0.315 & 0.449 & 0.297 & 0.402 & 0.379 & 0.423 & 0.379 & 0.3235 \\ 0.269 & 0.740 & 0.418 & 0.412 & 0.494 & 0.416 & 0.338 & 0.392 & 0.484 & 0.265\,. \end{array} \tag{40}$$

Table 4: Simulated values of the distribution of U_1 when $\mu = 0$ and $\sigma = 1$.

v	n	c	1%	2.50%	5%	10%	90%	95%	97.50%	99%
1	7	0	−3.686292	−3.686298	−3.68631	−1.23409	1.448849	1.811324	2.190922	3.049235
		1	−1.754612	−1.459681	−1.25981	−1.03561	1.307655	1.924963	2.499715	3.159464
		2	−1.794954	−1.508401	−1.28934	−1.05182	1.298412	1.895139	2.505579	3.260451
	10	0	−1.483964	−1.326313	−1.17203	−0.99761	1.349227	1.938966	2.549851	3.414044
		1	−1.525359	−1.336776	−1.18298	−0.9978	1.343092	1.914571	2.540299	3.359295
		2	−1.586863	−1.376637	−1.20636	−1.01423	1.338138	1.922457	2.520371	3.232563
		3	−1.630238	−1.425475	−1.23309	−1.01737	1.311007	1.917257	2.517179	3.225972
		4	−1.754612	−1.459681	−1.25981	−1.03561	1.307655	1.924963	2.499715	3.159464
2	7	0	−1.615403	−1.391141	−1.21902	−1.01619	1.319972	1.922683	2.491423	3.252833
		1	−1.663496	−1.448206	−1.25977	−1.03223	1.332836	1.918422	2.494936	3.207337
		2	−1.802659	−1.515264	−1.29324	−1.05587	1.319081	1.907491	2.448389	3.115657
	10	0	−1.479384	−1.299185	−1.15835	−0.98451	1.343612	1.984078	2.548237	3.182261
		1	−1.505824	−1.348861	−1.17004	−0.99502	1.335991	1.955258	2.533781	3.205181
		2	−1.548062	−1.357995	−1.18398	−1.00052	1.346091	1.950572	2.496046	3.253751
		3	−1.615403	−1.391141	−1.21902	−1.01619	1.319972	1.922683	2.491423	3.252833
		4	−1.663496	−1.448206	−1.25977	−1.03223	1.332836	1.918422	2.494936	3.207337
3	7	0	−1.638623	−1.422331	−1.23184	−1.01266	1.280785	1.903994	2.515901	3.268596
		1	−1.709657	−1.464361	−1.24497	−1.01847	1.276936	1.923978	2.499509	3.192122
		2	−1.805743	−1.519798	−1.30221	−1.05218	1.299622	1.909421	2.457119	3.214435
	10	0	−1.519226	−1.329914	−1.17265	−1.00041	1.324002	1.937141	2.541251	3.326594
		1	−1.550921	−1.350977	−1.18953	−1.00624	1.293752	1.912262	2.554841	3.373878
		2	−1.604826	−1.395428	−1.21302	−1.0188	1.294612	1.921591	2.512003	3.376578
		3	−1.638623	−1.422331	−1.23184	−1.01266	1.280785	1.903994	2.515901	3.268596
		4	−1.709657	−1.464361	−1.24497	−1.01847	1.276936	1.923978	2.499509	3.192122

Table 5: Simulated values of the distribution of U_2 when $\mu = 0$ and $\sigma = 1$.

