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Marshall–Olkin Power Generalized Weibull Distribution with Applications in Engineering and Medicine

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ABSTRACT

This paper proposes a new flexible four-parameter model called Marshall–Olkin power generalized Weibull (MOPGW) distribution which provides symmetrical, reversed-J shaped, left-skewed and right-skewed densities, and bathtub, unimodal, increasing, constant, decreasing, J shaped, and reversed-J shaped hazard rates. Some of the MOPGW structural properties are discussed. The maximum likelihood is utilized to estimate the MOPGW unknown parameters. Simulation results are provided to assess the performance of the maximum likelihood method. Finally, we illustrate the importance of the MOPGW model, compared with some rival models, via two real data applications from the engineering and medicine fields.

1. INTRODUCTION

The Weibull distribution has been used in modeling lifetime data with monotonic failure rates. It does not provide adequate fits to real data with unimodal or bathtub-shaped hazard rates, often encountered in engineering, medicine and reliability fields. Hence, several authors have constructed different generalizations and extended forms of the Weibull distribution to increase its flexibilty. For example, exponentiated Weibull [1], Marshall–Olkin extended Weibull [2], beta Weibull [3], Kumaraswamy Weibull [4], transmuted complementary Weibull geometric [5], Weibull-Weibull [6], odd log-logistic exponentiated Weibull [7], alpha logarithmic transformed Weibull [8], and odd Lomax Weibull [9] distributions.

Another Weibull extension called the power generalized Weibull (PGW) model pioneered by [10] and they used it in accelerated failure time models. The hazard rate function (HRF) of PGW model can be monotone, unimodal, and bathtub shaped.

The PGW model can be specified by the cumulative distribution function (CDF)

$$G_{PGW}(x;\lambda,\beta,\alpha) = 1 - e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}, \qquad x>0, \lambda,\beta,\alpha>0, \tag{1}$$

where λ represents a scale parameter, β and α represent shape parameters.

The corresponding probability density function (PDF) and HRF take the forms,

$$g_{PGW}(x;\lambda,\beta,\alpha) = \lambda\beta\alpha x^{\beta-1}\left(1+\lambda x^{\beta}\right)^{\alpha-1} e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}, \qquad x>0, \lambda,\beta,\alpha>0$$

and

$$h_{PGW}(x;\lambda,\beta,\alpha) = \lambda\beta\alpha x^{\beta-1}\left(1+\lambda x^{\beta}\right)^{\alpha-1}, \qquad x>0, \lambda,\beta,\alpha>0,$$

respectively.

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Some special cases of the PGW distribution are the Weibull distribution with parameters λ and β for $\alpha = 1$; the exponential distribution with parameter λ for $\alpha = \beta = 1$; the Rayleigh distribution with parameter λ for $\alpha = 1$ and $\beta = 2$; and the Nadarajah-Haghighi (NH) model [11] with parameters $\alpha = 1$ and λ for $\beta = 1$. Further applications of the PGW model can be explored in [12,13] and [14].

This paper is devoted to propose and study a new flexible extension of the PGW model called the MOPGW distribution, which has some desirable motivations as follows:

- The MOPGW contains a number of well-known lifetime sub-models called, Marshall–Olkin Weibull [15], Marshall–Olkin-NH [16], NH [11], PGW [10], Weibull [17], exponential, and Rayleigh distributions, among others, see Table 1.
- The PDF of the MOPGW distribution can be reversed-J shaped, right-skewed, concave down, left-skewed, or symmetric, see Figure 1. Further, its HRF takes some important forms such as, constant, monotone (decreasing or increasing), bathtub, upside down bathtub, and reversed-J shaped. Hence, the MOPGW distribution can be considered as a superior lifetime distribution to other models, which exhibit only monotonic and constant hazard rates, see Figure 2.
- The MOPGW distribution is considered as a suitable model for modeling skewed data that cannot be properly modeled by other extensions of the Weibull distribution. Further, it can be utilized to model real data in many applied areas, such as engineering, survival analysis, medicine, and industrial reliability.

Table 1 Special cases of the MOPGW distribution.

λ	β	α	γ	Distribution	Authors
–	–	1	–	MO-Weibull	[15]
–	1	1	–	MO-Exponential	[15]
–	2	1	–	MO-Rayleigh	–
–	1	–	–	MO-NH	[16]
–	–	–	1	PGW	[10]
–	–	1	1	Weibull	[17]
–	1	1	1	Exponential	–
–	2	1	1	Rayleigh	[21]
–	1	–	1	NH	[11]

MOPGW, Marshall-Olkin power generalized Weibull; MO, Marshall-Olkin; NH, Nadarajah-Haghighi; PGW, power generalized Weibull.

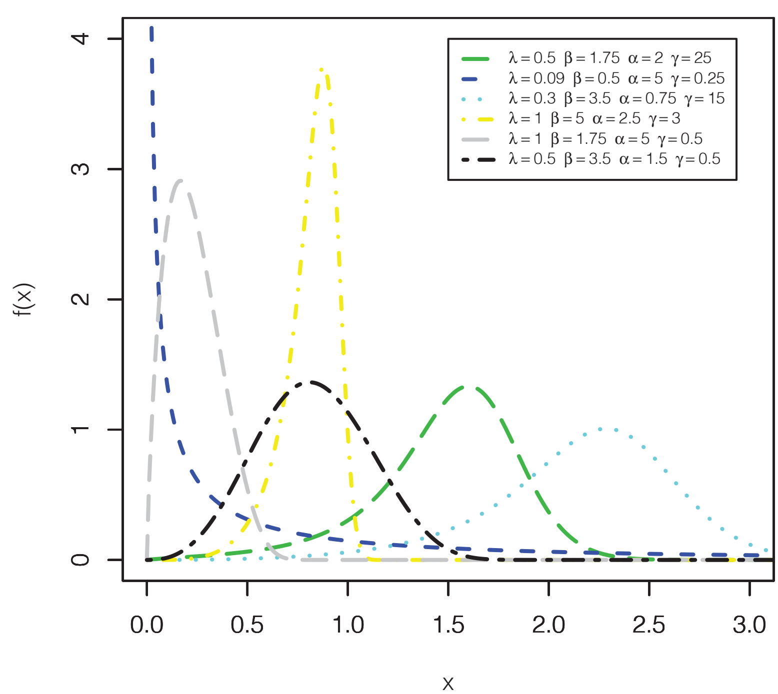

Figure 1 Probability density function (PDF) shapes for the Marshall–Olkin power generalized Weibull (MOPGW) distribution considering different values of its parameters.

- The kurtosis of the MOPGW distribution is more flexible as compared to the baseline PGW model, whereas its skewness varies within the interval (1.43529, 5.62470), whereas the PGW skewness can only range in the interval (-0.68927, 4.25756). Further, the MOPGW distribution can be leptokurtic (kurtosis > 3) or platykurtic (kurtosis < 3), see Table 2.
- Two real-life data applications from the engineering and medicine sciences, prove that the MOPGW distribution outperforms nine other well-known competing lifetime distributions, motivating its usage in applied fields.

