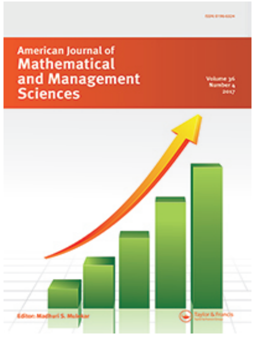

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Relations for Moments of Generalized Order Statistics from Extended Exponential Distribution

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SYNOPTIC ABSTRACT
The extended exponential distribution due to Nadarajah and Haghighi (2011) is an alternative, and always provides better fits than the gamma, Weibull, and the generalized exponential distributions whenever the data contains zero values. In this article, we consider the generalized order statistics (GOS) from this distribution. We obtain exact explicit expressions as well as recurrence relations for the single, product, and conditional moments of generalized order statistics from the extended exponential (EE) distribution. Then, we use these results to compute the means, variances, and covariances of order statistics and record values for samples of different sizes for various values of the shape and scale parameters. Further, we compute the mean and variances for progressively Type II censored order statistics.

KEY WORDS AND PHRASES
Extended exponential distribution; generalized order statistics; recurrence relations; single and product moments

1. Introduction

The concept of generalized order statistics (GOS) was introduced by Kamps (1995) as a general framework for models of ordered random variables. Moreover, many other models of ordered random variables, such as, order statistics, k-th upper record values, upper record values, progressively Type II censoring order statistics, Pfeifer records, and sequential order statistics are seen to be particular cases of GOS. These models can be effectively applied, e.g., in reliability theory.

For more details and some applications of GOS one may refer to Kamps and Gather (1997), Keseling (1999), Cramer and Kamps (2000), Ahsanullah (2000), Habibullah and Ahsanullah (2000), Pawlas and Szynal (2001), Raqab (2001), Ahmad and Fawzy (2003), AL-Hussaini and Ahmad (2003a, 2003b), AL-Hussaini (2004), Jaheen (2002, 2005), and Ahmad (2007, 2008), Kumar (2013, 2014, 2015a, 2015b) among others.

A new generalization of the exponential distribution was recently proposed by Nadarajah and Haghighi (2011). The new distribution has the probability density function (pdf),

$$f(x;\alpha,\lambda)=\alpha\lambda(1+\lambda x)^{\alpha-1}e^{1-(1+\lambda x)^{\alpha}},\quad x>0,\quad \alpha,\lambda>0 \tag{1}$$

and the corresponding cumulative distribution function (cdf) as,

$$F(x;\alpha,\lambda)=1-e^{1-(1+\lambda x)^{\alpha}},\quad x>0,\quad \alpha,\lambda>0. \tag{2}$$

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The hazard function is given by,

$$h(x) = \alpha\lambda(1+\lambda x)^{\alpha-1}, \quad x > 0, \quad \alpha, \lambda > 0. \tag{3}$$

Here, α and λ are the shape and scale parameters, respectively. We denote the EE distribution with the shape parameter α and the scale parameter λ as $EE(\alpha, \lambda)$. Nadarajah and Haghighi (2011) studied the properties of this distribution extensively and also pointed out that the density function (1) has a decreasing probability function like an exponential distribution, but its mode is at zero. They also showed that larger values of α in (1) will lead to faster decay of the upper tail. The distribution also allows for increasing, decreasing, and constant hazard rates like a Weibull distribution or an generalized exponential distribution. Further, this distribution is a particular member of the three-parameter Generalized Power Weibull distribution, introduced by Nikulin and Haghighi (2006). Moreover, this distribution is a special case of Gurvich, DiBenedetto, and Ranade (1997) class as $F(x) = 1 - exp(-\beta G(x))$, where $G(x)$ is a monotonically increasing function of x with the only limitation $G(x) \geq 0$. Lemonte (2013) extended the EE distribution using a similar idea with Gupta and Kundu (1999), called the Exponentiated Nadarajah and Haghighi (ENH) distribution. He had provided a comprehensive account of the mathematical properties of the ENH distributions, and the estimation of the unknown parameters of the distribution is obtained by the method of maximum likelihood for complete samples as well as for censored samples. The standard exponential distribution is the particular case of this distribution for $\alpha = \lambda = 1$.

One can observe from (1) and (2) that,

$$f(x; \alpha, \lambda) = \alpha \sum_{l=0}^{\alpha-1} \binom{\alpha-1}{l} \lambda^{l+1} x^l \, \overline{F}(x; \alpha, \lambda) \tag{4}$$

provided that $\alpha \geq 1$ is an integer. The relation in (4) will be exploited to derive recurrence relations for the moments of generalized order statistics for the extended exponential distribution.

In this article, we define generalized order statistics. It will be shown that order statistics, record values, and progressively Type II censored order statistics are special cases of generalized order statistics. First, we derive the explicit expressions for single moments, product moments, and conditional moments of order statistics and record values. Further, we also derive the expressions for single moments and product moments for progressively Type II censored order statistics. We also provide tabulations of means, variances, and covariances of order statistics and record values for samples of different sizes of the shape and scale parameters. Further more, we provide tabulations of mean and variances of progressively Type II censored order statistics with different sampling schemes.

The rest of the paper proceeds as follows. In Section 2, we describe briefly the preliminaries of generalized order statistics. In Section 3, we derive explicit expressions and recurrence relations for single moments, product moments, and conditional moments of generalized order statistics. Tabulations of means, variances, and covariances of order statistics, record values, and progressively Type II censored order statistics are given in Section 4. Finally, the article is concluded in Section 5.

2. Generalized Order Statistics and Preliminaries

Suppose $X(1, n, m, k), \ldots, X(n, n, m, k)$, ($k \geq 1$, m is a real number), are n GOS from an absolutely continuous cumulative distribution function cdf $F(x)$ with probability density function $pdf\ f(x)$, if their joint pdf is of the following form.

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^m f(x_i)\right)[1-F(x_n)]^{k-1}f(x_n), \tag{5}$$

for $F^{-1}(0) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1)$, where $\gamma_j = k + (n-j)(m+1) > 0$ for all j, $1 \leq j \leq n$, k is a positive integer and $m \geq -1$. If $m = 0$ and $k = 1$, we obtain the joint pdf of the order statistics. If $k = 1$ and $m = -1$, we obtain the joint pdf of the first n record values of the identically and independently distributed (iid) random variables with $cdf\ F(x)$ and corresponding $pdf\ f(x)$. Other statistics contained as particular cases include sequential order statistics, progressively type II censored order statistics and Pfeifer's record values.

In view of (5), the marginal pdf of the rth GOS, is given by,

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x)). \tag{6}$$

The joint pdf of r-th and s-th GOS is,

$$\begin{aligned} f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = &\frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m f(x)g_m^{r-1}(F(x)) \\ &\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y), \end{aligned} \tag{7}$$

for $x < y$, where $\bar{F}(x) = [1 - F(x)]$,

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x^{m+1}), & m \neq -1 \\ -ln(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \quad x \in [0,1).$$

3. Relations for Single and Product Moments of Generalized Order Statistics

In this section, we derive explicit expressions and recurrence relations for single and product moments of generalized order statistics from the extended exponential distribution.

3.1. Relations for Single Moments

We shall first establish explicit expressions for single moments of jth GOS, $E[X(r,n,m,k)^{(j)}] = \mu_{r,n,m,k}^{(j)}$. Theorem 1 gives an explicit expression for $1 \leq r \leq n$ and $j = 0, 1, 2, \ldots$. Theorem 2 gives an explicit expression for $1 \leq r \leq n$ and j a negative integer.

Theorem 1. *For* $1 \leq r \leq n$, $k \geq 1$, $m \geq -1$ *and* $j = 0, 1, 2, \ldots,$

$$\begin{aligned} \mu_{r,n,m,k}^{(j)} = &\frac{\lambda^{-j}C_{r-1}}{(r-1)!(s-r-1)!\,(m+1)^{r-1}}\sum_{u=0}^{r-1}\sum_{l=0}^{j}(-1)^{u+j-l}\binom{r-1}{u} \\ &\times \binom{j}{l}\frac{e^{\gamma_{r-u}}}{(\gamma_{r-u})^{1+\frac{l}{\alpha}}}\Gamma\left(\frac{l}{\alpha}+1,\ \gamma_{r-u}\right), \quad m \neq -1 \end{aligned} \tag{8}$$

and for $m = -1$

$$\mu^{(j)}_{r,n,-1,k} = \frac{e^k\, k^r}{\lambda^j\,(r-1)!} \sum_{u=0}^{r-1} \sum_{l=0}^{j} (-1)^{j+r-1-u-l} \binom{r-1}{u} \binom{j}{l} \times k^{-1-\frac{\alpha u + l}{\alpha}} \Gamma\left(\frac{\alpha u + l}{\alpha} + 1,\ k\right), \tag{9}$$

where $\Gamma(a, x)$ denotes the incomplete gamma function defined by $\Gamma(a, x) = \int_x^\infty t^{a-1}\, e^t\, dt$.

Proof. Using (6), we have,

$$\begin{aligned}
\mu^{(j)}_{r,n,m,k} &= \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_r - 1} g_m^{r-1}(F(x))\, f(x) dx. \\
&= \frac{C_{r-1}}{(r-1)!\,(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_{r-u}-1} f(x) dx \\
&= \frac{\alpha\lambda\, C_{r-1}}{(r-1)!\,(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} e^{\gamma_{r-u}} \\
&\quad \times \int_0^\infty x^j\, (1+\lambda x)^{\alpha-1}\, e^{-\gamma_{r-u}(1+\lambda x)^\alpha} dx \\
&= \frac{\alpha\lambda^{1-j}\, C_{r-1}}{(r-1)!\,(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} e^{\gamma_{r-u}} \\
&\quad \times \int_0^\infty (1+\lambda x - 1)^j\, (1+\lambda x)^{\alpha-1}\, e^{-\gamma_{r-u}(1+\lambda x)^\alpha} dx \\
&= \frac{\alpha\lambda^{1-j}\, C_{r-1}}{(r-1)!\,(m+1)^{r-1}} \sum_{u=0}^{r-1} \sum_{l=0}^{j} (-1)^{u+j-l} \binom{r-1}{u} \binom{j}{l} e^{\gamma_{r-u}} \\
&\quad \times \int_0^\infty (1+\lambda x)^{\alpha-1+l}\, e^{-\gamma_{r-u}(1+\lambda x)^\alpha} dx \\
&= \frac{\lambda^{-j}\, C_{r-1}}{(r-1)!\,(m+1)^{r-1}} \sum_{u=0}^{r-1} \sum_{l=0}^{j} (-1)^{u+j-l} \binom{r-1}{u} \binom{j}{l} \\
&\quad \times (\gamma_{r-u})^{-1-\frac{l}{\alpha}}\, e^{\gamma_{r-u}} \int_{\gamma_{r-u}}^\infty y^{\frac{l}{\alpha}}\, e^{-y} dy,
\end{aligned}$$

where $y = (1+\lambda x)^\alpha\, \gamma_{r-u}$. The result follows from the definition of the incomplete gamma function.

For $m = -1$,

$$\mu^{(j)}_{r,n,-1,k} = \frac{\lambda^{-j}\, e^k\, k^r}{(r-1)!} \sum_{u=0}^{r-1} \sum_{l=0}^{j} (-1)^{j+r-1-u-l} \binom{r-1}{u} \binom{j}{l} \times k^{-1-\frac{\alpha u + l}{\alpha}} \int_k^\infty y^{\frac{\alpha u + l}{\alpha}}\, e^{-y} dy,$$

where $y = k\,(1+\lambda x)^\alpha$. The result follows from the definition of the incomplete gamma function.

