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Solution and intuitionistic fuzzy stability of 3- dimensional cubic functional equation: using two different methods

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Abstract

In this article, we adopt fixed point method and direct method to find the solution and Intuitionistic fuzzy stability of 3-dimensional cubic functional equation

$$g(2u_1+u_2+u_3)=3g(u_1+u_2+u_3)+g(-u_1+u_2+u_3)+2g(u_1+u_2)+2g(u_1+u_3)-6g(u_1-u_2)-6g(u_1-u_3)$$
$$-3g(u_2+u_3)+2g(2u_1-u_2)+2g(2u_1-u_3)-18g(u_1)-6g(u_2)-6g(u_3).$$

Keywords: Functional equations, intuitionistic fuzzy Banach space, fixed point method, direct method.

2020 MSC: 39B52, 39B82, 46S40.

1. Introduction

The first stability problem was established by Ulam [29] in 1940. He raised a question "Is there an exact homomorphism close to approximate homomorphism?". An answer of Ulam problem was given by Hyers [15] in Banach spaces. Since then, the stability problems have been studied by several mathematicians.

In 1965, Zadeh [30] initialized the theory of fuzzy sets. Through the classical learning of Zadeh, there has been a large work to find fuzzy illustration of academic notions. Later on, various generalizations and extensions of Hyers' result were established by Rassias [22, 23], Găvruta [13], and Rassias [24] in different versions for the Cauchy (additive) functional equation $f(x+y)=f(x)+f(y)$ in Banach spaces. Since then, the stability problems of various functional equations on miscellaneous normed spaces have been extensively investigated by a number of authors; for instance, see [1–5, 7, 8, 11].

Authors [4, 9, 21] pointed out that a fixed point alternative method can be successfully used to solve the Ulam problem. The fixed point method in the study of the stability of functional equations firstly appears

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in [12]. In order to have more knowledge on the various stability problems of functional equations see [18–20, 25–27]. Jun et al. [16] introduced the following functional equation

$$g(2u+v)+g(2u-v)=2g(u+v)+2g(u-v)+12g(u) \tag{1.1}$$

and explored its ordinary solution and Hyers-Ulam stability respectively. The functional equation (1.1) is called cubic since $g(u)=cu^3$ is its solution. Many authors introduced the new types of functional equations which is cubic and one of the type of 3-dimensional cubic functional equation

$$\begin{aligned} g(2u_1+u_2+u_3) = {} & 3g(u_1+u_2+u_3)+g(-u_1+u_2+u_3)+2g(u_1+u_2) \\ & +2g(u_1+u_3)-6g(u_1-u_2)-6g(u_1-u_3)-3g(u_2+u_3) \\ & +2g(2u_1-u_2)+2g(2u_1-u_3)-18g(u_1)-6g(u_2)-6g(u_3) \end{aligned} \tag{1.2}$$

and introduced by Park et al. [14].

Preliminaries and definitions

Definition 1.1 ([10])**.** Let L^* be any set and the order relation $\leqslant_{L^*}$ defined by

$$L^*=\{(u_1,u_2):(u_1,u_2)\in[0,1]^2 \text{ and } u_1+u_2\leqslant 1\},$$
$$(u_1,u_2)\leqslant_{L^*}(v_1,v_2)\Leftrightarrow u_1\leqslant v_1, u_2\geqslant v_2, \quad \forall\ (u_1,u_2),(v_1,v_2)\in L^*.$$

Then the ordered pair $(L^*,\leqslant_{L^*})$ is a complete lattice.

Definition 1.2 ([6])**.** An intuitionistic fuzzy set $\mathcal{A}_{\sigma,\upsilon}$ in a universal set $\mathcal{U}$ is an object

$$\mathcal{A}_{\sigma,\upsilon}=\{(\sigma_{\mathcal{A}}(u),\upsilon_{\mathcal{A}}(u))\mid\ u\in\mathcal{U}\}$$

for all $u\in\mathcal{U}$, $\sigma_{\mathcal{A}}(u)\in[0,1]$ and $\upsilon_{\mathcal{A}}(u)\in[0,1]$ are called the membership degree and the non-membership degree, respectively, of u in $\mathcal{A}_{\sigma,\upsilon}$ and furthermore, they satisfy $\sigma_{\mathcal{A}}(u)+\upsilon_{\mathcal{A}}(u)\leqslant 1$. We denote its units by $0_{L^*}=(0,1)$ and $1_{L^*}=(1,0)$.

Definition 1.3 ([6])**.** A triangular norm (t'-norm) on L^* is a mapping $\mathcal{T}:(L^*)^2\to L^*$ satisfying the following

- (boundary condition): $\mathcal{T}(u,1_{L^*})=u$;
- (commutativity): $\mathcal{T}(u,v)=\mathcal{T}(v,u)$;
- (associativity): $\mathcal{T}(u,\mathcal{T}(v,w))=\mathcal{T}(\mathcal{T}(u,v),w)$;
- (monotonicity): $u\leqslant_{L^*}u'$ and $v\leqslant_{L^*}v'\Rightarrow\mathcal{T}(u,v)\leqslant_{L^*}\mathcal{T}(u',v')$.

If $(L^*,\leqslant_{L^*},\mathcal{T})$ is an Abelian topological monoid with unit 1_{L^*}, then L^* is said to be a continuous t'-norm.

Definition 1.4 ([28])**.** A negation on L^* is any decreasing mapping $\mathcal{N}:L^*\to L^*$ satisfying $\mathcal{N}(0_{L^*})=1_{L^*}$ and $\mathcal{N}(1_{L^*})=0_{L^*}$. If $\mathcal{N}(\mathcal{N}(u))=u$ for all $u\in L^*$, then $\mathcal{N}$ is called an involutive negator.

