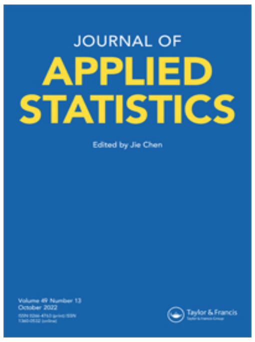

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Inverse Lindley power series distributions: a new compounding family and regression model with censored data

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ABSTRACT

This paper introduces a new class of distributions by compounding the inverse Lindley distribution and power series distributions which is called compound inverse Lindley power series (CILPS) distributions. An important feature of this distribution is that the lifetime of the component associated with a particular risk is not observable, rather only the minimum lifetime value among all risks is observable. Further, these distributions exhibit an unimodal failure rate. Various properties of the distribution are derived. Besides, two special models of the new family are investigated. The model parameters of the two sub-models of the new family are obtained by the methods of maximum likelihood, least square, weighted least square and maximum product of spacing and compared them using the Monte Carlo simulation study. Besides, the log compound inverse Lindley regression model for censored data is proposed. Three real data sets are analyzed to illustrate the flexibility and importance of the proposed models.

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Lindley distribution; inverse Lindley power series distributions; regression model; maximum-likelihood estimators; Monte Carlo simulation

1. Introduction

In the last 30 years or so, we see a spur in the efforts in constructing new univariate distributions which are used widely in statistics and allied areas. These efforts are largely motivated by the theoretical considerations or practical applications or both. However, probability distributions proposed in recent times are different from those proposed before 1997. Present-day researchers take a greater interest in formulating new generators or generalized classes of univariate continuous distributions by adding one or more parameters or by compounding to generate new distributions to a baseline distribution to make the generated distribution more flexible, especially for studying tail behavior. The utility of generating distributions by compounding is that it produces a very flexible class of continuous distribution functions, which in turn can have some interesting physical interpretations also. Besides, the suppleness of such compound distributions makes it possible to have one or

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more hazard rate shapes that may be monotonic decreasing, increasing, bathtub or upside-down bathtub shaped. Notable among such compound distributions introduced recently are as follows: Tahmasbi and Rezaei [38] introduced exponential-logarithmic (EL) distribution; Louzada *et al.* [20] introduced the complementary exponential-geometric (CEG) distribution; Tojeiro *et al.* [39] introduced the complementary Weibull geometric distribution; the Burr XII negative binomial distribution by Ramos *et al.* [31]; the complementary Lindley-geometric distribution by Gui and Guo [15]; a new three-parameter extension of the log-logistic distribution by Shakhatreh [32] and so on.

The power series class of distributions (PSCDs) was proposed and studied by Noack [28]. This class of distributions has some special cases, namely, binomial, geometric, logarithmic, Poisson and negative binomial distributions. For more details on these distributions, one may refer to [16]. Power series class of distributions have been used widely in recent years to develop new distributions. Prominent among these distributions are the exponential-power series (EPS) distribution by Chahkandi and Ganjali [9]; Weibull-power series (WPS) distribution by Moraisa and Barreto-Souz [21]; compound class of extended Weibull PSCDs by Silva *et al.* [35]; Lindley PSCDs by Warahena-Liyanage and Pararai [40]; Burr XII PSCDs by Silva and Cordeiro [36]; generalized modified Weibull PSCDs by Bagheri *et al.* [6]; exponentiated Burr XII PSCDs by Nasir *et al.* [24]; generalized Burr XII PSCDs by Elbatal *et al.* [13] and the references cited therein.

Sharma *et al.* [34] introduced one-parameter inverse Lindley distribution (ILD) to model data exhibiting an upside-down bathtub shaped hazard rate function. Note that the one-parameter Lindley distribution (LD) possesses an increasing hazard rate function, and hence, it cannot be used to fit non-monotone failure rate data. Additionally, it seems that limited attention has been given to studying upside-down bathtub shaped hazard rate function using LD. Even though various generalizations of the LD have been introduced to cover a wide range of shapes of hazard rate function including the unimodal ones. However, these generalizations usually involve four to five parameters and hence complexity arises for these distributions (see, e.g. [3,14] for further details on LD). The one-parameter ILD can be obtained by applying an inverse transformation to the Lindley random variable. The probability density function (PDF) and cumulative distribution function (CDF) of one-parameter ILD (for $x > 0$ and $\theta > 0$) are given, respectively:

$$g_{\mathrm{ILD}}(x;\theta) = \frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)\mathrm{e}^{-\frac{\theta}{x}} \quad \text{and} \quad G_{\mathrm{ILD}}(x;\theta) = \left(1+\frac{\theta}{(\theta+1)x}\right)\mathrm{e}^{-\frac{\theta}{x}}.$$

The ILD distribution is quite amenable as the CDF has a closed form, which eases the computation of the percentiles and the likelihood function for censored data. The hazard rate function (HRF) for the one-parameter ILD is

$$h_{\mathrm{ILD}}(x;\theta) = \frac{\theta^2(1+x)}{x^2[\theta + x(1+\theta)\{\exp(-\theta/x)-1\}]}.$$

It is worth mentioning that Sharma *et al.* [34] showed that the PDF and HRF functions of the ILD are unimodal. For applications of this distribution in a variety of fields, one can refer to [7,8].