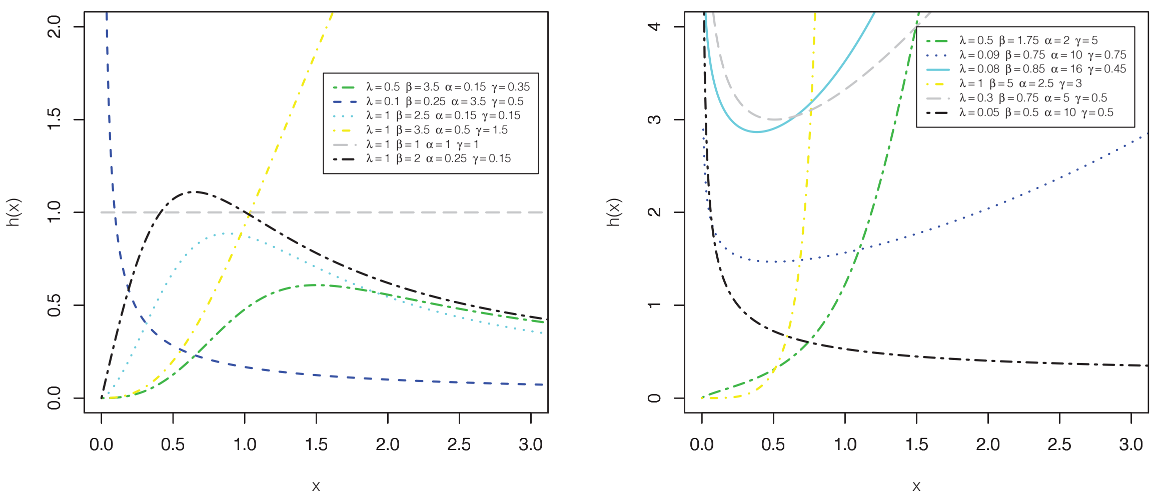

Figure 2 Hazard rate function (HRF) shapes for the Marshall–Olkin power generalized Weibull (MOPGW) distribution considering different values of its parameters.

Table 2 μ, σ^2, γ_1, and γ_2 of the MOPGW model for several values of its parameters with $\lambda = 1$.

α	β	γ	μ	σ^2	γ_1	γ_2
0.5	0.5	0.5	2.83608	30.1952	2.69914	10.2286
		1.5	4.22743	47.3561	1.92163	5.88874
		5.0	3.77437	51.0655	2.02701	6.09045
	1.5	0.5	0.32446	0.53667	5.62470	55.9161
		5.0	1.45888	2.59897	2.40956	13.0233
	5.0	0.5	0.01676	0.00084	3.39967	20.1131
		5.0	0.06375	0.00291	1.27168	5.17103
1.5	0.5	0.5	1.51528	2.77558	3.00036	18.1059
		1.5	2.54943	5.08931	2.11351	10.5501
		5.0	4.16067	8.48966	1.46293	6.80443
	1.5	0.5	0.47876	0.11798	1.11729	4.28686
		1.5	0.70459	0.15479	0.55711	3.02953
		5.0	0.98220	0.17173	0.07453	2.76358
	5.0	0.5	0.19101	0.01481	0.72482	3.05190
		1.5	0.27045	0.01725	0.16615	2.44296
		5.0	0.36019	0.01646	−0.35840	2.76453
5.0	0.5	0.5	1.02414	0.09972	0.50451	3.35065
		1.5	1.22535	0.11075	0.16575	2.98447
		5.0	1.45201	0.10985	−0.17619	3.10095
	1.5	0.5	0.75476	0.03309	−0.17730	2.74402
		1.5	0.86456	0.02914	−0.59094	3.26362
		5.0	0.97071	0.02195	−1.04886	4.61832
	5.0	0.5	0.57806	0.01664	−0.39356	2.82240
		1.5	0.65379	0.01351	−0.86979	3.74856
		5.0	0.72281	0.00917	−1.43529	5.93460

MOPGW, Marshall-Olkin power generalized Weibull.

The MOPGW distribution is constructed by incorporating the PGW as a baseline model in the Marshall–Olkin-G (MO-G) family proposed by [15]. This family has been widely used to provide more flexible extensions of the well-known classical distributions in the statistical literature. See, for example, [2,18,19], and the references therein.

The CDF of the MO-G family has the form

$$F(x;\gamma)=\frac{G(x)}{\gamma+(1-\gamma)G(x)},\qquad x>0,\gamma>0. \tag{2}$$

The corresponding PDF and HRF of the MO-G class are

$$f(x;\gamma)=\frac{\gamma g(x)}{\left[1-\overline{\gamma}\overline{G}(x)\right]^{2}},\qquad x>0,\gamma>0$$

and

$$h(x;\gamma)=\frac{\gamma g(x)}{\overline{G}(x)\left[1-\overline{\gamma}\overline{G}(x)\right]},\qquad x>0,\gamma>0,$$

where $\overline{\gamma}=(1-\gamma)$. The MO-G class reduces to the baseline distribution for $\gamma=1$. Further details about the MO-G class can be explored in [15] and [20].

The rest of the article is outlined as follows: The MOPGW distribution, its special cases, and PDF and HRF plots are provided in Section 2. Several structural properties of the MOPGW distribution are derived in Section 3. In Section 4, the estimation of the MOPGW parameters is demonstrated via the maximum likelihood and its performance is evaluated by simulation results. In Section 5, we illustrate the importance of the MOPGW model by using two real data applications. Finally, the paper is concluded in Section 6.

2. THE MOPGW DISTRIBUTION

In this section, we introduce the four-parameter MOPGW distribution. By inserting (1) in Equation (2), the CDF of the MOGPW distribution follows as

$$F(x;\lambda,\beta,\alpha,\gamma)=\frac{1-e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}}{\gamma+(1-\gamma)\left[1-e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}\right]},\quad x>0,\lambda,\beta,\alpha,\gamma>0. \tag{3}$$

The corresponding PDF of the MOPGW distribution takes the form

$$f(x;\lambda,\beta,\alpha,\gamma)=\frac{\lambda\beta\alpha\gamma x^{\beta-1}\left(1+\lambda x^{\beta}\right)^{\alpha-1}e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}}{\left[1-(1-\gamma)e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}\right]^{2}},\quad x>0,\lambda,\beta,\alpha,\gamma>0. \tag{4}$$

Henceforth, the random variable with PDF (4) is denoted by $X\sim$MOPGW (ϕ), where $\phi=(\lambda,\beta,\alpha,\gamma)$.

The HRF of the MOPGW distribution reduces to

$$h(x;\lambda,\beta,\alpha,\gamma)=\frac{\lambda\beta\alpha\gamma x^{\beta-1}\left(1+\lambda x^{\beta}\right)^{\alpha-1}}{1-\overline{\gamma}e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}},\quad x>0,\lambda,\beta,\alpha,\gamma>0.$$

Figures 1 and 2 display some plots of the probability density and hazard functions of the MOPGW distribution for some different values of its parameters. Figures 1 and 2 reveal that the MOGPW density exhibits reversed-J shaped, left-skewed, symmetric, or right-skewed shapes, whereas its HRF can exhibit constant, monotone (increasing or decreasing), or nonmonotone (unimodal or bathtub) hazard rate shapes.

Further, the MOPGW distribution contains some important special sub-models which are displayed in Table 1.

3. STATISTICAL PROPERTIES OF THE MOPGW DISTRIBUTION

Some properties of the MOPGW distribution including quantile function (QF), ordinary and conditional moments, generating function (MGF), mean deviation, Lorenz and Bonferroni curves, and residual life and reversed residual life moments are derived.