Definition 1.5 ([28])**.** Let α and β be membership and non-membership degree of an intuitionistic fuzzy set from $U\times(0,\infty)$ to $[0,1]$ such that $\alpha_u(t')+\beta_u(t')\leqslant 1$ for all $u\in U$ and $t'>0$. The triple $(U,\mathcal{P}_{\alpha,\beta},\mathcal{T})$ is said to be an intuitionistic fuzzy normed space (IFNS) if U is a vector space, $\mathcal{T}$ is a continuous t'-representable and $\mathcal{P}_{\alpha,\beta}$ is a mapping U x $(0,\infty)\to L^*$ satisfying the following

IFN1. $\mathcal{P}_{\alpha,\beta}(u,0)=0_{L^*}$;
IFN2. $\mathcal{P}_{\alpha,\beta}(u,t')=1_{L^*}\Leftrightarrow u=0$;
IFN3. $\mathcal{P}_{\alpha,\beta}(cu,t')=\mathcal{P}_{\alpha,\beta}\left(u,\frac{t'}{\|c\|}\right)$ for all $c\neq 0$;
IFN4. $\mathcal{P}_{\alpha,\beta}(u+v,t'+s')\geqslant_{L^*}\mathcal{T}(\mathcal{P}_{\alpha,\beta}(u,t'),\mathcal{P}_{\alpha,\beta}(v,s'))$ for all $u,v\in U$ and $t',s'>0$.

In this case, $\mathcal{P}_{\alpha,\beta}$ is called an intuitionistic fuzzy norm. Here, $\mathcal{P}_{\alpha,\beta}(u,t') = (\alpha_u(t'), \beta_u(t'))$.

Definition 1.6 ([28]). A sequence $\{u_m\}$ in an IFNS $(U, \mathcal{P}_{\alpha,\beta}, \mathcal{T})$ is called a Cauchy sequence if for any $\varepsilon > 0$ and $t' > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$\mathcal{P}_{\alpha,\beta}(u_m - u_{m'}, t') >_{L^*} (\mathcal{N}_s(\varepsilon), \varepsilon), \forall \ m, m' \geqslant m_0,$$

where $\mathcal{N}_s$ is the standard negator. The sequence $\{u_m\}$ is said to be convergent to a point $u \in U$ if

$$\mathcal{P}_{\alpha,\beta}(u_m - u, t') \to 1_{L^*}$$

as $m \to \infty$ for every $t' > 0$. An IFNS $(U, \mathcal{P}_{\alpha,\beta}, \mathcal{T})$ is said to be complete if every cauchy sequence in U is convergent to a point $u \in U$.

In this article, we adopt fixed point method and direct method to find the solution and Intuitionistic fuzzy stability of 3-dimensional cubic functional equation (1.2) in intuitionistic fuzzy normed space (IFNS).

2. General solution

In this segment, we find the solution of the functional equation (1.2) in intuitionistic fuzzy normed spaces.

Theorem 2.1. *Let U and V be real vector spaces. The mapping $g : U \to V$ satisfies the functional equation (1.1) for all $u, v \in U$ iff $g : U \to V$ satisfies the functional equation (1.2) for all $u_1, u_2, u_3 \in U$.*

Proof. Let $g : U \to V$ satisfies the functional equation (1.1). Putting $(u, v) = (0, 0)$ in (1.1), we get $g(0) = 0$. Substituting $(u, v) = (0, v)$ in (1.1), we get

$$g(-v) = -g(v)$$

for all $v \in U$. Therefore g is an odd function. Replacing (u, v) by $(u, 0), (u, u)$, and $(u, 2u)$, respectively in (1.1), we get

$$g(2u) = 2^3 g(u), \quad g(3u) = 3^3 g(u), \quad \text{and} \quad g(4u) = 4^3 g(u)$$

for all $u \in U$. In general for any positive integer c, we have

$$g(cu) = c^3 g(u)$$

for all $u \in U$. Putting $(u, v) = (u_1, u_2 + u_3)$ in (1.1), we get

$$g(2u_1 + u_2 + u_3) - 2g(u_1 + u_2 + u_3) = g(-2u_1 + u_2 + u_3) + 2g(u_1 - u_2 - u_3) + 12g(u_1). \tag{2.1}$$

Again putting $(u, v) = (u_2 + u_3, -2u_1)$ in (1.1), we get

$$4g(-u_1 + u_2 + u_3) + 4g(u_1 + u_2 + u_3) - g(2u_1 + u_2 + u_3) - 6g(u_2 + u_3) = g(-2u_1 + u_2 + u_3) \tag{2.2}$$

for all $u_1, u_2, u_3 \in U$. Substituting (2.2) in (2.1), we get

$$g(2u_1 + u_2 + u_3) - 3g(u_1 + u_2 + u_3) = g(-u_1 + u_2 + u_3) - 3g(u_2 + u_3) + 6g(u_1). \tag{2.3}$$

Now putting $(u, v) = (u_2, 2u_1)$ in (1.1), we get

$$4g(u_1 + u_2) - 4g(u_1 - u_2) - 6g(u_2) = g(2u_1 + u_2) - g(2u_1 - u_2). \tag{2.4}$$

Replacing (u,v) by $(u_3,2u_1)$ in (1.1), we get

$$4g(u_1+u_3)-4g(u_1-u_3)-6g(u_3)=g(2u_1+u_3)-g(2u_1-u_3). \tag{2.5}$$

Adding (2.4) and (2.5), we get

$$\begin{aligned}&4g(u_1+u_2)-4g(u_1-u_2)+4g(u_1+u_3)-4g(u_1-u_3)-6g(u_2)-6g(u_3)\\&-g(2u_1+u_2)+g(2u_1-u_2)-g(2u_1+u_3)+g(2u_1-u_3)=0.\end{aligned} \tag{2.6}$$

Again adding (2.3) and (2.6), we get

$$\begin{aligned}g(2u_1+u_2+u_3)=\;&3g(u_1+u_2+u_3)+g(-u_1+u_2+u_3)\\&-3g(u_2+u_3)+6g(u_1)+4g(u_1+u_2)-4g(u_1-u_2)\\&+4g(u_1+u_3)-4g(u_1-u_3)-6g(u_2)-6g(u_3)-g(2u_1+u_2)\\&+g(2u_1-u_2)-g(2u_1+u_3)+g(2u_1-u_3).\end{aligned} \tag{2.7}$$