Our main goal of this note is to introduce a new class of lifetime distributions called the compound inverse Lindley power series (CILPS) distributions. Several properties of the new distributions are derived. This new class includes several known lifetime distributions, such as the ILD, as a special case. The proposed distribution provides better fits than some well-known lifetime distributions. We are motivated to introduce the CILPS distributions because (i) the upside-down bathtub shaped hazard rate function is frequently encountered in real-life situations; (ii) this new class of distributions due to the stochastic representation $Z = \min(X_1, X_2, \ldots, X_N)$ may be suitable for modeling a complementary risk problem in the presence of latent risks which arise in several areas such as public health, actuarial science, biomedical studies, demography and industrial reliability; (iii) the CILPS distribution can be used to model the first failure of a system that is in a series; and (iv) three real data applications show that it compares well with other competing lifetime distributions in modeling survival and failure data. Besides, we discuss the estimation of the model parameters by four frequentist methods, namely, maximum-likelihood estimators (MLE), least-square estimators (LSE), weighted least-square estimators (WLSE) and maximum product of spacings estimators (MPSE) using two special submodels of the CILPS distribution, namely, compound inverse Lindley Poisson (CILP) distribution and compound inverse Lindley geometric (CILG) distribution, and compare them in terms of their mean-squared errors using extensive numerical simulations and to develop a guideline for choosing the best estimation method that gives better estimates for the model parameters, which we think would be of deep interest to applied statisticians. In this context, many authors have examined various frequentist estimators for estimating the parameters of different distributions. Recent among them, to cite a few, are Nassar *et al.* [25] for a new extension of the Weibull distribution, Afify *et al.* [1] for the heavy-tailed exponential distribution, Dey *et al.* [12] for the weighted inverted Weibull distribution, Al-Mofleh *et al.* [4] for a new two-parameter generalized Ramos–Louzada distribution, Afify and Mohamed [2] for the extended odd Weibull exponential distribution, Nassar *et al.* [26] for the alpha power exponential distribution, Nassar *et al.* [27] for the logarithm transformed Weibull distribution, and Kumar *et al.* [18] for the complementary exponentiated Lomax-Poisson distribution. Further, we obtain the MLEs of the log compound inverse Lindley regression model for censored data to show the flexibility of the log compound inverse Lindley regression model. To the best of our knowledge thus far, no attempt has been made to study the aforementioned methods of estimation for the considered distribution along with regression model for censored and uncensored data.

The rest of the paper is organized as follows. The new family defined by compounding the ILD and zero truncated power series distributions is presented in Section 2. In Section 3, we discuss some of its properties including a mixture representation. In Section 4, we introduce and study two special models of the CILPS family of distribution. In Section 5, the estimation of the model parameters is performed by the methods of maximum likelihood, least square, weighted least square and maximum product of spacing. In Section 6, a simulation study is carried out to compare the performance of the proposed classical estimators (MLE, LSE, WLSE and MPSE) using two special submodels of CILPS distribution, namely, the CILP and the CILG distributions. In Section 7, we present a new regression model, called the log compound inverse Lindley regression model for censored data. Three real data sets are analyzed and presented in Section 8. Finally, we conclude the paper in Section 9.

2. The new family of distributions

In this section, we propose a new family of probability distributions. The new family is defined as follows. Let N be a discrete random variable following a power series (PS) probability distribution with probability mass function (truncated at zero) given by

$$P(N = n) = \frac{b_n \lambda^n}{C(\lambda)}, \quad n = 1, 2, \ldots, \tag{1}$$

where the coefficients b_n's depends only on n, $C(\lambda) = \sum_{n=1}^{\infty} b_n \lambda^n$ is a convergent series and $\lambda > 0$. In Table 1, we give some power series distributions (truncated at zero) defined by Equation (1). Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables having ILD with parameter θ. Given N, let $X_{(1)} = \min(X_1, \ldots, X_N)$, then the conditional CDF of $X_{(1)}$ given $N = n$ is given as

$$G_{X_{(1)}|N=n}(x;\theta) = 1 - (1 - G(x))^n = 1 - \left[1 - \left(1 + \frac{\theta}{(\theta+1)x}\right) e^{-\frac{\theta}{x}}\right]^n$$

and

$$P(X_{(1)} \le x, N = n) = \frac{b_n \lambda^n}{C(\lambda)} \left[1 - \left\{1 - \left(1 + \frac{\theta}{(\theta+1)x}\right) e^{-\frac{\theta}{x}}\right\}^n\right].$$