v	n	c	1%	2.50%	5%	10%	90%	95%	97.50%	99%
1	7	0	−2.678889	−2.471173	−2.301446	−2.053213	0.454372	0.957141	1.421979	1.948598
		1	−2.554143	−2.375446	−2.212094	−1.974706	0.526451	1.017625	1.504291	2.077757
		2	−2.395787	−2.253001	−2.077632	−1.873422	0.630571	1.116881	1.612411	2.259385
	10	0	−3.019991	−2.759098	−2.548864	−2.288165	0.237174	0.720949	1.085917	1.604703
		1	−2.896556	−2.664619	−2.453476	−2.199037	0.315051	0.800666	1.187191	1.704299
		2	−2.807636	−2.583198	−2.374634	−2.120514	0.392716	0.853461	1.299217	1.812966
		3	−2.678889	−2.471173	−2.301446	−2.053213	0.454372	0.957141	1.421979	1.948598
		4	−2.554143	−2.375446	−2.212094	−1.974706	0.526451	1.017625	1.504291	2.077757
2	7	0	−2.710056	−2.505164	−2.309474	−2.070679	0.448602	0.949731	1.393545	1.913112
		1	−2.582256	−2.390288	−2.218548	−1.967492	0.523874	1.032095	1.508915	2.087237
		2	−2.393321	−2.230162	−2.073431	−1.862678	0.631468	1.147924	1.606034	2.213561
	10	0	−3.024276	−2.730335	−2.541782	−2.270712	0.275171	0.734777	1.150898	1.665428
		1	−2.898164	−2.655736	−2.463116	−2.217135	0.317773	0.801581	1.253501	1.728512
		2	−2.823059	−2.605918	−2.400323	−2.143586	0.369156	0.863872	1.325027	1.825864
		3	−2.710056	−2.505164	−2.309474	−2.070679	0.448602	0.949731	1.393545	1.913112
		4	−2.582256	−2.390288	−2.218548	−1.967492	0.523874	1.032095	1.508915	2.087237
3	7	0	−2.649813	−2.466518	−2.258944	−2.041399	0.498609	0.972135	1.437702	1.937601
		1	−2.519256	−2.344106	−2.173079	−1.946906	0.568205	1.095691	1.556311	2.072295
		2	−2.371208	−2.198918	−2.060219	−1.834712	0.659611	1.176077	1.635401	2.248324
	10	0	−2.926982	−2.724922	−2.529415	−2.255072	0.266124	0.745879	1.168265	1.687633
		1	−2.829318	−2.645105	−2.454084	−2.199288	0.346392	0.800395	1.295153	1.766573
		2	−2.748096	−2.559353	−2.350791	−2.123788	0.411725	0.895525	1.360834	1.909579
		3	−2.641983	−2.466518	−2.258944	−2.041399	0.498609	0.972135	1.437702	1.937601
		4	−2.519256	−2.344106	−2.173079	−1.946906	0.568205	1.095691	1.556311	2.072295

Now, a random sample of size 10 is selected from the given data set 0.418, 0.265, 0.613, 0.269, 0.654, 0.338, 0.315, 0.379, 0.297, 0.412. By the EPL distribution for the given sample, we have obtained the maximum likelihood estimate of $\delta_{\mathrm{ML}} = 3479.9773640$, $\gamma_{\mathrm{ML}} = 0.484329$, and $v_{\mathrm{ML}} = 14.626781$. By computation, the Kolmogorov–Smirnov (K–S) distance and the corresponding p values are 0.14282 and 0.9694, respectively, which implies that the EPL distribution provides a reasonable model for this data. Moreover, as a further illustration, the empirical cumulative

Table 6: Simulated values of the distribution of U_3 when $\mu = 0$ and $\sigma = 1$.