3.1. Quantile Function

The QF $Q(u)$, of the MOPGW distribution follows, by inverting its CDF, as

$$Q(u) = \left\{ \frac{1}{\lambda} \left[\left(1 - \ln \frac{1-u}{1-u(1-\gamma)} \right)^{\frac{1}{\alpha}} - 1 \right] \right\}^{\frac{-1}{\beta}}, \qquad u \in (0,1). \tag{5}$$

Equation (5) can be easily used to generate the MOPGW random variates, and the median of X follows from it with $u = 0.5$.

3.2. Moments

Here, we derive the rth moment of the MOPGW distribution.

Theorem 1. If $X \sim$MOPGW (ϕ), where $\phi = (\lambda, \beta, \alpha, \gamma)$, then the rth moment of X has the form

$$\mu_r'(x) = \gamma \lambda^{\frac{-r}{\beta}} \sum_{i,j=0}^{\infty} \frac{(1-\gamma)^j e^{j+1} (-1)^i}{(j+1)^{\frac{r-i\beta}{\alpha\beta}}} \binom{\frac{r}{\beta}}{i} \Gamma\left(\frac{r-\beta(i-\alpha)}{\alpha\beta}, j+1 \right).$$

Proof: The rth moment of X is defined by

$$\begin{aligned} \mu_r'(x) &= \int_0^\infty x^r f(x;\lambda,\beta,\alpha,\gamma) dx \\ &= \int_0^\infty \frac{\lambda\beta\alpha\gamma x^{r+\beta-1} \left(1+\lambda x^\beta\right)^{\alpha-1} e^{1-\left(1+\lambda x^\beta\right)^\alpha}}{\left[1-(1-\gamma)e^{1-\left(1+\lambda x^\beta\right)^\alpha}\right]^2} dx. \end{aligned} \tag{6}$$

Using the generalized binomial expression for $|d| < 1$ and $k > 0$ (a real non-integer)

$$(1-d)^{-k} = \sum_{i=0}^{\infty} \frac{\Gamma(k+i)}{\Gamma(k) i!} d^i. \tag{7}$$

Using Equation (7) in (6), we obtain

$$\mu_r'(x) = \lambda\beta\alpha\gamma \sum_{j=0}^{\infty} (j+1)(1-\gamma)^j e^{j+1} \int_0^\infty x^{r+\beta-1} \left(1+\lambda x^\beta\right)^{\alpha-1} e^{-(j+1)\left(1+\lambda x^\beta\right)^\alpha} dx.$$

Setting $y = (j+1)\left(1+\lambda x^\beta\right)^\alpha$, we get

$$x = \left\{ \frac{1}{\lambda} \left[\left(\frac{y}{j+1} \right)^{\frac{1}{\alpha}} - 1 \right] \right\}^{\frac{-1}{\beta}},$$

and after some algebra, we have

$$\mu_r'(x) = \gamma \lambda^{\frac{-r}{\beta}} \sum_{j=0}^{\infty} (1-\gamma)^j e^{j+1} \int_{j+1}^\infty \left[\left(\frac{y}{j+1} \right)^{\frac{1}{\alpha}} - 1 \right]^{\frac{r}{\beta}} dy. \tag{8}$$

Consider the power series with real number power w

$$(x+a)^w = \sum_{i=0}^{\infty} \binom{w}{i} x^w a^{w-i}, \tag{9}$$

where $\binom{w}{i}$ is a binomial coefficient. The above power series converges for $w \geq 0$ an integer, or $\left|\frac{x}{a}\right| < 1$ (see [22]). Applying (9) in Equation (8), the rth moment of X becomes

$$\mu_r'(x) = \gamma\lambda^{\frac{-r}{\beta}}\sum_{j=0}^{\infty}(1-\gamma)^j e^{j+1}\sum_{i=0}^{\infty}(-1)^i\binom{\frac{r}{\beta}}{i}\int_{j+1}^{\infty}\left(\frac{y}{j+1}\right)^{\frac{1}{\alpha}\left(\frac{r}{\beta}-i\right)}e^{-y}dy$$

$$= \gamma\lambda^{\frac{-r}{\beta}}\sum_{i,j=0}^{\infty}\frac{(1-\gamma)^j e^{j+1}(-1)^i}{(j+1)^{\frac{r-i\beta}{\alpha\beta}}}\binom{\frac{r}{\beta}}{i}\Gamma\left(\frac{r-\beta(i-\alpha)}{\alpha\beta}, j+1\right), \tag{10}$$

where the complementary incomplete gamma is defined by $\Gamma(s,t) = \int_t^{\infty} x^{s-1}e^{-x}dx$, for all real numbers except negative integers.

Table 2 shows some numerical values for the mean, μ, variance, σ^2, skewness, γ_1, and kurtosis, γ_2, of the MOPGW distribution which can be computed, using the R software, for several values of β, α, and γ, with $\lambda = 1$. Table 2 shows that the MOPGW skewness can range in the interval $(-1.43529, 5.62470)$, whereas the PGW skewness can range only in $(-0.68927, 4.25756)$ for $\lambda = 1$ and the shape parameters have values from 0.5 to 5. The kurtosis spread for the MOPGW model is much larger ranging in the interval $(2.76358, 55.9161)$, whereas the kurtosis spread for the PGW model can only vary from 2.54958 to 33.2167 for the same parameter values. Further, the MOPGW model can be right skewed and left skewed. Table 2 illustrates that the MOPGW model is leptokurtic (kurtosis > 3) or platykurtic (kurtosis < 3). Hence, the MOPGW distribution can be utilized in modeling skewed data.

3.3. Generating Function

The MGF is useful for several reasons, one of which is its application in analyzing the sums of random variables.

Theorem 2. If $X \sim$MOPGW (ϕ), where $\phi = (\lambda, \beta, \alpha, \gamma)$, hence the MGF of X follows as

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\gamma\lambda^{\frac{-r}{\beta}}\sum_{i,j=0}^{\infty}\frac{(1-\gamma)^j e^{j+1}(-1)^i}{(j+1)^{\frac{r-i\beta}{\alpha\beta}}}\binom{\frac{r}{\beta}}{i}\Gamma\left(\frac{r-\beta(i-\alpha)}{\alpha\beta}, j+1\right).$$

Proof.

The MGF can be defined as

$$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx}f(x;\lambda,\beta,\alpha,\gamma)dx.$$

Since the series expansion of e^{tx} is given by $e^{tx} = \sum_{k=0}^{\infty}\frac{(tx)^k}{k!}$. Thus,

$$M_X(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\int_0^{\infty} x^k f(x)dx = \sum_{k=0}^{\infty}\frac{t^k}{k!}\mu_k'(x). \tag{11}$$

Then, substituting from Equations (10) into (11), we get

$$M_X(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\gamma\lambda^{\frac{-k}{\beta}}\sum_{i,j=0}^{\infty}\frac{(1-\gamma)^j e^{j+1}(-1)^i}{(j+1)^{\frac{k-i\beta}{\alpha\beta}}}\binom{\frac{k}{\beta}}{i}\Gamma\left(\frac{k-\beta(i-\alpha)}{\alpha\beta}, j+1\right),$$

which completes the proof.