So, the marginal CDF of $X_{(1)}$ is reduced to

$$F_{\text{CILPS}}(x;\theta,\lambda) = 1 - \frac{C\left[\left(1 - \left(1 + \frac{\theta}{(\theta+1)x}\right) e^{-\frac{\theta}{x}}\right)\lambda\right]}{C(\lambda)}, \tag{2}$$

where $x > 0$, $\theta > 0$ and $\lambda > 0$, and hence, we can clearly say that X follows compound inverse Lindley power series family of distributions with parameter λ and θ and is denoted by $X \sim \text{CILPS}(\lambda, \theta)$. The PDF corresponding to Equation (2) is given by

$$f_{\text{CILPS}}(x,\theta,\lambda) = \frac{\lambda\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right) e^{-\frac{\theta}{x}} \frac{C'\left[\left(1 - \left(1 + \frac{\theta}{(\theta+1)x}\right) e^{-\frac{\theta}{x}}\right)\lambda\right]}{C(\lambda)}. \tag{3}$$

The survival function (SF) and the HRF of CILPS distribution are given, respectively, by

$$S_{\text{CILPS}}(x;\theta,\lambda) = \frac{C\left[\left(1 - \left(1 + \frac{\theta}{(\theta+1)x}\right) e^{-\frac{\theta}{x}}\right)\lambda\right]}{C(\lambda)},$$

Table 1. Useful quantities of some PS distributions.

Distribution	b_n	$C(\lambda)$	$C'(\lambda)$	$C''(\lambda)$	$C^{-1}(\lambda)$	λ
Poisson	$n!^{-1}$	$e^{\lambda} - 1$	e^{λ}	e^{λ}	$\log(\lambda + 1)$	$\lambda \in (0, \infty)$
Logarithmic	n^{-1}	$-\log(1 - \lambda)$	$(1 - \lambda)^{-1}$	$(1 - \lambda)^{-2}$	$1 - e^{-\lambda}$	$\lambda \in (0, 1)$
Geometric	1	$\lambda(1 - \lambda)^{-1}$	$(1 - \lambda)^{-2}$	$2\lambda(1 - \lambda)^{-3}$	$\lambda(\lambda + 1)^{-1}$	$\lambda \in (0, 1)$
Binomial	$\binom{m}{n}$	$(\lambda + 1)^m - 1$	$m(\lambda + 1)^{m-1}$	$\frac{m(m-1)}{(\lambda+1)^{2-m}}$	$(\lambda - 1)^{\frac{1}{m}} - 1$	$\lambda \in (0, \infty)$
Negative binomial	$\binom{n-1}{m-1}$	$\frac{\lambda^m}{(1-\lambda)^m}$	$\frac{m\lambda^{m-1}}{(1-\lambda)^{m+1}}$	$\frac{m(m+2\lambda-1)}{\lambda^{2-m}(1-\lambda)^{m+2}}$	$\frac{\lambda^{\frac{1}{m}}}{1+\lambda^{\frac{1}{m}}}$	$\lambda \in (0, 1)$

and

$$h_{\text{CILPS}}(x;\theta,\lambda) = \frac{\frac{\lambda\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)\mathrm{e}^{-\frac{\theta}{x}}C'\left[\left(1-\left(1+\frac{\theta}{(\theta+1)x}\right)\mathrm{e}^{-\frac{\theta}{x}}\right)\lambda\right]}{C\left[\left(1-\left(1+\frac{\theta}{(\theta+1)x}\right)\mathrm{e}^{-\frac{\theta}{x}}\right)\lambda\right]}. \tag{4}$$

In addition to the above mathematical derivation, the proposed family of distributions, i.e.CILPS distribution, has an interesting physical interpretation given as follows. Consider the failure of a certain mechanical component due to an unknown number of risk factors say, N, such that each risk factor can be determined once the failure has been occurred, in which case it is fixed immediately. If we assume that the failure lifetimes of these risk factors; $X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.) ILD random variables independent of N, which follows a PS distribution, then the time to the first failure is appropriately modeled by the CILPS family.

3. General properties

3.1. Mathematical properties

In this section, we study some of the mathematical properties of the CILPS family of distribution. The following proposition shows that the proposed family of distribution includes the ILD as a limiting distribution.

Proposition 3.1: *If $\lambda \to 0$, then the limiting case of the CILPS distribution is an ILD.*

Proof: See Appendix. ■

The following proposition reveals that the HRF of the CILPS distribution possesses an upside-down bathtub shaped hazard rate function.

Proposition 3.2: *The HR function of the CILPS distribution is unimodal.*

Proof: See Appendix. ■

Proposition 3.3: *The PDF of the CILPS family of distribution can be expressed as an infinite mixture of Lehmann type II (LTII) inverse Lindley densities with power parameter n and is given by*

$$f_{\text{CILPS}}(x;\theta,\lambda) = \frac{1}{C(\lambda)}\sum_{n=0}^{\infty} b_n\lambda^n\pi_n(x;\theta), \tag{5}$$

where $\pi_n(x;\theta) = n\,\bar{G}_{\text{ILD}}(x;\theta)^{n-1}g_{\text{ILD}}(x;\theta)$ denotes the LTII inverse Lindley density function with power parameter n, and $\bar{G}_{\text{ILD}}(x;\theta) = 1 - \bar{G}_{\text{ILD}}(x;\theta)$.

