Research Article Open Access

Arun Kajla and Serkan Araci*

Blending type approximation by Stancu-Kantorovich operators based on Pólya-Eggenberger distribution

DOI 10.1515/phys-2017-0037
Received Jan 21, 2017; accepted Mar 13, 2017

Abstract: In the paper the authors introduce the Kantorovich variant of Stancu operators based on Pólya-Eggenberger distribution. By making use of this new operator, we obtain some indispensable auxiliary results. We also deal with a Voronovskaja type asymptotic formula and some estimates of the rate of approximation involving modulus of smoothness, such as Ditzian-Totik modulus of smoothness. The rate of convergence for differential functions whose derivatives are bounded is also obtained.

Keywords: Pólya-Eggenberger distribution, Stancu operators, Euler functions, rate of convergence

PACS: 02.30.Mv, 02.30.Tb

1 Introduction

In the year 1923, Eggenberger and Pólya introduced originally Pólya-Eggenberger urn model to study processes such as the spread of contagious diseases. In one of its simplest form, the Pólya-Eggenberger urn model contains w white balls and b black balls. A ball is drawn at random and then replaced together with s balls of the same color. This procedure is repeated n times and noting the distribution of the random variable X representing the number of times a white ball is drawn (see [13]).

The distribution of X is given by

$$Pr(X=k)= \binom{n}{k}\frac{w(w+s)\cdot\ldots\cdot(w+\overline{k-1}s)b(b+s)\cdot\ldots\cdot(b+\overline{n-k-1}s)}{(w+b)(w+b+s)\cdot\ldots\cdot(w+b+\overline{n-1}s)}, \tag{1}$$

for $k=0,1,\cdots,n$ and $\overline{k-1}s=(k-1)s$. The distribution (1) is known as Pólya-Eggenberger distribution with parameters (n,w,b,s) and contains binomial, respectively hypergeometric distribution as particular cases, cf. [8].

Using (1), Stancu [25] constructed a new class of linear positive operators associated to a real-valued function $f:[0,1]\to\mathbb{R}$ as follows.

$$\begin{aligned} P_n^{[\alpha]}(f;x) &= \sum_{k=0}^{n} p_{n,k}^{[\alpha]}(x) f\left(\frac{k}{n}\right) \\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{\prod_{\nu=0}^{k-1}(x+\nu\alpha)\prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)}{(1+\alpha)(1+2\alpha)\cdots(1+(n-1)\alpha)} f\left(\frac{k}{n}\right), \end{aligned} \tag{2}$$

where $p_{n,k}^{[\alpha]}$ are usual Stancu polynomials and α is a non-negative parameter which may depend only on the natural number n. In the case when $\alpha=0$ operators (2) reduce to the known Bernstein operators [6] and for $\alpha=\frac{1}{n}$ we have

$$\begin{aligned} P_n^{[\frac{1}{n}]}(f;x) &= \sum_{k=0}^{n} p_{n,k}^{[\frac{1}{n}]}(x) f\left(\frac{k}{n}\right) \\ &= \frac{2(n!)}{(2n)!}\sum_{k=0}^{n}\binom{n}{k}\prod_{\nu=0}^{k-1}(nx+\nu)\prod_{\mu=0}^{n-k-1}(n-nx+\mu) f\left(\frac{k}{n}\right), \end{aligned} \tag{3}$$

given in [17]. Further information about the applications of (2) and (3), one can refer two recent papers [18], [19]. Taking into account the period in which the Stancu operators (2) were introduced, we remark that there exists a huge interest to study them. Some representative examples in this sense could be the papers of Razi [24], Finta [11], [12], Wang et al. [26], Abel et al. [1], Agrawal et al. [2], [3], [4], [5], Gupta et al. [15], [7], [16] and Deo et al [8].

For $\rho>0$, Özarslan and Duman [21] introduced a sequence of modified Bernstein-Kantorovich operators as follows:

$$K_{n,\rho}(f;x)=\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 f\left(\frac{k+t^{\rho}}{n+1}\right)dt \quad (x\in I), \tag{4}$$

known as modified Bernstein-Kantorovich operators, in which $p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$.

*Corresponding Author: Serkan Araci: Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, TR-27410 Gaziantep, Turkey; Email: mtsrkn@hotmail.com
Arun Kajla: Department of Mathematics, Central University of Haryana, Haryana-123029, India; Email: rachitkajla47@gmail.com

Motivated by above articles, for $f \in C[0,1]$, let us introduce

$$K_{n,\rho}^{[\alpha]}(f;x) = \sum_{k=0}^{n} p_{n,k}^{[\alpha]}(x) \int_0^1 f\left(\frac{k+t^\rho}{n+1}\right) dt.$$

We now call it as the Stancu-Kantorovich type operators arising from Pólya-Eggenberger distribution, where $\rho > 0$, $p_{n,k}^{[\alpha]}(x) = \binom{n}{k} \frac{1}{1^{[n,-\alpha]}} x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}$ are the known Stancu's fundamental polynomials and $t^{[n,h]} = t(t-h) \cdot \ldots \cdot (t-(n-1)h)$.

The aim of this paper is to introduce a new Kantorovich type modification of Stancu operators based on P´olya-Eggenberger distribution. For these new operators some indispensable auxiliary results are obtained in the second section. Our further study focuses on the qualitative part of these new operators involving the uniform convergence and asymptotic behavior. In order to get the degree of approximation, some quantitative theorems will be established. We are motivated to write this paper from Özarslan and Duman's paper [21].

2 Auxiliary results

Throughout of the paper, we make use of the notations: $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The monomials $e_k(x) = x^k$, for $k \in \mathbb{N}_0$ also called test functions play a key role in uniform approximation arising from linear positive operators. From (4) we present a useful form of these operators.

Lemma 1. *For $\rho > 0$, $\alpha > 0$ and $x \in (0,1)$, we get*

$$K_{n,\rho}^{[\alpha]}(f;x) = \frac{1}{\beta\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} K_{n,\rho}(f;t)dt,$$

where $K_{n,\rho}f$ are defined by (4).