Replacing (u,v) by $(-u_1,u_2)$ in (1.1), we get

$$-g(2u_1+u_2)=g(2u_1-u_2)-2g(u_1-u_2)-2g(u_1+u_2)-12g(u_1). \tag{2.8}$$

Replacing (u,v) by $(-u_1,u_3)$ in (1.1), we get

$$-g(2u_1+u_3)=g(2u_1-u_3)-2g(u_1-u_3)-2g(u_1+u_3)-12g(u_1). \tag{2.9}$$

Adding (2.8) and (2.9), we obtain

$$\begin{aligned}-g(2u_1+u_2)-g(2u_1+u_3)=\;&g(2u_1-u_2)-2g(u_1-u_2)-2g(u_1+u_2)-2g(u_1-u_3)\\&+g(2u_1-u_3)-2g(u_1-u_3)-g(u_1+u_3)-24g(u_1).\end{aligned} \tag{2.10}$$

Substituting (2.10) in (2.7), we have

$$\begin{aligned}g(2u_1+u_2+u_3)=\;&3g(u_1+u_2+u_3)+g(-u_1+u_2+u_3)+2g(u_1+u_2)+2g(u_1\\&+u_3)-6g(u_1-u_2)-6g(u_1-u_3)-3g(u_2+u_3)+2g(2u_1-u_2)\\&+2g(2u_1-u_3)-18g(u_1)-6g(u_2)-6g(u_3)\end{aligned}$$

for all $u_1,u_2,u_3\in U$. Conversely, let $g:U\to V$ satisfy the functional equation (1.2). Substituting $(u_1,u_2,u_3)=(0,u_2,0)$ in (1.2), we get

$$g(-u_2)=-g(u_2)$$

for all $u_2\in U$. Therefore g is an odd function. Replacing (u_1,u_2,u_3) by $(u,v,0)$ in (1.2), we have

$$g(2u+v)-2g(2u-v)=5g(u+v)-7g(u-v)-6g(u)-9g(v). \tag{2.11}$$

Again replacing (u_1,u_2,u_3) by $(u,0,-v)$ in (1.2), we have

$$g(2u-v)-2g(2u+v)=-7g(u+v)+5g(u-v)-6g(u)+9g(v) \tag{2.12}$$

for all $u,v\in U$. Adding (2.11) and (2.12), we get our results. □

3. Stability results

In this segment, we present the generalized Ulam-Hyers stability of the functional equation (1.2) in intuitionistic fuzzy normed spaces using direct method. Now, we use the following notation for a given mapping $g : U \to V$

$$\begin{aligned} Dg(u_1, u_2, u_3) = {} & g(2u_1 + u_2 + u_3) - 3g(u_1 + u_2 + u_3) - g(-u_1 + u_2 + u_3) \\ & - 2g(u_1 + u_2) - 2g(u_1 + u_3) + 6g(u_1 - u_2) + 6g(u_1 - u_3) + 3g(u_2 + u_3) \\ & - 2g(2u_1 - u_2) - 2g(2u_1 - u_3) + 18g(u_1) + 6g(u_2) + 6g(u_3). \end{aligned}$$

Theorem 3.1. *Let $\gamma \in (-1, 1)$. Let U be a linear space, $(\mathcal{Z}, \mathcal{P}'_{\alpha,\beta}, \mathcal{T})$ be an IFNS, $h : U^3 \to \mathcal{Z}$ be a mapping with $0 < (\frac{\tau}{2^3})^\gamma < 1$, for some $\tau > 0$,*

$$\mathcal{P}'_{\alpha,\beta}(h(2^\gamma u, 0, 0), \varepsilon) \geqslant_{L^*} \mathcal{P}'_{\alpha,\beta}(\tau^\gamma h(u, 0, 0), \varepsilon) \tag{3.1}$$

for all $u \in U$ and $\varepsilon > 0$ and

$$\lim_{m\to\infty} \mathcal{P}'_{\alpha,\beta}(h(2^{\gamma m}u_1, 2^{\gamma m}u_2, 2^{\gamma m}u_3), 2^{\gamma 3m}\varepsilon) = 1_{L^*} \tag{3.2}$$

for all $u_1, u_2, u_3 \in U$ and $\varepsilon > 0$. Assume that a mapping $g : U \to V$ satisfies the inequality

$$\mathcal{P}_{\alpha,\beta}(Dg(u_1, u_2, u_3), \varepsilon) \geqslant_{L^*} \mathcal{P}'_{\alpha,\beta}(h(u_1, u_2, u_3), \varepsilon). \tag{3.3}$$

Then the limit

$$\mathcal{P}_{\alpha,\beta}\left(Q(u) - \frac{g(2^{\gamma m}u)}{2^{\gamma 3m}}\right) \longrightarrow 1_{L^*} \text{ as } m \to \infty, \varepsilon > 0$$

exists for all $u \in U$ and the mapping $Q : U \to V$ is a unique cubic mapping satisfying (1.2) and

$$\mathcal{P}_{\alpha,\beta}(g(u) - Q(u), \varepsilon) \geqslant_{L^*} (h(u, 0, 0), 3\varepsilon|2^3 - \tau|) \tag{3.4}$$

for all $u \in U$ and $\varepsilon > 0$.