Proof: See Appendix. ■

Notice that Proposition 3.3 provides a useful expansion for the density function given in Equation (3), which can enable us to derive some statistical quantities of the CILPS distribution such as moments and generating function from those of the LTII inverse Lindley distribution.

The following proposition establishes the derivation for the mean residual life (MRL) of the CILPS distribution. The MRL function is another important reliability measure used in engineering reliability, particularly in maintenance scheduling.

Proposition 3.4: *The mean residual life (MRL) of the CILPS distribution can be obtained as follows:*

$$m(x) = \frac{1}{C(\lambda)S(t)} \sum_{n=1}^{\infty} b_n \sum_{k=0}^{n} \sum_{m=0}^{k} \sum_{l=0}^{\infty} (-1)^{k+l} \frac{\theta^k k^l \lambda^k}{l!} \binom{n}{k} \binom{k}{m} \left(\frac{\theta}{1+\theta}\right)^m \frac{1}{(m+l-1)t^{m+l-1}}.$$

Proof: See Appendix. ■

3.2. Moments

Moments for any distribution function, especially the first four moments are important in describing the distribution. In Proposition 3.5, we derive the kth moment of the CILPS distribution.

Proposition 3.5: *Let* $X \sim \text{CILPS}(\lambda, \theta)$. *The kth moment of X is*

$$\mu_k = E(X^k) = \frac{\theta^2}{\theta+1} \sum_{n=1}^{\infty} \frac{n a_n \lambda^n}{C(\lambda)} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i \psi_{i,k}, \tag{6}$$

where

$$\psi_{i,k} = \sum_{m=0}^{i} \binom{i}{m} \left(\frac{\theta}{\theta+1}\right)^{i-m} \left(\frac{\Gamma(k-m+i+2)}{((i+1)\theta)^{k-m+i+2}} + \frac{\Gamma(i-k-m+1)}{((i+1)\theta)^{i-k-m+1}} \right).$$

The proof of Proposition 3.5 can be obtained similar to the lines of the proof of Proposition 3.4. Notice that the kth moments are given in terms of a convergent infinite series. Therefore, in order to use these moments, one should consider some fixed terms in Equation (6). In the case of the two special sub-models considered in the following section, we use the first 50 terms in Equation (6). Alternatively, one may use Monte Carlo methods or numerical integration methods to approximate these moments, see, for example, Shakhatreh *et al.* [33] who used these methods to approximate the moments for the generalized extended exponential-Weibull distribution.

4. Two special sub-models

4.1. The CILP distribution

The CILP distribution is defined from the CDF given via Equation (2) with $C(\lambda) = e^{\lambda} - 1$ and is given by

$$F_{\text{CILP}}(x; \lambda, \theta) = \frac{e^{\lambda} - \exp\left[\left(1 - \left(1 + \frac{\theta}{(\theta+1)x}\right) e^{\frac{-\theta}{x}}\right) \lambda\right]}{e^{\lambda} - 1},$$

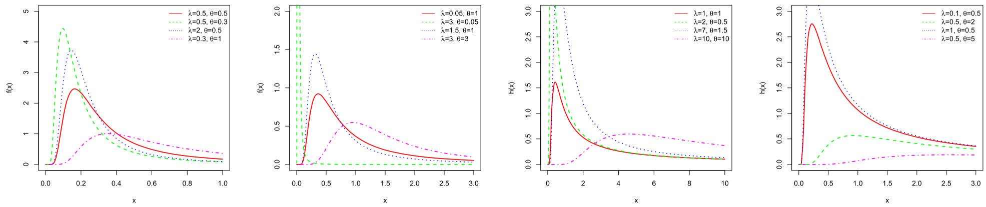

Figure 1. Plots of the PDFs and HRFs of CILP distribution for some parameter values.

Table 2. Mean, variance, skewness, and kurtosis for CILP distribution.

θ	λ	Mean	Variance	Skewness	Kurtosis
0.5	0.5	0.9371	12.2864	14.2616	274.4604
	1.5	0.6473	7.0413	18.7528	474.2531
	2.5	0.4502	3.7359	25.5623	884.1317
	3.5	0.3257	1.8591	35.8833	1754.415
1	0.5	2.1137	34.1699	8.5584	100.7417
	1.5	1.4694	20.0545	11.149	169.2361
	2.5	1.024	10.8766	15.0107	306.6268
	3.5	0.7385	5.5278	20.7429	590.374
1.5	0.5	3.2708	57.6727	6.5347	60.4529
	1.5	2.3026	34.5751	8.4568	99.0846
	2.5	1.6223	19.1373	11.2733	175.0051
	3.5	1.1789	9.9238	15.368	327.7227
2.5	0.5	5.3967	102.7195	4.7771	34.591
	1.5	3.8952	63.8246	6.1274	54.3548
	2.5	2.8088	36.6015	8.0461	91.9044
	3.5	2.0801	19.6748	10.7324	164.0487

where $x > 0$. Notice that the PDF given in Equation (7) is well defined for $\lambda \in \mathbb{R}$. The associated PDF, SF, and HRF are, respectively, given by

