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The Weibull Marshall–Olkin Lindley distribution: properties and estimation

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ABSTRACT
We obtain some properties of the Weibull Marshall–Olkin Lindley distribution. Its hazard rate function can be increasing, decreasing, bathtub-shaped, decreasing-increasing-decreasing and unimodal. We compare the performance of some methods to estimate its model parameters by means of extensive simulations. The potentiality of the new distribution to modeling real-life data is shown using two real data sets.

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1. Introduction

Modeling real data using generalized distributions remains strong nowadays. Many generalized distributions have been developed and applied in several fields. However, there still remain many important problems involving real data, which are not addressed by known models.

Lindley [1] proposed a distribution which bears his name defined by a mixture of exponential and gamma distributions to deal with lifetime data. Ghitany et al. [2] reported some of its structural properties and proved that it is better suited than the exponential distribution for some types of data. The statistical literature contains many extended forms of the Lindley distribution. For example, the generalized Lindley [3], negative binomial Lindley [4], generalized Poisson Lindley [5], generalized Lindley [6], extended Lindley [7], quasi-Lindley [8], power Lindley [9], weighted Lindley [10], gamma Lindley [11], transmuted Lindley [12], complementary geometric transmuted Lindley [13], Weibull Lindley (WL) [14] and extended odd Weibull Lindley [15] distributions.

For any baseline G distribution with parameter vector η, Korkmaz et al. [16] proposed the *Weibull Marshall–Olkin-G* (WMO-G) family based on the T-X generator [17]. Consider that $R(t)$ and $r(t)$ are the cumulative distribution function (CDF) and probability density function (PDF) of a random variable $T \in [a, b]$ (for $-\infty < a < b < \infty$), respectively. Let $W[G(x;\eta)]$ be a function of the CDF of another random variable X satisfying the following conditions: (i) $W[G(x;\eta)] \in [a, b]$; (ii) $W[G(x;\eta)] \to a$ if $x \to -\infty$ and $W[G(x;\eta)] \to b$ if $x \to \infty$; and (iii) $W[G(x;\eta)]$ is differentiable and monotonically non-decreasing.

The CDF of the T-X generator is defined by

$$F(x) = \int_a^{W[G(x;\eta)]} r(t)\, dt = R\left(W[G(x;\eta)]\right). \quad (1)$$

Setting $r(t) = \beta\, t^{\beta-1}\, e^{-t^{\beta}}$, $t > 0$, where $\beta > 0$ is a shape parameter, and $W[G(x;\eta)] = -\log[\alpha\bar{G}(x;\eta)/(G(x;\eta) + \alpha\bar{G}(x;\eta))]$, for $\alpha > 0$, the CDF of the WMO-G family has the form

$$F(x;\alpha,\beta,\eta) = 1 - \exp\left(-\left\{-\log\left[\frac{\alpha\bar{G}(x;\eta)}{1-\bar{\alpha}\bar{G}(x;\eta)}\right]\right\}^{\beta}\right). \quad (2)$$

The PDF corresponding to (2) is

$$f(x;\alpha,\beta,\eta) = \frac{\beta g(x;\eta)}{\bar{G}(x;\eta)\left[1-\bar{\alpha}\bar{G}(x;\eta)\right]} \times \left\{-\log\left[\frac{\alpha\bar{G}(x;\eta)}{1-\bar{\alpha}\bar{G}(x;\eta)}\right]\right\}^{\beta-1} \times \exp\left(-\left\{-\log\left[\frac{\alpha\bar{G}(x;\eta)}{1-\bar{\alpha}\bar{G}(x;\eta)}\right]\right\}^{\beta}\right), \quad (3)$$

where $g(x;\eta)$ is the baseline PDF, $\bar{\alpha} = 1 - \alpha$, and α and β are two extra positive shape parameters.

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The hazard rate function (HRF) of the WMO-G family is

$$\tau\left(x;\alpha,\beta,\boldsymbol{\eta}\right)=\frac{\beta\, w\left(x;\boldsymbol{\eta}\right)}{\left[1-\bar{\alpha}\bar{G}\left(x;\boldsymbol{\eta}\right)\right]}\times\left\{-\log\left[\frac{\alpha\bar{G}\left(x;\boldsymbol{\eta}\right)}{1-\bar{\alpha}\bar{G}\left(x;\boldsymbol{\eta}\right)}\right]\right\}^{\beta-1},\quad(1)$$

where $w(x;\boldsymbol{\eta})=g(x;\boldsymbol{\eta})/\bar{G}(x;\boldsymbol{\eta})$ is the baseline HRF.

For $\alpha=1$, we obtain the Weibull-X family [17,18] as a special case of the WMO-G family. For $\beta=1$, we have the MO-G family [19]. For $\alpha=\beta=1$, it follows the baseline distribution. Further details on the WMO-G family can be explored in Korkmaz et al. [16].

In this paper, we propose a new three-parameter model called the *Weibull Marshall–Olkin-Lindley* (WMOL) distribution. By setting the Lindley CDF (for $a>0$) $G(x;a)=1-((1+a+ax)/(1+a))\mathrm{e}^{-ax}$ in (2), we obtain

$$F\left(x\right)=1-\exp\left\{-\left[\log\left(\frac{\left(1+a\right)\mathrm{e}^{ax}}{\alpha\left(1+a+ax\right)}-\frac{\bar{\alpha}}{\alpha}\right)\right]^{\beta}\right\}.\quad(4)$$

The PDF corresponding to (4) is

$$f\left(x\right)=\frac{\beta a^{2}(1+x)\left(1+a+ax\right)^{-1}}{\left(1-\frac{1+a+ax}{1+a}\bar{\alpha}\mathrm{e}^{-ax}\right)}\times\left[\log\left(\frac{\left(1+a\right)\mathrm{e}^{ax}}{\alpha\left(1+a+ax\right)}-\frac{\bar{\alpha}}{\alpha}\right)\right]^{\beta-1}\times\exp\left\{-\left[\log\left(\frac{\left(1+a\right)\mathrm{e}^{ax}}{\alpha\left(1+a+ax\right)}-\frac{\bar{\alpha}}{\alpha}\right)\right]^{\beta}\right\},\quad(5)$$

where $\bar{\alpha}=1-\alpha$, $\alpha>0$ and $\beta>0$ are two shape parameters and $a>0$ is a shape parameter.

Henceforth, $X\sim\mathrm{WMOL}(\alpha,\beta,a)$ denotes a random variable with PDF (5). The HRF of X is

$$\tau\left(x\right)=\frac{\beta a^{2}(1+x)}{\left(1+a+ax\right)\left(1-\frac{1+a+ax}{1+a}\bar{\alpha}\mathrm{e}^{-ax}\right)}\times\left[\log\left(\frac{\left(1+a\right)\mathrm{e}^{ax}}{\alpha\left(1+a+ax\right)}-\frac{\bar{\alpha}}{\alpha}\right)\right]^{\beta-1}.$$

For $\alpha=1$, the WMOL model reduces to the WL distribution. We obtain the MO-Lindley [20] when $\beta=1$. For $\alpha=\beta=1$, it follows the Lindley distribution. Figures 1 and 2 show plots of the PDF and HRF of the WMOL distribution for some parameter values.

In fact, the WMOL distribution can be justified from the following reasons: (i) It generalizes some well-known models in the literature; (ii) Its PDF can be J shape, reversed-J shape, unimodal, symmetric, left-skewed or right-skewed; (iii) Its HRF can accommodate increasing, decreasing, bathtub-shaped, decreasing-increasing-decreasing and unimodal shapes; (iv) Its kurtosis can be more flexible compared to that one of the Lindley model; (v) It provides better fits than some generalized distributions under the Lindley baseline.

The remainder of this paper is structured as follows. We derive a linear representation for the WMOL density function in Section 2. Some of its properties are obtained in Section 3. Six methods to estimate the parameters of the new distribution are presented in Section 4. We perform a simulation study in Section 5 to compare these methods. We provide a guideline for choosing the best estimation method. The flexibility of the new distribution is illustrated via two real data sets in Section 6. We conclude the paper by some final remarks in Section 7.