3.1.1. Special Cases

1) If we put $m = 0, k = 1$ in (8), we get an explicit expression for ordinary order statistics $(\gamma_r = n - r + 1)$. We have,

$$\mu_{r:n}^{j} = C_{r:n}\lambda^{-j}\sum_{u=0}^{r-1}\sum_{l=0}^{j}(-1)^{u+j-l}\binom{r-1}{u}\binom{j}{l} \times \frac{e^{n-r+u+1}}{(n-r+u+1)^{1+\frac{l}{\alpha}}}\Gamma\left(\frac{l}{\alpha}+1,\ n-r+u+1\right),$$

as obtained by Kumar et al (2017). In particular, the mean order statistics and the variance order statistics are,

$$\mu_{r:n}^{(1)} = C_{r:n}\sum_{u=0}^{r-1}\binom{r-1}{u}\frac{e^{n-r+u+1}}{\lambda}\left\{\frac{(-1)^{u+1}}{n-r+u+1}\Gamma(1, n-r+u+1) + \frac{(-1)^{u}}{(n-r+u+1)^{1+\frac{l}{\alpha}}}\Gamma\left(\frac{1}{\alpha}+1,\ n-r+u+1\right)\right\}, \quad (10)$$

and

$$\sigma_{r:n}^{2} = \mu_{r:n}^{(2)} - \left[\mu_{r:n}^{(1)}\right]^2 = C_{r:n}\sum_{u=0}^{r-1}\binom{r-1}{u}\frac{e^{n-r+u+1}}{\lambda^2}\left\{\frac{(-1)^{u+2}}{n-r+u+1}\Gamma(1, n-r+u+1) + \frac{2(-1)^{u+1}}{(n-r+u+1)^{1+\frac{1}{\alpha}}}\Gamma\left(1+\frac{1}{\alpha},\ n-r+u+1\right) + \frac{(-1)^{u}}{(n-r+u+1)^{1+\frac{2}{\alpha}}}\Gamma\left(1+\frac{2}{\alpha},\ n-r+u+1\right)\right\} - \left[\mu_{r:n}^{(1)}\right]^2, \quad (11)$$

respectively. If $\alpha = \lambda = 1$, then (10) and (11) reduce for the mean and the variance of order statistics of the standard exponential distribution.

2) If we put $k = 1$ in (9) ordinary record values we have,

$$\mu_{r}^{(j)} = \frac{e^{(1)}}{\lambda^{j}\,(r-1)!}\sum_{u=0}^{r-1}\sum_{l=0}^{j}(-1)^{j+r-1-u-l}\binom{r-1}{u}\binom{j}{l}\Gamma\left(\frac{\alpha u+l}{\alpha}+1,\ 1\right). \quad (12)$$

In particular, the mean record statistics and the variance record statistics are,

$$\mu_{U(r)}^{(1)} = \frac{e^{(1)}}{\lambda\,(r-1)!}\sum_{u=0}^{r-1}\binom{r-1}{u}\left\{(-1)^{r-u}\Gamma\left(\frac{\alpha u}{\alpha}+1,\ 1\right) + (-1)^{r-u+1}\Gamma\left(\frac{\alpha u+1}{\alpha}+1,\ 1\right)\right\}. \quad (13)$$

and

$$\sigma_{U(r)}^{2} = \mu_{U(r)}^{(2)} - \left[\mu_{U(r)}^{(1)}\right]^2 = \frac{e^{(1)}}{\lambda\,(r-1)!}\sum_{u=0}^{r-1}\binom{r-1}{u}\left\{(-1)^{r-u+1}\Gamma\left(\frac{\alpha u}{\alpha}+1,\ 1\right) + 2(-1)^{r-u} \times\Gamma\left(\frac{\alpha u+1}{\alpha}+1,\ 1\right) + (-1)^{r-u-1}\Gamma\left(\frac{\alpha u+2}{\alpha}+1,\ 1\right)\right\} - \left[\mu_{U(r)}^{(1)}\right]^2, \quad (14)$$

respectively. If $\alpha = \lambda = 1$, then (13) and (14) reduce for the mean and the variance of record values of the standard exponential distribution.

Theorem 2. *For $1 \le r \le n$, $k \ge 1$, $m \ge -1$, and j a negative integer,*

$$\mu^{(j)}_{r,n,m,k} = \frac{\lambda^{-j} C_{r-1}}{(r-1)!(s-r-1)!\,(m+1)^{r-1}} \sum_{u=0}^{r-1} \sum_{l=0}^{\infty} (-1)^{u+j-l} \binom{r-1}{u} \times \binom{j}{l} \frac{e^{\gamma_{r-u}}}{(\gamma_{r-u})^{1+\frac{l}{\alpha}}} \Gamma\left(\frac{l}{\alpha}+1,\ \gamma_{r-u}\right), \quad m \neq -1 \tag{15}$$

and for $m = -1$,

$$\mu^{(j)}_{r,n,-1,k} = \frac{e^k k^r}{\lambda^j\,(r-1)!} \sum_{u=0}^{r-1} \sum_{l=0}^{\infty} (-1)^{j+r-1-u-l} \binom{r-1}{u} \binom{j}{l} \times k^{-1-\frac{\alpha u + l}{\alpha}} \Gamma\left(\frac{\alpha u + l}{\alpha}+1,\ k\right), \tag{16}$$

where $\Gamma(a, x)$ denotes the incomplete gamma function.

Proof. Similar to the proof of Theorem 1.

Theorem 3 establishes a recurrence relation for $\mu^{(j)}_{r,n,m,k}$. This result holds for positive, as well as negative, j.

Theorem 3. *For the extended exponential distribution in (1) and $1 \le r \le n$, $k \ge 1$, $m \ge -1$,*

$$\mu^{(j)}_{r,n,m,k} = \alpha\lambda\gamma_r \sum_{l=0}^{\alpha-1} \binom{\alpha-1}{l} \frac{\lambda^l}{j+l+1} \left[\mu^{(j+l+1)}_{r,n,m,k} - \mu^{(j+l+1)}_{r-1,n,m,k}\right]. \tag{17}$$

Throughout, we follow the conventions that $\mu^{(j)}_{0,n,m,k} = 0$ for $n \ge 1$ and $\mu^{(0)}_{r,n,m,k} = 1$ for $1 \le r \le n$.

Proof. For $1 \le r \le n$, we have from (4) and (6),

$$\mu^{(j)}_{r,n,m,k} = \frac{\alpha\, C_{r-1}}{(r-1)!} \sum_{l=0}^{\alpha-1} \binom{\alpha-1}{l} \lambda^{l+1} \int_0^\infty x^{j+l} [\bar{F}(x)]^{\gamma_r} g_m^{r-1}(F(x))dx.$$

By integrating by parts, we obtain,

$$\mu^{(j)}_{r,n,m,k} = \frac{\alpha\, C_{r-1}}{(r-1)!} \sum_{l=0}^{\alpha-1} \binom{\alpha-1}{l} \lambda^{l+1} \left\{ \frac{\gamma_r}{j+l+1} \int_0^\infty x^{j+l+1} [\bar{F}(x)]^{\gamma_r - 1} g_m^{r-1}(F(x)) f(x)dx - \frac{(r-1)}{j+l+1} \int_0^\infty x^{j+l+1} [\bar{F}(x)]^{\gamma_r + m} g_m^{r-2}(F(x)) f(x)dx \right\}.$$

The result follows.

In particular, upon setting $r = 1$ in Theorem 3, we deduce the following result.

Corollary 1. *For the extended exponential distribution given in (1),*

$$\mu^{(j)}_{1,n,m,k} = \alpha\lambda\gamma_1 \sum_{l=0}^{\alpha-1} \binom{\alpha-1}{l} \frac{\lambda^l}{j+l+1} \mu^{(j+l+1)}_{1,n,m,k}. \tag{18}$$

3.1.2. Special Cases

1) For $m = 0, k = 1$ in (17) and (18), we get the recurrence relations for ordinary order statistics,

$$\mu_{r:n}^{(j)} = \alpha\lambda(n-r+1)\sum_{l=0}^{\alpha-1}\binom{\alpha-1}{l}\frac{\lambda^l}{j+l+1}\left[\mu_{r:n}^{(j+l+1)} - \mu_{r-1:n}^{(j+l+1)}\right], \tag{19}$$

and

$$\mu_{1:n}^{(j)} = n\alpha\lambda\sum_{l=0}^{\alpha-1}\binom{\alpha-1}{l}\frac{\lambda^l}{j+l+1}\mu_{1:n}^{(j+l+1)}. \tag{20}$$

2) For $m = -1, k \geq 1$ in (17) and (18), we get recurrence relations for record values,

$$\mu_{U(r):k}^{(j)} = k\alpha\lambda\sum_{l=0}^{\alpha-1}\binom{\alpha-1}{l}\frac{\lambda^l}{j+l+1}\left[\mu_{U(r):k}^{(j+l+1)} - \mu_{U(r-1):k}^{(j+l+1)}\right], \tag{21}$$

and

$$\mu_{U(1):k}^{(j)} = k\alpha\lambda\sum_{l=0}^{\alpha-1}\binom{\alpha-1}{l}\frac{\lambda^l}{j+l+1}\mu_{U(1):k}^{(j+l+1)}. \tag{22}$$

3) If $m_i = R_i$ for $i = 1, 2, \ldots, m-1$ and $k = R_m + 1$, then (17) is reduced to progressively Type-II censored order statistics for $2 \leq m \leq n$ and $j \geq 0$,

$$\begin{aligned}\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(u+j+1)}} &= \frac{1}{(1+R_1)\alpha\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\lambda^{u+1}(u+j+1)^{-1}}\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(j)}} \\ &\quad - \frac{(n-R_1-1)}{(1+R_1)}\mu_{1:m-1:n}^{(R_1+1+R_2,\ldots,R_m)^{(u+j+1)}}. \end{aligned} \tag{23}$$

For $m = 1, n = 1, 2, \ldots$ and $j \geq 0$,

$$\mu_{1:1:n}^{(n-1)^{(u+j+1)}} = \frac{1}{n\alpha\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\lambda^{u+1}(u+j+1)^{-1}}\mu_{1:1:n}^{(n-1)^{(j)}}. \tag{24}$$

For $2 \leq i \leq m-1$, $m \leq n$ and $j \geq 0$,

$$\begin{aligned}\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)^{(u+j+1)}} &= \frac{1}{1+R_i}\left[\frac{1}{\alpha\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\lambda^{u+1}(u+j+1)^{-1}}\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)^{(j)}}\right. \\ &\quad - (n-R_1-R_2-\cdots-R_i-i) \\ &\quad \times \mu_{i:m-1:n}^{(R_1,R_2,\ldots,R_{i-1},R_i+R_{i+1}+1,R_{i+2},\ldots,R_m)^{(u+j+1)}} \\ &\quad + (n-R_1-R_2-\cdots-R_{i-1}-i+1) \\ &\quad \left.\times \mu_{i-1:m-1:n}^{(R_1,R_2,\ldots,R_{i-2},R_{i-1}+R_i+1,R_{i+1},\ldots,R_m)^{(u+j+1)}}\right]. \end{aligned} \tag{25}$$

For $2 \leq m \leq n$, and $j \geq 0$,

$$\begin{aligned}\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(u+j+1)}} &= \frac{1}{(1+R_m)\alpha\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\lambda^{u+1}(u+j+1)^{-1}}\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(j)}} \\ &\quad + \mu_{m-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-2},R_{m-1}+R_m+1,R_{i+1},\ldots,R_m)^{(u+j+1)}}. \end{aligned} \tag{26}$$

3.2. Relations for Product Moments

We shall first establish explicit expressions for the product moment of ith and jth generalized order statistics, $E[X^{(i,j)}_{r,s,n,m,k}] = \mu^{(i,j)}_{r,s,n,m,k}$. Theorem 4 gives an explicit expression for $1 \le r < s \le n$ and $i, j = 0, 1, 2, \ldots$. Theorem 5 gives an explicit expression for $1 \le r < s \le n$, $i = 0, 1, 2, \ldots$ and j a negative integer. Theorem 6 gives an explicit expression for $1 \le r < s \le n$, $j = 0, 1, 2, \ldots$ and i a negative integer. Theorem 7 gives an explicit expression for $1 \le r < s \le n$ and both i and j negative integers.