v	n	c	1%	2.50%	5%	10%	90%	95%	97.50%	99%
1	7	0	−1.291825	−1.262013	−1.219325	−1.149467	2.222395	3.523847	4.940092	7.134944
		1	−1.293906	−1.257118	−1.216151	−1.140551	2.350343	3.765088	5.389301	7.852344
		2	−1.252756	−1.219676	−1.181444	−1.106255	2.545932	4.159147	6.076095	8.899025
	10	0	−1.323036	−1.286828	−1.242329	−1.165381	2.110051	3.223612	4.498312	6.167443
		1	−1.308061	−1.270312	−1.231131	−1.161611	2.148975	3.262731	4.524937	6.631667
		2	−1.294854	−1.264969	−1.227509	−1.157487	2.252917	3.411704	4.704752	6.501364
		3	−1.291825	−1.262013	−1.219325	−1.149467	2.222395	3.523847	4.940092	7.134944
		4	−1.293906	−1.257118	−1.216151	−1.140551	2.350343	3.765088	5.389301	7.852344
2	7	0	−1.299635	−1.254525	−1.210859	−1.140081	2.320086	3.584004	5.171739	7.060132
		1	−1.269125	−1.231214	−1.188392	−1.119388	2.477281	3.871941	5.405274	7.898144
		2	−1.234267	−1.201231	−1.165376	−1.094119	2.620875	4.125688	5.952571	8.910766
	10	0	−1.316707	−1.279803	−1.234296	−1.155644	2.127752	3.257233	4.511755	5.975934
		1	−1.318328	−1.274231	−1.231598	−1.152525	2.158989	3.380708	4.554498	6.349453
		2	−1.306309	−1.266361	−1.220428	−1.142738	2.227079	3.486914	4.694881	6.876191
		3	−1.299635	−1.254525	−1.210859	−1.140081	2.320086	3.584004	5.171739	7.060132
		4	−1.269125	−1.231214	−1.188392	−1.119388	2.477281	3.871941	5.405274	7.898144
3	7	0	−1.289032	−1.249364	−1.210882	−1.131658	2.240714	3.457987	4.917726	7.291664
		1	−1.258874	−1.224641	−1.180973	−1.103054	2.319861	3.713039	5.304917	8.216081
		2	−1.243127	−1.210756	−1.171678	−1.099442	2.521328	4.071925	6.071664	9.205739
	10	0	−1.340722	−1.297027	−1.250248	−1.174011	2.080714	3.246739	4.459408	6.277229
		1	−1.326197	−1.285656	−1.237163	−1.155345	2.090644	3.291885	4.750722	6.487507
		2	−1.313368	−1.278006	−1.229561	−1.152197	2.173185	3.366241	4.764053	6.650645
		3	−1.289032	−1.249364	−1.210882	−1.131658	2.240714	3.457987	4.917726	7.291664
		4	−1.258874	−1.224641	−1.180973	−1.103054	2.319861	3.713039	5.304917	8.216081

Table 7: Mean, variance, and coefficient of skewness and kurtosis of U_1^* and U_2^* when $\mu = 0$ and $\sigma = 1$.

v	n	c	U_1				U_2			
			Mean	W_1	$\sqrt{\alpha_1}$	α_2	Mean	W_1	$\sqrt{\alpha_1}$	α_2
1	7	0	0	0.039518	1.293415	2.911714	−0.869812	0.077551	0.742414	0.979527
		1	0	0.053594	1.192561	2.520105	−0.797493	0.092252	0.779777	1.008172
		2	0	0.081318	1.120099	2.226028	−0.711487	0.115903	0.855671	1.063987
	10	0	0	0.019474	1.423328	3.072102	−1.076674	0.050613	0.577731	0.656743
		1	0	0.024152	1.390529	2.929123	−1.010657	0.057441	0.646992	0.849696
		2	0	0.030666	1.313859	2.779625	−0.937868	0.066703	0.704326	0.996806
		3	0	0.039518	1.293415	2.911714	−0.869812	0.077551	0.742414	0.979527
		4	0	0.053594	1.192561	2.520105	−0.797493	0.092252	0.779777	1.008172
2	7	0	0	0.040446	1.346119	2.959134	−0.872235	0.077554	0.657331	0.621787
		1	0	0.056396	1.290511	2.950683	−0.791361	0.094215	0.711686	0.724204
		2	0	0.084088	1.154996	2.724035	−0.702471	0.119568	0.845766	1.076625
	10	0	0	0.020031	1.520931	3.976296	−1.070904	0.051448	0.563839	0.452005
		1	0	0.024299	1.464689	3.627171	−1.008042	0.058065	0.589949	0.502475
		2	0	0.030976	1.446371	3.457844	−0.947609	0.065707	0.625893	0.561158
		3	0	0.040446	1.346119	2.959134	−0.872235	0.077554	0.657331	0.621787
		4	0	0.056396	1.290511	2.950683	−0.791361	0.094215	0.711686	0.724204
3	7	0	0	0.041336	1.406169	3.581796	−0.862483	0.079136	0.678438	0.457074
		1	0	0.057864	1.338369	3.284062	−0.783676	0.095853	0.757721	0.647221
		2	0	0.081859	1.167649	2.806261	−0.696523	0.121341	0.870486	0.981312
	10	0	0	0.019406	1.420998	3.140231	−1.057713	0.052619	0.563401	0.372104
		1	0	0.024194	1.468318	3.741814	−0.994451	0.059526	0.590185	0.356138
		2	0	0.030713	1.375474	3.216648	−0.928065	0.068347	0.631802	0.412006
		3	0	0.041336	1.406169	3.581796	−0.862483	0.079136	0.678438	0.457074
		4	0	0.057864	1.338369	3.284062	−0.783676	0.095853	0.757721	0.647221