2. Linear representation

Here, we provide a linear representation for the WMOL PDF. Based on the power series

$$\mathrm{e}^{x}=\sum_{i=0}^{\infty}\frac{x^{i}}{i!},$$

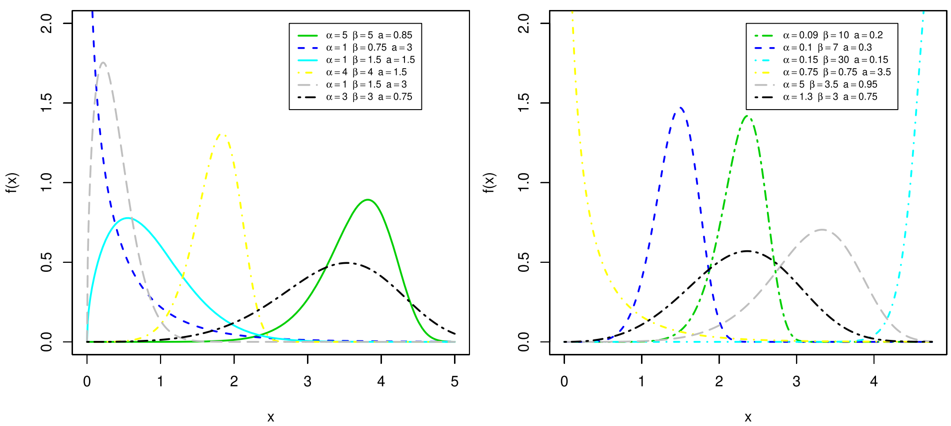

Figure 1. The PDF plots of the WMOL distribution for selected parameter values.

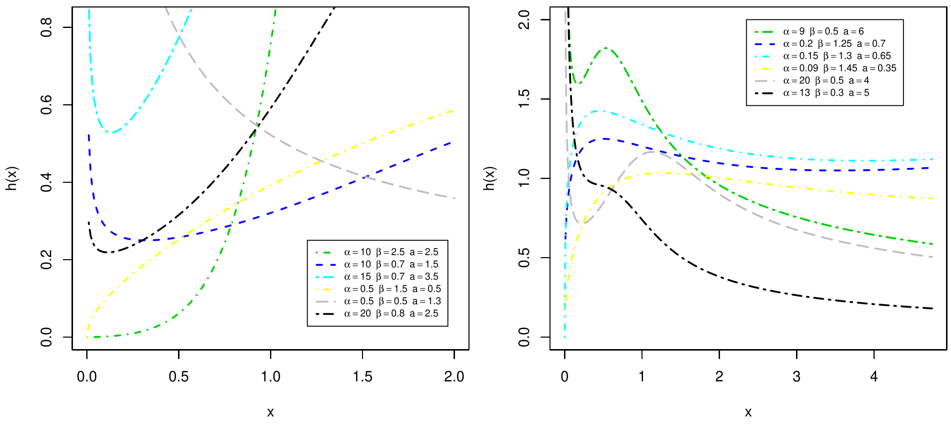

Figure 2. The HRF plots of the WMOL distribution for selected parameter values.

the CDF of X can be expressed from (4) as

$$F(x)=\sum_{i=1}^{\infty}\frac{(-1)^i}{i!}\times\left[-\log\left\{1-\left(1-\frac{\frac{\alpha(1+a+ax)}{1+a}\,\mathrm{e}^{-ax}}{1-\bar{\alpha}\left(\frac{1+a+ax}{1+a}\right)\mathrm{e}^{-ax}}\right)\right\}\right]^{i\beta}. \tag{6}$$

For $z\in(0,1)$ and any real parameter b, the formula holds

$$[-\log(1-z)]^b=x^b+\sum_{j=0}^{\infty}\phi_j(b)\,z^{j+b+1}, \tag{7}$$

where

$$\phi_0(b)=\frac{1}{2}b,\quad \phi_1(b)=\frac{1}{24}[b(3b+5)],$$
$$\phi_2(b)=\frac{1}{48}[b(b^2+5b+6)],$$
$$\phi_3(b)=\frac{1}{5760}[b(15b^3+150b^2+485b+502)],\ldots,$$

are Stirling polynomials. The proof can be found in Flajonet and Odlyzko [21, Theorem 3A] and Flajonet and Sedgewick [22, Theorem VI.2]. The previous results have been used by Cordeiro et al. [23]. We can write

$$[-\log(1-x)]^{i\beta}=\sum_{j=0}^{\infty}\phi_{j-1}(i\,\beta)\,x^{j+i\beta}, \tag{8}$$

where $\phi_{-1}(i\,\beta)=0$ by convention and $\phi_j(i\,\beta)$ (for $j\geq 0$) follows from (7). Equation (8) comes from Balakrishnan and Cohan [24] and Shawky and Bakoban [25]. Then, the CDF (6) can be expressed by (8) as

$$F(x)=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i}{i!}\phi_{j-1}(i\,\beta)\times\left[1-\frac{\frac{\alpha(1+a+ax)}{1+a}\,\mathrm{e}^{-ax}}{1-\bar{\alpha}\left(\frac{1+a+ax}{1+a}\right)\mathrm{e}^{-ax}}\right]^{j+i\beta}.$$

For $|z|<1$ and a real non-integer b, the power series holds

$$(1-z)^b=\sum_{k=0}^{\infty}(-1)^k\binom{b}{k}z^k,$$

where $\binom{b}{k}=b(b-1)(b-2)(b-k+1)/k!$ is defined for any real b.

Hence, we can write

$$F(x)=\sum_{i=1}^{\infty}\sum_{j,k=0}^{\infty}\frac{(-1)^{i+k}\alpha^k\phi_{j-1}(i\,\beta)}{i!}\binom{i\,\beta+k}{k}\times\left(\frac{1+a+ax}{1+a}\mathrm{e}^{-ax}\right)^k\times\left[1-\bar{\alpha}\frac{1+a+ax}{1+a}\mathrm{e}^{-ax}\right]^{-k}. \tag{9}$$

Consider the convergent power series expression (for $|x|<1$ and $p>0$)

$$(1-x)^{-p}=\sum_{l=0}^{\infty}(-1)^l\binom{-p}{l}x^l.$$

For $\alpha\in(0,1)$, we can rewrite $F(x)$ as

$$F(x)=\sum_{k,l=0}^{\infty}w_{k,l}\,\bar{G}(x;a)^{k+l},$$

where

$$w_{k,l}=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+k+l}\alpha^k\phi_{j-1}(i\,\beta)}{i!(1-\alpha)^{-l}}\binom{i\,\beta+k}{k}\binom{-k}{l}$$

and $\bar{G}(x;a)=1-G(x;a)$ is the Lindley survival function.

Consider the Lehmann type II (LTII) CDF $\Pi_c(x)=1-\{1-G(x)\}^c$ [26] with power parameter $c>0$ defined from the baseline $G(x)$. Thus, the LTII density is given by $\pi_c(x)=c\,\bar{G}(x)^{c-1}\,g(x)$, where $g(x)=dG(x)/dx$.

We define the set of non-negative integers $J=\{(k,l);k,l=0,1,2,\ldots;k+l\geq 1\}$. By differentiating the last equation for $F(x)$, the PDF of X follows as

$$f(x) = \sum_{(k,l)\in J} w_{k,l}\,\pi_{k+l}(x;a), \tag{10}$$

where $\pi_{k+l}(x) = (k+l)\,\bar{G}(x;a)^{k+l-1}\,g(x;a)$ denotes the LTII Lindley density function with power parameter $k+l$.

Otherwise, if $\alpha > 1$, we can rewrite (9) as

$$F(x) = \sum_{i=1}^{\infty}\sum_{j,k=0}^{\infty} \frac{(-1)^{i+k}\alpha^k \phi_{j-1}(i\beta)}{i!} \times \binom{i\beta+k}{k}\left(\frac{1+a+ax}{1+a}e^{-ax}\right)^k \times \alpha^{-k}\left[1-(1-\alpha^{-1})\frac{1+a+ax}{1+a}e^{-ax}\right]^{-k}.$$

By using previous series expressions, we obtain

$$F(x) = \sum_{k,l=0}^{\infty} \nu_{k,l}\,\bar{G}(x;a)^{k+l},$$

where

$$\nu_{k,l} = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+k+l}\phi_{j-1}(i\beta)}{i!(1-\alpha^{-1})^{-l}}\binom{i\beta+k}{k}\binom{-k}{l}.$$

Hence, the density function of X follows as

$$f(x) = \sum_{(k,l)\in J} \nu_{k,l}\,\pi_{k+l}(x;a), \tag{11}$$

Equations (10) and (11) reveal that the WMOL density function for both cases are linear combinations of LTII Lindley densities.