$$f_{\text{CILP}}(x;\lambda,\theta) = \frac{\theta^2}{\theta+1}\left(\frac{x+1}{x^3}\right)\lambda e^{\frac{-\theta}{x}}\,\frac{\exp\left[\left(1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{\frac{-\theta}{x}}\right)\lambda\right]}{e^{\lambda}-1},$$

$$S_{\text{CILP}}(x) = \frac{\exp\left[1-\left(\left(1+\frac{\theta}{(\theta+1)x}\right)e^{\frac{-\theta}{x}}\right)\lambda\right]-1}{e^{\lambda}-1},$$

$$h_{\text{CILP}}(x;\lambda,\theta) = \frac{\frac{\theta^2}{\theta+1}\lambda\, e^{-\frac{\theta}{x}}\left(\frac{x+1}{x^3}\right)\exp\left[\left(1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\lambda\right]}{\exp\left[\left(1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right)\lambda\right]-1}, \tag{7}$$

$x > 0$ and $\lambda \in \mathbb{R}$. Figure 1 displays plots of PDFs and HRFs for the CILP distribution for some selected values of λ and θ. The figure reveals that both PDF and HR functions of the CILP distribution are unimodal.

Table 2 reports the mean, variance, skewness, and kurtosis for the CILP distribution for some selected values of λ and θ. From Table 2, we observe that both mean and variance decrease when λ and θ increase, while skewness and kurtosis increase when λ and θ increase.

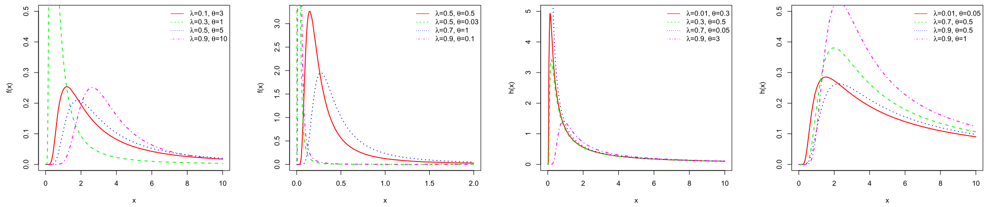

Figure 2. Plots of the PDFs and HRFs of CILG distribution for some parameters.

4.2. The CILG distribution

The CILG distribution is defined from the CDF given in Equation (2) with $C(\lambda) = \lambda(1-\lambda)^{-1}$. Therefore, the CDF of the CILG distribution is

$$F_{\text{CILG}}(x;\lambda,\theta) = \frac{\left(1-\left(1+\frac{\theta}{(\theta+1)x}\right)\mathrm{e}^{-\frac{\theta}{x}}\right)(1-\lambda)}{1-\left(1-\left(1+\frac{\theta}{(\theta+1)x}\right)\mathrm{e}^{-\frac{\theta}{x}}\right)\lambda}, \tag{8}$$

$x > 0$ and $\theta,\ \lambda > 0$. However, the CDF of the CILG distribution in Equation (8) is well defined for $\lambda < 1$. The associated PDF, SF and HRF are given for $x > 0$, respectively, by

$$f_{\text{CILG}}(x;\lambda,\theta) = \frac{\left(\frac{\theta^2}{1+\theta}\right)\left(\frac{1+x}{x^3}\right)\mathrm{e}^{-\frac{\theta}{x}}(1-\lambda)}{\left[1-\left(1-\left(1+\frac{\theta}{(\theta+1)x}\right)\mathrm{e}^{-\frac{\theta}{x}}\right)\lambda\right]^2},$$

$$S_{\text{CILG}}(x;\lambda,\theta) = \frac{1-\lambda^2\left(1-\left(1+\frac{\theta}{(\theta+1)x}\right)\mathrm{e}^{-\frac{\theta}{x}}\right)}{1-\left(1-\left(1+\frac{\theta}{(\theta+1)x}\right)\mathrm{e}^{-\frac{\theta}{x}}\right)\lambda}$$

and $$h_{\text{CILG}}(x;\lambda,\theta) = \frac{\left(\frac{\theta^2}{1+\theta}\right)\left(\frac{1+x}{x^3}\right)\mathrm{e}^{-\frac{\theta}{x}}(1-\lambda)}{1-\lambda^2\left(1-\left(1+\frac{\theta}{(\theta+1)x}\right)\mathrm{e}^{-\frac{\theta}{x}}\right)},$$

where $x > 0; \theta > 0, \lambda < 1$. Figure 2 displays plots of PDF and HRF of the CILG distribution for some selected values of λ and θ. Similarly, the figure reveals that both PDF and HRF of the CILG distribution are unimodal. Table 3 reports some basic statistical measures such as the mean, variance, skewness, and kurtosis for the CILG distribution for some selected values of λ and θ. Similarly, we observe that both mean and variance decrease when λ and θ increase, while skewness and kurtosis increase when λ and θ increase.

5. Parameter estimation

In this section, different classical methods of estimation, namely maximum likelihood, weighted least square, ordinary least square and maximum product of spacing, are considered to get the estimates of the parameters of the model.