Table 8: Average width of the simulated CIs.

v	n	c	U_1 90%	U_1 95%	U_2 90%	U_2 95%	U_3 90%	U_3 95%
1	7	0	3.141698	3.882563	3.259204	3.898709	3.018612	3.606341
		1	3.178194	3.943143	3.250642	3.899202	3.181415	3.910418
		2	3.200727	3.963654	3.221355	3.836194	3.274709	3.101615
	10	0	3.142429	3.847422	3.276559	3.881234	3.475781	3.902103
		1	3.125297	3.882641	3.264698	3.909236	3.981026	3.154185
		2	3.134547	3.854041	3.264195	3.930944	3.560345	3.925101
		3	3.141698	3.882563	3.259204	3.898709	3.670165	3.258404
		4	3.178194	3.943143	3.250642	3.899202	3.413724	3.135169
2	7	0	3.150345	3.942654	3.258585	3.893152	3.436196	3.853183
		1	3.184769	3.959394	3.229719	3.879738	3.066534	3.860473
		2	3.184477	4.013981	3.194513	3.865411	3.949676	3.866606
	10	0	3.110997	3.876164	3.269813	3.845016	3.435706	3.873179
		1	3.097545	3.877075	3.254142	3.851881	3.846834	3.109103
		2	3.128814	3.897008	3.228095	3.882415	3.489336	3.951775
		3	3.150345	3.942654	3.258585	3.893152	3.369341	3.785608
		4	3.184769	3.959394	3.229719	3.879738	3.313953	3.148138
3	7	0	3.135834	3.938231	3.231079	3.904221	3.889142	3.348875
		1	3.168948	3.963871	3.268771	3.900417	3.971983	3.771145
		2	3.211631	3.976917	3.236295	3.834328	3.097121	3.096853
	10	0	3.109786	3.871166	3.275329	3.893187	3.419819	3.801354
		1	3.101788	3.905817	3.254481	3.940258	3.876775	3.313559
		2	3.134611	3.907431	3.246316	3.920187	3.421646	3.916297
		3	3.135834	3.938231	3.231079	3.904221	3.601593	3.065644
		4	3.168948	3.963871	3.268771	3.900417	3.242489	3.029839