Every LTII Lindley can be expressed in terms of exponentiated Lindley (EL) densities. By expanding $\Pi_c(x) = 1-\{1-G(x)\}^c$ (for c real), the power series converges everywhere

$$\Pi_c(x) = \sum_{r=1}^{\infty}(-1)^{r+1}\binom{c}{r}G(x)^r.$$

By differentiating the last equation, we have

$$\pi_c(x) = \sum_{r=0}^{\infty}(-1)^r\binom{c}{r+1}\rho_{r+1}(x), \tag{12}$$

where $\rho_{r+1}(x) = (r+1)\,G(x)^r\,g(x)$ is the EL density with power parameter $r+1$. If c is a positive integer, the last sum stops at c.

Hence, some structural properties of the WMOL distribution can be determined from those of the EL distribution reported by Nadarajah et al. [6].

3. Properties of the WMOL distribution

Some mathematical properties of the WMOL distribution are presented in this section. We consider only the case $0 < \alpha < 1$ since for $\alpha > 1$ all equations derived hold by changing the coefficients $w_{k,l}$ by $\nu_{k,l}$.

3.1. Moments and generating functions

We obtain ordinary moments and the moment generating function (MGF) of $X \sim$WMOL(α,β,a). Nadarajah et al. [6] defined and computed

$$K(p,q,c,\delta) = \int_0^{\infty} x^c(1+x) \times \left[1-\frac{1+q+qx}{1+q}e^{-qx}\right]^{p-1} e^{-\delta x}\,dx,$$

which can be used to produce the rth ordinary moment $\mu'_r = E(X^r)$. We can write

$$K(p,q,c,\delta) = \sum_{u=0}^{\infty}\sum_{v=0}^{u}\sum_{w=0}^{v+1}\binom{p-1}{u}\binom{u}{v}\binom{v+1}{w} \times \frac{(-1)^u q^v \Gamma(w+c+1)}{(1+q)^u(qu+\delta)^{w+c+1}}. \tag{13}$$

Therefore, from (10), (12) and (13), we obtain

$$\mu'_n = E(X^n) = \frac{a^2}{1+a}\sum_{(k,l)\in J}\sum_{r=0}^{k+l}(-1)^r\binom{k+l}{r+1} \times (r+1)\,w_{k,l}\,K((r+1),a,n,a).$$

The mean, variance, skewness and kurtosis of the WMOL distribution are computed numerically for different values of the parameters α, β and a using the R software. Table 1 gives these numerical values which indicate that the skewness of the WMOL distribution can range in the interval (−1.049, 5.656), whereas the skewness of the L distribution can only range in the interval (1.512, 1.989) when the parameter a takes the values 0.5, 1.5, 3.5, 5.0, 10, 15 and 20. The spread for the WMOL kurtosis is much larger ranging from 2.746 to 65.17, whereas the kurtosis of the L distribution can only varies from 6.343 to 8.913 for these values of the parameter a. Further, the WMOL model can be negative skewed or positive skewed. Hence, the WMOL distribution is a flexible distribution which can be used in modelling skewed data.

Analogously, the MGF of X can be determined (for $t < a$) as

$$M(t) = \frac{a^2}{1+a}\sum_{(k,l)\in J}\sum_{r=0}^{k+l}(-1)^r\binom{k+l}{r+1} \times (r+1)\,w_{k,l}\,K((r+1),a,0,a-t).$$

3.2. Conditional moments, mean residual life and mean deviations

We obtain the conditional moments of the WMOL distribution. Nadarajah et al. [6] defined the following

Table 1. Mean, variance, skewness and kurtosis of the WMOL distribution for various values of the parameters α, β and a.

Parameters	Mean	Variance	Skewness	Kurtosis
($\alpha = 0.5$, $\beta = 0.5$, $a = 0.5$)	4.630	94.45	5.656	57.37
($\alpha = 0.5$, $\beta = 1.5$, $a = 0.5$)	2.198	2.072	1.073	4.445
($\alpha = 0.5$, $\beta = 3.5$, $a = 0.5$)	2.193	0.438	0.019	2.746
($\alpha = 0.5$, $\beta = 5.0$, $a = 0.5$)	2.236	0.238	−0.257	2.906
($\alpha = 1.5$, $\beta = 0.5$, $a = 1.5$)	1.812	11.31	5.616	65.17
($\alpha = 1.5$, $\beta = 1.5$, $a = 1.5$)	1.070	0.371	0.688	3.362
($\alpha = 1.5$, $\beta = 3.5$, $a = 1.5$)	1.103	0.086	−0.190	2.807
($\alpha = 1.5$, $\beta = 5.0$, $a = 1.5$)	1.127	0.047	−0.427	3.116
($\alpha = 3.5$, $\beta = 0.5$, $a = 3.5$)	0.874	2.130	5.404	62.11
($\alpha = 3.5$, $\beta = 1.5$, $a = 3.5$)	0.593	0.084	0.422	2.982
($\alpha = 3.5$, $\beta = 3.5$, $a = 3.5$)	0.622	0.020	−0.360	3.021
($\alpha = 3.5$, $\beta = 5.0$, $a = 3.5$)	0.635	0.010	−0.561	3.396
($\alpha = 5.0$, $\beta = 0.5$, $a = 5.0$)	0.644	1.051	5.325	61.01
($\alpha = 5.0$, $\beta = 1.5$, $a = 5.0$)	0.461	0.044	0.318	2.909
($\alpha = 5.0$, $\beta = 3.5$, $a = 5.0$)	0.484	0.010	−0.424	3.140
($\alpha = 5.0$, $\beta = 5.0$, $a = 5.0$)	0.494	0.005	−0.610	3.523
($\alpha = 10$, $\beta = 10$, $a = 15$)	0.199	$1.6 \cdot 10^{-5}$	−0.886	4.357
($\alpha = 20$, $\beta = 10$, $a = 20$)	0.182	$9.5 \cdot 10^{-5}$	−0.911	4.459
($\alpha = 20$, $\beta = 20$, $a = 10$)	0.382	$1.1 \cdot 10^{-5}$	−1.049	8.298

equation

$$L(p,q,c,\delta,t) = \int_t^x x^c(1+x) \times \left[1 - \frac{1+q+qx}{1+q}\, e^{-qx}\right]^{p-1} e^{-\delta x}\, dx,$$

which can be used to find conditional moments. We can write

$$L(p,q,c,\delta,t) = \sum_{u=0}^{\infty}\sum_{v=0}^{u}\sum_{w=0}^{v+1} \binom{p-1}{u}\binom{u}{v}\binom{v+1}{w} \times \frac{(-1)^u q^v\, \Gamma[(w+c+1,\, qu+\delta)t}{(1+q)^u(qu+\delta)^{w+c+1}}, \tag{14}$$

where $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\,dt$ denotes the upper incomplete gamma function. Hence, we obtain the conditional moments of the WMOL distribution from (10), (12) and (14) as

$$\mu'_n(t) = E[X^n|x > t] = \frac{a^2}{(1+a)S(x)} \sum_{(k,l)\in J}\sum_{r=0}^{k+l}(-1)^r \times \binom{k+l}{r+1}(r+1)\, w_{k,l}\, L((r+1),a,n,a,t).$$

The mean residual life is the expected remaining life $X-x$ assuming that the item has survived to time x, say $m_X(x) = E(X-x|X > x)$. We have

$$m_X(x) = \frac{a^2}{(1+a)S(x)} \sum_{(k,l)\in J}\sum_{r=0}^{k+l}(-1)^r \binom{k+l}{r+1} \times (r+1)\, w_{k,l}\, L((r+1),a,1,a,x).$$

Let M denote the median of X. The mean deviations about the mean ($\delta_{\mu'_1}$) and the median (δ_M) used to measure the variation in a population from the center are given by

$$\delta_{\mu'_1} = \int_0^\infty |x-\mu'_1|\, f(x)\,dx = 2\mu'_1 F(\mu'_1) - 2\mu'_1 + \frac{2a^2}{(1+a)} \sum_{(k,l)\in J}\sum_{r=0}^{k+l}(-1)^r \times \binom{k+l}{r+1}(r+1)\, w_{k,l}\, L((r+1),a,1,a,\mu'_1)$$

and

$$\delta_M = \int_0^\infty |x-M|\, f(x)\,dx = 2MF(M) - M - \mu'_1 + \frac{2a^2}{(1+a)} \sum_{(k,l)\in J}\sum_{r=0}^{k+l}(-1)^r \times \binom{k+l}{r+1}(r+1)\, w_{k,l}\, L((r+1),a,1,a,M),$$

where $F(\mu'_1)$ and $F(M)$ can be easily calculated from (4).