Proof. Using the relationship between Euler's functions

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

in which $\Gamma(r)$ is usual Gamma function given by

$$\Gamma(r) = \int_0^\infty u^{r-1}e^{-u}du, \quad (r>0),$$

and, for $n \in \mathbb{N}$, it satisfies the following relation

$$\Gamma(r+n) = r(r+1) \cdot \ldots \cdot (r+n-1)\Gamma(r)$$

thus we get

$$\beta\left(\frac{x}{\alpha}+k, \frac{1-x}{\alpha}+n-k\right) = \frac{\Gamma\left(\frac{x}{\alpha}+k\right)\Gamma\left(\frac{1-x}{\alpha}+n-k\right)}{\Gamma\left(\frac{1}{\alpha}+n\right)}$$
$$= p_{n,k}^{[\alpha]}(x)\binom{n}{k}^{-1} \beta\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right).$$

Since

$$p_{n,k}^{[\alpha]}(x) = \binom{n}{k}\left(\beta\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)\right)^{-1} \beta\left(\frac{x}{\alpha}+k, \frac{1-x}{\alpha}+n-k\right)$$

we readily see that

$$K_{n,\rho}^{[\alpha]}(f;x)$$
$$= \sum_{k=0}^{n} \binom{n}{k} \frac{\beta\left(\frac{x}{\alpha}+k, \frac{1-x}{\alpha}+n-k\right)}{\beta\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \int_0^1 f\left(\frac{k+s^\rho}{n+1}\right) ds$$
$$= \frac{1}{\beta\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \left(\sum_{k=0}^{n} \binom{n}{k} \int_0^1 t^{\frac{x}{\alpha}+k-1}(1-t)^{\frac{1-x}{\alpha}+n-k-1} dt \right.$$
$$\left. \times \int_0^1 f\left(\frac{k+s^\rho}{n+1}\right) ds\right)$$
$$= \frac{1}{\beta\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} K_{n,\rho}(f;t)dt.$$

□

Below, we present four results involving Stancu-Kantorovich type operators (4) without proof. The images of the test functions $e_k(x) = x^k$, for $k \in \mathbb{N}_0$ by operators (4) are given in the following Lemma 2.

Lemma 2. *For the Stancu-Kantorovich type operators hold*
$K_{n,\rho}^{[\alpha]}(e_0;x) = 1$; $K_{n,\rho}^{[\alpha]}(e_1;x) = \frac{nx}{n+1} + \frac{1}{(n+1)(1+\rho)}$;
$K_{n,\rho}^{[\alpha]}(e_2;x) = \frac{x^2 n(n-1)}{(1+\alpha)(1+n)^2} + \frac{nx(3+\rho+\alpha(2+n+n\rho))}{(1+\rho)(1+\alpha)(1+n)^2} + \frac{1}{(1+n)^2(1+2\rho)}$;
$K_{n,\rho}^{[\alpha]}(e_3;x) = \frac{n(n-1)(n-2)x(x+\alpha)(x+2\alpha)}{(1+\alpha)(1+2\alpha)(1+n)^3} + \frac{3n(n-1)(2+\rho))x(x+\alpha)}{(1+\alpha)(1+\rho)(1+n)^3} + \frac{nx(7+2\rho(6+\rho))}{(1+\rho)(1+2\rho)(1+n)^3} + \frac{1}{(1+3\rho)(1+n)^3}$;
$K_{n,\rho}^{[\alpha]}(e_4;x) = \frac{n(n-1)(n-2)(n-3)x^4}{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+n)^4}$
$+ \frac{2n(n-1)(n-2)x^3(5+3\rho+3\alpha(2+n+n\rho))}{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+\rho)(1+n)^4}$
$\frac{n(n-1)x^2(25+\rho(51+14\rho)+\alpha(65+(99-2\rho)\rho+6n(1+2\rho)(5+3\rho))}{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+\rho)(1+2\rho)(1+n)^4}$
$+ \frac{\alpha^2(36(1+\rho)+n(1+2\rho)(35-\rho+11n(1+\rho))))}{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+\rho)(1+2\rho)(1+n)^4}$
$+ nx\left(\frac{(1+n\alpha)(1+\alpha(-1+6n(1+n\alpha)))}{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+n)^4} + \frac{4(1+n\alpha)(1+2n\alpha)}{(1+\alpha)(1+2\alpha)(1+\rho)(1+n)^4}\right.$
$\left. + \frac{4}{(1+3\rho)(1+n)^4} + \frac{6(1+n\alpha)}{(1+\alpha)(1+2\rho)(1+n)^4}\right) + \frac{1}{(1+4\rho)(1+n)^4}$.

From the expression of (4), for brevity we will write in the sequel $\mathcal{P}_{n,\rho,r}^{[\alpha]}(x) := K_{n,\rho}^{[\alpha]}\left((e_1-x)^r;x\right)$, where $n \geq 1$, $r \geq 0$ and $x \in [0,1]$.

Lemma 3. *For the Stancu-Kantorovich type operators hold* $\mathcal{P}^{[\alpha]}_{n,\rho,1}(x) = \frac{-x}{n+1} + \frac{1}{(n+1)(1+\rho)}$; $\mathcal{P}^{[\alpha]}_{n,\rho,2}(x) = \frac{(1-n)(1+\alpha+n\alpha)x^2}{(1+\alpha)(1+n)^2} + \frac{x}{(n+1)^2}\left(\frac{n(1+n\alpha)}{(1+\alpha)} - \frac{2}{(1+\rho)}\right) + \frac{1}{(n+1)^2(1+2\rho)}$.

Lemma 4. *For any* $n \in \mathbb{N}$*, we have*

$$\mathcal{P}^{[\alpha]}_{n,\rho,2}(x) = K^{[\alpha]}_{n,\rho}((e_1 - x)^2; x) \leq \frac{\mathcal{D}^{[\alpha]}_{\rho}\ x(1-x)}{(1+n)},$$

where $\mathcal{D}^{[\alpha]}_{\rho}$ *is a positive fixed based on* ρ *and* α.