Proof. Firstly, assume $\gamma = 1$. Replacing (u_1, u_2, u_3) by $(u, 0, 0)$ in (3.3), we get

$$\mathcal{P}_{\alpha,\beta}(3g(2u) - 24g(u), \varepsilon) \geqslant_{L^*} \mathcal{P}'_{\alpha,\beta}(h(u, 0, 0), \varepsilon)$$

for all $u \in U$ and $\varepsilon > 0$. Replacing u by $2^m u$ in (1.2) and using (IFN3), we have

$$\mathcal{P}_{\alpha,\beta}\left(\frac{g(2^{m+1}u)}{2^3} - g(2^m u), \frac{\varepsilon}{24}\right) \geqslant_{L^*} \mathcal{P}'_{\alpha,\beta}(h(2^m u, 0, 0), \varepsilon). \tag{3.5}$$

Again using (3.1), (IFN3) in (3.5), we find

$$\mathcal{P}_{\alpha,\beta}\left(\frac{g(2^{m+1}u)}{2^3} - g(2^m u), \frac{\varepsilon}{24}\right) \geqslant_{L^*} \mathcal{P}'_{\alpha,\beta}\left(h(u, 0, 0), \frac{\varepsilon}{\tau^m}\right). \tag{3.6}$$

It is simple to check from (3.6), that

$$\mathcal{P}_{\alpha,\beta}\left(\frac{g(2^{m+1}u)}{2^{3(m+1)}} - \frac{g(2^m u)}{2^{3m}}, \frac{\varepsilon}{24.2^{3m}}\right) \geqslant_{L^*} \mathcal{P}'_{\alpha,\beta}\left(h(u, 0, 0), \frac{\varepsilon}{\tau^m}\right) \tag{3.7}$$

holds for all $u \in U$ and $\varepsilon > 0$. Replacing ε by $\tau^m\varepsilon$ in (3.7), we obtain

$$\mathcal{P}_{\alpha,\beta}\left(\frac{g(2^{m+1}u)}{2^{3(m+1)}} - \frac{g(2^m u)}{2^{3m}}, \frac{\varepsilon\tau^m}{24.2^{3m}}\right) \geqslant_{L^*} \mathcal{P}'_{\alpha,\beta}(h(u, 0, 0), \varepsilon) \tag{3.8}$$

for all $u \in U$ and $\varepsilon > 0$. It is simple to see that

$$\frac{g(2^m u)}{2^{3m}} - g(u) = \Sigma_{i=0}^{m-1} \frac{g(2^{i+1}u)}{2^{3(i+1)}} - \frac{g(2^i u)}{2^{3i}} \tag{3.9}$$

for all $u \in U$. From equations (3.8) and (3.9), we get

$$\begin{aligned} \mathcal{P}_{\alpha,\beta}\Big(\frac{g(2^m u)}{2^{3m}} - g(u), \Sigma_{i=0}^{m-1} \frac{\tau^i \varepsilon}{3(2^{3i}.2^3)}\Big) &\geqslant_{L^*} \mathcal{T}_{i=0}^{m-1}\{\mathcal{P}'_{\alpha,\beta}\Big(\frac{g(2^{i+1}u)}{2^{3(i+1)}} - \frac{g(2^i u)}{2^{3i}}, \frac{\tau^i \varepsilon}{3(2^{3i}.2^3)}\Big)\} \\ &\geqslant_{L^*} \mathcal{T}_{i=0}^{m-1}\{\mathcal{P}'_{\alpha,\beta}(h(u,0,0),\varepsilon)\} \geqslant_{L^*} \mathcal{P}'_{\alpha,\beta}(h(u,0,0),\varepsilon) \end{aligned} \tag{3.10}$$

for all $u \in U$ and $\varepsilon > 0$. Replacing u by $2^{m'}u$ in (3.10) and using (3.1), we get

$$\mathcal{P}_{\alpha,\beta}\Big(\frac{g(2^{m+m'}u)}{2^{3(m+m')}} - \frac{g(2^{m'}u)}{2^{3m'}}, \Sigma_{i=0}^{m-1} \frac{\tau^i \varepsilon}{24.2^{3(i+m')}}\Big) \geqslant_{L^*} \mathcal{P}'_{\alpha,\beta}\Big(h(u,0,0), \frac{\varepsilon}{\tau^{m'}}\Big) \tag{3.11}$$

for all $u \in U$, $\varepsilon > 0$, and $m, m' \geqslant 0$. Replacing ε by $\tau^{m'}\varepsilon$ in (3.11), we get

$$\mathcal{P}_{\alpha,\beta}\Big(\frac{g(2^{m+m'}u)}{2^{3(m+m')}} - \frac{g(2^{m'}u)}{2^{3m'}}, \Sigma_{i=m'}^{m+m'-1} \frac{\tau^i \varepsilon}{24.2^{3i}}\Big) \geqslant_{L^*} \mathcal{P}'_{\alpha,\beta}(h(u,0,0),\varepsilon) \tag{3.12}$$

for all $u \in U$, $\varepsilon > 0$, and $m, m' \geqslant 0$. Using (IFN3) in (3.12), we get

$$\mathcal{P}_{\alpha,\beta}\Big(\frac{g(2^{m+m'}u)}{2^{3(m+m')}} - \frac{g(2^{m'}u)}{2^{3m'}}, \varepsilon\Big) \geqslant_{L^*} \mathcal{P}'_{\alpha,\beta}\Big(h(u,0,0), \frac{\varepsilon}{\Sigma_{i=m'}^{m+m'-1} \frac{\tau^i \varepsilon}{24.2^{3i}}}\Big) \tag{3.13}$$

for all $u \in U$, $\varepsilon > 0$ and $m, m' \geqslant 0$. Since $0 < \tau < 2^3$ and $\Sigma_{i=0}^{m}(\frac{\tau}{2^3})^i < \infty$. Thus $\{\frac{g(2^m u)}{2^{3m}}\}$ is a Cauchy sequence in V. Since $(V, \mathcal{P}_{\alpha,\beta}, \mathcal{T})$ is a complete IFNS, the sequence converges to some point $Q(u) \in V$. So, define the mapping $Q : U \to V$ by

$$\mathcal{P}_{\alpha,\beta}\Big(Q(u) - \frac{g(2^{\gamma m}u)}{2^{\gamma 3m}}\Big) \longrightarrow 1_{L^*} \quad \text{as} \quad m \longrightarrow \infty, \quad \varepsilon > 0$$

for all $u \in U$. Letting $m' \to 0$ in (3.13), we obtain

$$\mathcal{P}_{\alpha,\beta}\Big(\frac{g(2^m u)}{2^{3m}} - g(u), \varepsilon\Big) \geqslant_{L^*} \mathcal{P}'_{\alpha,\beta}\Big(h(u,0,0), \frac{\varepsilon}{\Sigma_{i=0}^{m-1} \frac{\tau^i \varepsilon}{24.2^{3i}}}\Big) \tag{3.14}$$

for all $u \in U$ and $\varepsilon > 0$. Letting $m \to \infty$ in (3.14), we find

$$\mathcal{P}_{\alpha,\beta}(g(u) - Q(u), \varepsilon) \geqslant_{L^*} \mathcal{P}'_{\alpha,\beta}(h(u,0,0), 3\varepsilon(2^3 - \tau)).$$