3.3. Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves are very common in several fields. These curves can be constructed varying p from 0 to 1 by

$$B(p) = \frac{1}{p\mu'_1}\int_0^q x\, f(x)\,dx \quad \text{and} \quad L(p) = p\,B(p),$$

respectively, where $\mu'_1 = E(X)$ and $q = F^{-1}(p)$. For the WMOL distribution, the Bonferroni and Lorenz curves of X are expressed as

$$B(p) = \frac{1}{p\mu'_1}\left\{\mu - \frac{a^2}{(1+a)} \sum_{(k,l)\in J}\sum_{r=0}^{k+l}(-1)^r \times \binom{k+l}{r+1}(r+1)\, w_{k,l}\, L((r+1),a,1,a,q)\right\}$$

and $L(p) = p\,B(p)$, respectively.

4. Methods of estimation

In the following, we utilize six methods, usually known as maximum likelihood, ordinary least squares, weighted least squares, maximum product of spacing, Cramér-von-Mises and Anderson–Darling, to estimate the parameters of the new distribution.

4.1. Maximum likelihood

The maximum likelihood estimation is the most important method to estimate parameters of a distribution due to its good properties. Let $x_1, \ldots, x_n$ be a random sample of size n from the new distribution. The log-likelihood function comes from (5) as

$$\begin{aligned}\ell(\alpha,\beta,a) &\\ \propto\; & n\log(\beta) + 2n\log(a) - \sum_{i=1}^{n}\log(1+a+ax_i) \\ & - \sum_{i=1}^{n}\log\left[1-\left(1+\frac{ax_i}{1+a}\right)\bar{\alpha}e^{-ax_i}\right] \\ & + (\beta-1)\sum_{i=1}^{n}\log\left\{\log\left[\frac{(1+a)e^{ax_i}}{\alpha(1+a+ax_i)} - \frac{\bar{\alpha}}{\alpha}\right]\right\} \\ & - \sum_{i=1}^{n}\left\{\log\left[\frac{(1+a)e^{ax_i}}{\alpha(1+a+ax_i)} - \frac{\bar{\alpha}}{\alpha}\right]\right\}^{\beta}.\end{aligned}$$

Let $\hat{\alpha}_{MLE}$, $\hat{\beta}_{MLE}$ and $\hat{a}_{MLE}$ be the maximum likelihood estimates (MLEs) of the model parameters. They can be determined numerically by maximizing $\ell(\alpha,\beta,a)$ or by solving the nonlinear equations:

$$\begin{aligned}\frac{\partial\ell(\alpha,\beta,a)}{\partial\alpha} &= -\sum_{i=1}^{n}\frac{\left(1+\frac{ax_i}{1+a}\right)e^{-ax_i}}{1-\left(1+\frac{ax_i}{1+a}\right)\bar{\alpha}e^{-ax_i}} \\ &\quad + \sum_{i=1}^{n}\frac{(1+a)(1-e^{ax_i})+ax_i}{\alpha^2\xi_i(1+a+ax_i)} \\ &\quad \times\left[\frac{\beta-1}{\log(\xi_i)} - \beta(\log(\xi_i))^{\beta-1}\right] = 0, \\ \frac{\partial\ell(\alpha,\beta,a)}{\partial\beta} &= \frac{n}{\beta} + \sum_{i=1}^{n}\log(\log(\xi_i))[1-(\log(\xi_i))^{\beta}] = 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\ell(\alpha,\beta,a)}{\partial a} &= \frac{2n}{a} - \sum_{i=1}^{n}\frac{1+x_i}{1+a+ax_i} \\ &\quad + \bar{\alpha}\sum_{i=1}^{n}\frac{x_i(1+a)(1+a+ax_i)e^{ax_i} - x_ie^{ax_i}}{(1+a)^2\left[1-\left(1+\frac{ax_i}{1+a}\right)\bar{\alpha}e^{-ax_i}\right]} \\ &\quad + \sum_{i=1}^{n}\frac{ax_ie^{ax_i}(a+ax_i+x_i+2)}{\alpha\xi_i(1+a+ax_i)^2} \\ &\quad \times\left[\frac{\beta-1}{\log(\xi_i)} - \beta(\log(\xi_i))^{\beta-1}\right] = 0,\end{aligned}$$

where

$$\xi_i = \xi(\alpha,a;x_i) = \frac{(1+a)e^{ax_i}}{\alpha(1+a+ax_i)} - \frac{\bar{\alpha}}{\alpha}. \tag{15}$$

4.2. Ordinary least squares

Let $x_{1:n} < x_{2:n} < \cdots < x_{n:n}$ be the order observations of a sample of size n from the WMOL distribution with CDF (4). The ordinary least squares (OLS) estimates $\hat{\alpha}_{OLS}$, $\hat{\beta}_{OLS}$ and $\hat{a}_{OLS}$ of α, β and a can be calculated by minimizing numerically the function

$$\begin{aligned}S(\alpha,\beta,a) &\\ = \sum_{i=1}^{n}&\left[\varrho(i,n) - \exp\left\{-\left\{\log\left[\frac{(1+a)e^{ax_{i:m}}}{\alpha(1+a+ax_{i:m})} - \frac{\bar{\alpha}}{\alpha}\right]\right\}^{\beta}\right\}\right]^2,\end{aligned}$$

where $\varrho(i,n) = (n+1-i)/(n+1)$, with respect to α, β and a. Alternatively, the estimates can be found by solving the equations:

$$\begin{aligned}\frac{\partial S(\alpha,\beta,a)}{\partial\alpha} &= \sum_{i=1}^{n}\left\{\varrho(i,n) - \exp\left\{-[\log(\xi_{i:n})]^{\beta}\right\}\right\} \\ &\quad \times\psi_1(x_{i:n}|\alpha,\beta,a) = 0, \\ \frac{\partial S(\alpha,\beta,a)}{\partial\beta} &= \sum_{i=1}^{n}\left\{\varrho(i,n) - \exp\left\{-[\log(\xi_{i:n})]^{\beta}\right\}\right\} \\ &\quad \times\psi_2(x_{i:n}|\alpha,\beta,a) = 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial S(\alpha,\beta,a)}{\partial a} &= \sum_{i=1}^{n}\left\{\varrho(i,n) - \exp\left\{-[\log(\xi_{i:n})]^{\beta}\right\}\right\} \\ &\quad \times\psi_3(x_{i:n}|\alpha,\beta,a) = 0,\end{aligned}$$

where $\xi_{i:n}$ is the order transformed observation of ξ_i given by (15), and

$$\psi_1(x_{i:n}|\alpha,\beta,a) = \frac{\beta[(1+a+ax_{i:n}) - (1+a)e^{ax_{i:m}}]\,(\log(\xi_{i:n}))^{\beta-1}\exp\{-\log(\xi_{i:n})^{\beta}\}}{\alpha^2\xi_{i:n}(1+a+ax_{i:n})}, \tag{16}$$

$$\begin{aligned}\psi_2(x_{i:n}|\alpha,\beta,a) &= \exp\{-\log(\xi_{i:n})^{\beta}\} \\ &\quad \times(\log(\xi_{i:n}))^{\beta}\log(\log(\xi_{i:n}))\end{aligned} \tag{17}$$

and

$$\psi_3(x_{i:n}|\alpha,\beta,a) = \frac{\beta ax_ie^{ax_{i:n}}(\log(\xi_{i:n}))^{\beta-1}\,(a+ax_{i:n}+x_i+2)}{\alpha\xi_{i:n}(1+a+ax_{i:n})^2}. \tag{18}$$

4.3. Weighted least squares

Using the same notation as in the previous section, we can obtain the weighted least squares (WLS) estimates of α, β and a denoted by $\hat{\alpha}_{WLS}$ and $\hat{\beta}_{WLS}$ and $\hat{a}_{WLS}$

for the WMOL distribution by minimizing the following function in relation to the parameters

$$W(\alpha,\beta,a)=\sum_{i=1}^{n}\kappa(i,n)\left[\varrho(i,n)-\exp\left\{-\left\{\log\left[\frac{(1+a)e^{ax_{i:m}}}{\alpha(1+a+ax_{i:m})}-\frac{\bar{\alpha}}{\alpha}\right]\right\}^{\beta}\right\}\right]^{2},$$

where $\kappa(i,n)=(n+1)^2(n+2)/i(n-i+1)$, or equivalently by solving the nonlinear equations

$$\frac{\partial W(\alpha,\beta,a)}{\partial\alpha}=\sum_{i=1}^{n}\kappa(i,n)\left\{\varrho(i,n)-\exp\left\{-[\log(\xi_{i:n})]^{\beta}\right\}\right\}\times\psi_1(x_{i:n}|\alpha,\beta,a)=0,$$

$$\frac{\partial W(\alpha,\beta,a)}{\partial\beta}=\sum_{i=1}^{n}\kappa(i,n)\left\{\varrho(i,n)-\exp\left\{-[\log(\xi_{i:n})]^{\beta}\right\}\right\}\times\psi_2(x_{i:n}|\alpha,\beta,a)=0$$

and

$$\frac{\partial W(\alpha,\beta,a)}{\partial a}=\sum_{i=1}^{n}\kappa(i,n)\left\{\varrho(i,n)-\exp\left\{-[\log(\xi_{i:n})]^{\beta}\right\}\right\}\times\psi_3(x_{i:n}|\alpha,\beta,a)=0,$$

where $\xi_{i:n}$ and $\psi_j(x_{i:n}|\alpha,\beta,a)$ $(j=1,\ 2,\ 3)$ are given by (15), (16), (17) and (18), respectively.