Since α is a non-negative parameter which may depend only on the natural number n, we state the following Lemma.

Lemma 5. *If* $\alpha \to 0$ *as* $n \to \infty$ *and* $\lim_{n\to\infty} n\alpha = c \in \mathbb{R}$*, then*

$$\lim_{n\to\infty} n \cdot \mathcal{P}^{[\alpha]}_{n,\rho,1}(x) = -x + \frac{1}{1+\rho},$$

$$\lim_{n\to\infty} n \cdot \mathcal{P}^{[\alpha]}_{n,\rho,2}(x) = (1+c)x(1-x),$$

$$\lim_{n\to\infty} n^2 \cdot \mathcal{P}^{[\alpha]}_{n,\rho,4}(x) = 3(1+c)^2x^2(1-x)^2.$$

3 Main results

Our studies focuses on the qualitative part of Stancu-Kantorovich type operators, involving the uniform convergence and asymptotic behavior.

Theorem 1. *Let* $f \in C[0,1]$ *and* $\alpha \in \mathbb{N}_0$ *depending on* $n \in \mathbb{N}$*, with* $\alpha \to 0$*, as* $n \to \infty$*, then* $\lim_{n\to\infty} K^{[\alpha]}_{n,\rho}(f;x) = f(x)$ *uniformly on* $[0,1]$.

The next result provides a Voronovskaja type result for the Stancu-Kantorovich type operators.

Theorem 2. *Let* $f : [0,1] \to \mathbb{R}$*,* $\alpha \to 0$ *as* $n \to \infty$ *and* $\lim_{n\to\infty} n\alpha = c \in \mathbb{R}$*. If* $f \in C^2[0,1]$*, then*

$$\lim_{n\to\infty} n\left(K^{[\alpha]}_{n,\rho}(f;x) - f(x)\right) = \left(-x + \frac{1}{1+\rho}\right)f'(x) + \frac{(1+c)x(1-x)}{2}f''(x).$$

Proof. In order to prove this theorem, we first use Taylor's expansion formula for a function f, as follows.

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \varpi(t,x)(t-x)^2, \tag{5}$$

where $\varpi(t,x) := \varpi(t-x)$ is a bounded function and $\lim_{t\to x}\varpi(t,x) = 0$. By the linearity of Stancu-Kantorovich type operators, and then applying the operators $K^{[\alpha]}_{n,\rho}$ to the both side of above equation (5), we derive

$$K^{[\alpha]}_{n,\rho}(f;x) - f(x) = K^{[\alpha]}_{n,\rho}((e_1 - x);x)f'(x) + \frac{1}{2}K^{[\alpha]}_{n,\rho}\left((e_1-x)^2;x\right)f''(x) + K^{[\alpha]}_{n,\rho}\left(\varpi(t,x)\cdot(e_1-x)^2;x\right).$$

Further, from Lemma 3, we obtain

$$\lim_{n\to\infty} n\left(K^{[\alpha]}_{n,\rho}(f;x) - f(x)\right) = \left(-x + \frac{1}{1+\rho}\right)f'(x) + \frac{(1+c)x(1-x)}{2}f''(x) + \lim_{n\to\infty} n\left(K^{[\alpha]}_{n,\rho}\left(\varpi(t,x)\cdot(e_1-x)^2;x\right)\right). \tag{6}$$

Thus, we can estimate the last term on the right-hand side of the above equality, applying the Cauchy-Schwarz inequality, that is:

$$K^{[\alpha]}_{n,\rho}\left(\varpi(t,x)\cdot(e_1-x)^2;x\right) \leq \sqrt{K^{[\alpha]}_{n,\rho}\left(\varpi^2(t,x);x\right)}\sqrt{K^{[\alpha]}_{n,\rho}\left((e_1-x)^4;x\right)}. \tag{7}$$

Since $\varpi^2(x,x) = 0$ and $\varpi^2(\cdot,x) \in C[0,1]$, by Theorem 1, we get

$$\lim_{n\to\infty} K^{[\alpha]}_{n,\rho}\left(\varpi^2(t,x);x\right) = \varpi^2(x,x) = 0. \tag{8}$$

Therefore, taking Lemma 5 into account and from (7), and (8) yields

$$\lim_{n\to\infty} n\left(K^{[\alpha]}_{n,\rho}\left(\varpi(t,x)\cdot(e_1-x)^2;x\right)\right) = 0$$

and using (6), we arrive at the desired result (4). □

The main tools to measure the degree of approximation of linear positive operators towards the identity operators are moduli of smoothness. For $f \in C[0,1]$ and $\delta \geq 0$ we know the definition of the moduli of smoothness of first, and second order, given by

$$\omega_1(f,\delta) := \sup\{|f(x+h) - f(x)| : x, x+h \in [0,1],\ 0 \leq h \leq \delta\}$$

and

$$\omega_2(f,\delta) := \sup\{|f(x+h) - 2f(x) + f(x-h)| : x, x\pm h \in [0,1],\ 0 \leq h \leq \delta\}$$

respectively.

Definition 1. Let $f \in C_B[0,1]$ be the space of all real-valued functions continuous and bounded on $[0,1]$ endowed with the norm $\|f\| = \sup_{x\in[0,1]}|f(x)|$ and let us consider Peetre's K-functional

$$K_2(f,\delta) = \inf\{\|f-g\| + \delta\|g''\| \, : \, g \in C^2[0,1]\}, \quad (9)$$

or $\delta > 0$.

There exists an absolute fixed $M > 0$, such that

$$K_2(f,\delta) \le M \cdot \omega_2\left(f, \sqrt{\delta}\right), \quad (10)$$

conformable ([9], p. 177, Theorem 2.4).