Theorem 4. *For the distribution given in (1) and $1 \le r < s \le n$, $n \ge 1$, $k = 1, 2, \ldots$ and $i,\ j = 0, 1, 2, \ldots$*

$$\mu^{(i,j)}_{r,s;n,m,k} = \frac{\alpha\lambda^{-i-j} C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{i}\sum_{q=0}^{j}(-1)^{u+v+i+j-p-q}$$
$$\times \binom{r-1}{u}\binom{s-r-1}{v}\binom{i}{p}\binom{j}{q} \frac{e^{\gamma_{r-u}}\Gamma\left(2+\frac{p+q}{\alpha}\right)}{(p+\alpha)\ (\gamma_{r-u})^{2+\frac{p+q}{\alpha}}}$$
$$\times {}_2F_1\left(1,\ 2+\frac{p+q}{\alpha};\ 2+\frac{p}{\alpha};\ \frac{(u+s-r-v)(m+1)}{\gamma_{r-u}}\right),\ m \ne -1 \qquad (27)$$

and for $m = -1$

$$\mu^{(i,j)}_{r,s;n,-1,k} = \frac{\alpha\lambda^{-i-j}\, e^{k}}{(r-1)!(s-r-1)!} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{i}\sum_{q=0}^{j}(-1)^{i+j+r+v+u-p-q-1}$$
$$\times \binom{r-1}{u}\binom{s-r-1}{v}\binom{i}{p}\binom{j}{q} \frac{\Gamma\left(\frac{\alpha(u+s-r-1)+p+q}{\alpha}+1\right)}{[\alpha(u+v)+p]\,k^{\frac{\alpha(u+v)+p}{\alpha}-1}}$$
$$\times {}_2F_1\left(1,\ u+s-r+\frac{p+q}{\alpha};\ u+v+1+\frac{p}{\alpha};0\right), \qquad (28)$$

where ${}_2F_1(a, b;\ c;\ x)$ denotes the Gauss hypergeometric function defined by,

$${}_2F_1(a, b;\ c;\ x) = \sum_{k=0}^{\infty}\frac{(a)_k\,(b)_k}{(c)_k}\frac{x^k}{k!},$$

where $(e)_k = e(e+1)\ldots(e+k-1)$ denotes the ascending fractorial.

Proof. Using (7), we have for $m \ne -1$,

$$\mu^{(i,j)}_{r,s;n,m,k} = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^{\infty}\int_x^{\infty} x^i y^j[\bar F(x)]^m f(x) g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar F(y)]^{\gamma_s-1} f(y)dydx$$
$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$
$$\times \int_0^{\infty}\int_x^{\infty} x^i y^j[\bar F(x)]^{(s-r+u-v)(m+1)}[\bar F(y)]^{\gamma_{s-v}-1} f(x)f(y)dydx$$
$$= \frac{\alpha^2\lambda^2\, e^{\gamma_{r-u}}\, C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$

$$\times \int_0^\infty \int_x^\infty x^i y^j (1+\lambda x)^{\alpha-1}(1+\lambda y)^{\alpha-1} e^{-\gamma_{s-v}(1+\lambda y)^\alpha}$$
$$\times e^{-(u+s-r-v)(m+1)(1+\lambda x)^\alpha} dydx$$
$$= \frac{\alpha^2 \lambda^{2-i-j} e^{\gamma_{r-u}} C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1} (-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$
$$\times \int_0^\infty \int_x^\infty (1+\lambda x-1)^i (1+\lambda y-1)^j (1+\lambda x)^{\alpha-1}(1+\lambda y)^{\alpha-1}$$
$$\times e^{-\gamma_{s-v}(1+\lambda y)^\alpha} e^{-(u+s-r-v)(m+1)(1+\lambda x)^\alpha} dydx$$
$$= \frac{\alpha^2 \lambda^{2-i-j} e^{\gamma_{r-u}} C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1} (-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$
$$\times \sum_{p=0}^{i}\sum_{q=0}^{j}(-1)^{i+j-p-q}\binom{i}{p}\binom{j}{q}\int_0^\infty \int_x^\infty (1+\lambda x)^{p+\alpha-1}(1+\lambda y)^{q+\alpha-1}$$
$$\times e^{-\gamma_{s-v}(1+\lambda y)^\alpha} e^{-(u+s-r-v)(m+1)(1+\lambda x)^\alpha} dydx$$
$$= \frac{\alpha \lambda^{1-i-j} e^{\gamma_{r-u}} C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1} (-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$
$$\times \sum_{p=0}^{i}\sum_{q=0}^{j}(-1)^{i+j-p-q}\binom{i}{p}\binom{j}{q}\int_0^\infty (1+\lambda x)^{p+\alpha-1} e^{-(u+s-r-v)(m+1)(1+\lambda x)^\alpha}$$
$$\times (\gamma_{s-v})^{-1-\frac{q}{\alpha}} \int_{\gamma_{s-v}}^\infty z^{\frac{q}{\alpha}} e^{-z}\, dzdx$$
$$= \frac{\alpha \lambda^{1-i-j} e^{\gamma_{r-u}} C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1} (-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$
$$\times \sum_{p=0}^{i}\sum_{q=0}^{j}(-1)^{i+j-p-q}\binom{i}{p}\binom{j}{q}\int_0^\infty (1+\lambda x)^{p+\alpha-1} e^{-(u+s-r-v)(m+1)(1+\lambda x)^\alpha}$$
$$\times (\gamma_{s-v})^{-1-\frac{q}{\alpha}}\, \Gamma\left(\frac{q}{\alpha}+1,\ \gamma_{s-v}(1+\lambda x)^\alpha\right) dx$$
$$= \frac{\lambda^{-i-j} e^{\gamma_{r-u}} C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1} (-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$
$$\times \sum_{p=0}^{i}\sum_{q=0}^{j}(-1)^{i+j-p-q}\binom{i}{p}\binom{j}{q} (\gamma_{s-v})^{-1-\frac{q}{\alpha}} [(u+s-r-v)(m+1)]^{-1-\frac{p}{\alpha}}$$
$$\times \int_0^\infty w^{\frac{p}{\alpha}} e^{-w}\, \Gamma\left(\frac{q}{\alpha}+1,\ \gamma_{s-v}(1+\lambda x)^\alpha\right) dx, \tag{29}$$

where $z = \gamma_{s-v}(1+\lambda y)^\alpha$ and $w = (u+s-r-v)(m+1)(1+\lambda x)^\alpha$. The result follows by using Equation (6.455.1) in Gradshteyn and Ryzhik (2000) to calculate the integral in (29).

The proof is complete.

For $m = -1$,

$$\mu_{r,s;n,m,k}^{(i,j)} = \frac{k^s}{(r-1)!(s-r-1)!}\int_0^\infty \int_x^\infty x^i y^j [-ln\bar{F}(x)]^{r-1}\frac{f(x)}{\bar{F}(x)}$$
$$\times [-ln\bar{F}(y) + ln\bar{F}(x)]^{s-r-1}[\bar{F}(y)]^{k-1} f(y)dy\,dx$$

$$= \frac{\alpha^2 \lambda^{2-i-j} k^s e^k}{(r-1)!(s-r-1)!} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{r-1+v-u} \binom{r-1}{u} \binom{s-r-1}{v}$$
$$\times \sum_{p=0}^{i} \sum_{q=0}^{j} (-1)^{i+j-p-q} \binom{i}{p} \binom{j}{q} \int_0^\infty \int_x^\infty x^i y^j e^{-k(1+\lambda y)^\alpha}$$
$$\times (1+\lambda x)^{\alpha(u+v)+p+\alpha-1} (1+\lambda y)^{\alpha(s-r-1-v)+q+\alpha-1} dydx$$
$$= \frac{\alpha \lambda^{1-i-j} k^s e^k}{(r-1)!(s-r-1)!} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{r-1+v-u} \binom{r-1}{u} \binom{s-r-1}{v}$$
$$\times \sum_{p=0}^{i} \sum_{q=0}^{j} (-1)^{i+j-p-q} \binom{i}{p} \binom{j}{q} k^{-\frac{\alpha(s-r-1-v)+q}{\alpha}}$$
$$\times \int_0^\infty (1+\lambda x)^{\alpha(u+v)+p+\alpha-1} \Gamma\left(\frac{\alpha(s-r-1-v)+q}{\alpha}, k(1+\lambda x)^\alpha\right) dx$$
$$= \frac{\lambda^{-i-j} k^s e^k}{(r-1)!(s-r-1)!} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{r-1+v-u} \binom{r-1}{u} \binom{s-r-1}{v}$$
$$\times \sum_{p=0}^{i} \sum_{q=0}^{j} (-1)^{i+j-p-q} \binom{i}{p} \binom{j}{q} k^{-\frac{\alpha(s-r-1-v)+q}{\alpha}}$$
$$\times \int_0^\infty t^{\frac{\alpha(u+u)+p}{\alpha}} \Gamma\left(\frac{\alpha(s-r-1-v)+q}{\alpha}, kt\right) dt, \tag{30}$$

where $t = (1+\lambda x)^\alpha$. The result follows by using Equation (6.455.1) in Gradshteyn and Ryzhik (2000) to calculate the integral in (30).

The proof is complete.

3.2.1. Special Cases

1) For $m = 0$ and $k = 1$ in (27), we get the explicit expression for product moments of ordinary order statistics,

$$\mu_{r,s:n}^{(i,j)} = \alpha\, C_{r,s:n}\, \lambda^{-i-j} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} \sum_{p=0}^{i} \sum_{q=0}^{j} (-1)^{u+v+i+j-p-q} \binom{r-1}{u}$$
$$\times \binom{s-r-1}{v} \binom{i}{p} \binom{j}{q} \frac{e^{u-r+n+1}\, \Gamma\left(2+\frac{p+q}{\alpha}\right)}{(p+\alpha)\ (u+n-r+1)^{2+\frac{p+q}{\alpha}}}$$
$$\times {}_2F_1\left(1,\ 2+\frac{p+q}{\alpha};\ 2+\frac{p}{\alpha};\ \frac{(u+s-r-v)}{u+n-r+1}\right),$$

as obtained by Kumar et al (2017). In particular, the covariance of order statistics is,

$$\mu_{r,s:n}^{(1,1)} = \alpha\, C_{r,s:n}\, \lambda^{-2} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} \binom{r-1}{u} \binom{s-r-1}{v}$$
$$\times e^{u-r+n+1} \left\{ \frac{(-1)^{u+v+2}}{\alpha(u+n-r+1)^2}\, {}_2F_1\left(1,\ 2;\ 2;\ \frac{u+s-r-v}{u+n-r+1}\right)\right.$$
$$+ \frac{(-1)^{u+v+1}\, \Gamma\left(2+\frac{1}{\alpha}\right)}{(u+n-r+1)^{2+\frac{1}{\alpha}} (1+\alpha)}\, {}_2F_1\left(1,\ 2+\frac{1}{\alpha};\ 2+\frac{1}{\alpha};\ \frac{u+s-r-v}{u+n-r+1}\right)$$

$$+\frac{(-1)^{u+v+1}\,\Gamma\left(2+\frac{1}{\alpha}\right)}{\alpha(u+n-r+1)^{2+\frac{1}{\alpha}}}\,{}_2F_1\left(1,\ 2+\frac{1}{\alpha};\ 2;\ \frac{u+s-r-v}{u+n-r+1}\right)$$
$$\left.+\frac{(-1)^{u+v}\,\Gamma\left(2+\frac{2}{\alpha}\right)}{(u+n-r+1)^{2+\frac{1}{\alpha}}(1+\alpha)}\,{}_2F_1\left(1,\ 2+\frac{2}{\alpha};\ 2+\frac{1}{\alpha};\ \frac{u+s-r-v}{u+n-r+1}\right)\right\}. \tag{31}$$