3.4. Conditional Moments

The rth lower incomplete moment of X can be defined (for any real $s > 0$) as $\nu_s(t) = E(x^s \mid X < t) = \int_0^t x^s f(x,\phi)dx$. Hence, $\nu_s(t)$ for the MOPGW distribution takes the form

$$\nu_s(t) = \int_0^t x^s f(x)dx$$

$$= \int_0^t \frac{\lambda\beta\alpha\gamma x^{s+\beta-1}\left(1+\lambda x^{\beta}\right)^{\alpha-1}e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}}{\left[1-(1-\gamma)e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}\right]^2}dx.$$

After some algebra, we get

$$\nu_s(t) = \gamma\lambda^{\frac{-s}{\beta}} \sum_{i,j=0}^{\infty} \frac{(1-\gamma)^j e^{j+1}(-1)^i}{(j+1)^{\frac{s-i\beta}{\alpha\beta}}} \binom{\frac{s}{\beta}}{i} \Lambda\left(\frac{s-\beta(i-\alpha)}{\alpha\beta}, (j+1)\left(1+\lambda t^{\beta}\right)^{\alpha}\right), \tag{12}$$

where the lower incomplete gamma function is denoted by $\Lambda(s,t) = \int_0^t x^{s-1}e^{-x}dx$. The first incomplete moment of X, $\nu_1(t)$, is computed using Equation (12) by setting $s=1$, and it takes the form

$$\nu_1(t) = \gamma\lambda^{\frac{-1}{\beta}} \sum_{i,j=0}^{\infty} \frac{(1-\gamma)^j e^{j+1}(-1)^i}{(j+1)^{\frac{1-i\beta}{\alpha\beta}}} \binom{\frac{1}{\beta}}{i} \Lambda\left(\frac{1-\beta(i-\alpha)}{\alpha\beta}, (j+1)\left(1+\lambda t^{\beta}\right)^{\alpha}\right).$$

Similarly, the rth upper incomplete moment of X can be defined (for any real $s>0$) as $\eta_s(t) = E(x^s \mid X > t) = \int_t^{\infty} x^s f(x,\phi)dx$.

The rth upper incomplete moment of MOPGW distribution is

$$\begin{aligned}
\eta_s(t) &= \int_t^{\infty} x^s f(x)dx \\
&= \int_t^{\infty} \frac{\lambda\beta\alpha\gamma x^{s+\beta-1}\left(1+\lambda x^{\beta}\right)^{\alpha-1} e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}}{\left[1-(1-\gamma)e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}\right]^2}dx \\
&= \gamma\lambda^{\frac{-s}{\beta}} \sum_{i,j=0}^{\infty} \frac{(1-\gamma)^j e^{j+1}(-1)^i}{(j+1)^{\frac{s-i\beta}{\alpha\beta}}} \binom{\frac{s}{\beta}}{i} \Gamma\left(\frac{s-\beta(i-\alpha)}{\alpha\beta}, (j+1)\left(1+\lambda t^{\beta}\right)^{\alpha}\right),
\end{aligned}$$

where the upper incomplete gamma function is denoted by $\Gamma(s,t) = \int_t^{\infty} x^{s-1}e^{-x}dx$.

3.5. Mean Deviation, Lorenz, and Bonferroni Curves

This section is devoted to derive the mean deviation about the mean, denoted by $\delta_1(x)$, and the mean deviation about the median, denoted by $\delta_2(x)$, for the MOPGW model.

The mean deviations about the mean of the MOPGW distribution is

$$\delta_1(x) = E\left(\mid x - \mu_1' \mid\right) = 2\mu_1' F(\mu_1') - 2\nu_1(\mu_1').$$

The mean deviations about the median of the MOPGW distribution has the form

$$\delta_2(x) = E\left(\mid x - M \mid\right) = \mu_1' - 2\nu_1(M).$$

Using Equation (10), we get the mean of X, $\mu_1' = E(X)$, the median of X is $M = Q(0.5)$, $F(\mu_1')$ follows simply from Equation (3) and $\nu_1(\mu_1')$ denotes the first incomplete moment.

Further, the Lorenz and Bonferroni curves have useful applications in income inequality measures, demography, reliability, insurance, and medicine.

Lorenz curve has the form

$$L(p) = \frac{\nu_1(q)}{\mu_1'},$$

where $q = G^{-1}(p)$.

The Bonferroni curve takes the form

$$B(p) = \frac{\nu_1(q)}{p\mu_1'}.$$

3.6. Moments of Residual and Reversed Residual Lives

The rth moment of the residual life is

$$\mu_r(t) = E[(X-t)^r \mid X > t] = \frac{1}{\bar{F}(t)} \int_t^\infty (x-t)^r f(x)dx, r \geq 1.$$

Using the binomial series to the term $(x-t)^r$ and the PDF of the MOPGW in (4), the $\mu_r(t)$ reduces to

$$\begin{aligned}\mu_r(t) &= \frac{1}{\bar{F}(t)} \sum_{h=0}^{r} (-t)^{r-h} \binom{r}{h} \int_t^\infty x^r f(x)dx \\ &= \frac{(1-\gamma)^j e^{j+1} (-1)^i}{\bar{F}(t)} \sum_{h=0}^{r} \sum_{i,j=0}^{\infty} \frac{(-t)^{r-h} \binom{r}{h} \gamma \lambda^{\frac{-r}{\beta}}}{(j+1)^{\frac{r-i\beta}{\alpha\beta}}} \binom{\frac{r}{\beta}}{i} \\ &\times \Gamma\left(\frac{r-\beta(i-\alpha)}{\alpha\beta}, (j+1)\left(1+\lambda t^\beta\right)^\alpha\right),\end{aligned}$$

where upper incomplete gamma is $\Gamma(s,t) = \int_t^\infty x^{s-1} e^{-x} dx$.

The mean residual life (MRL) of the MOPGW distribution follows from the last equation, with $r = 1$, as

$$\mu_1(t) = \frac{\gamma \lambda^{\frac{-1}{\beta}}}{\bar{F}(t)} \sum_{i,j=0}^{\infty} \frac{(1-\gamma)^j e^{j+1} (-1)^i}{(j+1)^{\frac{1-i\beta}{\alpha\beta}}} \binom{\frac{1}{\beta}}{i} \Gamma\left(\frac{1-\beta(i-\alpha)}{\alpha\beta}, (j+1)\left(1+\lambda t^\beta\right)^\alpha\right) - t.$$

The variance residual life of the MOPGW distribution is obtained directly using $\mu_2(t)$ and $\mu(t)$.

Further, the rth moment of reversed residual life is

$$m_r(t) = E[(t-X)^r \mid X \leq t] = \frac{1}{F(t)} \int_0^t (t-x)^r f(x)dx, r \geq 1.$$

Using the binomial series to the term $(t-x)^r$ and the MOPGW PDF in (4), the $m_r(t)$ follows as

$$\begin{aligned}m_r(t) &= \frac{1}{F(t)} \sum_{h=0}^{r} (-t)^{r-h} \binom{r}{h} \int_0^t x^r f(x)dx \\ &= \frac{(1-\gamma)^j e^{j+1} (-1)^i}{F(t)} \sum_{h=0}^{r} \sum_{i,j=0}^{\infty} \frac{(-t)^{r-h} \binom{r}{h} \gamma \lambda^{\frac{-r}{\beta}}}{(j+1)^{\frac{r-i\beta}{\alpha\beta}}} \binom{\frac{r}{\beta}}{i} \\ &\times \Lambda\left(\frac{r-\beta(i-\alpha)}{\alpha\beta}, (j+1)\left(1+\lambda t^\beta\right)^\alpha\right),\end{aligned}$$

where the lower incomplete gamma has the form $\Lambda(s,t) = \int_0^t x^{s-1} e^{-x} dx$.