To show Q satisfies (1.2), replacing (u_1, u_2, u_3) by $(2^m u_1, 2^m u_2, 2^m u_3)$ in (3.3), respectively, we get

$$\mathcal{P}_{\alpha,\beta}\Big(\frac{1}{2^{3m}} Dg((2^m u_1, 2^m u_2, 2^m u_3), \varepsilon\Big) \geqslant_{L^*} \mathcal{P}'_{\alpha,\beta}(h(2^m u_1, 2^m u_2, 2^m u_3), 2^{3m}\varepsilon) \tag{3.15}$$

for all $u_1, u_2, u_3 \in U$ and $\varepsilon > 0$. Now,

$$\begin{aligned} &\mathcal{P}_{\alpha,\beta}(Q(2u_1 + u_2 + u_3) - 3Q(u_1 + u_2 + u_3) - Q(-u_1 + u_2 + u_3) \\ &\qquad - 2Q(u_1 + u_2) - 2Q(u_1 + u_3) + 6Q(u_1 - u_2) + 6Q(u_1 - u_3) \\ &\qquad + 3Q(u_2 + u_3) - 2Q(2u_1 - u_2) - 2Q(2u_1 - u_3) + 18Q(u_1) + 6Q(u_2) + 6Q(u_3), \varepsilon) \\ &\geqslant_{L^*} \mathcal{T}\Big\{\mathcal{P}_{\alpha,\beta}\Big(Q(2u_1 + u_2 + u_3) - \frac{1}{2^{3m}} g(2^m(2u_1 + u_2 + u_3)), \frac{\varepsilon}{2^3}\Big), \end{aligned}$$

$$\begin{aligned}
&\mathcal{P}_{\alpha,\beta}\big(-3Q(u_1+u_2+u_3)+\frac{1}{2^{3m}}3g(2^m(u_1+u_2+u_3)),\frac{\varepsilon}{2^3}\big),\\
&\mathcal{P}_{\alpha,\beta}\big(-Q(-u_1+u_2+u_3)+\frac{1}{2^{3m}}g(2^m(-u_1+u_2+u_3)),\frac{\varepsilon}{2^3}\big),\\
&\mathcal{P}_{\alpha,\beta}\big(-2Q(u_1+u_2)+\frac{1}{2^{3m}}2g(2^m(2Q(u_1+u_2)),\frac{\varepsilon}{2^3}\big),\\
&\mathcal{P}_{\alpha,\beta}\big(-2Q(u_1+u_3)+\frac{1}{2^{3m}}2g(2^m(2Q(u_1+u_3)),\frac{\varepsilon}{2^3}\big),\\
&\mathcal{P}_{\alpha,\beta}\big(6Q(u_1-u_2)-\frac{1}{2^{3m}}6g(2^m(u_1-u_2)),\frac{\varepsilon}{2^3}\big),\\
&\mathcal{P}_{\alpha,\beta}\big(6Q(u_1-u_3)-\frac{1}{2^{3m}}6g(2^m(u_1-u_3)),\frac{\varepsilon}{2^3}\big),\\
&\mathcal{P}_{\alpha,\beta}\big(3Q(u_2+u_3)-\frac{1}{2^{3m}}3g(2^m(u_2+u_3)),\frac{\varepsilon}{2^3}\big),\\
&\mathcal{P}_{\alpha,\beta}\big(-2Q(2u_1-u_2)+\frac{1}{2^{3m}}2g(2^m(2u_1-u_2)),\frac{\varepsilon}{2^3}\big),\\
&\mathcal{P}_{\alpha,\beta}\big(-2Q(2u_1-u_3)+\frac{1}{2^{3m}}2g(2^m(2u_1-u_3)),\frac{\varepsilon}{2^3}\big),\qquad \mathcal{P}_{\alpha,\beta}\big(18Q(u_1)-\frac{1}{2^{3m}}18g(2^m(u_1)),\frac{\varepsilon}{2^3}\big),\\
&\mathcal{P}_{\alpha,\beta}\big(6Q(u_2)-\frac{1}{2^{3m}}6g(2^m(u_2)),\frac{\varepsilon}{2^3}\big),\mathcal{P}_{\alpha,\beta}\big(6Q(u_3)-\frac{1}{2^{3m}}6g(2^m(u_3)),\frac{\varepsilon}{2^3}\big),\\
&\mathcal{P}_{\alpha,\beta}\big(\frac{1}{2^{3m}}g(2^m(2u_1+u_2+u_3))-\frac{1}{2^{3m}}3g(2^m(u_1+u_2+u_3))\\
&-\frac{1}{2^{3m}}g(2^m(-u_1+u_2+u_3))-\frac{1}{2^{3m}}2g(2^m(2Q(u_1+u_2))-\frac{1}{2^{3m}}2g(2^m(2Q(u_1+u_3))\\
&+\frac{1}{2^{3m}}6g(2^m(u_1-u_2))+\frac{1}{2^{3m}}6g(2^m(u_1-u_3))+\frac{1}{2^{3m}}3g(2^m(u_2+u_3))\\
&-\frac{1}{2^{3m}}2g(2^m(2u_1-u_2))-\frac{1}{2^{3m}}2g(2^m(2u_1-u_3))\\
&+\frac{1}{2^{3m}}18g(2^m(u_1))+\frac{1}{2^{3m}}6g(2^m(u_2))+\frac{1}{2^{3m}}6g(2^m(u_3))\big)\Big\}
\end{aligned}\tag{3.16}$$