2) For $k=1$ in (28), we get explicit expression for record values,

$$\mu^{(i,j)}_{r,s;n,-1,k}=\frac{\alpha\,\lambda^{-i-j}\,e}{(r-1)!(s-r-1)!}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{i}\sum_{q=0}^{j}(-1)^{i+j+r+v+u-p-q-1}$$
$$\times\binom{r-1}{u}\binom{s-r-1}{v}\binom{i}{p}\binom{j}{q}\frac{\Gamma\left(\frac{\alpha(u+s-r-1)+p+q}{\alpha}+1\right)}{[\alpha(u+v)+p]}$$
$$\times\,{}_2F_1\left(1,\ u+s-r+\frac{p+q}{\alpha};\ u+v+1+\frac{p}{\alpha};0\right), \tag{32}$$

Theorem 5. *For the distribution given in (1) and* $1\le r<s\le n,\ n\ge 1,\ k=1,2,\dots,\ i=0,1,2,\dots,$ *and* j *a negative integer,*

$$\mu^{(i,j)}_{r,s;n,m,k}=\frac{\alpha\lambda^{-i-j}\,C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{i}\sum_{q=0}^{\infty}(-1)^{u+v+i+j-p-q}$$
$$\times\binom{r-1}{u}\binom{s-r-1}{v}\binom{i}{p}\binom{j}{q}\frac{e^{\gamma_{r-u}}\Gamma\left(2+\frac{p+q}{\alpha}\right)}{(p+\alpha)\,(\gamma_{r-u})^{2+\frac{p+q}{\alpha}}}$$
$$\times\,{}_2F_1\left(1,\ 2+\frac{p+q}{\alpha};\ 2+\frac{p}{\alpha};\ \frac{(u+s-r-v)(m+1)}{\gamma_{r-u}}\right),\ m\neq -1 \tag{33}$$

and for $m=-1$,

$$\mu^{(i,j)}_{r,s;n,-1,k}=\frac{\alpha\lambda^{-i-j}\,e^{k}}{(r-1)!(s-r-1)!}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{i}\sum_{q=0}^{\infty}(-1)^{i+j+r+v+u-p-q-1}$$
$$\times\binom{r-1}{u}\binom{s-r-1}{v}\binom{i}{p}\binom{j}{q}\frac{\Gamma\left(\frac{\alpha(u+s-r-1)+p+q}{\alpha}+1\right)}{[\alpha(u+v)+p]\,k^{\frac{\alpha(u+v)+p}{\alpha}-1}}$$
$$\times\,{}_2F_1\left(1,\ u+s-r+\frac{p+q}{\alpha};\ u+v+1+\frac{p}{\alpha};0\right), \tag{34}$$

Proof. Similar to the proof of Theorem 4.

Theorem 6. *For the distribution given in (1) and* $1\le r<s\le n,\ n\ge 1,\ k=1,2,\dots,\ j=0,1,2,\dots,$ *and* i *a negative integer,*

$$\mu^{(i,j)}_{r,s;n,m,k}=\frac{\alpha\lambda^{-i-j}\,C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{\infty}\sum_{q=0}^{j}(-1)^{u+v+i+j-p-q}$$
$$\times\binom{r-1}{u}\binom{s-r-1}{v}\binom{i}{p}\binom{j}{q}\frac{e^{\gamma_{r-u}}\,\Gamma\left(2+\frac{p+q}{\alpha}\right)}{(p+\alpha)\ (\gamma_{r-u})^{2+\frac{p+q}{\alpha}}}$$
$$\times\,{}_2F_1\left(1,\ 2+\frac{p+q}{\alpha};\ 2+\frac{p}{\alpha};\ \frac{(u+s-r-v)(m+1)}{\gamma_{r-u}}\right),\ m\neq -1 \tag{35}$$

and for $m=-1$,

$$\mu^{(i,j)}_{r,s;n,-1,k}=\frac{\alpha\lambda^{-i-j}e^{k}}{(r-1)!(s-r-1)!}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{\infty}\sum_{q=0}^{j}(-1)^{i+j+r+v+u-p-q-1}$$
$$\times\binom{r-1}{u}\binom{s-r-1}{v}\binom{i}{p}\binom{j}{q}\frac{\Gamma\left(\frac{\alpha(u+s-r-1)+p+q}{\alpha}+1\right)}{[\alpha(u+v)+p]\,k^{\frac{\alpha(u+v)+p}{\alpha}-1}}$$
$$\times{}_2F_1\left(1,\;u+s-r+\frac{p+q}{\alpha};\;u+v+1+\frac{p}{\alpha};0\right),\tag{36}$$

Proof. Similar to the proof of Theorem 4.

Theorem 7. *For the distribution given in (1) and* $1\le r<s\le n$, $n\ge 1$, $k=1,2,\ldots,$ *and both* i *and* j *negative integers,*

$$\mu^{(i,j)}_{r,s;n,m,k}=\frac{\alpha\lambda^{-i-j}C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{u+v+i+j-p-q}$$
$$\times\binom{r-1}{u}\binom{s-r-1}{v}\binom{i}{p}\binom{j}{q}\frac{e^{\gamma_{r-u}}\Gamma\left(2+\frac{p+q}{\alpha}\right)}{(p+\alpha)\,(\gamma_{r-u})^{2+\frac{p+q}{\alpha}}}$$
$$\times{}_2F_1\left(1,\;2+\frac{p+q}{\alpha};\;2+\frac{p}{\alpha};\;\frac{(u+s-r-v)(m+1)}{\gamma_{r-u}}\right),\;m\neq-1\tag{37}$$

and for $m=-1$,

$$\mu^{(i,j)}_{r,s;n,-1,k}=\frac{\alpha\lambda^{-i-j}e^{k}}{(r-1)!(s-r-1)!}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{i+j+r+v+u-p-q-1}$$
$$\times\binom{r-1}{u}\binom{s-r-1}{v}\binom{i}{p}\binom{j}{q}\frac{\Gamma\left(\frac{\alpha(u+s-r-1)+p+q}{\alpha}+1\right)}{[\alpha(u+v)+p]\,k^{\frac{\alpha(u+v)+p}{\alpha}-1}}$$
$$\times{}_2F_1\left(1,\;u+s-r+\frac{p+q}{\alpha};\;u+v+1+\frac{p}{\alpha};0\right),\tag{38}$$

Proof. Similar to the proof of Theorem 4.

Theorem 8 establishes a recurrence relation for $\mu^{(i,j)}_{r,s;n,m,k}$. This result holds for positive as well as negative values of i and j.

Theorem 8. *For the distribution given in (1) and* $1\le r<s\le n$, $n\ge 1$, $k=1,2,\ldots,$

$$\mu^{(i,j)}_{r,s,n,m,k}=\alpha\lambda\sum_{l=0}^{\alpha-1}\binom{\alpha-1}{l}\lambda^{l}\left[\frac{\gamma_s}{j+l+1}\mu^{(i,j+l+1)}_{r,s,n,m,k}-\frac{s-r-1}{j+l+1}\mu^{(i,j+l+1)}_{r,s-1,n,m,k}\right].\tag{39}$$

Proof. Using (7), we have,

$$\mu^{(i,j)}_{r,s,n,m,k}=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^{\infty}x^i[\bar F(x)]^m g_m^{r-1}(F(x))G(x)f(x)dx\tag{40}$$

where,

$$G(x) = \int_x^\infty y^j [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} f(y) dy$$
$$= \alpha\lambda \sum_{l=0}^{\alpha-1} \binom{\alpha-1}{l} \lambda^l \int_x^\infty y^{j+l} [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s} dy.$$

Integrating by parts with respect to y, we obtain,

$$G(x) = \alpha\lambda \sum_{l=0}^{\alpha-1} \binom{\alpha-1}{l} \lambda^l$$
$$\times \left\{ \frac{\gamma_s}{j+l+1} \int_x^\infty y^{j+l+1} [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} f(y) dy \right.$$
$$\left. - \frac{s-r-1}{j+l+1} \int_x^\infty y^{j+l+1} [h_m(F(y)) - h_m(F(x))]^{s-r-2} [\bar{F}(y)]^{\gamma_s + m} f(y) dy \right\}. \quad (41)$$

The result follows by combining (32) and (33).

In particular, upon setting $s = r+1$ in Theorem 8, we deduce the following result.

Corollary 2. *For the distribution given in (1) and* $1 \le r \le n$, $n \ge 1$, $k = 1, 2, \ldots,$

$$\mu_{r,r+1,n,m,k}^{(i,j)} = \alpha\lambda \sum_{l=0}^{\alpha-1} \binom{\alpha-1}{l} \lambda^l \left[\frac{\gamma_{r+1}}{j+l+1} \mu_{r,r+1,n,m,k}^{(i,j+l+1)} - \frac{s-r-1}{j+l+1} \mu_{r,n,m,k}^{(i+j+l+1)} \right]. \quad (42)$$

3.2.2. Special Cases

1) For $m = 0$, $k = 1$ in (39) and (42), we get the recurrence relations for ordinary order statistics,

$$\mu_{r,s:n}^{(i,j)} = \alpha\lambda \sum_{l=0}^{\alpha-1} \binom{\alpha-1}{l} \lambda^l \left[\frac{n-s+1}{j+l+1} \mu_{r:n}^{(i,j+l+1)} - \frac{s-r-1}{j+l+1} \mu_{r,s-1:n}^{(i,j+l+1)} \right] \quad (43)$$

and

$$\mu_{r,r+1:n}^{(i,j)} = \alpha\lambda \sum_{l=0}^{\alpha-1} \binom{\alpha-1}{l} \lambda^l \left[\frac{n-r}{j+l+1} \mu_{r,r+1:n}^{(i,j+l+1)} - \frac{s-r-1}{j+l+1} \mu_{r:n}^{(i+j+l+1)} \right]. \quad (44)$$

2) For $m = -1$, $k \ge 1$, (39), and (42), we get the recurrence relation for record values,

$$\mu_{U(r,s):k}^{(i,j)} = \alpha\lambda \sum_{l=0}^{\alpha-1} \binom{\alpha-1}{l} \lambda^l \left[\frac{k}{j+l+1} \mu_{U(r,s):n}^{(i,j+l+1)} - \frac{s-r-1}{j+l+1} \mu_{U(r,s):k}^{(i,j+l+1)} \right]$$

and

$$\mu_{U(r,r+1):k}^{(i,j)} = \alpha\lambda \sum_{l=0}^{\alpha-1} \binom{\alpha-1}{l} \lambda^l \left[\frac{k}{j+l+1} \mu_{U(r,r+1):k}^{(i,j+l+1)} - \frac{s-r-1}{j+l+1} \mu_{U(r):k}^{(i+j+l+1)} \right].$$

3) If $m_i = R_i$ for $i = 1, 2, \ldots, m-1$, and $k = R_m + 1$, then (39) is reduced to to progressively Type-II censored order statistics for $1 \le i < j \le m-1$ and $m \le n$,

Table 1. Means of order statistics for different values of parameters.