The mean waiting time (or mean reversed residual life) of the MOPGW distribution reduces to

$$m(t) = t - \frac{\gamma \lambda^{\frac{-1}{\beta}}}{F(t)} \sum_{i,j=0}^{\infty} \frac{(1-\gamma)^j e^{j+1} (-1)^i}{(j+1)^{\frac{1-i\beta}{\alpha\beta}}} \binom{\frac{1}{\beta}}{i} \Lambda\left[\frac{1-\beta(i-\alpha)}{\alpha\beta}, (j+1)\left(1+\lambda t^\beta\right)^\alpha\right].$$

The variance and coefficient of variation of reversed residual life for the MOPGW distribution follow simply using $m(t)$ and $m_2(t)$.

4. ESTIMATION AND SIMULATION

In this section, we discuss the estimation of the MOPGW parameters using the maximum likelihood. Let $x_1, x_2, \ldots, x_n$ be a random sample from the MOPGW distribution with parameters vector $\phi = (\lambda, \beta, \alpha, \gamma)^T$.

The log likelihood function, $\log \ell(\phi)$, has the form

$$\log \ell(\phi) = n \log(\lambda\beta\alpha\gamma) + (\beta - 1)\sum_{i=1}^{n} \log(x_i) + (\alpha - 1)\sum_{i=1}^{n} \log\left(1 + \lambda x_i^{\beta}\right)$$
$$- \sum_{i=1}^{n}\left(1 + \lambda x_i^{\beta}\right)^{\alpha} - 2\sum_{i=1}^{n} \log\left[1 - (1-\gamma)e^{1-\left(1+\lambda x_i^{\beta}\right)^{\alpha}}\right]. \tag{13}$$

The maximum likelihood estimates (MLEs) of λ, β, α, and γ are obtained by maximizing Equation (13) with respect to these parameters. Further, the MLEs of the MOPGW parameters can be obtained by solving the following four nonlinear equations, which represent the score vector elements given by

$$\frac{\partial \log \ell(\phi)}{\partial \lambda} = \frac{n}{\lambda} + (\alpha - 1)\sum_{i=1}^{n} \frac{x_i^{\beta}}{\left(1 + \lambda x_i^{\beta}\right)} - \alpha \sum_{i=1}^{n} x_i^{\beta}\left(1 + \lambda x_i^{\beta}\right)^{\alpha-1}$$
$$-2\sum_{i=1}^{n} \frac{(1-\gamma)\beta x_i^{\beta}\left(1 + \lambda x_i^{\beta}\right)^{\alpha-1} e^{1-\left(1+\lambda x_i^{\beta}\right)^{\alpha}}}{1 - (1-\gamma)e^{1-\left(1+\lambda x_i^{\beta}\right)^{\alpha}}},$$

$$\frac{\partial \log \ell(\phi)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log x_i + (\alpha - 1)\sum_{i=1}^{n} \frac{\lambda \log(x_i) x_i^{\beta}}{\left(1 + \lambda x_i^{\beta}\right)}$$
$$-\alpha \sum_{i=1}^{n} \lambda \log(x_i) x_i^{\beta}\left(1 + \lambda x_i^{\beta}\right)^{\alpha-1}$$
$$-2\alpha\lambda \sum_{i=1}^{n} \frac{(1-\gamma)e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}} \log(x_i) x_i^{\beta}\left(1 + \lambda x_i^{\beta}\right)^{\alpha-1}}{1 - (1-\gamma)e^{1-\left(1+\lambda x_i^{\beta}\right)^{\alpha}}},$$

$$\frac{\partial \log \ell(\phi)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left(1 + \lambda x_i^{\beta}\right) - \sum_{i=1}^{n} \log\left(1 + \lambda x_i^{\beta}\right)\left(1 + \lambda x_i^{\beta}\right)^{\alpha}$$
$$-2\lambda \sum_{i=1}^{n} \frac{(1-\gamma)e^{1-\left(1+\lambda x_i^{\beta}\right)^{\alpha}} \log\left(1 + \lambda x_i^{\beta}\right)\left(1 + \lambda x_i^{\beta}\right)^{\alpha}}{1 - (1-\gamma)e^{1-\left(1+\lambda x_i^{\beta}\right)^{\alpha}}}$$

and

$$\frac{\partial \log \ell(\phi)}{\partial \gamma} = \frac{n}{\gamma} - 2\sum_{i=1}^{n} \frac{e^{1-\left(1+\lambda x_i^{\beta}\right)^{\alpha}}}{1 - (1-\gamma)e^{1-\left(1+\lambda x_i^{\beta}\right)^{\alpha}}}.$$

Further, there are some programs called, Ox program (sub-routine MaxBFGS), SAS (PROCNLMIXED), R (optim function), Mathcad, and Newton–Rapshon method, which can be utilized to maximize the log-likelihood function in order to determine the MLEs.

Now, we perform a Monte Carlo simulation to evaluate the performance of the MLEs of the MOPGW parameters λ, β, α, and γ based on their mean squares errors (MSEs). The simulation results are obtained via the Mathcad software, version 14.0. The MOPGW variates are generated for several values of the parameters and for different sample sizes, $n = 50$ and 100. The following parameters values are considered, $\lambda = (0.5, 1.0)$, $\beta = (0.1, 0.3)$, $\alpha = (0.5, 1.0, 1.5, 2.0)$, and $\gamma = (1.0, 2.0)$. The average estimates (AVEs) and their corresponding MSEs are calculated for each setting. The AVEs and MSEs are listed in Tables 3 and 4.

Table 3 AVEs and their associated MSEs ($n = 50$) .