for all $u_1, u_2, u_3 \in U$ and $\varepsilon > 0$. Using (3.15) in (3.16), we obtain

$$\begin{aligned}
&\mathcal{P}_{\alpha,\beta}\big(Q(2u_1+u_2+u_3)-3Q(u_1+u_2+u_3)\\
&\quad -Q(-u_1+u_2+u_3)-2Q(u_1+u_2)-2Q(u_1+u_3)+6Q(u_1-u_2)\\
&\quad +6Q(u_1-u_3)+3Q(u_2+u_3)-2Q(2u_1-u_2)-2Q(2u_1-u_3)\\
&\quad +18Q(u_1)+6Q(u_2)+6Q(u_3),\varepsilon\big)\geqslant_{L^*}\mathcal{T}\{1_{L^*},1_{L^*},1_{L^*},1_{L^*},1_{L^*},1_{L^*},1_{L^*},\\
&\quad\ 1_{L^*},1_{L^*},1_{L^*},1_{L^*},1_{L^*},1_{L^*},\mathcal{P}'_{\alpha,\beta}(h(2^m u_1,2^m u_2,2^m u_3),2^{3m}\varepsilon)\}\\
&\geqslant_{L^*}\mathcal{P}'_{\alpha,\beta}(h(2^m u_1,2^m u_2,2^m u_3),2^{3m}\varepsilon)
\end{aligned}\tag{3.17}$$

for all $u_1, u_2, u_3 \in U$ and $\varepsilon > 0$. Letting $m \to \infty$ in (3.17) and using (3.2), we observe that

$$\begin{aligned}
&\mathcal{P}_{\alpha,\beta}\big(Q(2u_1+u_2+u_3)-3Q(u_1+u_2+u_3)-Q(-u_1+u_2+u_3)-2Q(u_1+u_2)\\
&\quad -2Q(u_1+u_3)+6Q(u_1-u_2)+6Q(u_1-u_3)+3Q(u_2+u_3)-2Q(2u_1-u_2)\\
&\quad -2Q(2u_1-u_3)+18Q(u_1)+6Q(u_2)+6Q(u_3),\varepsilon\big)=1_{L^*}
\end{aligned}$$

for all $u_1, u_2, u_3 \in U$ and $\varepsilon > 0$. Using (IFN2) in the above inequality, which gives

$$\begin{aligned}
Q(2u_1+u_2+u_3)&=3Q(u_1+u_2+u_3)+Q(-u_1+u_2+u_3)+2Q(u_1+u_2)\\
&\quad +2Q(u_1+u_3)-6Q(u_1-u_2)-6Q(u_1-u_3)-3Q(u_2+u_3)\\
&\quad +2Q(2u_1-u_2)+2Q(2u_1-u_3)-18Q(u_1)-6Q(u_2)-6Q(u_3)
\end{aligned}$$

for all $u_1, u_2, u_3 \in U$. Hence Q satisfies the cubic functional equation (1.2). In order to show $Q(u)$ is unique, let $Q'(u)$ be another cubic mapping satisfying (1.2) and (3.4). Hence

$$\begin{aligned}\mathcal{P}_{\alpha,\beta}(Q(u)-Q'(u),\varepsilon) &= \mathcal{P}_{\alpha,\beta}\big(\frac{Q(2^m u)}{2^{3m}}-\frac{Q'(2^m u)}{2^{3m}},\varepsilon\big)\\ &\geqslant_{L^*} \mathcal{T}\{P_{\alpha,\beta}\big(\frac{Q(2^m u)}{2^{3m}}-\frac{g(2^m u)}{2^{3m}},\frac{\varepsilon}{2}\big), P_{\alpha,\beta}\big(\frac{g(2^m u)}{2^{3m}}-\frac{Q'(2^m u)}{2^{3m}},\frac{\varepsilon}{2}\big)\}\\ &\geqslant_{L^*} P'_{\alpha,\beta}\big(h(2^m u,0,0),\frac{3(2^m)\varepsilon(2^3-\tau)}{2}\big)\\ &\geqslant_{L^*} P'_{\alpha,\beta}\big(h(u,0,0),\frac{3(2^m)\varepsilon(2^3-\tau)}{2.\tau^m}\big)\end{aligned}$$

for all $u \in U$ and $\varepsilon > 0$. Since

$$\lim_{m\to\infty}\frac{3(2^m)\varepsilon(2^3-\tau)}{2.\tau^m}=\infty,$$

we get

$$\lim_{m\to\infty}P'_{\alpha,\beta}\big(h(u,0,0),\frac{3(2^m)\varepsilon(2^3-\tau)}{2.\tau^m}\big)=1_{L^*}.$$

Thus,

$$P_{\alpha,\beta}(Q(u)-Q'(u),\varepsilon)=1_{L^*}$$

for all $u \in U$ and $\varepsilon > 0$, hence $Q(u) = Q'(u)$. Therefore, $Q(u)$ is unique. For $\gamma = -1$, we can show the result by a similar method. This completes the proof of the theorem. □

From above theorem, we have the following corollary concerning the Hyers-Ulam-Rassias and JMRassias stabilities for the functional equation (1.2).