Proposition 1. *Let f be a real-valued function continuous and bounded on* $[0,1]$, *with* $\|f\| = \sup_{x\in[0,1]}|f(x)|$, *then*

$$\left|K_{n,\rho}^{[\alpha]}(f;x)\right| \le \|f\|.$$

Proof. It is proved by making use of the definition of Stancu-Kantorovich type operators and Lemma 2, as follows.

$$\left|K_{n,\rho}^{[\alpha]}(f;x)\right| = \left|\sum_{k=0}^{n} p_{n,k}^{[\alpha]}(x)\int_0^1 f\left(\frac{k+t^\rho}{n+1}\right)dt\right| \le \|f\| \cdot K_{n,\rho}^{[\alpha]}(e_0;x) = \|f\|.$$

□

Theorem 3. *For* $f \in C[0,1]$ *and* $x \in [0,1]$. *Then, there exists a constant* $M > 0$ *such that*

$$|\, K_{n,\rho}^{[\alpha]}(f;x) - f(x)\,| \le M\omega_2\left(f, (n+1)^{-1/2}\delta_n(x)\right) + \omega\left(f, \left|\frac{-x}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right|\right),$$

where $\delta_{n,\rho}^2(x) = \frac{\mathcal{D}_\rho^{[\alpha]}\; x(1-x)}{(1+n)} + \left(\frac{-x}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right)^2$.

Proof. We first consider the auxiliary operator in this form:

$$T_{n,\rho}^{[\alpha]}(f;x) = K_{n,\rho}^{[\alpha]}(f;x) + f(x) - f\left(\frac{nx}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right).$$

Then, by Corollary 3, it becomes

$$T_{n,\rho}^{[\alpha]}(1;x) = K_{n,\rho}^{[\alpha]}(1;x) = 1$$

and

$$T_{n,\rho}^{[\alpha]}(t;x) = K_{n,\rho}^{[\alpha]}(t;x) + x - \left(\frac{nx}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right) = x.$$

Let $g \in C^2[0,1]$ and $t \in [0,1]$. Applying Taylor's expansion we derive

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Using the operator $T_{n,\rho}^{[\alpha]}$ on both sides of the above equation, we have

$$\begin{aligned}
T_{n,\rho}^{[\alpha]}(g;x) &= g(x) + T_{n,\rho}^{[\alpha]}\left(\int_x^t (t-u)g''(u)du\right) \\
&= g(x) + K_{n,\rho}^{[\alpha]}\left(\int_x^t (t-u)g''(u)du, x\right) \\
&\quad - \int_x^{\frac{nx}{(n+1)}+\frac{1}{(n+1)(1+\rho)}} \left(\frac{nx}{(n+1)} + \frac{1}{(n+1)(1+\rho)} - u\right)g''(u)du.
\end{aligned}$$

Hence

$$\begin{aligned}
|\, T_{n,\rho}^{[\alpha]}(g;x) - g(x)\,| &\le K_{n,\rho}^{[\alpha]}\left(\left|\int_x^t |t-u||g''(u)|du\right|, x\right) \\
&\quad + \left|\int_x^{\frac{nx}{(n+1)}+\frac{1}{(n+1)(1+\rho)}} \left|\frac{nx}{(n+1)} + \frac{1}{(n+1)(1+\rho)} - u\right||g''(u)|du\right| \\
&\le \left\{K_{n,\rho}^{[\alpha]}((t-x)^2;x) + \left(\frac{nx}{(n+1)} + \frac{1}{(n+1)(1+\rho)} - x\right)^2\right\}||g''|| \\
&= \left\{K_{n,\rho}^{[\alpha]}((t-x)^2;x) + \left(\frac{-x}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right)^2\right\}||g''||.
\end{aligned}$$

From Lemma 4, we get

$$|\, T_{n,\rho}^{[\alpha]}(g;x) - g(x)\,| \le \frac{\mathcal{D}_\rho^{[\alpha]}\; x(1-x)}{(1+n)} + \left(\frac{-x}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right)^2$$

Hence

$$|\, T_{n,\rho}^{[\alpha]}(g;x) - g(x)\,| \le \delta_{n,\rho}^2(x)||g''||,$$

where $\delta_{n,\rho}^2(x) = \frac{\mathcal{D}_\rho^{[\alpha]}\; x(1-x)}{(1+n)} + \left(\frac{-x}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right)^2$. In view of Proposition 1, we have

$$|\, T_{n,\rho}^{[\alpha]}(f;x)\,| \le 3||f||,$$

for all $f \in C[0,1]$.

Now, for $f \in C[0,1]$ and $g \in C^2[0,1]$, we get

$$\begin{aligned}
&|\, K_{n,\rho}^{[\alpha]}(f;x) - f(x)\,| \\
&\le |\, T_{n,\rho}^{[\alpha]}(f;x) - f(x) + f\left(\frac{nx}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right) - f(x)\,| \\
&\le |T_{n,\rho}^{[\alpha]}(f-g;x)| + |T_{n,\rho}^{[\alpha]}(g;x) - g(x)| + |g(x) - f(x)| \\
&\quad + \left|f\left(\frac{nx}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right) - f(x)\right| \\
&\le 4||f-g|| + \frac{M}{n+1}\delta_n^2(x)||g''|| + \omega\left(f, \left|\frac{-x}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right|\right).
\end{aligned}$$