		$\lambda = 0.5$			
n	r	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$
1	1	0.194059	0.067237	0.034172	0.018808
2	1	0.370476	0.122249	0.062131	0.037616
	2	0.209354	0.046364	0.017696	0.008574
3	1	0.289199	0.183373	0.093196	0.056423
	2	0.131368	0.069547	0.026545	0.012861
	3	0.056669	0.024953	0.007124	0.002757
4	1	0.705668	0.244497	0.124261	0.075231
	2	0.398769	0.092729	0.035393	0.017148
	3	0.215459	0.033271	0.009498	0.003675
	4	0.113284	0.011579	0.002471	0.000763
5	1	0.882085	0.305621	0.155326	0.094039
	2	0.498461	0.115911	0.044241	0.021436
	3	0.269324	0.041588	0.011873	0.004594
	4	0.141605	0.014474	0.003088	0.000954
	5	0.073179	0.004950	0.000790	0.000195
		$\lambda = 1.0$			
1	1	0.352834	0.122249	0.062131	0.037616
2	1	0.705668	0.244497	0.124261	0.075231
	2	0.398769	0.092729	0.035393	0.017148
3	1	1.058502	0.366746	0.186392	0.112847
	2	0.598153	0.139093	0.053089	0.025723
	3	0.323189	0.049906	0.014248	0.005513
4	1	1.411336	0.488994	0.248522	0.150462
	2	0.797538	0.185458	0.070786	0.034297
	3	0.430918	0.066541	0.018997	0.007351
	4	0.226567	0.023159	0.004941	0.001527
5	1	1.764170	0.611243	0.310653	0.188078
	2	0.996922	0.231822	0.088482	0.042871
	3	0.538648	0.083177	0.023746	0.009188
	4	0.283209	0.028948	0.006177	0.001908
	5	0.146358	0.009910	0.001579	0.000390

$$\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)^{(1,u+1)}} = \frac{1}{R_j+1}\Bigg[\frac{1}{\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\alpha\lambda^{u+1}(u+1)^{-1}}\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)} - (n-R_1-1-\cdots-R_j-j)\mu_{i,j:m-1:n}^{(R_1,R_2,\ldots,R_{j-1},R_j+R_{j+1}+1,\cdots R_m)^{(1,u+1)}} + (n-R_1-1-\cdots-R_{j-1}-j+1) \times \mu_{i,j-1:m-1:n}^{(R_1,R_2,\ldots,R_{j-1}+R_j+1,\ldots,R_m)^{(1,u+1)}}\Bigg]. \tag{45}$$

For $1 \le i \le m-1$ and $m \le n$,

$$\mu_{i,m:m:n}^{(R_1,R_2,\ldots,R_m)^{(1,u+1)}} = \frac{1}{R_m+1}\Bigg[\frac{1}{\sum_{u=0}^{\alpha-1}\binom{\alpha-1}{u}\alpha\lambda^{u+1}(u+1)^{-1}}\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)} + (n-R_1-1-\cdots-R_{m-1}-m+1) \times \mu_{i,m-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-1}+R_m+1,\ldots,R_m)^{(1,u+1)}}\Bigg]. \tag{46}$$

3.3. Relations for Conditional Moments

Let $X(r,n,m,k)$, $r = 1,2,\ldots,n$ be *gos*, then from a continuous population with cdf $F(x)$ and pdf $f(x)$, then the conditional pdf of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \le r < s \le$

Table 2. Variances and covariances of order statistics for $\lambda = 0.5$ and 1.0.

			$\lambda = 0.5$			$\lambda = 1.0$		
n	s	r	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$
1	1	1	3.315561	2.930030	1.967140	1.078890	1.482507	1.241785
2	1	1	3.63112	2.86006	1.934281	1.157781	1.965015	1.483570
	2	1	2.860891	1.658041	1.752746	1.465223	1.914510	1.688186
	2	2	1.183150	1.418909	0.943085	0.795787	0.354727	0.235771
3	1	1	1.946680	1.79009	1.90142	1.236671	1.447522	1.725355
	2	1	1.79134	1.48706	1.12912	1.697834	1.871765	1.532280
	2	2	1.774725	1.128364	1.414627	1.193681	0.532091	0.353657
	3	1	1.74887	1.571811	1.608844	1.937219	1.392953	1.152211
	3	2	1.106164	1.686650	1.187098	0.776541	0.421663	0.296775
	3	3	1.019533	0.354379	0.192927	0.254883	0.088595	0.048232
4	1	1	1.26225	1.72012	1.868560	1.315561	1.930030	1.967140
	2	1	1.721780	1.316080	1.505491	1.930445	1.829020	1.376373
	2	2	1.366299	1.837819	1.886170	1.591575	0.709455	0.471542
	3	1	1.665170	1.762422	1.478463	1.916293	1.190604	1.869615
	3	2	1.141552	1.248867	1.582798	1.035388	0.562217	0.395699
	3	3	1.359377	0.472506	0.257236	0.339844	0.118126	0.064309
	4	1	1.916593	1.605231	1.474252	2.479149	1.901307	1.618562
	4	2	1.456323	1.035751	1.458000	0.864081	0.508938	0.364500
	4	3	1.032447	0.422185	0.237631	0.258112	0.105546	0.059408
	4	4	0.333961	0.094078	0.041518	0.083490	0.023520	0.010379
5	1	1	1.577813	1.650150	1.835703	1.3944522	1.412537	1.208925
	2	1	1.652231	1.145103	1.881870	1.163057	1.786275	1.220466
	2	2	1.957874	1.547273	1.357712	1.989469	0.886818	0.589428
	3	1	1.581463	1.953020	1.348071	1.895366	1.988255	1.587018
	3	2	1.176940	1.811084	1.978497	1.294235	0.702771	0.494624
	3	3	1.699221	0.590632	0.321545	0.424805	0.147658	0.080386
	4	1	1.395743	1.506531	1.092810	1.348936	1.626634	3.273202
	4	2	1.320404	1.544689	1.822500	1.080101	0.636172	0.455625
	4	3	1.290559	0.527731	0.297038	0.322640	0.131933	0.074260
	4	4	0.417452	0.117598	0.051897	0.104363	0.029399	0.012974
	5	1	1.305143	1.712991	1.331450	1.076284	1.428247	1.082863
	5	2	1.951343	1.416041	1.734231	0.987836	0.604010	0.433558
	5	3	1.142553	0.501426	0.284630	0.285638	0.125356	0.071157
	5	4	0.355083	0.111510	0.049950	0.088771	0.027877	0.012488
	5	5	0.114665	0.025749	0.009099	0.028666	0.006437	0.002275

n, in view of (6) and (7), is as follows.

$$f_{s|r}(y|x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}} \frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}}{[\bar{F}(x)]^{\gamma_{r+1}}} f(y) \tag{47}$$

The conditional pdf of $X(r,n,m,k)$ given $X(s,n,m,k) = y$, $1 \le r < s \le n$ is as follows.

$$f_{r|s}(x|y) = \frac{(s-1)!(m+1)[\bar{F}(x)]^m[1-(\bar{F}(x))^{m+1}]^{r-1}}{(r-1)!(s-r-1)!} \times \frac{[(\bar{F}(x))^{m+1} - (\bar{F}(x))^{m+1}]^{s-r-1}}{[1-(\bar{F}(y))^{m+1}]^{s-1}} f(y). \tag{48}$$

We shall first establish conditional moments of GOS, $X(s,n,m,k)$ given $X(r,n,m,k) = x$ $E[X^{(j)}(s,n,m,k)|X(r,n,m,k) = x] = \mu^{(j)}_{s,n,m,k|r,n,m,k}$ and $X(r,n,m,k)$ given $X(s,n,m,k) = y$ $E[X^{(j)}(r,n,m,k)|X(s,n,m,k) = y] = \mu^{(j)}_{r,n,m,k|s,n,m,k}$. Theorem 9 gives the conditional moments of GOS, $X(s,n,m,k)$ given $X(r,n,m,k) = x$ for $1 \le r < s \le n$ and $j = 0, 1, 2, \ldots$. Theorem 10 gives the conditional moments of GOS, $X(r,n,m,k)$ given $X(s,n,m,k) = y$ for $1 \le r < s \le n$ and $j = 0, 1, 2, \ldots$.

Table 3. Means of record statistics for $k = 1$ and different values of parameters.

$r \downarrow$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$
		$\lambda = 0.5$		
1	0.447782	0.895563	1.343345	1.791126
2	0.332034	0.664068	0.996101	1.328135
3	0.238880	0.477761	0.716641	0.955521
4	0.168331	0.336661	0.504992	0.673323
5	0.116888	0.233775	0.350663	0.467551
6	0.080311	0.160622	0.240933	0.321244
7	0.054755	0.029140	0.164264	0.219019
8	0.037119	0.074238	0.111357	0.148476
9	0.025057	0.050114	0.075172	0.100229
10	0.016862	0.033724	0.050586	0.067447
		$\lambda = 1.0$		
1	0.176417	0.352834	0.529251	0.705668
2	0.099692	0.199384	0.299077	0.398769
3	0.053865	0.107730	0.161594	0.215459
4	0.028321	0.056642	0.084963	0.113284
5	0.014636	0.029272	0.043907	0.058543
6	0.007480	0.014959	0.022439	0.029918
7	0.003794	0.007589	0.011383	0.015178
8	0.001916	0.003831	0.005747	0.007663
9	0.000964	0.001928	0.002892	0.003856
10	0.000484	0.000968	0.001452	0.001936

Theorem 9. *For the distribution given in (1) and $1 \leq r < s \leq n, n \geq 1, k = 1, 2, \ldots$ and $i, \; j = 0, 1, 2, \ldots,$*

$$\mu^{(j)}_{s,n,m,k|r,n,m,k} = \frac{C_{s-1}}{\lambda^j (s-r-1)! C_{r-1} (m+1)^{s-r-1} [e^{1-(1+\lambda x)^\alpha}]^{\gamma_s}} \sum_{u=0}^{s-r-1} (-1)^u \binom{s-r-1}{u} \times \sum_{l=0}^{j} (-1)^{j-l} \binom{j}{l} \frac{e^{\gamma_{s-u}} \Gamma\left(\frac{l}{\alpha}+1, \; (1+\lambda x)^\alpha \gamma_{s-u}\right)}{\gamma_{s-u}^{\frac{l}{\alpha}+1}}. \tag{49}$$

Proof. Similar to the proof of Theorem 1.

Theorem 10. *For the distribution given in (1) and $1 \leq r < s \leq n, \; n \geq 1, \; k = 1, 2, \ldots$ and $i, \; j = 0, 1, 2, \ldots,$*

$$\mu^{(j)}_{r,n,m,k|s,n,m,k} = \frac{(s-1)!(m+1)}{\lambda^j (r-1)!(s-r-1)![1-(e^{1-(1+\lambda y)^\alpha})^{m+1}]^{s-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \times \sum_{v=0}^{s-r-1} (-1)^{s-r-1-v} \binom{s-r-1}{u} \sum_{l=0}^{j} (-1)^{j-l} \binom{j}{l} e^{\xi} \times \frac{[e^{1-(1+\lambda y)^\alpha}]^{s-r-1-v} \gamma\left(\frac{l}{\alpha}+1, \; \xi\,(1+\lambda y)^\alpha\right)}{\xi^{\frac{l}{\alpha}+1}}, \tag{50}$$

where $\xi = u + v + m(1+u) + 1$ and $\gamma(a, x)$ denotes the incomplete gamma function defined by $\gamma(a, x) = \int_0^y t^{a-1} e^t \, dt$.

Proof. Similar to the proof of Theorem 1.

Remark 1. For $k = 1, m = 0$, and $k = 1, m = -1$, in (49) and (50), we obtain the conditional momemts of order statistics and record values, respectively.

Table 4. Variances and covariances of record statistics.