4.4. Maximum product of spacing

Cheng and Amin [27,28] pioneered the maximum product of spacing (MPS) method to estimate parameters based on the differences between the CDF values evaluated at consecutive ordered observations. This method (applied to a random sample of size n from the WMOL distribution) is based on the expression

$$D_i(\alpha,\beta,a)=F(x_{i:n}|\alpha,\beta,a)-F(x_{i-1:n}|\alpha,\beta,a),\quad i=1,\ldots,n,$$

where $F(x_{0:n}|\alpha,\beta,a)=0$ and $F(x_{n+1:n}|\alpha,\beta,a)=1$. The MPS estimates $\hat{\alpha}_{MPS}$, $\hat{\beta}_{MPS}$ and $\hat{a}_{MPS}$ of α, β and a can be determined by maximizing the function

$$M(\alpha,\beta,a)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i(\alpha,\beta,a),$$

in relation to α, β and a. The estimates can be also calculated numerically from the equations:

$$\frac{\partial M(\alpha,\beta,a)}{\partial\alpha}=\sum_{i=1}^{n+1}\frac{\psi_1(x_{i:n}|\alpha,\beta,a)-\psi_1(x_{i-1:n}|\alpha,\beta,a)}{D_i(\alpha,\beta,a)}=0,$$

$$\frac{\partial M(\alpha,\beta,a)}{\partial\lambda}=\sum_{i=1}^{n+1}\frac{\psi_2(x_{i:n}|\alpha,\beta,a)-\psi_2(x_{i-1:n}|\alpha,\beta,a)}{D_i(\alpha,\beta,a)}=0$$

and

$$\frac{\partial M(\alpha,\beta,a)}{\partial a}=\sum_{i=1}^{n+1}\frac{\psi_3(x_{i:n}|\alpha,\beta,a)-\psi_3(x_{i-1:n}|\alpha,\beta,a)}{D_i(\alpha,\beta,a)}=0,$$

where $\psi_j(.|\alpha,\beta,a)$ $(j=1,2,3)$ are defined by (16), (17) and (18).

4.5. Cramér-von-Mises

The Cramér-von-Mises' method is based on the differences between the estimated CDF at the ordered observations and the empirical distribution function [29]. For the WMOL distribution, the Cramér-von-Mises estimates (CMEs) $\hat{\alpha}_{CME}$, $\hat{\beta}_{CME}$ and $\hat{a}_{CME}$ of the unknown parameters can be found numerically by minimizing the function

$$C(\alpha,\beta,a)=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{2(n-i)+1}{2n}-\exp\left\{-\left\{\log\left[\frac{(1+a)e^{ax_{i:m}}}{\alpha(1+a+ax_{i:m})}-\frac{\bar{\alpha}}{\alpha}\right]\right\}^{\beta}\right\}\right]^{2},$$

with respect to α, β and a.

These estimates also can be found numerically from the equations:

$$\frac{\partial C(\alpha,\beta,a)}{\partial\alpha}=\sum_{i=1}^{n}\left\{\frac{2(n-i)+1}{2n}-\exp\left\{-[\log(\xi_{i:n})]^{\beta}\right\}\right\}\times\psi_1(x_{i:n}|\alpha,\beta,a)=0,$$

$$\frac{\partial C(\alpha,\beta,a)}{\partial\beta}=\sum_{i=1}^{n}\left\{\frac{2(n-i)+1}{2n}-\exp\left\{-[\log(\xi_{i:n})]^{\beta}\right\}\right\}\times\psi_2(x_{i:n}|\alpha,\beta,a)=0$$

and

$$\frac{\partial C(\alpha,\beta,a)}{\partial a}=\sum_{i=1}^{n}\left\{\frac{2(n-i)+1}{2n}-\exp\left\{-[\log(\xi_{i:n})]^{\beta}\right\}\right\}\times\psi_3(x_{i:n}|\alpha,\beta,a)=0,$$

where $\xi_{i:n}$ and $\psi_j(x_{i:n}|\alpha,\beta,a)$ $(j=1,\ 2,\ 3)$ are given by (15)–(18), respectively.

4.6. Anderson–Darling

We can obtain the Anderson–Darling estimates (ADEs) of the parameters α, β and a, denoted by $\hat{\alpha}_{ADE}$, $\hat{\beta}_{ADE}$ and $\hat{a}_{ADE}$, by minimizing the function

$$AD(\alpha,\beta,a)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log F(x_{i:n}|\alpha,\beta,a)+\log\bar{F}(x_{n-i+1:n}|\alpha,\beta,a)\right\},$$

in relation to α, β and a, or by solving the equations

$$\frac{\partial AD(\alpha,\beta,a)}{\partial\alpha} = \sum_{i=1}^{n}(2i-1)\left\{\frac{\psi_1(x_{i:n}|\alpha,\beta,a)}{F(x_{i:n}|\alpha,\beta,a)} - \frac{\psi_1(x_{n-i+1:n}|\alpha,\beta,a)}{\bar{F}(x_{n-i+1:n}|\alpha,\beta,a)}\right\} = 0,$$

$$\frac{\partial AD(\alpha,\beta,a)}{\partial\beta} = \sum_{i=1}^{n}(2i-1)\left\{\frac{\psi_2(x_{i:n}|\alpha,\beta,a)}{F(x_{i:n}|\alpha,\beta,a)} - \frac{\psi_2(x_{n-i+1:n}|\alpha,\beta,a)}{\bar{F}(x_{n-i+1:n}|\alpha,\beta,a)}\right\} = 0$$

and

$$\frac{\partial AD(\alpha,\beta,a)}{\partial a} = \sum_{i=1}^{n}(2i-1)\left\{\frac{\psi_3(x_{i:n}|\alpha,\beta,a)}{F(x_{i:n}|\alpha,\beta,a)} - \frac{\psi_3(x_{n-i+1:n}|\alpha,\beta,a)}{\bar{F}(x_{n-i+1:n}|\alpha,\beta,a)}\right\} = 0,$$

where $\psi_j(.|\alpha,\beta,a)$, $j = 1,\ 2,\ 3$ are given by (16), (17) and (18).

5. Simulation results

We now provide some simulations by comparing the performance of the six methods discussed in Section 4. We consider three different parameter configurations (Conf), say: Conf 1 ($\alpha = 0.8, \beta = 1.5, a = 1.5$), Conf 2 ($\alpha = 1.5, \beta = 2, a = 2$) and Conf 3 ($\alpha = 2, \beta = 1, a = 0.8$). The data are generated from the WMOL distribution under these configurations by choosing $n = 10, 50$, 100 and 200. For each setting, we generate 1,000 random samples from the WMOL distribution. The inverse CDF of the WMOL distribution can not be obtained in closed-form to generate random samples. So, we find the numerical solution for x of $F(x|\alpha,\beta,a) = u$, where u is a uniform variate in $(0, 1)$. The simulations are performed using the Mathcad program (version 2007). We obtain the average values of the biases and root mean-squared errors (RMSEs) of the estimates, namely

$$Bias(\hat{\theta}_j) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\theta}_{j,i} - \theta_j) \quad \text{and}$$

$$RMSE(\hat{\theta}_j) = \sqrt{\frac{1}{1000}\sum_{i=1}^{1000}(\hat{\theta}_{j,i} - \theta_j)^2},$$

where $\boldsymbol{\theta} = (\alpha, \beta, a)$ and $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{a})$. The values obtained are displayed in Figures 3–8. Based on these plots, we can conclude:

(1) All the estimators are asymptotically unbiased since their biases converge to zero when n increases.
(2) All estimators are consistent since their RMSEs tend to zero for large n.
(3) The MPS estimates perform better than the other estimates in terms of minimum biases and RMSEs, except for few cases of the parameter β.
(4) In most cases, the ADEs have the highest biases and RMSEs comparing with the other methods.
(5) It is reasonable the use of the MPS method for estimating the unknown parameters of the WMOL distribution.