Taking the infimum on the right hand side over all $g \in C^2[0,1]$, we obtain

$$| K_{n,\rho}^{[\alpha]}(f;x) - f(x) | \leq 4K_2\left(f, \frac{1}{n+1}\delta_n^2(x)\right) + \omega\left(f, \left|\frac{-x}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right|\right),$$

and by the inequality (10), we get

$$| K_{n,\rho}^{[\alpha]}(f;x) - f(x) | \leq M\omega_2\left(f, (n+1)^{-1/2}\delta_n(x)\right) + \omega\left(f, \left|\frac{-x}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right|\right),$$

which completes the proof. □

4 Global approximation

Let $f \in C[0,1]$ and $\phi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. The second order Ditzian-Totik Modulus of smoothness and corresponding K- functional are given by, respectively,

$$\omega_2^{\phi}(f, \sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x\pm h\phi(x)\in[0,1]} | f(x+h\phi(x)) - 2f(x) + f(x-h\phi(x)) |,$$

$$\tilde{K}_{2,\phi(x)}(f,\delta) = \inf\{||f-g|| + \delta||\phi^2 g''|| + \delta^2||g''|| : g \in W^2(\phi)\}, (\delta > 0),$$

where $W^2(\phi) = \{g \in C[0,1] : g' \in AC_{loc}[0,1], \phi^2 g'' \in C[0,1]\}$ and $g' \in AC_{loc}[0,1]$ means that g is differentiable and g' is absolutely continuous on every closed interval $[a,b] \subset (0,1)$. It is known ([10],Theorem 1.3.1) that there exists a positive constant $C > 0$, such that

$$\tilde{K}_{2,\phi(x)}(f,\delta) \leq C\omega_2^{\phi}(f,\sqrt{\delta}). \tag{11}$$

Also, the Ditzian -Totik moduli of first order is given by

$$\overrightarrow{\omega_{\psi}}(f,\delta) = \sup_{0<h\leq\delta} \sup_{x\pm\frac{h}{2}\psi(x)\in[0,1]} \left| f\left(x+\frac{h}{2}\psi(x)\right) - f\left(x-\frac{h}{2}\psi(x)\right)\right|$$

where ψ is an admissible step -weight function on $[0,1]$.

Theorem 4. *Let $f \in C[0,1]$. Then, for $x \in [0.1]$,*

$$||K_{n,\rho}^{[\alpha]}f - f|| \leq C\omega_2^{\phi}(f,(n+1)^{-1/2}) + \overrightarrow{\omega_{\psi_\rho}}\left(f,(n+1)^{-1}\right),$$

where $C > 0$ is an absolute constant, $\phi(x) = \sqrt{x(1-x)}$ and $\psi_\rho(x) = (\rho+1)x + 1$.

Proof. We introduce the auxiliary operators as follows:

$$T_{n,\rho}^{[\alpha]}(f;x) = K_{n,\rho}^{[\alpha]}(f;x) + f(x) - f\left(\frac{nx}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right).$$

Let $g \in W^2(\phi)$ then by using Taylor's expansion of g, on proceeding as in the proof of Theorem 3, we get

$$| T_{n,\rho}^{[\alpha]}(g;x) - g(x) | \leq K_{n,\rho}^{[\alpha]}\left(\left|\int_x^t |t-u||g''(u)|du\right|, x\right) + \int_x^{\frac{nx}{(n+1)}+\frac{1}{(n+1)(1+\rho)}} \left|\frac{nx}{(n+1)} + \frac{1}{(n+1)(1+\rho)} - u\right||g''(u)|du. \tag{12}$$

Now let,

$$\zeta_n^2(x) := x(1-x) + \frac{1}{(n+1)}.$$

Because the function ζ_n^2 is concave on $x \in [0,1]$, for $u = \lambda x + (1-\lambda)t$, $\lambda \in [0,1]$, we get

$$\frac{|t-u|}{\zeta_n^2(u)} = \frac{\lambda|t-x|}{\zeta_n^2(\lambda x + (1-\lambda)t)} \leq \frac{\lambda|t-x|}{\zeta_n^2(x)\lambda + \zeta_n^2(t)(1-\lambda)} \leq \frac{|t-x|}{\zeta_n^2(x)}.$$

Thus, the inequality (12) leads us to

$$\begin{aligned} | T_{n,\rho}^{[\alpha]}(g;x) - g(x) | &\leq K_{n,\rho}^{[\alpha]}\left(\left|\int_x^t \frac{|t-u|}{\zeta_n^2(u)}du\right|, x\right)||\zeta_n^2 g''|| \\ &+ \left(\int_x^{\frac{nx}{(n+1)}+\frac{1}{(n+1)(1+\rho)}} \frac{\left|\frac{nx}{(n+1)} + \frac{1}{(n+1)(1+\rho)} - u\right|}{\zeta_n^2(u)}du\right)||\zeta_n^2 g''||. \\ &\leq \frac{1}{\zeta_n^2(x)}||\zeta_n^2 g''|| \left[K_{n,\rho}^{[\alpha]}((t-x)^2;x) + \left(\frac{-x}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right)^2\right]. \end{aligned} \tag{13}$$

Since

$$| T_{n,\rho}^{[\alpha]}(g;x) - g(x) | \leq \frac{C}{n+1}||\zeta_n^2 g''|| \leq \frac{C}{n+1}\left(||\phi^2 g''|| + \frac{1}{n+1}||g''||\right).$$

Using (13), we have for $f \in C[0,1]$,

$$\begin{aligned} | K_{n,\rho}^{[\alpha]}(f;x) - f(x) | \leq& | T_{n,\rho}^{[\alpha]}(f-g,x) | \\ &+ | T_{n,\rho}^{[\alpha]}(g;x) - g(x) | + | g(x) - f(x) | \\ &+ \left|f\left(\frac{nx}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right) - f(x)\right| \end{aligned}$$

$$\leq 4||f-g|| + \frac{C}{n+1}||\phi^2 g''|| + \frac{C}{(n+1)^2}||g''|| + \left|f\left(\frac{nx}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right) - f(x)\right|$$

Taking the infimum on the right hand side over all $g \in W^2(\phi)$, we obtain

$$|K_{n,\rho}^{[\alpha]}(f;x) - f(x)| \leq C\tilde{K}_{2,\phi}\left(f, \frac{1}{(n+1)}\right) + \left|f\left(\frac{nx}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right) - f(x)\right|. \quad (14)$$