Parameters				AVEs				MSEs			
λ	β	α	γ	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\gamma}$
0.5	0.1	0.5	1	0.9202	0.0500	0.8640	0.9930	1.6582	0.9753	0.0500	2.3723
		1		1.2811	0.0580	1.0871	2.9701	1.1396	1.3785	0.0541	1.4844
		1.5		1.9702	0.1011	0.8217	2.4601	0.4943	1.3842	0.4189	1.1360
		2		1.3065	0.1214	0.9389	2.7378	1.4620	1.7830	0.1344	1.7093
	0.3	0.5		2.2057	0.1235	0.6703	2.7709	1.0459	1.9742	0.0815	1.9016
		1		0.9405	0.2207	1.0513	1.8933	0.7882	1.4132	0.3502	2.1485
		1.5		1.2020	0.2977	1.2730	2.5025	0.7177	1.4006	0.3133	1.6867
		2		1.1791	0.3759	1.0358	2.6856	0.5435	1.1479	0.4736	1.3190
1	0.1	0.5		1.5039	0.0500	0.9729	2.7398	1.1576	0.7366	0.2676	1.4537
		1		1.9530	0.7891	0.7961	1.7431	1.4313	0.1478	0.9689	1.5849
		1.5		2.0288	0.3393	1.2637	2.9830	1.0931	0.0707	2.0806	1.5534
		2		2.5381	1.1032	0.9184	2.9873	1.4982	0.0679	1.5117	2.3054
	0.3	0.5		1.9103	0.2057	0.6087	1.8260	1.5463	0.7388	0.9302	1.1497
		1		2.2046	0.5422	0.7175	2.9335	1.9484	0.5651	0.6425	1.8055
		1.5		3.0120	1.3879	0.5676	3.1894	1.5879	0.4016	1.1681	2.2586
		2		2.1232	2.5133	1.7885	2.9748	0.9844	0.3422	1.8977	0.9037
0.5	0.1	0.5	2	0.3270	0.0500	0.6942	0.8281	0.9630	0.4918	0.6203	1.7827
		1		1.1391	0.0533	1.1584	2.7994	0.1380	1.0959	0.5341	2.5354
		1.5		0.9781	0.0737	1.1984	2.9578	0.1363	2.1470	0.0547	2.2515
		2		1.7858	0.1049	0.7978	2.9440	0.2439	1.9954	0.1420	2.9927
	0.3	0.5		2.2858	0.1029	0.6213	2.6937	0.4588	1.4278	0.1588	2.0491
		1		2.1103	0.1975	0.5946	2.3362	0.3802	1.2081	0.4836	2.1916
		1.5		2.2977	0.2504	0.6448	2.9056	0.3301	1.5568	0.3173	1.7513
		2		1.5577	0.3140	1.0508	2.8405	0.4583	1.5468	0.4170	1.3579
1	0.1	0.5		0.9596	0.0510	1.0642	1.7048	1.0923	0.6854	0.1785	1.4263
		1		1.0611	0.1292	0.9846	1.5859	1.3129	0.2748	0.8653	0.7359
		1.5		1.6313	0.3118	1.1332	2.6599	1.5714	0.1743	1.1237	1.5683
		2		2.5047	0.8806	1.0001	2.9519	1.3696	0.0676	1.5708	1.4902
	0.3	0.5		0.9017	0.1320	1.0972	1.9979	1.1166	0.9673	0.7618	1.7971
		1		1.2100	0.4056	0.7544	1.6171	1.4462	0.4686	0.6188	1.4686
		1.5		0.8302	0.5522	1.7014	2.8225	1.5482	0.6256	0.8528	1.7732
		2		2.0142	1.6809	1.5975	2.9999	1.3241	0.4989	1.3067	1.7950

AVE, average estimate; MSE, mean squares error.

Table 4 AVEs and their associated MSEs ($n = 100$).

Parameters				AVEs				MSEs			
λ	β	α	γ	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\gamma}$
0.5	0.1	0.5	1	0.3595	0.0500	0.6929	0.4942	1.0292	0.7498	0.2392	2.1435
		1		0.9676	0.0580	1.2379	2.7996	1.0687	0.9651	0.3836	1.4553
		1.5		1.3876	0.0985	0.9082	2.2925	0.9067	1.8129	0.1981	1.3743
		2		1.3068	0.1283	0.9110	2.0583	1.0018	2.0171	0.2020	1.2384
	0.3	0.5		1.5576	0.2141	0.9024	2.4123	0.7984	2.0533	0.2072	1.8196
		1		1.6017	0.2043	0.9302	2.5424	0.6430	1.5325	0.3686	1.1215
		1.5		1.4011	0.2943	1.0464	2.3157	0.8278	1.4305	0.3485	1.6112
		2		1.6386	0.3782	0.9007	2.4975	0.7073	1.9542	0.3109	1.2240
1	0.1	0.5		1.0035	0.0519	1.0910	2.4792	1.2406	0.8302	0.2642	1.1904
		1		2.3296	0.5527	0.6585	2.0968	1.8703	0.2951	0.4097	0.7702
		1.5		2.7437	0.4934	0.6510	2.9925	1.3197	0.0946	1.7443	1.5855
		2		2.0756	0.9331	1.4044	2.9889	1.3320	0.0658	2.0452	1.4687
	0.3	0.5		1.6041	0.2006	0.6433	1.5800	2.3048	1.3419	0.2079	0.8049
		1		1.7475	0.4622	1.0189	2.5818	1.6972	0.4632	0.8230	1.5771
		1.5		2.1441	0.9269	1.1965	2.9988	1.3120	0.4003	1.3964	1.8371
		2		1.9626	1.7117	1.7679	2.9993	1.2800	0.3688	1.9772	1.6761

(*continued*)

Table 4 AVEs and their associated MSEs ($n = 100$). *(Continued)*

Parameters				AVEs				MSEs			
λ	β	α	γ	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\gamma}$
0.5	0.1	0.5	2	0.2979	0.0500	0.7382	0.8526	0.9125	0.5190	0.3053	2.1986
		1		1.0792	0.0508	1.1367	2.9312	0.4648	1.2058	0.3080	2.4279
		1.5		1.1903	0.0761	1.0201	2.5770	0.5441	1.4865	0.3016	2.2644
		2		1.5855	0.1078	0.8191	2.1929	0.7954	2.2353	0.1865	2.1491
	0.3	0.5		1.3581	0.0998	0.9561	2.2802	0.5436	2.0943	0.1089	2.3885
		1		1.2990	0.1881	0.9117	2.1297	0.3253	1.8123	0.2371	2.0485
		1.5		1.1645	0.2507	1.1015	2.4764	0.4894	1.9389	0.2591	2.2461
		2		1.4522	0.3199	0.9606	2.6714	0.5056	1.6968	0.3897	1.8849
1	0.1	0.5		1.5041	0.0501	0.9779	1.6841	0.9834	0.7566	0.3622	1.7222
		1		2.2392	0.1842	0.4764	1.3209	1.9527	0.6950	0.3336	0.5380
		1.5		2.2205	0.3257	0.8010	2.7536	1.8079	0.2037	0.9809	1.8040
		2		2.3423	0.7380	1.0303	2.9997	1.4461	0.0953	1.6802	1.7298
	0.3	0.5		1.5499	0.1429	0.8390	2.1318	1.4969	1.5425	0.3213	1.4146
		1		1.9087	0.4567	0.6987	1.6015	1.9548	0.9360	0.3987	1.1856
		1.5		2.1028	0.8042	0.9827	2.9339	1.3663	0.5518	1.0519	1.6054
		2		1.9990	1.4741	1.5191	2.9987	1.1484	0.5384	1.7617	1.6857

AVE, average estimate; MSE, mean squares error.

The values of AVEs and MSEs reveal that

- All parameters estimates show consistency property, that is, the MSEs decrease when the sample size increases.
- For fixed λ, β, and γ, the MSEs of $\hat{\alpha}$ increase, in most cases when α increases.
- For fixed λ, α, and γ, the MSEs of $\hat{\beta}$ increase, when β increases.
- For fixed β, α, and γ, the MSEs of $\hat{\lambda}$ increase, when λ increases.
- For fixed λ, β, and α, the MSEs of $\hat{\gamma}$ increase, when γ increases.