		$\lambda = 0.5$			$\lambda = 1.0$		
r	s	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$
1	1	9.0283462	8.315561	6.775821	3.648953	2.078890	1.693955
1	2	5.267103	4.930445	4.218403	2.055083	1.232611	1.054601
1	3	4.134393	3.916293	3.452603	1.060843	0.979073	0.863151
1	4	3.986392	3.479149	3.118780	0.965401	0.869787	0.779695
1	5	3.749502	3.261027	2.946530	0.957263	0.815257	0.736633
1	6	3.520278	3.143586	2.848129	0.893164	0.785897	0.712032
1	7	3.319361	3.077308	2.787888	0.820892	0.769327	0.696972
1	8	3.125013	3.038634	2.749184	0.812618	0.759659	0.687296
1	9	3.107148	3.015473	2.723474	0.806329	0.753868	0.680869
2	2	1.212775	1.591575	0.963569	0.303194	0.397894	0.240892
2	3	1.353388	1.035388	0.718258	0.338347	0.258847	0.17956
2	4	1.392593	0.864081	0.634816	0.348148	0.216020	0.158704
2	5	1.407665	0.790269	0.597443	0.351916	0.197567	0.149361
2	6	1.414247	0.754026	0.578157	0.353562	0.188506	0.144539
2	7	1.417306	0.735006	0.567199	0.354326	0.183751	0.141800
2	8	1.418774	0.724603	0.560513	0.354694	0.181151	0.140128
2	9	1.419492	0.718736	0.556216	0.354873	0.179684	0.139054
3	3	0.071090	0.084961	0.044993	0.284359	0.339844	0.179972
3	4	0.297832	0.258112	0.153716	0.074458	0.064528	0.038429
3	5	0.302410	0.228511	0.143493	0.075603	0.057128	0.035873
3	6	0.304325	0.215197	0.138739	0.076081	0.053799	0.034685
3	7	0.305199	0.208620	0.136260	0.076300	0.052155	0.034065
3	8	0.305616	0.205204	0.134846	0.076404	0.051301	0.033712
3	9	0.305819	0.203369	0.133981	0.076455	0.050842	0.033495
4	4	0.091612	0.083490	0.040061	0.022903	0.020873	0.010015
4	5	0.093130	0.071017	0.036969	0.023282	0.017754	0.009242
4	6	0.093731	0.065853	0.035655	0.023433	0.016463	0.008914
4	7	0.093999	0.063425	0.035024	0.023500	0.015856	0.008756
4	8	0.094126	0.062212	0.034689	0.023531	0.015553	0.008672
4	9	0.094187	0.061584	0.034490	0.023547	0.015396	0.008624
5	5	0.033398	0.022933	0.009890	0.008349	0.005733	0.002472
5	6	0.033594	0.020897	0.009503	0.008399	0.005224	0.002376
5	7	0.033680	0.019980	0.009330	0.008420	0.004995	0.002332
5	8	0.033720	0.019536	0.009244	0.008430	0.004884	0.002311
5	9	0.033739	0.019311	0.009198	0.008435	0.004828	0.002299
6	6	0.013079	0.006776	0.002589	0.003270	0.001694	0.000647
6	7	0.013107	0.006428	0.002539	0.003277	0.001607	0.000635
6	8	0.013120	0.006263	0.002516	0.003280	0.001566	0.000629
6	9	0.013126	0.006181	0.002504	0.003281	0.001545	0.000626
7	7	0.005369	0.002096	0.000701	0.001342	0.000524	0.000175
7	8	0.005373	0.002034	0.000694	0.001343	0.000509	0.000174
7	9	0.005369	0.002004	0.000691	0.001344	0.000501	0.000173
8	8	0.002278	0.000667	0.000193	0.000569	0.000167	0.000048
8	9	0.002258	0.000656	0.000183	0.000570	0.000164	0.000043
9	9	0.001180	0.000216	0.000125	0.000247	0.000122	0.000034
10	10	0.000438	0.000171	0.000014	0.000109	0.000018	0.000001

4. Numerical Results

4.1. Tabulations of Means, Variances, and Covariances

The recurrence relations obtained in the preceding sections allow us to evaluate the means, variances, and covariances of all order statistics and record values for all sample sizes in a simple recursive manner. Starting with the initial values given by (10) and (11), the recurrence relations in (17) and (18) can be used recursively to compute the first two moments of all order statistics for sample sizes $n = 2, 3, \ldots$. From the resulting values, variances of order statistics can be readily computed. In Table 1, we have presented the means of all order statistics, up to six decimal places, for sample sizes $n = 1(1)5$ and $\alpha = 1(1)4$. These moments were checked by using the identities,

$$\sum_{r=1}^{n} \mu_{r:n}^{(j)} = n\mu_{1:1}^{(j)}, \quad j = 1, 2.$$

Table 5. Means of progressively Type-II right censored order statistics.

$\lambda = 0.5, \alpha = 2$							
$m \downarrow$	$n \downarrow$	Scheme	Mean				
5	5(0,3)	(0,3)	0.030068	0.067654			
5	5(3,0)	(3,0)	0.030068	0.180410			
8	8(6,0)	(6,0)	0.018792	0.169135			
8	8(0,6)	(0,6)	0.018792	0.040270			
10	10(8,0)	(8,0)	0.015034	0.165376			
10	10(0,8)	(0,8)	0.015034	0.031738			
12	12(10,0)	(10,0)	0.012528	0.162870			
12	12(0,10)	(0,10)	0.012528	0.026195			
15	15(13,0)	(13,0)	0.010022	0.160365			
15	15(0,13)	(0,13)	0.010022	0.020761			
18	18(16,0)	(16,0)	0.008352	0.158694			
18	18(0,16)	(0,16)	0.008352	0.017196			
20	20(18,0)	(18,0)	0.007517	0.157859			
20	20(0,18)	(0,18)	0.007517	0.015429			
5	5(2,0,0)	(2,0,0)	0.030068	0.105239	0.255581		
5	5(0,0,2)	(0,0,2)	0.030068	0.067654	0.117768		
8	8(5,0,0)	(5,0,0)	0.018792	0.093963	0.244306		
8	8(0,0,5)	(0,0,5)	0.018792	0.040270	0.065327		
10	10(7,0,0)	(7,0,0)	0.015034	0.090205	0.240547		
10	10(0,0,7)	(0,0,7)	0.015034	0.031738	0.050531		
12	12(9,0,0)	(9,0,0)	0.012528	0.087699	0.238041		
12	12(0,0,9)	(0,0,9)	0.012528	0.026195	0.041230		
15	15(12,0,0)	(12,0,0)	0.010022	0.085193	0.235536		
15	15(0,0,12)	(0,0,12)	0.010022	0.020761	0.032326		
18	18(15,0,0)	(15,0,0)	0.008352	0.083523	0.233865		
18	18(0,0,15)	(0,0,15)	0.008352	0.017196	0.026592		
20	20(17,0,0)	(17,0,0)	0.007517	0.082688	0.233030		
20	20(0,0,17)	(0,0,17)	0.007517	0.015429	0.023782		
5	5(1,0,0,0)	(1,0,0,0)	0.030068	0.080182	0.155353	0.305695	
5	5(0,0,0,1)	(0,0,0,1)	0.030068	0.067654	0.117768	0.192939	
8	8(4,0,0,0)	(4,0,0,0)	0.018792	0.068906	0.144077	0.294420	
8	8(0,0,0,4)	(0,0,0,4)	0.018792	0.040270	0.065327	0.095395	
10	10(6,0,0,0)	(6,0,0,0)	0.015034	0.065148	0.140319	0.290661	
10	10(0,0,0,6)	(0,0,0,6)	0.015034	0.031738	0.050531	0.072009	
12	12(8,0,0,0)	(8,0,0,0)	0.012528	0.062642	0.137813	0.288155	
12	12(0,0,0,8)	(0,0,0,8)	0.012528	0.026195	0.041230	0.057934	
15	15(11,0,0,0)	(11,0,0,0)	0.010022	0.060136	0.135308	0.285650	
15	15(0,0,0,11)	(0,0,0,11)	0.010022	0.020761	0.032326	0.044854	
18	18(14,0,0,0)	(14,0,0,0)	0.008352	0.058466	0.133637	0.283979	
18	18(0,0,0,14)	(0,0,0,14)	0.008352	0.017196	0.026592	0.036615	
20	20(16,0,0,0)	(16,0,0,0)	0.007517	0.057631	0.132802	0.283144	
20	20(0,0,0,16)	(0,0,0,16)	0.007517	0.015429	0.023782	0.032625	
5	5(0,0,0,0,0)	(0,0,0,0,0)	0.030068	0.067654	0.117768	0.192939	0.343281
8	8(3,0,0,0,0)	(3,0,0,0,0)	0.018792	0.056378	0.106492	0.181663	0.332005
8	8(0,0,0,0,3)	(0,0,0,0,3)	0.018792	0.040270	0.065327	0.095395	0.132981
10	10(5,0,0,0,0)	(5,0,0,0,0)	0.015034	0.052619	0.102733	0.177904	0.328247
10	10(0,0,0,0,5)	(0,0,0,0,5)	0.015034	0.031738	0.050531	0.072009	0.097066
12	12(7,0,0,0,0)	(7,0,0,0,0)	0.012528	0.050114	0.100228	0.175399	0.325741
12	12(0,0,0,0,7)	(0,0,0,0,7)	0.012528	0.026195	0.041230	0.057934	0.076727
15	15(10,0,0,0,0)	(10,0,0,0,0)	0.010022	0.047608	0.097722	0.172893	0.323235
15	15(0,0,0,0,10)	(0,0,0,0,10)	0.010022	0.020761	0.032326	0.044854	0.058522
18	18(13,0,0,0,0)	(13,0,0,0,0)	0.008352	0.045937	0.096051	0.171223	0.321565
18	18(0,0,0,0,13)	(0,0,0,0,13)	0.008352	0.017196	0.026592	0.036615	0.047353
20	20(15,0,0,0,0)	(15,0,0,0,0)	0.007517	0.045102	0.095216	0.170387	0.320730
20	20(0,0,0,0,15)	(0,0,0,0,15)	0.007517	0.015429	0.023782	0.032625	0.042022

One can see that the means are decreasing with respect to n, but increasing with respect to α.

For computing covariances of order statistics, all the product moments $\mu_{r,s:n}^{(1,1)}$ can be computed in a systematic manner. Starting with the initial values given by (31), the recurrence

Table 6. Means of progressively Type-II right censored order statistics.