On the other hand,

$$\begin{aligned}
&\left|f\left(\frac{nx}{(n+1)} + \frac{1}{(n+1)(1+\rho)}\right) - f(x)\right| \quad (15)\\
&= \left|f\left(x + \psi_\rho(x)\frac{\frac{nx}{(n+1)} + \frac{1}{(n+1)(1+\rho)} - x}{\psi_\rho(x)}\right) - f(x)\right|\\
&\leq \sup_{t\in[0,1]}\left|f\left(t + \psi_\rho(x)\frac{\frac{1}{(1+\rho)} - x}{(n+1)\psi_\rho(x)}\right) - f(t)\right|\\
&\leq \overrightarrow{\omega_{\psi_\rho}}\left(f, \frac{\left|\frac{1}{(1+\rho)} - x\right|}{(n+1)\psi_\rho(x)}\right) \leq \overrightarrow{\omega_{\psi_\rho}}\left(f, \frac{1}{(n+1)}\right).
\end{aligned}$$

Hence, combining (11), (14) and (15), the desired result is immediate. □

Let us give the Lipschitz-type space with two parameters defined in [20]: For $a_1 \geq 0$, $a_2 > 0$ and $\eta \in (0,1]$,

$$Lip_M^{(a_1,a_2)}(\eta) := \left\{f \in C[0,1] : |f(t) - f(x)| \leq M\frac{|t-x|^\eta}{(t + a_1x^2 + a_2x)^{\frac{\eta}{2}}}; t \in [0,1], x \in (0,1]\right\},$$

where M is a positive constant.

Theorem 5. *If $f \in Lip_M^{(a_1,a_2)}(\eta)$ and $x \in (0,1]$, then we have*

$$\left|K_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| \leq M\left(\frac{\mathcal{P}_{n,\rho,2}^{[\alpha]}(x)}{a_1x^2 + a_2x}\right)^{\eta/2},$$

where $\mathcal{P}_{n,\rho,2}^{[\alpha]}(x)$ is given in Lemma 3.

Proof. Let we prove the theorem for the case $0 < \eta \leq 1$, applying Holder's inequality with $p = \frac{2}{\eta}$, $q = \frac{2}{2-\eta}$

$$\begin{aligned}
&\left|K_{n,\rho}^{[\alpha]}(f;x) - f(x)\right|\\
&\leq \sum_{k=0}^n p_{n,k}^{[\alpha]}(x)\int_0^1\left|f\left(\frac{k+t^\rho}{n+1}\right) - f(x)\right|dt\\
&\leq \sum_{k=0}^n p_{n,k}^{[\alpha]}(x)\left(\int_0^1\left|f\left(\frac{k+t^\rho}{n+1}\right) - f(x)\right|^{\frac{2}{\eta}}dt\right)^{\frac{\eta}{2}}\\
&\leq \left\{\sum_{k=0}^n p_{n,k}^{[\alpha]}(x)\int_0^1\left|f\left(\frac{k+t^\rho}{n+1}\right) - f(x)\right|^{\frac{2}{\eta}}dt\right\}^{\frac{\eta}{2}} \times \left(\sum_{k=0}^n p_{n,k}^{[\alpha]}(x)\right)^{\frac{2-\eta}{2}}\\
&= \left(\sum_{k=0}^n p_{n,k}^{[\alpha]}(x)\int_0^1\left|f\left(\frac{k+t^\rho}{n+1}\right) - f(x)\right|^{\frac{2}{\eta}}dt\right)^{\frac{\eta}{2}}\\
&\leq M\left(\sum_{k=0}^n p_{n,k}^{[\alpha]}(x)\int_0^1\frac{\left(\frac{k+t^\rho}{n+1} - x\right)^2}{\left(\frac{k+t^\rho}{n+1} + a_1x^2 + a_2x\right)}dt\right)^{\frac{\eta}{2}}\\
&\leq \frac{M}{(a_1x^2 + a_2x)^{\frac{\eta}{2}}}\left(\sum_{k=0}^n p_{n,k}^{[\alpha]}(x)\int_0^1\left(\frac{k+t^\rho}{n+1} - x\right)^2dt\right)^{\frac{\eta}{2}}\\
&= \frac{M}{(a_1x^2 + a_2x)^{\frac{\eta}{2}}}K_{n,\rho}^{[\alpha]}((t-x)^2;x)^{\frac{\eta}{2}}\\
&= \frac{M}{(a_1x^2 + a_2x)^{\frac{\eta}{2}}}(\mathcal{P}_{n,\rho,2}^{[\alpha]}(x))^{\frac{\eta}{2}}.
\end{aligned}$$

Therefore, the proof is completed. □

We establish the rate of convergence for differential functions whose derivatives are of bounded variation on [0, 1]. Let $DBV[0,1]$ be the class of differentiable functions f defined on [0, 1], whose derivatives f' are of bounded variation on [0, 1]. The functions $f \in DBV[0,1]$ could be represented

$$f(x) = \int_0^x g(t)dt + f(0),$$

where $g \in BV[0,1]$, which means that g is a function of bounded variation on [0, 1]. Also, the operators $K_{n,\rho}^{[\alpha]}f$ admit the integral representation

$$K_{n,\rho}^{[\alpha]}(f;x) = \int_0^1 \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)f(t)dt, \quad (16)$$

where the kernel $\mathcal{S}_{n,\rho}^{[\alpha]}$ is given by

$$\mathcal{S}_{n,\rho}^{[\alpha]}(x,t) = \sum_{k=0}^n p_{n,k}^{[\alpha]}(x)\chi_{n,k}^\rho(t),$$

where $\chi_{n,k}^\rho(t)$ is the characteristic function of the interval $[k/(n+1),(k+1)/(n+1)]$ with respect to [0, 1].

Lemma 6. *Let α be a non-negative parameter which may depend on $n \in \mathbb{N}$, with $\alpha \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} n\alpha = c \in \mathbb{R}$. For a fixed $x \in (0,1)$ and sufficiently large n, it follows*

i) $\lambda_{n,\rho}^{[\alpha]}(x,y) = \int_0^y \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt \le \frac{\mathcal{D}_\rho^{[\alpha]}}{(1+n)}\frac{x(1-x)}{(x-y)^2}$, $0 \le y < x$;

ii) $1 - \lambda_{n,\rho}^{[\alpha]}(x,z) = \int_z^1 \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt \le \frac{\mathcal{D}_\rho^{[\alpha]}}{(1+n)}\frac{x(1-x)}{(z-x)^2}$, $x < z < 1$.