Corollary 3.2. *Assume that a function $g : U \to V$ satisfies the inequality*

$$P_{\alpha,\beta}(Dg(u_1,u_2,u_3),\varepsilon) \geqslant_{L^*} \begin{cases} P'_{\alpha,\beta}(\delta\ \ \Sigma_{i=1}^3\|u_i\|^s,\varepsilon),\\ P'_{\alpha,\beta}(\delta(\Pi_{i=1}^3\|u_i\|^s+\Sigma_{i=1}^3\|u_i\|^{3s}),\varepsilon)\end{cases}$$

for all $u_1, u_2, u_3 \in U$ and $\varepsilon > 0$, where δ, s are constants with $\delta > 0$. Then there exists a unique cubic mapping $Q : U \to V$ such that

$$P_{\alpha,\beta}(Dg(u_1,u_2,u_3),\varepsilon) \geqslant_{L^*} \begin{cases} P'_{\alpha,\beta}(\delta\ \ \|u\|^s, 3\varepsilon|2^3-2^s), & s\neq 3,\\ P'_{\alpha,\beta}(\delta\|u_i\|^{3s}, 3\varepsilon|2^3-2^{ms}|), & s\neq \frac{3}{m}\end{cases}$$

for all $u \in U$ and $\varepsilon > 0$. If we define

$$\psi(u_1,u_2,u_3)=\begin{cases}\delta\Sigma_{i=1}^3\|u_i\|^s,\\ \delta(\Pi_{i=1}^3\|u_i\|^s+\delta\Sigma_{i=1}^3\|u_i\|^{3s}),\end{cases}$$

then the corollary is followed from above theorem, by taking $\tau = \begin{cases} 2^s,\\ 2^{3s}.\end{cases}$

4. Stability result for the functional equation: fixed point method

In this segment, we examine the generalized Ulam-Hyers stability of the functional equation (1.2) in IFNS using fixed point method. Now, we recall the fundamental results in fixed point theory.

Theorem 4.1 (The alternative of fixed point, [12])**.** *Suppose that for a complete generalized metric space* $(\mathcal{U}, \tau)$ *and a strictly contractive mapping* $\mathcal{T} : \mathcal{U} \to \mathcal{U}$ *with Lipschitz constant* L*. Then, for each given element* $u \in \mathcal{U}$*, either*

$$\tau(\mathcal{T}^m u, \mathcal{T}^{m+1} u) = \infty, \ \forall \ m \geqslant 0,$$

or there exists a natural number m_0 *such that*

(1) $\tau(\mathcal{T}^m u, \mathcal{T}^{m+1} u) < \infty$ *for all* $m \geqslant m_0$;
(2) *The sequence* $(\mathcal{T}^m u)$ *is convergent to a fixed point* $\nu*$ *of* $\mathcal{T}$;
(3) $\nu*$ *is the unique fixed point of* $\mathcal{T}$ *in the set* $V = \{\nu \in \mathcal{U} : \tau(\mathcal{T}^{m_0} u, \nu) < \infty\}$;
(4) $\tau(\nu*, \nu) \leqslant \frac{1}{1-L} \tau(\nu, \mathcal{T}\nu)$ *for all* $\nu \in V$.

Proof. For proving the stability result, we define: ξ_i is a constant such that

$$\xi_i = \begin{cases} 2, & \text{if } i = 0, \\ \frac{1}{2}, & \text{if } i = 1, \end{cases}$$

and Λ is the set such that $\Lambda = \{\mu | \mu : \mathcal{U} \to V, \mu(0) = 0\}$. □

Theorem 4.2. *Let* $g : \mathcal{U} \longrightarrow V$ *be a mapping for which there exist a function* $h : \mathcal{U}^3 \to \mathcal{Z}$ *with the condition*

$$\lim_{l \to \infty} P'_{\alpha,\beta}(h(\xi_i^l u_1, \xi_i^l u_2, \xi_i^l u_3), \xi_i^{3l} \varepsilon) = 1_{L^*}$$

for all $u_1, u_2, u_3 \in \mathcal{U}$, $\varepsilon > 0$ *and satisfying the inequality*

$$P_{\alpha,\beta}(Dg(u_1, u_2, u_3), \varepsilon) \geqslant_{L^*} P_{\alpha,\beta}(h(u_1, u_2, u_3), \varepsilon) \tag{4.1}$$

for all $u_1, u_2, u_3 \in \mathcal{U}$, $\varepsilon > 0$*. If there exists* $L = L(i)$ *such that the function*

$$u \to \gamma(u) = \frac{1}{3} h\Big(\frac{u}{2}, 0, 0\Big)$$

has the property

$$P'_{\alpha,\beta}\Big(L \frac{1}{\xi_i^3} \gamma(\xi_i u), \varepsilon\Big) = P'_{\alpha,\beta}(\gamma(u), \varepsilon) \tag{4.2}$$

for all $u \in \mathcal{U}$, $\varepsilon > 0$*, then there exists unique cubic function* $Q : \mathcal{U} \to V$ *satisfying the functional equation* (1.2) *and*

$$P_{\alpha,\beta}(g(u) - Q(u), \varepsilon) \geqslant_{L^*} P'_{\alpha,\beta}\Big(\gamma(u), \frac{L^{1-i}}{1-L} \varepsilon\Big)$$

for all $u \in \mathcal{U}$, $\varepsilon > 0$.

Proof. Let τ be a general metric on Λ, such that

$$\tau(\mu, \nu) = \inf\{M \in (0, \infty) | P_{\alpha,\beta}(\mu(u) - \nu(u), \varepsilon) \geqslant_{L^*} P'_{\alpha,\beta}(\gamma(u), M\varepsilon), u \in \mathcal{U}, \varepsilon > 0\}.$$

It is simple to verify that (Λ, τ) is complete. Define $T : \Lambda \to \Lambda$ by

$$T(\mu(u)) = \frac{1}{\xi_i^3} \mu(\xi_i u)$$

for all $u \in \mathcal{U}$. For $\mu, \nu \in \Lambda$, we have $\tau(\mu, \nu) \leqslant M$,

$$P_{\alpha,\beta}(\mu(u) - \nu(u), \varepsilon) \geqslant_{L^*} P'_{\alpha,\beta}(\gamma(u), M\varepsilon),$$

$$\begin{aligned}
P_{\alpha,\beta}\Big(\frac{\mu(\xi_i u)}{\xi_i^3} - \frac{\nu(\xi_i u)}{\xi_i^3}, \varepsilon\Big) &\geqslant_{L^*} P'_{\alpha,\beta}(\gamma(\xi_i u), M\xi_i^3\varepsilon),\\
P_{\alpha,\beta}(\mathcal{T}\mu(u) - \mathcal{T}\nu(u), \varepsilon) &\geqslant_{L^*} P'_{\alpha,\beta}(\gamma(u), ML\varepsilon),\\
\tau(\mathcal{T}\mu(u), \mathcal{T}\nu(u), \varepsilon) &\leqslant ML,\\
\tau(\mathcal{T}\mu, \mathcal{T}\nu) &\leqslant L\tau(\mu, \nu)
\end{aligned}$$

for all $\mu, \nu \in \Lambda$. Therefore $\mathcal{T}$ is strictly contractive mapping on Λ with Lipschitz constant L. Replacing (u_1, u_2, u_3) by $(u, 0, 0)$ in (4.1), we obtain

$$P_{\alpha,\beta}(3g(2u) - 24g(u), \varepsilon) \geqslant_{L^*} P'_{\alpha,\beta}(h(u, 0, 0), \varepsilon). \tag{4.3}$$