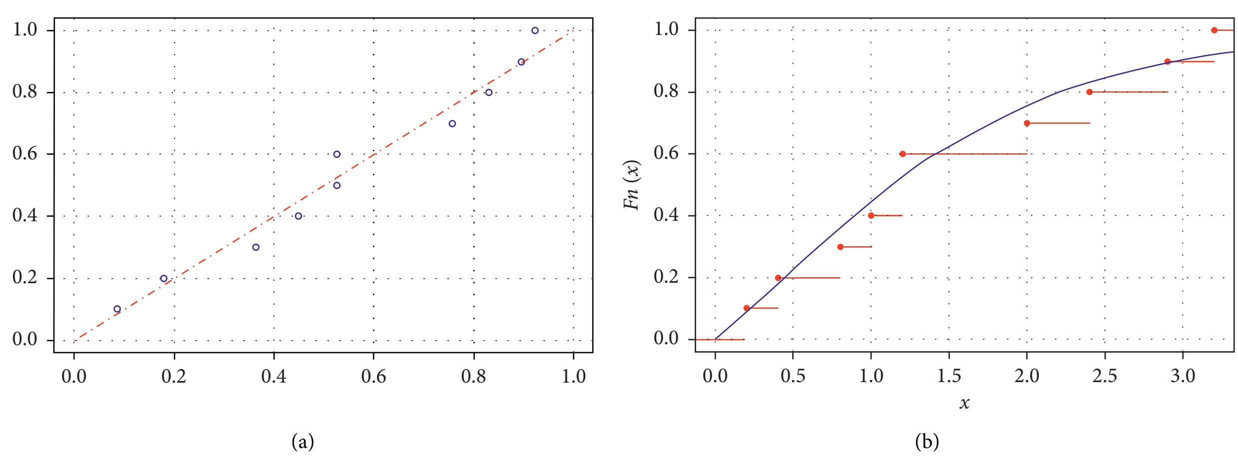
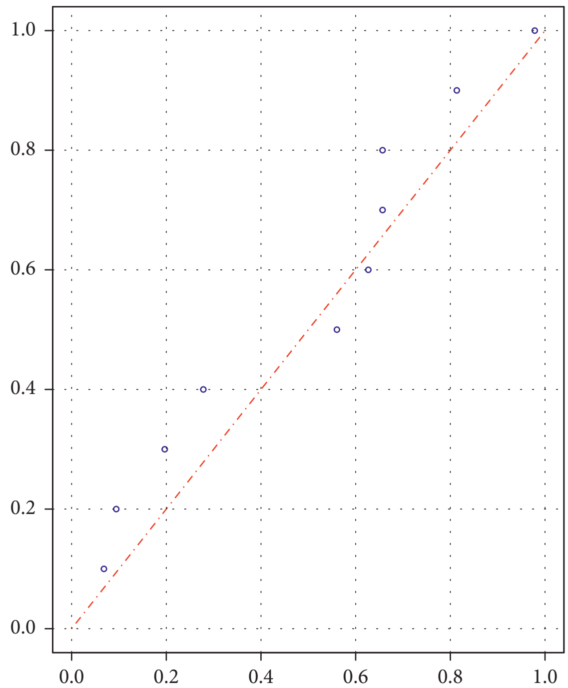


Figure 1: QQ plot and CDF of the real data set I on the EPL distribution. (a) QQ plot of sample vs theoretical distribution; alpha = 1, beta = 1.1, and theta = 1. (b) CDF for sample and theoretical CDF.

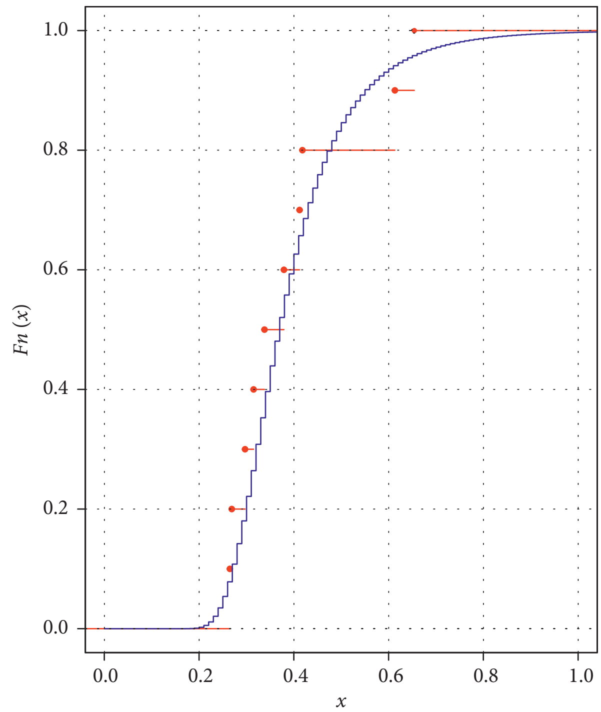
distributions plot and overlay the theoretical EPL distribution, the quantile-quantile (Q-Q) plots are also presented under the maximum flood level data, as shown in Figure 2, which also imply that the EPL distribution can be used as a proper model to fit this data.