Table 3. Mean, variance, skewness, and kurtosis for CILG distribution.

θ	λ	Mean	Varinace	Skewness	Kurtosis
0.5	0.3	0.8754	11.2076	14.926	300.4469
	0.5	0.6976	8.1173	17.5027	412.7296
	0.7	0.5003	4.9514	22.333	672.0396
	0.9	0.2624	1.6889	37.9566	1947.798
1	0.3	1.9761	31.2938	8.9481	109.7963
	0.5	1.5783	22.9574	10.4493	148.6349
	0.7	1.1313	14.2181	13.2553	237.7219
	0.9	0.5858	4.9512	22.2947	671.2451
1.5	0.3	3.0633	53.0021	6.8287	65.6352
	0.5	2.4617	39.3271	7.9533	87.7262
	0.7	1.7761	24.6937	10.0478	138.1144
	0.9	0.9244	8.7671	16.7621	380.919
3	0.3	5.993	114.432	4.4878	31.204
	0.5	4.9061	87.2476	5.2062	40.3956
	0.7	3.6205	56.6687	6.5312	61.1433
	0.9	1.9431	21.1464	10.7126	158.8445

5.1. Maximum-likelihood estimators

Among the different statistical methods of estimation, the maximum-likelihood (ML) method is widely used due to its desirable properties including consistency, asymptotic efficiency, and invariance. Here, the unknown parameters of the CILPS distribution are estimated by using the method of maximum likelihood. Suppose that $X_1, X_2, \dots, X_n$ is a random sample from the CILPS distribution with unknown parameters θ and λ. The likelihood function of CILPS is given by

$$L(\theta,\lambda \mid x_1,\dots,x_n) = \left(\frac{\lambda\theta^2}{\theta+1}\right)^n \prod_{i=1}^{n}\left(\frac{1+x_i}{x_i^3}\right) e^{-\frac{\theta}{x_i}} \frac{C'\left[\left(1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right)\lambda\right]}{C(\lambda)}. \tag{9}$$

The corresponding log likelihood function ($\ell(\theta,\lambda)$ is given by

$$\ell(\theta,\lambda) = n\log\lambda + 2n\log\theta - n\log(\theta+1) + \sum_{i=1}^{n}\log(1+x_i) - \sum_{i=1}^{n}\log x_i^3 - \sum_{i=1}^{n}\frac{\theta}{x_i} + \sum_{i=1}^{n}\log\left[\frac{C'\left(\left(1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right)\lambda\right)}{C(\lambda)}\right]. \tag{10}$$

The score functions are $\mathbf{S} = (\frac{\partial\ell(\theta,\lambda)}{\partial\theta}, \frac{\partial\ell(\theta,\lambda)}{\partial\lambda})^T$, where its elements are given by, respectively,

$$\frac{\partial\ell(\theta,\lambda)}{\partial\theta} = \frac{2n}{\theta} - \frac{n}{1+\theta} - \sum_{i=1}^{n}\frac{1}{x_i} + \sum_{i=1}^{n}\frac{C''\left(\left(1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right)\lambda\right)}{C\prime\left(\left(1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right)\lambda\right)}$$

$$\times \lambda \left(\frac{e^{-\frac{\theta}{x_i}}}{x_i} \left(-\frac{1}{(\theta+1)^2} + \frac{\theta}{(\theta+1)x_i} + 1 \right) \right)$$

and

$$\frac{\partial \ell(\theta,\lambda)}{\partial \lambda} = \frac{n}{\lambda} - \frac{nc\prime(\lambda)}{c(\lambda)} + \sum_{i=1}^{n} \frac{C''\left(\left(1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right)\lambda\right)}{C'\left(\left(1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right)\lambda\right)}$$
$$\times \left(1-\left(1+\frac{\theta}{(\theta+1)x_i}\right)e^{-\frac{\theta}{x_i}}\right).$$

On equating the score equations to 0, i.e. $\mathbf{S} = 0$, and solving these equations simultaneously, the ML estimates can be obtained.

5.2. Ordinary least-square estimators

Here, regression-based method estimators of the parameters are obtained: namely, the ordinary least-square (OLS) estimators. This approach was suggested by Swain *et al.* [37] to estimate the parameters of the beta distribution but of course it can be used to any other continuous distribution as well. The OLS method is described as follows. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the order statistics of a random sample of size n from a population with CDF $F(x)$. It is well known that $E(F(X_{i:n})) = i/(n+1)$ (see, e.g. [37]). Therefore, the OLS estimate can be obtained by minimizing the sum of the squared differences between the CDF to each order statistic; $F(X_{i:n})$ and the corresponding expected value; $E(F(X_{i:n}))$, i.e. by minimizing $\sum_{i=1}^{n}[F(x_{i:n}) - \frac{i}{n+1}]^2$. For the CILPS distribution with CDF given in Equation (2), the OLS estimates $\hat{\theta}_{OLS}$ and $\hat{\lambda}_{OLS}$ of θ and λ can be determined by minimizing the following function:

$$S(\lambda,\theta) = \sum_{i=1}^{n} \left[F_{\mathrm{CILPS}}(x_{i:n};\theta,\lambda) - \frac{i}{n+1} \right]^2, \tag{11}$$