			$\lambda = 1, \alpha = 4$				
$m \downarrow$	$n \downarrow$	Scheme	Mean				
5	5(0,3)	(0,3)	0.046260	0.104086			
5	5(3,0)	(3,0)	0.046260	0.277564			
8	8(6,0)	(6,0)	0.028912	0.260216			
8	8(0,6)	(0,6)	0.028912	0.061956			
10	10(8,0)	(8,0)	0.023130	0.254433			
10	10(0,8)	(0,8)	0.023130	0.048830			
12	12(10,0)	(10,0)	0.019275	0.250578			
12	12(0,10)	(0,10)	0.019275	0.040302			
15	15(13,0)	(13,0)	0.015420	0.246723			
15	15(0,13)	(0,13)	0.015420	0.031941			
18	18(16,0)	(16,0)	0.012850	0.244153			
18	18(0,16)	(0,16)	0.012850	0.026456			
20	20(18,0)	(18,0)	0.011565	0.242868			
20	20(0,18)	(0,18)	0.011565	0.023739			
5	5(2,0,0)	(2,0,0)	0.046260	0.161912	0.393215		
5	5(0,0,2)	(0,0,2)	0.046260	0.104086	0.181187		
8	8(5,0,0)	(5,0,0)	0.028912	0.144564	0.375868		
8	8(0,0,5)	(0,0,5)	0.028912	0.061956	0.100506		
10	10(7,0,0)	(7,0,0)	0.023130	0.138782	0.370085		
10	10(0,0,7)	(0,0,7)	0.023130	0.048830	0.077743		
12	12(9,0,0)	(9,0,0)	0.019275	0.134927	0.366230		
12	12(0,0,9)	(0,0,9)	0.019275	0.040302	0.063433		
15	15(12,0,0)	(12,0,0)	0.015420	0.131071	0.362375		
15	15(0,0,12)	(0,0,12)	0.015420	0.031941	0.049734		
18	18(15,0,0)	(15,0,0)	0.012850	0.128501	0.359805		
18	18(0,0,15)	(0,0,15)	0.012850	0.026456	0.040912		
20	20(17,0,0)	(17,0,0)	0.011565	0.127216	0.358520		
20	20(0,0,17)	(0,0,17)	0.011565	0.023739	0.036589		
5	5(1,0,0,0)	(1,0,0,0)	0.046260	0.123361	0.239013	0.470317	
5	5(0,0,0,1)	(0,0,0,1)	0.046260	0.104086	0.181187	0.296839	
8	8(4,0,0,0)	(4,0,0,0)	0.028912	0.106014	0.221665	0.452969	
8	8(0,0,0,4)	(0,0,0,4)	0.028912	0.061956	0.100506	0.146767	
10	10(6,0,0,0)	(6,0,0,0)	0.023130	0.100231	0.215883	0.447186	
10	10(0,0,0,6)	(0,0,0,6)	0.023130	0.048830	0.077743	0.110787	
12	12(8,0,0,0)	(8,0,0,0)	0.019275	0.096376	0.212028	0.443331	
12	12(0,0,0,8)	(0,0,0,8)	0.019275	0.040302	0.063433	0.089133	
15	15(11,0,0,0)	(11,0,0,0)	0.015420	0.092521	0.208173	0.439476	
15	15(0,0,0,11)	(0,0,0,11)	0.015420	0.031941	0.049734	0.069009	
18	18(14,0,0,0)	(14,0,0,0)	0.012850	0.089951	0.205603	0.436906	
18	18(0,0,0,14)	(0,0,0,14)	0.012850	0.026456	0.040912	0.056332	
20	20(16,0,0,0)	(16,0,0,0)	0.011565	0.088666	0.204318	0.435621	
20	20(0,0,0,16)	(0,0,0,16)	0.011565	0.023739	0.036589	0.050195	
5	5(0,0,0,0,0)	(0,0,0,0,0)	0.046260	0.104086	0.181187	0.296839	0.528143
8	8(3,0,0,0,0)	(3,0,0,0,0)	0.028912	0.086738	0.163839	0.279491	0.510795
8	8(0,0,0,0,3)	(0,0,0,0,3)	0.028912	0.061956	0.100506	0.146767	0.204593
10	10(5,0,0,0,0)	(5,0,0,0,0)	0.023130	0.080956	0.158057	0.273709	0.505012
10	10(0,0,0,0,5)	(0,0,0,0,5)	0.023130	0.048830	0.077743	0.110787	0.149337
12	12(7,0,0,0,0)	(7,0,0,0,0)	0.019275	0.077101	0.154202	0.269854	0.501157
12	12(0,0,0,0,7)	(0,0,0,0,7)	0.019275	0.040302	0.063433	0.089133	0.118046
15	15(10,0,0,0,0)	(10,0,0,0,0)	0.015420	0.073246	0.150347	0.265999	0.497302
15	15(0,0,0,0,10)	(0,0,0,0,10)	0.015420	0.031941	0.049734	0.069009	0.090037
18	18(13,0,0,0,0)	(13,0,0,0,0)	0.012850	0.070676	0.147777	0.263429	0.494732
18	18(0,0,0,0,13)	(0,0,0,0,13)	0.012850	0.026456	0.040912	0.056332	0.072854
20	20(15,0,0,0,0)	(15,0,0,0,0)	0.011565	0.069391	0.146492	0.262143	0.493447
20	20(0,0,0,0,15)	(0,0,0,0,15)	0.011565	0.023739	0.036589	0.050195	0.064651

relations in (39) and (42) can be used recursively to compute $\mu_{r,s:n}^{(1,1)}$ for all r, s, and $n = 2, 3, \ldots$. The variances and covariances, up to six decimal places, are presented in Table 2. The accuracy of their computation was checked by the well-known identities for any

Table 7. Variances of progressively Type-II right censored order statistics.

				$\lambda = 0.5, \alpha = 2$				
$m\downarrow$	$n\downarrow$	Scheme		Variance				
2	5	5(0,3)	(0,3)	0.000904	0.002316			
2	5	5(3,0)	(3,0)	0.000904	0.023506			
2	8	8(6,0)	(6,0)	0.000353	0.022955			
2	8	8(0,6)	(0,6)	0.000353	0.000814			
2	10	10(8,0)	(8,0)	0.000226	0.022828			
2	10	10(0,8)	(0,8)	0.000226	0.000505			
2	12	12(10,0)	(10,0)	0.000156	0.022759			
2	12	12(0,10)	(0,10)	0.000156	0.000343			
2	15	15(13,0)	(13,0)	0.000100	0.022703			
2	15	15(0,13)	(0,13)	0.000100	0.000215			
2	18	18(16,0)	(16,0)	6.976169	0.022672			
2	18	18(0,16)	(0,16)	6.976169	0.000147			
2	20	20(18,0)	(18,0)	5.650697	0.022659			
2	20	20(0,18)	(0,18)	5.650697	0.000119			
3	5	5(2,0,0)	(2,0,0)	0.000904	0.006554	0.029157		
3	5	5(0,0,2)	(0,0,2)	0.000904	0.002316	0.004828		
3	8	8(5,0,0)	(5,0,0)	0.000353	0.006003	0.028606		
3	8	8(0,0,5)	(0,0,5)	0.000353	0.000814	0.001442		
3	10	10(7,0,0)	(7,0,0)	0.000226	0.005876	0.028479		
3	10	10(0,0,7)	(0,0,7)	0.000226	0.000505	0.000858		
3	12	12(9,0,0)	(9,0,0)	0.000156	0.005807	0.028410		
3	12	12(0,0,9)	(0,0,9)	0.000156	0.000343	0.000569		
3	15	15(12,0,0)	(12,0,0)	0.000100	0.005751	0.028353		
3	15	15(0,0,12)	(0,0,12)	0.000100	0.000215	0.000349		
3	18	18(15,0,0)	(15,0,0)	6.976169	0.005720	0.028323		
3	18	18(0,0,15)	(0,0,15)	6.976169	0.000147	0.000236		
3	20	20(17,0,0)	(17,0,0)	5.650697	0.005707	0.028309		
3	20	20(0,0,17)	(0,0,17)	5.650697	0.000119	0.000188		
4	5	5(1,0,0,0)	(1,0,0,0)	0.000904	0.003415	0.009066	0.031669	
4	5	5(0,0,0,1)	(0,0,0,1)	0.000904	0.002316	0.004828	0.010478	
4	8	8(4,0,0,0)	(4,0,0,0)	0.000353	0.002864	0.008515	0.031118	
4	8	8(0,0,0,4)	(0,0,0,4)	0.000353	0.000814	0.001442	0.002346	
4	10	10(6,0,0,0)	(6,0,0,0)	0.000226	0.002737	0.008388	0.030990	
4	10	10(0,0,0,6)	(0,0,0,6)	0.000226	0.000505	0.000858	0.001319	
4	12	12(8,0,0,0)	(8,0,0,0)	0.000156	0.002668	0.008319	0.030921	
4	12	12(0,0,0,8)	(0,0,0,8)	0.000156	0.000343	0.000569	0.000848	
4	15	15(11,0,0,0)	(11,0,0,0)	0.000100	0.002611	0.008262	0.030865	
4	15	15(0,0,0,11)	(0,0,0,1)	0.000100	0.000215	0.000349	0.000506	
4	18	18(14,0,0,0)	(14,0,0,0)	6.976169	0.002581	0.008231	0.030834	
4	18	18(0,0,0,14)	(0,0,0,1)	6.976169	0.000147	0.000236	0.000336	
4	20	20(16,0,0,0)	(16,0,0,0)	5.650697	0.002567	0.008218	0.030821	
4	20	20(0,0,0,16)	(0,0,0,1)	5.650697	0.000119	0.000188	0.000267	
5	5	5(0,0,0,0,0)	(0,0,0,0)	0.000904	0.002316	0.004828	0.010478	0.033081
5	8	8(3,0,0,0,0)	(3,0,0,0)	0.000353	0.001765	0.004277	0.009927	0.032530
5	8	8(0,0,0,0,3)	(0,0,0,0)	0.000353	0.000814	0.001442	0.002346	0.003759
5	10	10(5,0,0,0,0)	(5,0,0,0)	0.000226	0.001638	0.004150	0.009800	0.032403
5	10	10(0,0,0,0,5)	(0,0,0,0)	0.000226	0.000505	0.000858	0.001319	0.001947
5	12	12(7,0,0,0,0)	(7,0,0,0)	0.000156	0.001569	0.004081	0.009731	0.032334
5	12	12(0,0,0,0,7)	(0,0,0,0)	0.000156	0.000343	0.000569	0.000848	0.001202
5	15	15(10,0,0,0,0)	(10,0,0,0)	0.000100	0.001513	0.004024	0.009675	0.032278
5	15	15(0,0,0,0,10)	(0,0,0,0)	0.000100	0.000215	0.000349	0.000506	0.000693
5	18	18(13,0,0,0,0)	(13,0,0,0)	6.976169	0.001482	0.003993	0.009644	0.032247
5	18	18(0,0,0,0,13)	(0,0,0,0)	6.976169	0.000147	0.000236	0.000336	0.000452
5	20	20(15,0,0,0,0)	(15,0,0,0)	5.650697	0.001469	0.003980	0.009631	0.032234
5	20	20(0,0,0,0,15)	(0,0,0,0)	5.650697	0.000119	0.000188	0.000267	0.000355

arbitrary continuous distribution.

$$\sum_{r=1}^{n}\sum_{s=1}^{n}\sigma_{r,s:n} = n\sigma_{1,1:1},$$

Table 8. Variances of progressively Type-II right censored order statistics.