Using (IFN2) in (4.3), we obtain

$$P_{\alpha,\beta}\Big(\frac{g(2u)}{2^3} - g(u), \varepsilon\Big) \geqslant_{L^*} P'_{\alpha,\beta}(h(u, 0, 0), 24\varepsilon) \tag{4.4}$$

with the help of (4.2), when $i = 0$, it follows from (4.4), we get

$$\begin{aligned}
P_{\alpha,\beta}\Big(\frac{g(2u)}{2^3} - g(u), \varepsilon\Big) &\geqslant_{L^*} P'_{\alpha,\beta}(\gamma(u), L\varepsilon),\\
\tau(\mathcal{T}g, g) &\leqslant L = L' = L^{1-i}.
\end{aligned} \tag{4.5}$$

Replacing u by $\frac{u}{2}$ in (4.3), we get

$$P_{\alpha,\beta}\Big(g(u) - 2^3 g\Big(\frac{u}{2}\Big), \varepsilon\Big) \geqslant_{L^*} P'_{\alpha,\beta}\Big(h\Big(\frac{u}{2}, 0, 0\Big), 3\varepsilon\Big) \tag{4.6}$$

with the help of (4.2) when $i = 1$, it follows from (4.6), we get

$$\begin{aligned}
P_{\alpha,\beta}\Big(g(u) - 2^3 g\Big(\frac{u}{2}\Big), \varepsilon\Big) &\geqslant_{L^*} P'_{\alpha,\beta}(\gamma(u), \varepsilon),\\
\tau(g, \mathcal{T}g) &\leqslant 1 = L^0 = L^{1-i}.
\end{aligned} \tag{4.7}$$

Then from (4.5) and (4.7), we can conclude

$$\tau(g, \mathcal{T}g) \leqslant L^{1-i} < \infty.$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point Q of $\mathcal{T}$ in Λ such that

$$\lim_{m\to\infty} P_{\alpha,\beta}\Big(\frac{g(\xi_i^m u)}{\xi_i^m} - Q(u), \varepsilon\Big) \to 1_{L^*}$$

for all $u \in U$, $\varepsilon > 0$. Replacing (u_1, u_2, u_3) by $(\xi_i u_1, \xi_i u_2, \xi_i u_3)$ in (4.1), we get

$$P_{\alpha,\beta}\Big(\frac{1}{\xi_i^{3m}} Dg(\xi_i u_1, \xi_i u_2, \xi_i u_3), \varepsilon\Big) \geqslant_{L^*} P'_{\alpha,\beta}\big(h(\xi_i u_1, \xi_i u_2, \xi_i u_3), \xi_i^{3m}\varepsilon\big)$$

for all $u_1, u_2, u_3 \in U$ and $\varepsilon > 0$. By proceeding the same procedure as in the above theorem (3.1), we can show the function $Q : U \to V$ satisfies the functional equation (1.2). By fixed point alternative, since Q is unique fixed point of $\mathcal{T}$ in the set

$$\Gamma = \{g \in \Lambda | \tau(g, Q) < \infty\},$$

hence Q is a unique function such that

$$P_{\alpha,\beta}(g(u) - Q(u), \varepsilon) \geqslant_{L^*} P'_{\alpha,\beta}(\gamma(u), M\varepsilon)$$

for all $u \in U$, $\varepsilon > 0$, and $M > 0$. Again using the fixed point alternative, we obtain

$$\tau(g, Q) \leqslant \frac{1}{1-L}\tau(g, \mathcal{T}g), \qquad \tau(g, Q) \leqslant \frac{L^{1-i}}{1-L}, \qquad P_{\alpha,\beta}(g(u) - Q(u), \varepsilon) \geqslant_{L^*} P'_{\alpha,\beta}(\gamma(u), \frac{L^{1-i}}{1-L}\varepsilon)$$

for all $u \in U$, $\varepsilon > 0$. This completes the proof of the theorem. □

From above theorem, we have the following corollary concerning the stability for the functional equation (1.2).

Corollary 4.3. *Assume that a function $g : U \to V$ satisfies the inequality*

$$P_{\alpha,\beta}(Dg(u_1, u_2, u_3)) \geqslant \begin{cases} P'_{\alpha,\beta}(\xi\Sigma_{i=1}^{3}\|u_i\|^s, \varepsilon), \\ P'_{\alpha,\beta}(\xi\Pi_{i=1}^{3}\|u_i\|^s + \Sigma_{i=1}^{3}\|u_i\|^{3s}, \varepsilon) \end{cases}$$

for all $u_1, u_2, u_3 \in U$ and $\varepsilon > 0$, where ε, s are constants with $\varepsilon > 0$. Then there exists a unique cubic mapping $Q : U \to V$ such that

$$P_{\alpha,\beta}(g(u) - Q(u), \varepsilon) \geqslant_{L^*} \begin{cases} P'_{\alpha,\beta}(\xi\|u\|^s, \frac{2^{s+2}}{2^3-2^s}\varepsilon), & s < 3 \text{ or } s > 3, \\ P'_{\alpha,\beta}(\xi\|u\|^{3s}, \frac{2^{3s+2}}{2^3-2^{ms}}\varepsilon), & s < \frac{3}{m} \text{ or } s > \frac{3}{m} \end{cases}$$

for all $u \in U$ and $\varepsilon > 0$.

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