with respect to λ and θ. These estimates can be obtained by solving the following equations:

$$\frac{\partial S(\lambda,\theta)}{\partial \theta} = \sum_{i=1}^{n} \left[F_{\mathrm{CILPS}}(x_{i:n};\theta,\lambda) - \frac{i}{n+1} \right] \psi_1(x_{i:n} \mid \lambda,\theta) = 0,$$
$$\frac{\partial S(\lambda,\theta)}{\partial \lambda} = \sum_{i=1}^{n} \left[F_{\mathrm{CILPS}}(x_{i:n};\theta,\lambda) - \frac{i}{n+1} \right] \psi_2(x_{i:n} \mid \lambda,\theta) = 0,$$

where $\psi_1(x_{i:n} \mid \lambda,\theta)$ and $\psi_2(x_{i:n} \mid \lambda,\theta)$ are given by

$$\psi_1(x_{i:n} \mid \lambda,\theta) = \frac{C'\left(\left(1-\left(1+\frac{\theta}{(\theta+1)x_{i:n}}\right)e^{-\frac{\theta}{x_{i:n}}}\right)\lambda\right)}{C(\lambda)}$$

$$\times \left[-\lambda \left(\frac{e^{-\frac{\theta}{x_{i:n}}}}{x_{i:n}} \left(\frac{1}{(\theta+1)^2} - \frac{\theta}{(\theta+1)x_{i:n}} - 1 \right) \right) \right], \tag{12}$$

$$\psi_2(x_{i:n} \mid \lambda, \theta) = \frac{\left(1 + \frac{\theta}{(\theta+1)x_{i:n}}\right) e^{-\frac{\theta}{x_{i:n}}} C' \left[\left(1 - \left(1 + \frac{\theta}{(\theta+1)x_{i:n}}\right) e^{-\frac{\theta}{x_{i:n}}} \right) \lambda \right]}{[C(\lambda)]} - \frac{C'(\lambda) C \left[\left(1 - \left(1 + \frac{\theta}{(\theta+1)x_{i:n}}\right) e^{-\frac{\theta}{x_{i:n}}} \right) \lambda \right]}{[C(\lambda)]^2}. \tag{13}$$

5.3. Weighted least-squares estimators

Similar to the OLS method and on using the same notation, the weighted least-squares (WLS) estimators can be obtained by minimizing $\sum_{i=1}^{n} w(n,i)[F(x_{i:n}) - \frac{i}{n+1}]^2$, where $w(n,i)$ is a weight function. Note that the only difference between OLS and WLS is the weight function. This weight function is taken to be the inverse of the variance of $F(x_{i:n})$ (see [37] for further details). So the WLS estimates of λ and θ denoted by $\hat{\lambda}_{\text{WLS}}$ and $\hat{\theta}_{\text{WLS}}$ for the CILPS distribution can be obtained by minimizing the following function with respect to λ and θ

$$W(\lambda, \theta) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F_{\text{CILPS}}(x_{i:n}; \theta, \lambda) - \frac{i}{n+1} \right]^2. \tag{14}$$

The WLS estimates, namely, $\hat{\lambda}_{\text{WLS}}$ and $\hat{\theta}_{\text{WLS}}$ of the parameters can be obtained by solving the following nonlinear equations:

$$\frac{\partial W(\lambda, \theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F_{\text{CILPS}}(x_{i:n}; \theta, \lambda) - \frac{i}{n+1} \right] \psi_1(x_{i:n} \mid \lambda, \theta) = 0,$$

$$\frac{\partial W(\lambda, \theta)}{\partial \lambda} = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F_{\text{CILPS}}(x_{i:n}; \theta, \lambda) - \frac{i}{n+1} \right] \psi_2(x_{i:n} \mid \lambda, \theta) = 0,$$

where $\psi_1(x_{i:n} \mid \lambda, \theta)$ and $\psi_2(x_{i:n} \mid \lambda, \theta)$ are provided in Equations (12) and (13), respectively.

5.4. Maximum product of spacing estimators

Cheng and Amin [10,11] presented an alternative method to the MLE method to estimate the parameters of continuous univariate distributions. The MPS method gives consistent estimators and they are asymptotically normal and efficient as the ML estimates. Additionally, Anatolyev and Kosenok [5] investigated the invariance property of the MPS estimators and showed that they have the same property as the ML estimates. For the CILPS distribution define

$$D_i(\lambda, \theta) = F_{\text{CILPS}}(x_{i:n}; \theta, \lambda) - F_{\text{CILPS}}(x_{i-1:n}; \theta, \lambda), \quad i = 1, \ldots, n,$$

where $F_{\text{CILPS}}(x_{0:n};\theta,\lambda)=0$ and $F_{\text{CILPS}}(x_{n+1:n};\theta,\lambda)=1$. The MPS estimates $\hat{\lambda}_{\text{MPS}}$ and $\hat{\theta}_{\text{MPS}}$ of λ and θ can be determined by maximizing the function