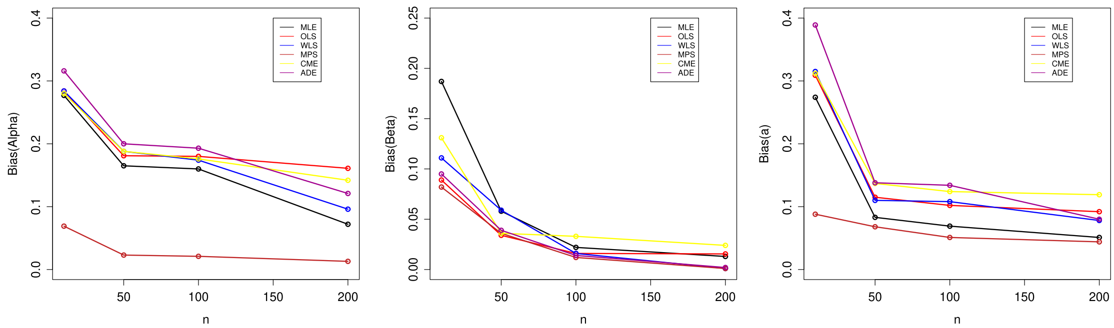

Figure 3. Biases for the estimates of $\alpha = 0.8$, $\beta = 1.5$ and $a = 1.5$ for each sample size for different methods of estimation.

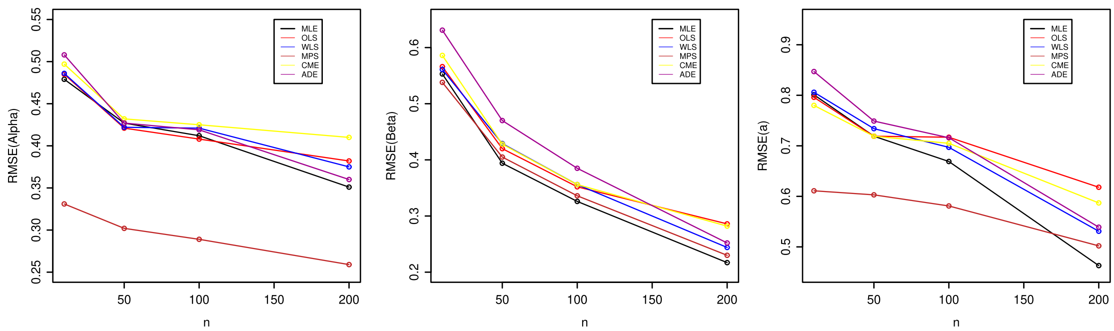

Figure 4. RMSEs for the estimates of $\alpha = 0.8$, $\beta = 1.5$ and $a = 1.5$ for each sample size for different methods of estimation.

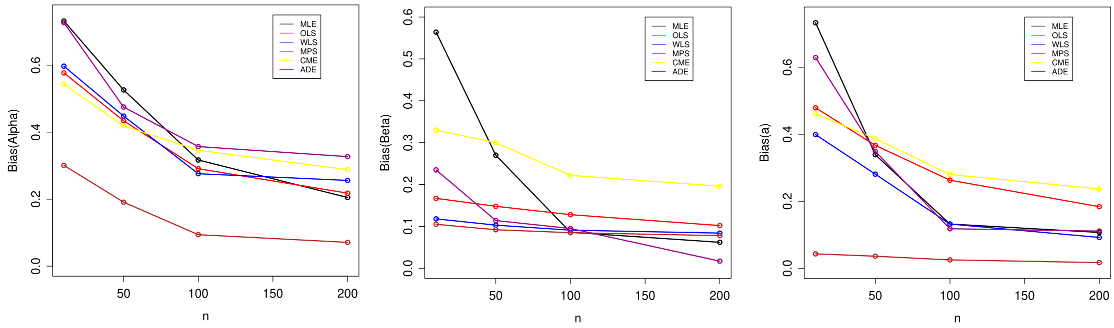

Figure 5. Biases for the estimates of $\alpha = 1.5$, $\beta = 2$ and $a = 2$ for each sample size for different methods of estimation.

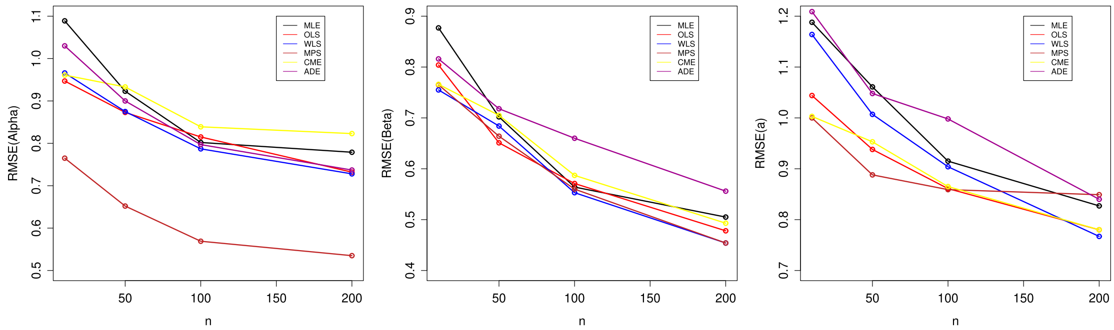

Figure 6. RMSEs for the estimates of $\alpha = 1.5$, $\beta = 2$ and $a = 2$ for each sample size for different methods of estimation.

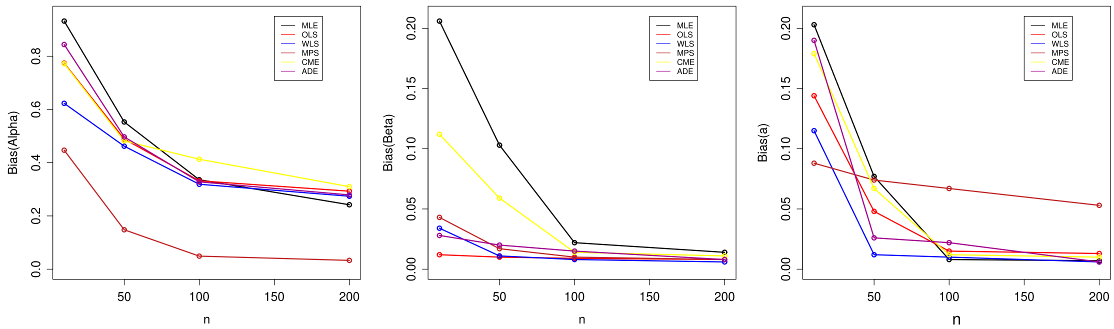

Figure 7. Biases for the estimates of $\alpha = 2$, $\beta = 1$ and $a = 0.8$ for each sample size for different methods of estimation.

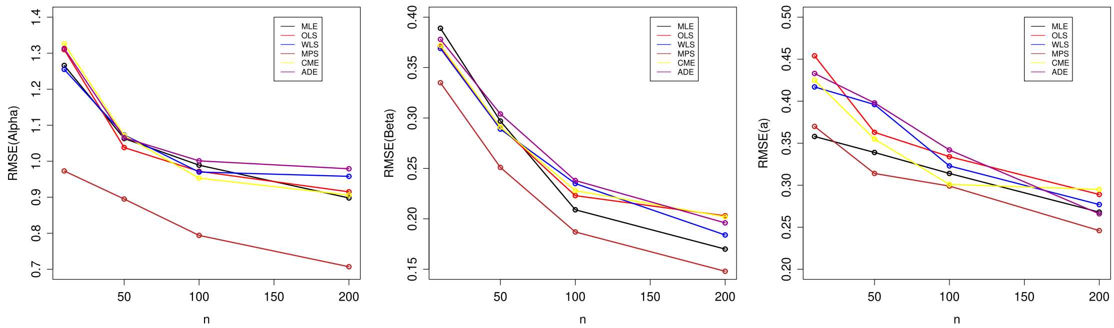

Figure 8. RMSEs for the estimates of $\alpha = 2$, $\beta = 1$ and $a = 0.8$ for each sample size for different methods of estimation.

6. Two applications

We prove the flexibility of the WMOL distribution using two real data sets. The fits of the WMOL distribution to the data sets are compared to those of some competitive distributions listed in Table 2. The density functions of the fitted models are given in Appendix.

The first data set consists of 101 failure times in hours of Kevlar 49/epoxy strands with pressure at 90%. These data have been reported in Barlow et al. [41] and have been analysed by Domma et al. [42]. The WMOL distribution gives a better fit to these data than the generalized half-normal [43], gamma, lognormal, Weibull and Birnbaum-Saunders distributions (see Table 5 in [43]).