$\lambda = 1.0, \alpha = 4$								
$m\downarrow$	$n\downarrow$	Scheme		Variance				
2	5	5(0,3)	(0,3)	0.002140	0.005483			
2	5	5(3,0)	(3,0)	0.002140	0.055641			
2	8	8(6,0)	(6,0)	0.000835	0.054337			
2	8	8(0,6)	(0,6)	0.000835	0.001927			
2	10	10(8,0)	(8,0)	0.000535	0.054036			
2	10	10(0,8)	(0,8)	0.000535	0.001195			
2	12	12(10,0)	(10,0)	0.000371	0.053872			
2	12	12(0,10)	(0,10)	0.000371	0.000813			
2	15	15(13,0)	(13,0)	0.000237	0.053739			
2	15	15(0,13)	(0,13)	0.000237	0.000510			
2	18	18(16,0)	(16,0)	0.000165	0.053666			
2	18	18(0,16)	(0,16)	0.000165	0.000350			
2	20	20(18,0)	(18,0)	0.000133	0.053635			
2	20	20(0,18)	(0,18)	0.000133	0.000281			
3	5	5(2,0,0)	(2,0,0)	0.002140	0.015515	0.069016		
3	5	5(0,0,2)	(0,0,2)	0.002140	0.005483	0.011428		
3	8	8(5,0,0)	(5,0,0)	0.000835	0.014211	0.067712		
3	8	8(0,0,5)	(0,0,5)	0.000835	0.001927	0.003413		
3	10	10(7,0,0)	(7,0,0)	0.000535	0.013910	0.067411		
3	10	10(0,0,7)	(0,0,7)	0.000535	0.001195	0.002031		
3	12	12(9,0,0)	(9,0,0)	0.000371	0.013746	0.067248		
3	12	12(0,0,9)	(0,0,9)	0.000371	0.000813	0.001348		
3	15	15(12,0,0)	(12,0,0)	0.000237	0.013613	0.067114		
3	15	15(0,0,12)	(0,0,12)	0.000237	0.000510	0.000827		
3	18	18(15,0,0)	(15,0,0)	0.000165	0.013540	0.067041		
3	18	18(0,0,15)	(0,0,15)	0.000165	0.000350	0.000559		
3	20	20(17,0,0)	(17,0,0)	0.000133	0.013509	0.067010		
3	20	20(0,0,17)	(0,0,17)	0.000133	0.000281	0.000447		
4	5	5(1,0,0,0)	(1,0,0,0	0.002140	0.008084	0.021459	0.074961	
4	5	5(0,0,0,1)	(0,0,0,1	0.002140	0.005483	0.011428	0.024803	
4	8	8(4,0,0,0)	(4,0,0,0	0.000835	0.006780	0.020155	0.073657	
4	8	8(0,0,0,4)	(0,0,0,4	0.000835	0.001927	0.003413	0.005554	
4	10	10(6,0,0,0)	(6,0,0,0	0.000535	0.006479	0.019854	0.073356	
4	10	10(0,0,0,6)	(0,0,0,6	0.000535	0.001195	0.002031	0.003123	
4	12	12(8,0,0,0)	(8,0,0,0	0.000371	0.006316	0.019691	0.073192	
4	12	12(0,0,0,8)	(0,0,0,8	0.000371	0.000813	0.001348	0.002009	
4	15	15(11,0,0,0)	(11,0,0,	0.000237	0.006182	0.019557	0.073059	
4	15	15(0,0,0,11)	(0,0,0,1	0.000237	0.000510	0.000827	0.001198	
4	18	18(14,0,0,0)	(14,0,0,	0.000165	0.006109	0.019485	0.072986	
4	18	18(0,0,0,14)	(0,0,0,1	0.000165	0.000350	0.000559	0.000797	
4	20	20(16,0,0,0)	(16,0,0,	0.000133	0.006078	0.019453	0.072954	
4	20	20(0,0,0,16)	(0,0,0,1	0.000133	0.000281	0.000447	0.000632	
5	5	5(0,0,0,0,0)	(0,0,0,0	0.002140	0.005483	0.011428	0.024803	0.078305
5	8	8(3,0,0,0,0)	(3,0,0,0	0.000835	0.004179	0.010124	0.023499	0.077001
5	8	8(0,0,0,0,3)	(0,0,0,0	0.000835	0.001927	0.003413	0.005554	0.008897
5	10	10(5,0,0,0,0)	(5,0,0,0	0.000535	0.003878	0.009823	0.023198	0.076700
5	10	10(0,0,0,0,5)	(0,0,0,0	0.000535	0.001195	0.002031	0.003123	0.004609
5	12	12(7,0,0,0,0)	(7,0,0,0	0.000371	0.003715	0.009659	0.023035	0.076536
5	12	12(0,0,0,0,7)	(0,0,0,0	0.000371	0.000813	0.001348	0.002009	0.002845
5	15	15(10,0,0,0,0)	(10,0,0,	0.000237	0.003581	0.009526	0.022901	0.076402
5	15	15(0,0,0,0,10)	(0,0,0,0	0.000237	0.000510	0.000827	0.001198	0.001641
5	18	18(13,0,0,0,0)	(13,0,0,	0.000165	0.003508	0.009453	0.022828	0.076330
5	18	18(0,0,0,0,13)	(0,0,0,0	0.000165	0.000350	0.000559	0.000797	0.001069
5	20	20(15,0,0,0,0)	(15,0,0,	0.000133	0.003477	0.009422	0.022797	0.076298
5	20	20(0,0,0,0,15)	(0,0,0,0	0.000133	0.000281	0.000447	0.000632	0.000841

and

$$\sum_{s=r+1}^{n} \sigma_{r,s:n} + \sum_{i=1}^{r} \sigma_{i,r+1:n} = \left(r\mu_{1:1}^{(1)} - \sum_{i=1}^{r} \mu_{i:n}^{(1)} \right) \left(\mu_{r+1:n}^{(1)} - \mu_{r:n}^{(1)} \right), \quad 1 \le r \le n-1,$$

where $\sigma_{r,s:n} = \mu_{r,s:n}^{(1,1)} - \mu_{r:n}^{(1)}\mu_{s:n}^{(1)}$, see Joshi and Balakrishnan (1982). We can see that variances and covariances decrease as both α and λ increase.

The relation in (21) can be used in a simple recursive process to obtain all the jth single moments of all record statistics. The computations of these moments can be done based on the jth single moment of the first record value i.e. μ_1^j. We have computed the values of μ_r and σ_r^2 for $r = 1, \ldots, 10$ and $\alpha = 1(1)4$, and these are presented in Tables 3 and 4, respectively.

The means and variances of all progressive type-II right censored order statistics from the extended exponential distribution can be readily computed as follows.

All the first and second order moments with $m = 1$ for all sample sizes n can be obtained by setting $j = 0$ in Equation (24) and then again setting $k = 1$ in the same equation. Next, using Equation (23), we can determine all the moments of the form $\mu_{1,2:n}^{(R_1,R_2)}$, $n = 2, 3 \ldots$, which can in turn be used again with (23), to determine all moments of the form $\mu_{1,2:n}^{(R_1,R_2)^2}$, $n = 2, 3 \ldots$. Equation (26) can then be used to obtain $\mu_{2,2:n}^{(R_1,R_2)}$ for all R_1, R_2 and $n \ge 2$, and these values can be used to obtain all moments of the form $\mu_{1,2:n}^{(R_1,R_2)^2}$ by using Equation (26) again. Equation (23) can now be used again to obtain $\mu_{1,3:n}^{(R_1,R_2,R_3)}$, $\mu_{1,3:n}^{(R_1,R_2,R_3)^2}$ for all n, R_1, R_2, and R_3 and Equation (25) can be used next to obtain all moments of the form $\mu_{2,3:n}^{(R_1,R_2,R_3)}$, $\mu_{2,3:n}^{(R_1,R_2,R_3)^2}$. Finally, Equation (26) can be used to obtain all moments of the form $\mu_{3,3:n}^{(R_1,R_2,R_3)}$, $\mu_{3,3:n}^{(R_1,R_2,R_3)^2}$.

This process can be continued until all desired first and second order moments, and hence, all variances are obtained. In Table 5–6 and Table 7–8, we have presented the means and variances of progressively Type-II right censored order statistics, respectively, up to six decimal places.

5. Conclusion

We have derived explicit expressions for single moments, product moments, and conditional moments of order statistics and record values from the extended exponential distribution. Further, we also derive the expressions for single moments and product moments for progressively Type II censored order statistics. We have given tabulations of the explicit expressions for single moments and product moments of order statistics and record values. We have also provided tabulations of means and variances of progressively Type II censored order statistics.

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References

Ahmad, A. A., & Fawzy, M. (2003). Recurrence relations for single moments of generalized order statistics from doubly truncated distributions. *J. Stat. Plann. and Inference*, *117*, 241–249.

Ahmad, A. A. (2007). Relations for single and product moments of generalized order statistics from doubly truncated Burr type XII distribution. *J. Egypt. Math. Soc.*, *15*, 117–128.

Ahmad, A. A. (2008). Single and product moments of generalized order statistics from linear exponential distribution. *Commun. Stat. Theory Methods*, *37*, 1162–1172.

Ahsanullah, M. (2000). Generalized order statistics from exponential distribution. *J. Stat. Plann. Inferace*, *85*, 85–91.

AL-Hussaini, E. K., & Ahmad, A. A. (2003a). On Bayesian interval prediction of future records. *Test*, *12*, 79–99.

AL-Hussaini, E. K., & Ahmad, A. A. (2003b). On Bayesian predictive distributions of generalized order statistics. *Metrika*, *57*, 165–176.

AL-Hussaini, E. K. (2004). Generalized order statistics: Prospective and Applications. *J. Appl. Stat. Sci.*, *13*, 59–85.

Cramer, E., & Kamps, U. (2000). Relations for expectations of functions of generalized order statistics. *J. Stat. Plann. Inference*, *89*, 79–89.

Gradshteyn, I. S., & Ryzhik, I. M. (2000). *Table of Integrals, Series, and Products*. Sixth edition. Academic Press.

Gupta, R. D., & Kundu, D. (1999). Generalized exponential distribution. *Aust. N. Z. J. Stat.*, *41*, 173–188.

Gurvich, M. R., DiBenedetto, A. T., & Ranade, S. V. (1997). A new statistical distribution for characterizing the random strength of brittle materials. *Journal of Materials Science*, *32*, 2559–2564.

Habibullah, M., & Ahsanullah, M. (2000). Estimation of parameters of a Pareto distribution by generalized order statistics. *Commun. Stat. Theory Methods*, *29*, 1597–1609.

Joshi, P. C., & Balakrishnan, N. (1982). Moments of order statistics from doubly truncated Pareto distribution. *Journal of Indian Statistical Association*, *20*, 109–117.

Jaheen, Z. F. (2002). On Bayesian prediction of generalized order statistics. *J. Stat. Theory Appl.*, *1*, 191–204.

Jaheen, Z. F. (2005). Estimation based on generalized order statistics from the Burr model. *Commun. Stat. Theory Methods*, *34*, 785–794.

Kamps, U. (1995). *A Concept of Generalized Order Statistics*. B.G. Teubner Stuttgart.

Kamps, U., & Gather, U. (1997). Characteristic property of generalized order statistics for exponential distributions. *Appl. Math. (Warsaw)*, *24*, 383–391.

Keseling, C. (1999). Conditional distributions of generalized order statistics and some characterizations. *Metrika*, *49*, 27–40.

Kumar, D. (2013). On moments of lower generalized order statistics from exponentiated lomax distribution and characterization. *American Journal of Mathematical and Management Sciences*, *32*, 238–256.

Kumar, D. (2014). Moment generating functions of generalized order statistics from extended type II generalized logistic distribution. *Journal of Statistical Theory and Applications*, *13*, 273–288.

Kumar, D. (2015a). Exact moments of generalized order statistics from type II exponentiated log-logistic distribution. *Hacettepe Journal of Mathematics and Statistics*, *44*, 715–733.

Kumar, D. (2015b). Lower generalized order statistics based on inverse burr distribution. *American Journal of Mathematical and Management Sciences*, *35*, 15–35.

Kumar, D., Dey, S., & Nadarajah, S. (2017). Extended exponential distribution based on order statistics. *Communications in Statistics - Theory and Methods*, *86*(18), 9166–9184.

Lemonte, J. A. (2013). A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. *Computational Statistics and Data Analysis*, *62*, 149–170.

Nadarajah, S., & Haghighi, F. (2011). An extension of the exponential distribution. *Statistics*, *45*, 543–558.

Nikulin, M., & Haghighi, F. (2006). A chi-squared test for the generalized power Weibull family for the head-and-neck cancer censored data. *Journal of Mathematical Sciences*, *133*, 1333–1341.

Pawlas, P., & Szynal, D. (2001). Recurrence relations for single and product moments of generalized order statistics from Pareto and Burr distributions. *Commun. Stat. Theory Methods*, *30*, 739–746.

Raqab, M. Z. (2001). Optimal prediction-intervals for the exponential distribution, based on generalized order statistics. *IEEE. Trans. Reliab.*, *50*, 112–115.