Proof.

i) Using Lemma 4, we get

$$\begin{aligned}\lambda_{n,\rho}^{[\alpha]}(x,y) &= \int_0^y \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt \le \int_0^y \left(\frac{x-t}{x-y}\right)^2 \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt \\ &= \frac{1}{(x-y)^2}\cdot K_{n,\rho}^{[\alpha]}\left((e_1-x)^2;x\right) \\ &\le \frac{\mathcal{D}_\rho^{[\alpha]}}{(1+n)}\frac{x(1-x)}{(x-y)^2}.\end{aligned}$$

ii) The proof is immediately, hence the details are omitted. □

Theorem 6. *Let $f \in DBV[0,1]$, $\alpha \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} n\alpha = c \in \mathbb{R}$. Then for every $x \in (0,1)$ and sufficiently large n, we have*

$$\begin{aligned}\left|K_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| &\le \left|-x + \frac{1}{1+\rho}\right| \frac{|f'(x+)+f'(x-)|}{2} \\ &+ \sqrt{\frac{\mathcal{D}_\rho^{[\alpha]}x(1-x)}{(1+n)}}\frac{|f'(x+)-f'(x-)|}{2} \\ &+ \frac{\mathcal{D}_\rho^{[\alpha]}(1-x)}{(1+n)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^{x}(f_x') + \frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f_x') \\ &+ \frac{\mathcal{D}_\rho^{[\alpha]}x}{(1+n)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+((1-x)/k)}(f_x') + \frac{(1-x)}{\sqrt{n}}\bigvee_{x}^{x+((1-x)/\sqrt{n})}(f_x'),\end{aligned}$$

where $\bigvee_a^b(f_x')$ denotes the total variation of f_x' on $[a,b]$ and f_x' is defined by

$$f_x'(t) = \begin{cases} f'(t) - f'(x-), & 0 \le t < x \\ 0, & t = x \\ f'(t) - f'(x+) & x < t < 1. \end{cases} \tag{17}$$

Proof. The Stancu-Durrmeyer type operators preserve constants and using (16), for every $x \in (0,1)$ we have

$$K_{n,\rho}^{[\alpha]}(f;x) - f(x) = \int_0^1 \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)(f(t)-f(x))dt \tag{18}$$

$$= \int_0^1 \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)\int_x^t f'(u)dudt.$$

For any $f \in DBV[0,1]$, from (17) we may write

$$\begin{aligned}f'(u) &= f_x'(u) + \frac{f'(x+)+f'(x-)}{2} \\ &+ \frac{f'(x+)-f'(x-)}{2}sgn(u-x) \\ &+ \delta_x(u)\left(f'(u) - \frac{f'(x+)+f'(x-)}{2}\right),\end{aligned} \tag{19}$$

where

$$\delta_x(u) = \begin{cases} 1, & u = x \\ 0, & u \ne x. \end{cases}$$

Obviously,

$$\int_0^1\left(\int_x^t\left(f'(u) - \frac{f'(x+)+f'(x-)}{2}\right)\delta_x(u)du\right)\mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt = 0$$

and

$$\begin{aligned}&\int_0^1\left(\int_x^t \frac{f'(x+)+f'(x-)}{2}du\right)\mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt \\ &= \frac{f'(x+)+f'(x-)}{2}\int_0^1 (t-x)\mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt \\ &= \frac{f'(x+)+f'(x-)}{2}\cdot K_{n,\rho}^{[\alpha]}(e_1-x;x).\end{aligned}$$

Applying Chauchy-Schwarz inequality for linear positive operators, it follows

$$\begin{aligned}&\left|\int_0^1 \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)\left(\int_x^t \frac{f'(x+)-f'(x-)}{2}sgn(u-x)du\right)dt\right| \\ &\le \frac{|f'(x+)-f'(x-)|}{2}\int_0^1 |t-x|\mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt \\ &\le \frac{|f'(x+)-f'(x-)|}{2}\cdot K_{n,\rho}^{[\alpha]}(|t-x|;x) \\ &\le \frac{|f'(x+)-f'(x-)|}{2}\left(K_{n,\rho}^{[\alpha]}((t-x)^2;x)\right)^{1/2}.\end{aligned}$$

Using Lemma 3, respectively Lemma 4 and the relations (18), (19) yields

$$\begin{aligned}&\left|K_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| \\ &\le \frac{|f'(x+)-f'(x-)|}{2}\sqrt{\frac{\mathcal{D}_\rho^{[\alpha]}x(1-x)}{(1+n)}} \\ &+ \left|\int_0^x\left(\int_x^t f_x'(u)du\right)\mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt\right. \\ &+ \left.\int_x^1\left(\int_x^t f_x'(u)du\right)\mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt\right|.\end{aligned} \tag{20}$$

Let be

$$\mathcal{G}_{n,\rho}^{[\alpha]}(f_x', x) = \int_0^x \left(\int_x^t f_x'(u)du \right) \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt,$$

$$\mathcal{F}_{n,\rho}^{[\alpha]}(f_x', x) = \int_x^1 \left(\int_x^t f_x'(u)du \right) \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt.$$