The second data set consists of 128 remission times (in months) of bladder cancer patients. These data have been reported in Lee and Wang [44] and have been analysed by Cordeiro et al. [23,45]. The new distribution provides a more adequate fit to this data set than

Table 2. Some competitive models to the WMOL distribution.

Distribution	Author(s)
Lindley (L)	Lindley [1]
Quasi xgamma-geometric (QXGGc)	Sen et al. [30]
Quasi xgamma-Poisson (QXGPo)	Sen et al. [31]
Alpha logarithmic transformed Weibull (ALTW)	Nassar et al. [32]
Complementary Weibull geometric (CWGc)	Tojeiro et al. [33]
Generalized odd Burr III L (GBIIIL)	Haq et al. [34]
Extended odd Weibull L (EOWL)	Alizadeh et al. [15]
Weibull L (WL)	Asgharzadeh et al. [14]
Generalized L (GL)	Nadarajah et al. [6]
Exponentiated Weibull (EW)	Mudholkar and Srivastava [35]
Transmuted two-parameter L (TTL)	Kemaloglu and Yilmaz [12]
L Weibull (LW)	Cordeiro et al. [36,37]
Weibull–Poisson (WPo)	Lu and Shi [38]
L geometric (LGc)	Zakerzadeh and Mahmoudi [39]
Quasi L (QL)	Shanker and Mishra [8]
New weighted L (NWL)	Asgharzadeh et al. [10]
Alpha power inverse Weibull (APIW)	Basheer [40]

the exponentiated Weibull Lindley [23], Weibull Lindley (special case of the WMOL model), exponentiated exponential Lindley [46], extended Lindley [7] and power Lindley [9] distributions (see Table 6 in [23]).

The fitted distributions are compared based on the following statistics: the maximized log-likelihood ($-\hat{\ell}$), Cramér-Von Mises (CVM), Anderson–Darling (AD), Kolmogorov-Smirnov (KS) and its p-value (PV). The results are carried out in the R environment. Tables 3 and 4 give the MLEs and their standard errors (SEs) (in parentheses) and the values of the four statistics for the fitted WMOL model and other fitted distributions to the data sets I and II, respectively. Some plots of the estimated densities are displayed for both data sets in Figure 9. We also use some estimation methods discussed in Section 4 to estimate the unknown parameters from both data sets. The estimates of the WMOL parameters obtained from the six methods and the KS and PV values are listed in Tables 5 and 6 for both data sets. The figures in these tables indicate that

Table 5. The parameter estimates under different methods, KS statistics and the corresponding p-values for data set I.

Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	KS	PV
MLE	7.185	0.620	2.973	0.069	0.710
OLS	15.482	0.510	3.671	0.062	0.830
WLS	9.024	0.583	3.224	0.063	0.819
MPS	7.187	0.593	2.970	0.064	0.801
CME	14.382	0.525	3.613	0.059	0.873
ADE	9.744	0.571	3.298	0.061	0.845

Table 3. MLEs, SEs (in parentheses) and goodness-of-fit measures for data set I.

Model	Estimates				$-\hat{\ell}$	CVM	AD	KS	PV
WMOL(α,β,a)	7.185(6.493)	0.620(0.097)	2.973(0.788)		100.589	0.075	0.525	0.069	0.710
QXGGc(α,β,a)	0.081(0.202)	0.715(1.034)	0.954(0.166)		101.358	0.175	0.984	0.078	0.567
QXGPo(α,β,a)	0.307(0.111)	1.005(0.275)	4.428(1.799)		101.143	0.111	0.664	0.086	0.445
ALTW(α,β,a)	14.39(25.27)	0.743(0.110)	2.028(0.821)		101.793	0.107	0.686	0.073	0.657
CWGc(α,β,a)	0.255(0.251)	0.693(0.169)	2.709(2.275)		101.842	0.119	0.737	0.070	0.699
GBIIIL(α,β,k,a)	39.33(36.30)	0.469(0.241)	0.045(0.030)	1.314(0.156)	101.080	0.158	0.893	0.084	0.473
EOWL(α,β,a)	0.848(0.103)	1.053(0.457)	1.516(0.282)		102.771	0.143	0.863	0.078	0.563
WL(α,β,a)	0.748(0.254)	0.366(0.609)	0.888(0.552)		102.627	0.147	0.879	0.078	0.561
GL(α,a)	0.774(0.100)	1.204(0.134)			102.633	0.132	0.815	0.072	0.662
EW(α,β,a)	0.811(0.313)	1.060(0.239)	0.793(0.287)		102.787	0.165	0.959	0.084	0.468
TTL(α,β,a)	0.334(0.645)	0.846(0.209)	0.760(0.307)		102.961	0.185	1.040	0.088	0.408
LW(α,β,a)	1.607(2.641)	0.837(0.109)	1.045(1.168)		102.597	0.166	0.960	0.081	0.512
WPo(α,β,a)	0.931(0.073)	0.009(0.006)	106.9(71.23)		102.982	0.201	1.123	0.091	0.373
LGc(α,a)	0.437(0.259)	1.088(0.246)			103.739	0.200	1.128	0.084	0.460
QL(α,a)	174.4(4410)	0.981(0.170)			103.479	0.180	1.028	0.088	0.403
NWL(α,a)	254.6(382.3)	1.387(0.107)			104.487	0.143	0.848	0.108	0.189
L(a)	1.384(0.106)				104.655	0.137	0.834	0.106	0.204
APIW(α,β,a)	43.23(42.82)	0.779(0.053)	0.116(0.032)		124.366	0.955	5.143	0.164	0.009

Table 4. MLEs, SEs (in parentheses) and goodness-of-fit measures for data set II.

Model	Estimates				$-\ell$	CVM	AD	KS	PV
WMOL(α,β,a)	0.059(0.073)	1.067(0.078)	0.054(0.035)		409.268	0.013	0.088	0.031	0.999
LGc(α,a)	0.889(0.099)	0.074(0.035)			409.593	0.015	0.103	0.040	0.984
GBIIIL(α,β,k,a)	15.65(11.30)	0.410(0.113)	0.152(0.097)	0.228(0.034)	409.850	0.020	0.122	0.038	0.993
EW(α,β,a)	0.454(0.239)	0.654(0.134)	2.796(1.262)		410.680	0.044	0.288	0.045	0.958
WPo(α,β,a)	1.269(0.086)	0.016(0.006)	4.264(1.691)		410.189	0.043	0.255	0.046	0.950
CWGc(α,β,a)	0.004(0.001)	0.319(0.026)	31.38(18.38)		410.976	0.059	0.343	0.043	0.969
QXGGc(α,β,a)	0.028(0.019)	0.083(0.033)	0.971(0.029)		410.493	0.056	0.314	0.046	0.949
LW(α,β,a)	46.92(59.82)	0.737(0.047)	0.024(0.020)		411.520	0.069	0.424	0.055	0.827
EOWL(α,β,a)	1.215(0.190)	2.879(0.987)	0.299(0.048)		412.041	0.067	0.431	0.063	0.689
TTL(α,β,a)	0.157(0.167)	0.712(0.206)	0.117(0.029)		412.941	0.117	0.687	0.063	0.677
ALTW(α,β,a)	0.181(0.225)	1.280(0.202)	0.031(0.028)		413.270	0.098	0.611	0.069	0.582
WL(α,β,a)	1.047(0.067)	0.104(0.009)	0.001(0.017)		414.088	0.131	0.786	0.070	0.556
QL(α,a)	117.9(1483)	0.107(0.014)			414.343	0.119	0.716	0.084	0.318
QXGPo(α,β,a)	0.290(0.111)	0.123(0.036)	4.028(1.403)		415.015	0.188	1.068	0.080	0.390
GL(α,a)	0.733(0.091)	0.164(0.016)			416.285	0.192	1.147	0.092	0.220
NWL(α,a)	235.1(558.7)	0.196(0.012)			419.464	0.169	1.012	0.116	0.061
L(a)	0.196(0.012)				419.529	0.171	1.025	0.116	0.062
APIW(α,β,a)	650.8(861.1)	1.002(0.057)	0.636(0.145)		427.132	0.402	2.531	0.096	0.191

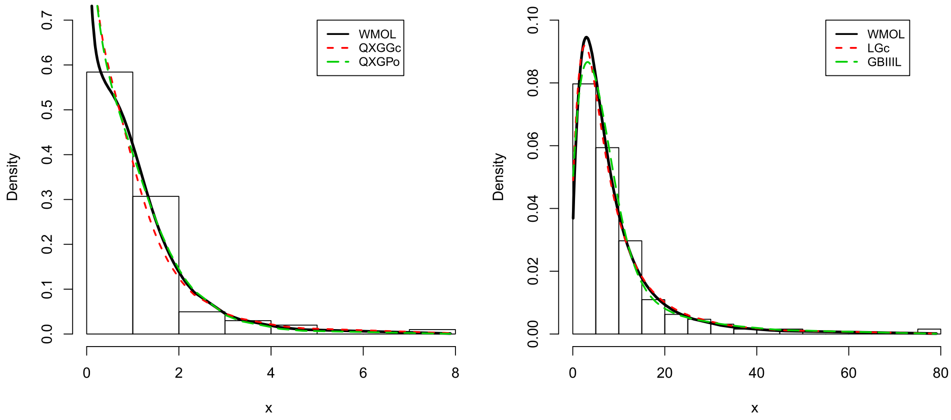

Figure 9. The fitted WMOL PDF for data set I (left panel) and data set II (right panel).