Then, by using the BLUE coefficients in Tables 1 and 2, we have

$$\mu^* = \sum_{j=1}^{n} p_j X_{j:n} = 0.246173,$$
$$\sigma^* = \sum_{j=1}^{n} q_j X_{j:n} = 0.099934. \quad (41)$$



(a)



(b)

FIGURE 2: QQ plot and CDF of the real data set II on the EPL distribution.

7. Conclusion

In this paper, we have considered the EPL distribution when data are available in the form of order statistics. We first presented expressions for the single, double, triple, and quadruple moments. By using these moments, we have calculated the BLUEs of the location and scale parameters and the coefficient of skewness and kurtosis for some linear pivotal quantities. In the simulation study, we observed that the variance of the BLUEs increases when a high censoring level is taken into account; however, it decreases when a large value of parameter υ is considered corresponding to a fixed value of parameters δ and γ. Also, the covariance of the BLUEs decreases as the censoring level increases while it increases when the sample size increases and it decreases as υ increases. We next considered the distributions of the pivotal quantities in terms of Edgeworth approximation. These pivotal quantities are used to construct the interval estimation for the location and scale parameters. From our finding, we see that the moments of order statistics of the distribution are well behaved. This will encourage the study of the other properties of order statistics for a future research.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] S. K. Ashour and M. A. Eltehiwy, “Exponentiated power Lindley distribution,” *Journal of Advanced Research*, vol. 6, no. 6, pp. 895–905, 2015.

[2] S. Nadarajah, H. S. Bakouch, and R. Tahmasbi, “A generalized lindley distribution,” *Sankhya B*, vol. 73, no. 2, pp. 331–359, 2011.

[3] M. E. Ghitany, D. K. Al-Mutairi, N. Balakrishnan, and L. J. Al-Enezi, “Power Lindley distribution and associated inference,” *Computational Statistics & Data Analysis*, vol. 64, pp. 20–33, 2013.

[4] V. K. Sharma, S. K. Singh, U. Singh, and F. Merovci, “The generalized inverse lindley distribution: a new inverse statistical model for the study of upside-down bathtub data,” *Communications in Statistics-Theory and Methods*, vol. 45, no. 19, pp. 5709–5729, 2016.

[5] S. Nedjar and H. Zeghdoudi, “On Pseudo lindley distribution: properties and applications,” *New Trends in Mathematical Science*, vol. 1, no. 5, pp. 59–65, 2017.

[6] A. Beghriche and H. Zeghdoud, “A size biased gamma lindley distribution,” *Thailand Statistician*, vol. 17, pp. 179–189, 2019.

[7] H. A. David and H. N. Nagaraja, *Order Statistics*, Wiley, New York, NY, USA, 3rd edition, 2003.

[8] B. C. Arnold, N. Balakrishnan, and H. N. Nagaraja, *A First Course in Order Statistics*, John Wiley, New York, NY, USA, 2003.

[9] N. Balakrishnan and A. C. Cohen, *Order Statistics and Inference: Estimation Methods*, Academic Press, San Diego, CA, USA, 1991.

[10] N. Balakrishnan and M. Ahsanullah, "Relations for single and product moments of record values from exponential distribution," *Journal of Applied Statistical Science*, vol. 2, pp. 73–87, 1995.

[11] N. Balakrishnan and K. S. Sultan, "Recurrence relations and identities for moments of order statistics," in *Handbook of Statistics*, N. Balakrishnan and C. R. Rao, Eds., vol. 16, pp. 149–228, Elsevier, Amsterdam, Netherlands, 1998.