5. APPLICATIONS

Two real data applications are provided in this section to study empirically the importance and flexibility of the MOPGW distribution. The first set of data represents the gauge lengths of 20 mm and it consists of 74 observations. This data is reported in [23], and it is studied by [24,25] and [26]. The second set of data consists of $n = 128$ observations of remission times (months) of bladder cancer patients studied by [27]. The MOPGW provides better fits to the cancer data than the odd Lindley Burr XII [7], generalized odd Lindley Burr XII [29], quasi xgamma-geometric [30], and Weibull Marshall–Olkin Lindley [31] distributions.

For the gauge lengths and cancer data, the fits of the MOPGW model is compared with some competitive distributions called, the exponentiated power generalized Weibull (EPGW) [32], alpha power exponentiated W (APEW) [33], Kumaraswamy W (KwW) [4], beta W (BW) [3], transmuted geometric W (TGcW [34], transmuted exponentiated generalized W (TEGW) [35], odd Lomax W (OLxW) [9], W Burr XII (WBXII) [36], and W Fréchet(WFr) [37].

The W^* (Cramér–Von Mises), A^* (Anderson–Darling), KS (Kolmogorov–Smirnov), and its PV (p-value) measures, are considered to compare the fits of the MOPGW distribution with other aforementioned models.

Tables 5 and 6 provide the MLEs and associated standard errors (SEs) (between parentheses) of the parameters of all fitted models and the values of W^*, A^*, KS, and PV, for both data sets, respectively. They show that the MOPGW distribution gives the best fit to the given data sets and it can be considered a very competitive distribution to aforementioned extensions of the Weibull distribution.

Figure 3 displays the histogram plots for the gauge lengths and cancer data, and the fitted PDF, CDF, SF, and PP plots of the MOPGW distribution.

The TTT plot of gauge lengths data and the HRF plot of the MOPGW distribution are displayed in Figure 4, whereas the TTT plot of cancer data and MOPGW HRF plot are shown in Figure 5. It is clear that the MOPGW HRF is increasing for gauge lengths data, whereas it is unimodal (upside down bathtub) for cancer data. Furthermore, the scaled TTT plot for gauge lengths data is concave which indicates an increasing HRF, whereas it is concave then convex for cancer data which indicates a unimodal HRF. Hence, the MOPGW distribution is a suitable for modeling both data sets.

Table 5 MLEs, associated SEs, W^*, A^*, KS, and PV for gauge lengths data.

Distribution	**Estimates (SEs)**				W^*	A^*	KS	PV
MOPGW $(\lambda, \beta, \alpha, \gamma)$	0.0287	7.1320	0.3624	6.3451	0.0244	0.1839	0.0503	0.9921
	(0.0895)	(6.2545)	(0.6108)	(21.620)				
EPGW $(\lambda, \beta, \alpha, \gamma)$	0.0320	3.5114	1.4106	2.1014	0.0269	0.2119	0.0580	0.9648
	(0.0661)	(4.4837)	(3.1836)	(2.7709)				
APEW $(\alpha, \beta, \lambda, \beta)$	5.5281	3.3500	0.0863	2.1428	0.0256	0.1911	0.0521	0.9880
	(16.298)	(1.5888)	(0.1855)	(1.5643)				
KwW (α, β, a, b)	0.2914	1.3671	6.0143	60.246	0.0265	0.2084	0.0573	0.9684
	(0.7269)	(14.118)	(79.223)	(1576.5)				
BW (α, β, a, b)	0.4099	4.5309	1.5816	0.9338	0.0267	0.2102	0.0578	0.9656
	(0.3434)	(2.3153)	(1.1844)	(4.5321)				
TGcW $(\alpha, \beta, \lambda, \beta)$	0.3363	6.5104	0.7005	1.3899	0.0253	0.1950	0.0538	0.9829
	(0.0432)	(1.2878)	(0.4511)	(1.5504)				
TEGW (β, λ, a, b)	3.8225	−0.4156	0.0436	1.8658	0.0265	0.1998	0.0541	0.9819
	(1.1864)	(0.7320)	(0.0668)	(1.1250)				
OLxW (α, β, a, b)	0.1101	2.1569	0.5848	5.1896	0.0266	0.2126	0.0608	0.9470
	(0.2148)	(2.5461)	(0.1790)	(1.1192)				
WBXII (α, β, a, b)	3.5400	0.3785	0.0404	3.1760	0.0278	0.2256	0.0606	0.9487
	(5.0510)	(0.3761)	(0.1375)	(2.1929)				
WFr (α, β, a, b)	2.4251	0.9826	4.5360	3.7590	0.0278	0.2243	0.0602	0.9514
	(49.622)	(14.716)	(604.17)	(25.468)				

MLE, maximum likelihood estimate; SE, standard error; MOPGW, Marshall–Olkin power generalized Weibull; EPGW, exponentiated power generalized Weibull; APEW, alpha power exponentiated W; KwW, Kumaraswamy W, BW, beta W; TGcW, transmuted geometric W; TEGW, transmuted exponentiated generalized W; OlxW, odd Lomax W; WBXII, W Burr XII; WFr, W Fréchet; KS, Kolmogorov–Smirnov; PV, p-value.

Table 6 MLEs, associated SEs, W^*, A^*, KS, and PV for cancer data.

Distribution	**Estimates (SEs)**				W^*	A^*	KS	PV
MOPGW $(\lambda, \beta, \alpha, \gamma)$	2375.7	0.5804	0.2726	25323.6	0.0138	0.0855	0.0284	0.9999
	(1427.3)	(0.0858)	(0.0238)	(14785)				
EPGW $(\lambda, \beta, \alpha, \gamma)$	0.0072	2.9130	0.2416	0.4518	0.0159	0.1103	0.0319	0.9995
	(0.0078)	(0.5370)	(0.0604)	(0.1001)				
APEW $(\alpha, \beta, \lambda, \beta)$	0.0080	0.7009	0.1535	2.2892	0.0203	0.1445	0.0348	0.9978
	(0.0283)	(0.6411)	(0.3522)	(2.8493)				
KwW (α, β, a, b)	0.2162	0.4588	4.1198	2.9402	0.0415	0.2732	0.0447	0.9603
	(0.2491)	(0.5283)	(5.9761)	(8.3513)				
BW (α, β, a, b)	0.3218	0.6662	2.7348	0.9076	0.0436	0.2882	0.0450	0.9582
	(0.4365)	(0.2450)	(1.5996)	(1.5117)				
TGcW $(\alpha, \beta, \lambda, \beta)$	0.0307	1.5319	−0.4581	19.176	0.0265	0.1888	0.0331	0.9990
	(0.0157)	(0.2577)	(0.5461)	(18.941)				
TEGW (β, λ, a, b)	0.7241	0.7225	0.2523	2.2424	0.0310	0.2045	0.0409	0.9830
	(0.1998)	(0.3437)	(0.1874)	(1.0744)				
OLxW (α, β, a, b)	0.2987	29.936	3.2692	0.5736	0.0117	0.0734	0.0326	0.9992
	(0.2229)	(61.348)	(5.1876)	(0.2744)				
WBXII (α, β, a, b)	0.7448	0.1575	19.089	2.6464	0.0474	0.3134	0.0481	0.9280
	(0.2979)	(0.1960)	(97.974)	(0.9554)				
WFr (α, β, a, b)	106.09	0.1918	42.274	2.7145	0.0671	0.4277	0.0548	0.8363
	(350.22)	(0.1560)	(160.04)	(2.2077)				

MLE, maximum likelihood estimate; SE, standard error; MOPGW, Marshall–Olkin power generalized Weibull; EPGW, exponentiated power generalized Weibull; APEW, alpha power exponentiated W; KwW, Kumaraswamy W, BW, beta W; TGcW, transmuted geometric W; TEGW, transmuted exponentiated generalized W; OlxW, odd Lomax W; WBXII, W Burr XII; WFr, W Fréchet; KS, Kolmogorov–Smirnov; PV, p-value.