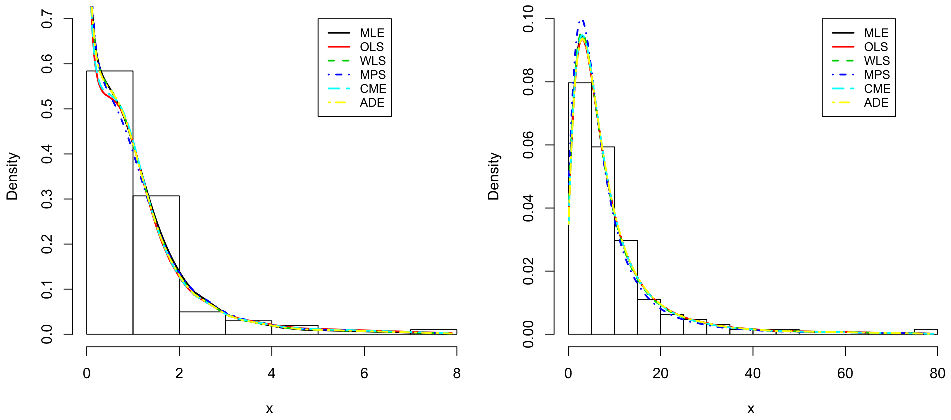

Figure 10. The fitted PDF of the WMOL distribution for various methods for data set I (left panel) and data set II (right panel).

Table 6. The parameter estimates under different methods, KS statistics and the corresponding p-values for data set II.

Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	KS	PV
MLE	0.059	1.067	0.054	0.0319	0.99947
OLS	0.040	1.061	0.043	0.0304	0.99979
WLS	0.038	1.058	0.043	0.0371	0.99449
MPS	0.062	1.070	0.055	0.0275	0.99998
CME	0.051	1.077	0.050	0.0320	0.99944
ADE	0.050	1.073	0.050	0.0375	0.99369

the CME method can be used to estimate the WMOL parameters for data set I and the MPS method for data set II. However, all estimation methods perform well for both data sets. The histograms and the estimated densities from the estimation methods are displayed in Figure 10 for both data sets. These plots support the figures in Tables 5 and 6.

7. Conclusions

We introduce the three-parameter *Weibull Marshall–Olkin Lindley* (WMOL) distribution that includes, as special models, some known distributions. The WMOL failure rate function can have the four classical forms and then it can be used quite effectively in analysing lifetime data. The new distribution serves as an alternative model to some generalized forms of the Lindley and Weibull distributions. The model parameters are estimated by six different methods and a simulation study is conducted to compare the performance of the different estimators. We show by means of two applications to real data that the proposed distribution can yield better fits than some other distributions.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendix

The PDFs of the competitive models used in the application section:

$$\text{QXGGc}: f(x) = \frac{\beta(1-a)}{(1+\alpha)}\left(\alpha + \frac{\beta^2}{2}x^2\right)e^{-\beta x} \times \left[1 - \frac{ae^{-\beta x}\left(1+\alpha+\beta x+\frac{\beta^2}{2}x^2\right)}{(1+\alpha)}\right]^{-2},$$

$$\text{QXGPo}: f(x) = \frac{a\beta(1+\alpha)^{-1}}{[\exp(a)-1]}\left(\alpha+\frac{\beta^2}{2}x^2\right) \times \exp\left[\left(1+\alpha+\beta x+\frac{\beta^2}{2}x^2\right) \times \frac{a\exp(-\beta x)}{1+\alpha}\right]\exp(-\beta x),$$

$$\text{ALTW}: f(x) = \frac{\alpha-1}{\log(\alpha)}a\beta x^{\beta-1}\mathrm{e}^{-ax^{\beta}} \times [\alpha-(\alpha-1)(1-\mathrm{e}^{-ax^{\beta}})]^{-1},$$

$$\text{CWGc}: f(x) = \alpha\beta a^{\beta}x^{\beta-1}\mathrm{e}^{-(ax)^{\beta}}[\alpha+(1-\alpha)\mathrm{e}^{-(ax)^{\beta}}]^{-2},$$

$$\text{EOWL}: f(x) = \frac{\alpha a^2(1+x)\mathrm{e}^{-ax}\left(1-\frac{1+a+ax}{1+a}\mathrm{e}^{-ax}\right)^{\alpha-1}}{(1+a)\left[\frac{1+a+ax}{1+a}\mathrm{e}^{-ax}\right]^{\alpha+1}} \times \left[1+\beta\left(\frac{1-\frac{1+a+ax}{1+a}\mathrm{e}^{-ax}}{\frac{1+a+ax}{1+a}\mathrm{e}^{-ax}}\right)^{\alpha}\right]^{\frac{-1}{\beta}-1},$$

$$\text{WL}: f(x) = \frac{\mathrm{e}^{-\lambda x-(\beta x)^{\alpha}}}{1+\lambda}\left[\alpha\lambda(\beta x)^{\alpha}+\alpha\beta(1+\lambda)(\beta x)^{\alpha-1} + \lambda^2(1+x)\right],$$

$$\text{GL}: f(x) = \frac{\alpha a^2}{1+a}(1+x)\mathrm{e}^{-ax} \times \left[1-\frac{1+a+ax}{1+a}\mathrm{e}^{-ax}\right]^{\alpha-1},$$

$$\text{EW}: f(x) = \alpha\beta a x^{\beta-1}\mathrm{e}^{-\alpha x^{\beta}}\left(1-\mathrm{e}^{-\alpha x^{\beta}}\right)^{a-1},$$

$$\text{TTL}: f(x) = \frac{a^2}{\alpha+a}(1+\alpha x)\mathrm{e}^{-ax} \times \left(1-\beta+2\beta\frac{\alpha+a+\alpha ax}{\alpha+a}\mathrm{e}^{-ax}\right),$$

$$\text{LW}: f(x) = \frac{\beta a^2}{a+1}\left[\alpha^{\beta}x^{\beta-1}+\alpha^{2\beta}x^{2\beta-1}\right]\mathrm{e}^{-a(\alpha x)^{\beta}},$$

$$\text{WPo}: f(x) = \frac{\alpha\beta a}{1-\mathrm{e}^{-a}}x^{\alpha-1}\mathrm{e}^{-a-\beta x^{\alpha}+a\mathrm{e}^{-\beta x^{\alpha}}},$$

$$\text{LGc}: f(x) = \frac{a^2}{a+1}(1-\alpha)(1+x)\mathrm{e}^{-ax} \times \left[1-\alpha(1+\frac{ax}{a+1})\mathrm{e}^{-ax}\right]^{-2},$$

$$\text{QL}: f(x) = \frac{a}{\alpha+1}(\alpha+ax)\mathrm{e}^{-ax},$$

$$\text{NWL}: f(x) = \frac{a^2(1+\alpha)^2(1+x)\left(1-\mathrm{e}^{-\alpha ax}\right)\mathrm{e}^{-ax}}{a\alpha(1+\alpha)+\alpha(2+\alpha)}$$

and

$$\text{APIW}: f(x) = \frac{\log(\alpha)}{\alpha-1}a\beta x^{-\beta-1}\mathrm{e}^{-ax^{-\beta}}\alpha^{\mathrm{e}^{-ax^{-\beta}}}.$$

The parameters of the above densities are all positive real numbers except $a \in (0,1)$ for the QXGGc model, $|\beta| \leq 1$ for the TTL model, $\alpha > -1$ for the QL model, and $\alpha \in (0,1)$ for the CWGc and LGc models.