To complete the proof, it is sufficient to estimate $\mathcal{G}_{n,\rho}^{[\alpha]}$ and $\mathcal{F}_{n,\rho}^{[\alpha]}$. Since $\int_e^f d_t\lambda_{n,\rho}^{[\alpha]}(x,t) \le 1$ for all $[a,b] \subseteq [0,1]$, applying the integration formula by parts and using Lemma 6 with $y = x - (x/\sqrt{n})$, we may write

$$\begin{aligned}
\left|\mathcal{G}_{n,\rho}^{[\alpha]}(f_x', x)\right| &= \left| \int_0^x \left(\int_x^t f_x'(u)du \right) d_t\lambda_{n,\rho}^{[\alpha]}(x,t) \right| \\
&= \left| \int_0^x \lambda_{n,\rho}^{[\alpha]}(x,t) f_x'(t)dt \right| \\
&\le \left(\int_0^y + \int_y^x \right) |f_x'(t)| \cdot |\lambda_{n,\rho}^{[\alpha]}(x,t)|dt \\
&\le \frac{\mathcal{D}_\rho^{[\alpha]} x(1-x)}{(1+n)} \int_0^y \bigvee_t^x (f_x')(x-t)^{-2}dt \\
&\quad + \int_y^x \bigvee_t^x (f_x')dt \\
&\le \frac{\mathcal{D}_\rho^{[\alpha]} x(1-x)}{(1+n)} \int_0^y \bigvee_t^x (f_x')(x-t)^{-2}dt \\
&\quad + \frac{x}{\sqrt{n}} \bigvee_{x-(x/\sqrt{n})}^x (f_x').
\end{aligned}$$

By the substitution of $u = x/(x-t)$, we get

$$\begin{aligned}
&\frac{\mathcal{D}_\rho^{[\alpha]} x(1-x)}{(1+n)} \int_0^{x-(x/\sqrt{n})} (x-t)^{-2} \bigvee_t^x (f_x')dt \\
&= \frac{\mathcal{D}_\rho^{[\alpha]}(1-x)}{(1+n)} \int_1^{\sqrt{n}} \bigvee_{x-(x/u)}^x (f_x')du \\
&\le \frac{\mathcal{D}_\rho^{[\alpha]}(1-x)}{(1+n)} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_{x-(x/k)}^x (f_x')du \\
&\le \frac{\mathcal{D}_\rho^{[\alpha]}(1-x)}{(1+n)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-(x/k)}^x (f_x').
\end{aligned}$$

Thus

$$\left|\mathcal{G}_{n,\rho}^{[\alpha]}(f_x', x)\right| \le \frac{\mathcal{D}_\rho^{[\alpha]}(1-x)}{(1+n)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-(x/k)}^x (f_x') \quad (21)$$

$$+ \frac{x}{\sqrt{n}} \bigvee_{x-(x/\sqrt{n})}^x (f_x').$$

Using the integration formula by parts and applying Lemma 6 with $z = x + ((1-x)/\sqrt{n})$, we get

$$\begin{aligned}
\left|\mathcal{F}_{n,\rho}^{[\alpha]}(f_x', x)\right| &= \left| \int_x^1 \left(\int_x^t f_x'(u)du \right) \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt \right| \\
&= \left| \int_x^z \left(\int_x^t f_x'(u)du \right) d_t(1-\lambda_{n,\rho}^{[\alpha]}(x,t)) \right. \\
&\quad \left. + \int_z^1 \left(\int_x^t f_x'(u)du \right) d_t(1-\lambda_{n,\rho}^{[\alpha]}(x,t)) \right| \\
&= \left| \left[\int_x^t f_x'(u)(1-\lambda_{n,\rho}^{[\alpha]}(x,t))du \right]_x^z - \int_x^z f_x'(t)(1-\lambda_{n,\rho}^{[\alpha]}(x,t))dt \right. \\
&\quad \left. + \int_z^1 \left(\int_x^t f_x'(u)du \right) d_t(1-\lambda_{n,\rho}^{[\alpha]}(x,t)) \right| \\
&= \left| \int_x^z f_x'(u)du(1-\lambda_{n,\rho}^{[\alpha]}(x,z)) - \int_x^z f_x'(t)(1-\lambda_{n,\rho}^{[\alpha]}(x,t))dt \right. \\
&\quad + \left[\int_x^t f_x'(u)du(1-\lambda_{n,\rho}^{[\alpha]}(x,t)) \right]_z^1 \\
&\quad \left. - \int_z^1 f_x'(t)(1-\lambda_{n,\rho}^{[\alpha]}(x,t))dt \right| \\
&= \left| \int_x^z f_x'(t)(1-\lambda_{n,\rho}^{[\alpha]}(x,t))dt + \int_z^1 f_x'(t)(1-\lambda_{n,\rho}^{[\alpha]}(x,t))dt \right| \\
&\le \frac{\mathcal{D}_\rho^{[\alpha]} x(1-x)}{(1+n)} \int_z^1 \bigvee_x^t (f_x')(t-x)^{-2}dt + \int_x^z \bigvee_x^t (f_x')dt \\
&= \frac{\mathcal{D}_\rho^{[\alpha]} x(1-x)}{(1+n)} \int_{x+((1-x)/\sqrt{n})}^1 \bigvee_x^t (f_x')(t-x)^{-2}dt \\
&\quad + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x)/\sqrt{n})} (f_x').
\end{aligned}$$

By the substitution of $v = (1-x)/(t-x)$, we get

$$\begin{aligned}
&\left|\mathcal{F}_{n,\rho}^{[\alpha]}(f_x', x)\right| \quad (22) \\
&\le \frac{\mathcal{D}_\rho^{[\alpha]} x(1-x)}{(1+n)} \int_1^{\sqrt{n}} \bigvee_x^{x+((1-x)/v)} (f_x')(1-x)^{-1}dv \\
&\quad + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x)/\sqrt{n})} (f_x')
\end{aligned}$$

$$\leq \frac{\mathcal{D}_{\rho}^{[\alpha]}x}{(1+n)}\sum_{k=1}^{[\sqrt{n}]}\int_{k}^{k+1}\bigvee_{x}^{x+((1-x)/v)}(f_x')dv+\frac{(1-x)}{\sqrt{n}}\bigvee_{x}^{x+((1-x)/\sqrt{n})}(f_x')$$

$$=\frac{\mathcal{D}_{\rho}^{[\alpha]}x}{(1+n)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+((1-x)/k)}(f_x')+\frac{(1-x)}{\sqrt{n}}\bigvee_{x}^{x+((1-x))/\sqrt{n}}(f_x').$$

Collecting the estimates (20)-(22), we get the required result. □

Acknowledgement: The second author of this paper is supported by the Research fund of *Hasan Kalyoncu University* in 2017.

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