[12] H. J. Malik, N. Balakrishnan, and S. E. Ahmed, "Recurrence relations and identities for moments of order statistics. I: arbitrary continuous distribution," *Communications in Statistics-Theory and Methods*, vol. 17, pp. 2623–2655, 1988.

[13] M. A. W. Mahmoud, K. S. Sultan, and S. M. Amer, "Order statistics from inverse weibull distribution and associated inference," *Computational Statistics & Data Analysis*, vol. 42, no. 1-2, pp. 149–163, 2003.

[14] A. İ. Genç, "Moments of order statistics of topp-leone distribution," *Statistical Papers*, vol. 53, no. 1, pp. 117–131, 2012.

[15] S. M. T. K. MirMostafaee, "On the moments of order statistics coming from the Topp-Leone distribution," *Statistics & Probability Letters*, vol. 95, pp. 85–91, 2014.

[16] N. Balakrishnan, X. Zhu, and B. Al-Zahrani, "Recursive computation of the single and product moments of order statistics from the complementary exponential-geometric distribution," *Journal of Statistical Computation and Simulation*, vol. 85, no. 11, pp. 2187–2201, 2015.

[17] D. Kumar and S. Dey, "Power generalized Weibull distribution based on order statistics," *Journal of Statistical Research*, vol. 51, pp. 61–78, 2017.

[18] K. S. Sultan, A. Childs, and N. Balakrishnan, "Higher order moments of order statistics from the power function distribution and Edgeworth approximate inference," in *Advances in Stochastic Simulation and Methods*, N. Balakrishnan, V. B. Melas, and S. Ermakove, Eds., pp. 245–282, Birkhauser, Boston, MA, USA, 2000.

[19] K. S. Sultan and W. S. AL-Thubyani, "Higher order moments of order statistics from the Lindley distribution and associated inference," *Journal of Statistical Computation and Simulation*, vol. 86, no. 17, pp. 3432–3445, 2016.

[20] D. Kumar, S. Dey, and S. Nadarajah, "Extended exponential distribution based on order statistics," *Communications in Statistics-Theory and Methods*, vol. 46, no. 18, pp. 9166–9184, 2017.

[21] M. Ahsanullah and A. Alzaatreh, "Parameter estimation for the log-logistic distribution based on order statistics," *REVSTAT Statistical Journal*, vol. 16, pp. 429–443, 2018.

[22] D. Kumar and A. Goyal, "Order Statistics from the power Lindley distribution and associated inference with application," *Annals of Data Science*, vol. 6, no. 1, pp. 153–177, 2019.

[23] D. Kumar and A. Goyal, "Generalized lindley distribution based on order statistics and associated inference with application," *Annals of Data Science*, vol. 6, no. 4, pp. 707–736, 2019.

[24] D. Kumar, M. Kumar, and J. P. S. Joorel, "Estimation with modified power function distribution based on order statistics with application to evaporation data," *Annals of Data Science*, 2020.

[25] D. Kumar, M. Nassar, and S. Dey, "Inference for generalized inverse Lindley distribution based on generalized order statistics," *Afrika Matematika*, 2020.

[26] D. K. Bhaumik and R. D. Gibbons, "One-sided approximate prediction intervals for at Leastpofm observations from a gamma population at each ofrLocations," *Technometrics*, vol. 48, no. 1, pp. 112–119, 2006.

[27] K. Krishnamoorthy, T. Mathew, and S. Mukherjee, "Normal-based methods for a gamma distribution," *Technometrics*, vol. 50, no. 1, pp. 69–78, 2008.

[28] D. K. Bhaumik, K. Kapur, and R. D. Gibbons, "Testing parameters of a gamma distribution for small samples," *Technometrics*, vol. 51, no. 3, pp. 326–334, 2009.

[29] R. Dumonceaux and C. E. Antle, "Discrimination between the log-normal and the weibull distributions," *Technometrics*, vol. 15, no. 4, pp. 923–926, 1973.