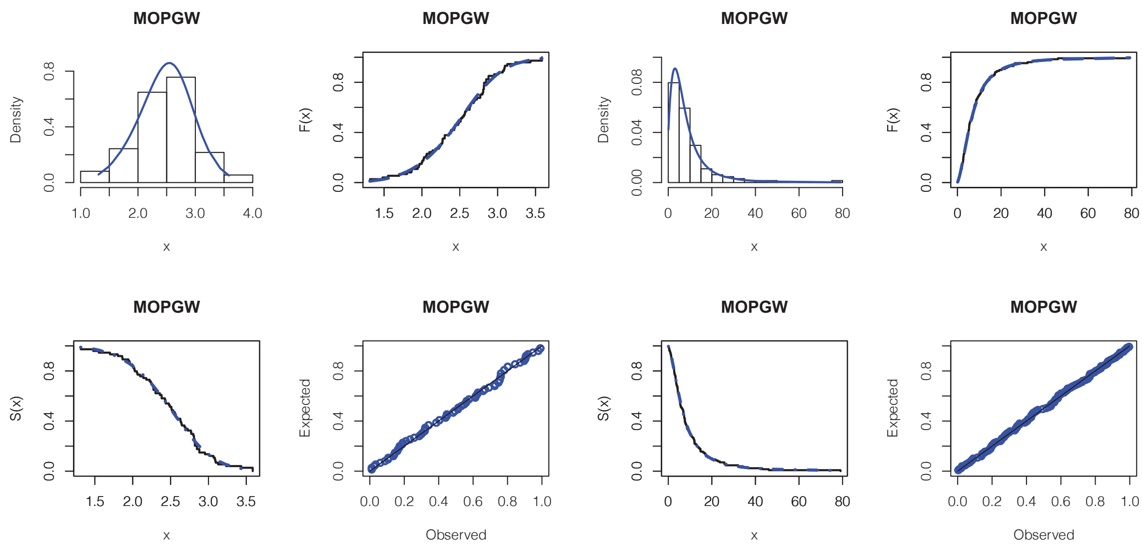

Figure 3 Fitted Marshall–Olkin power generalized Weibull (MOGPW) probability density function (PDF), estimated cumulative distribution function (CDF), SF, and PP plots (left panel) for gauge lengths data and (right panel) for cancer data.

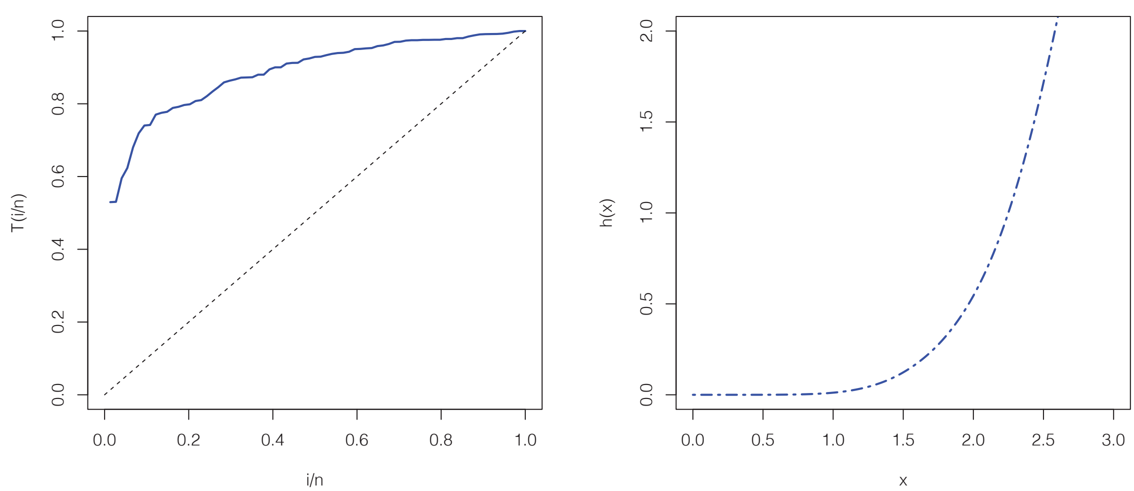

Figure 4 TTT plot for gauge lengths data (left panel) and the Marshall–Olkin power generalized Weibull (MOPGW) hazard rate function (HRF) plot (right panel).

6. CONCLUSIONS

We propose and study the flexible four-parameter Marshall–Olkin power generalized Weibull (MOPGW) distribution. Some mathematical quantities of the MOPGW distributions are derived in explicit expressions. The maximum likelihood is utilized to estimate the MOPGW parameters and the simulation results show its performance. The MOPGW distribution can provide better fits than some other generalized extensions of the Weibull model for two real data sets from the engineering and medicine fields.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

AUTHORS' CONTRIBUTIONS

All authors contributes equally.

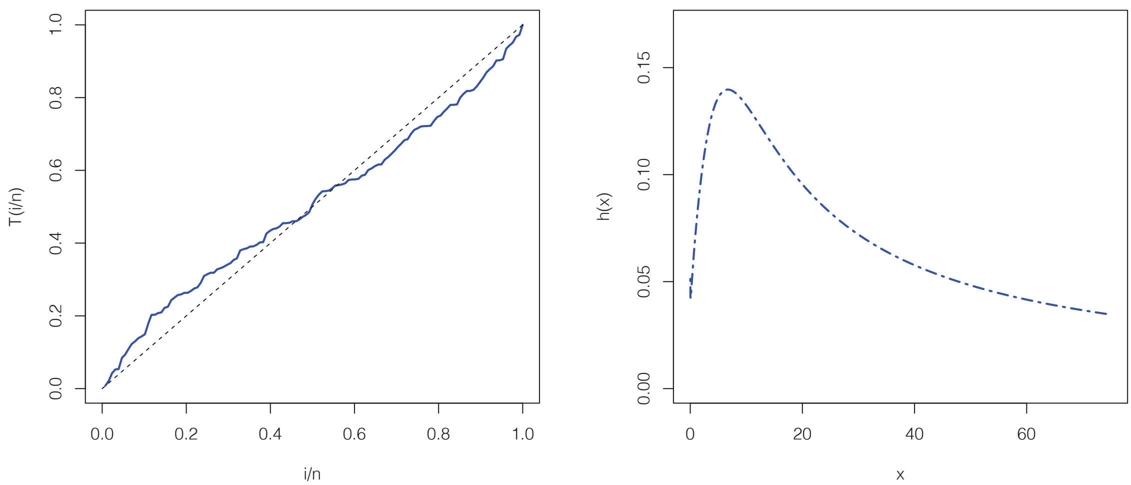

Figure 5 TTT plot for cancer data (left panel) and the Marshall–Olkin power generalized Weibull (MOPGW) hazard rate function (HRF) plot (right panel).

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