$$M(\lambda,\theta)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i(\lambda,\theta), \tag{15}$$

in relation to λ and θ. Equivalently, these estimates can also be obtained by solving the equations

$$\frac{\partial M(\lambda,\theta)}{\partial\theta}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{\psi_1(x_{i:n}\,|\,\lambda,\theta)-\psi_1(x_{i-1:n}\,|\,\lambda,\theta)}{D_i(\lambda,\theta)}=0,$$

$$\frac{\partial M(\lambda,\theta)}{\partial\lambda}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{\psi_2(x_{i:n}\,|\,\lambda,\theta)-\psi_2(x_{i-1:n}\,|\,\lambda,\theta)}{D_i(\lambda,\theta)}=0,$$

where $\psi_1(x_{i:n}\,|\,\lambda,\theta)$ and $\psi_2(x_{i:n}\,|\,\lambda,\theta)$ are given in Equations (12) and (13), respectively.

5.5. Computational issues

Apparently, solutions to the equations (10), (11), (14), and (15) cannot be obtained in closed forms. Thus, we utilize the differential evolution technique (see [29]), which is operated in the R software to obtain the estimates for all considered estimation methods.

6. Simulation study

In this section, we provide two algorithms for simulating random samples from the CILPS distribution given via Equation (3). Additionally, Monte Carlo simulation experiments are conducted in order to evaluate the performance of the parameters of CILPS.

6.1. Generation algorithm

Algorithm 6.1: *In this algorithm, generating random samples from the CILPS distribution can be carried out based on the mixture form of the ILD. Observe that the pdf of the ILD is composed of a two-component mixture of an inverse gamma distribution with shape 2 and scale θ parameters, denoted as* INGAMMAD$(2,\theta)$ *and by an inverse exponential distribution with scale parameter θ, denoted as* IED(θ). *Specifically, $g(x)=\gamma g_1(x)+(1-\gamma)g_2(x)$, where $\gamma=\theta/(1+\theta)$, $g_1=\text{IED}(\theta)$ and $g_2=\text{INGAMMAD}(2,\theta)$. (1) generate N_i from the PS(λ) given via Equation* (1), *for $i=1,\dots,n$;* (2) *for $j=1,\dots,N_i$, and $i=1,\dots,n$, do; generate $U_{i,j}\sim\text{Uniform}(0,1)$; generate $E_{i,j}\sim\text{IED}(\theta)$; generate $G_{i,j}\sim\text{INGAMMAD}(2,\theta)$; if $U_{i,j}\le\gamma$, then set $X_{i,j}=E_{i,j}$, else set $X_{i,j}=G_{i,j}$;* (3) *put $Y_i=\min(X_{i,1},\dots,X_{i,N_i})$, $i=1,\dots,n$.*

Algorithm 6.2: *This algorithm can be used for generating random samples from CILPS using quantile functions of the CILPS distribution.* (1) *Generate $u_i\sim\text{Uniform}(0,1)$, for $i=1,\dots,n$; (2) solve*

$$x_i-\frac{1}{\theta}\ln\left[1+\frac{\theta}{(1+\theta)x_i}\right]=\frac{1}{\theta}\left(\frac{1}{\lambda}[1-C^{-1}(C(\lambda)(1-u_i))]-1\right).$$

6.2. Monte Carlo simulation study

In this subsection, we assess the behavior of the estimators obtained from the four different estimation methods in estimating parameters of the CILPS distribution, by conducting simulation experiments. The R function `DEoptim` given in [22], which takes into account the procedure called *differential evolution*, is used in order to obtain the estimates from the objective functions given in Equations (10), (11), (14) and (15). Steps of the simulation studies are explained as follows: (i) set initial values for the parameters λ and θ, and designate the sample size n; (ii) use Algorithm 6.1 to obtain a random sample of size n from the CILPS model; (iii) compute the estimates of the CILPS distribution parameters; and (iv) redo steps (ii) and (iii) N times. We take $n = 20, 50, 80, 100, 150$, and 200 and different values for the distribution parameters. Additionally, we take $N = 1000$. The finite sample behavior of the ML, OLS, WLS, and MPS estimates are assessed based on the following quantities for each sample size: the average estimate (AE) and the root-mean-square error (RMSE). Table 4 report the results of the Monte Carlo simulation for the sub-model CILP distribution, whereas Table 5 list the results of the Monte Carlo simulation for the sub-model CILG distribution. The average values of estimates and RMSEs of ML, LSE, WLSE and MPSE estimates corresponding to the submodels namely, CILP and CILG are obtained and reported in Tables 4 and 5, respectively. These findings demonstrate interesting facts. The behavior of the estimates of the parameters for the two sub-models namely, CILP and CILG distributions obtained from the four estimation methods is quite satisfactory, showing little bias and satisfactory RMSEs in all cases studied; that is, these estimates are reliable, and particularly quite near to the true values, indicating that these estimates behaved asymptotically unbiased. Moreover, the RMSE decays to zero as $n \to \infty$. Hence, it implies that these estimates are consistent. It is worth mentioning that further computational effort is necessary for universal results regarding the behavior of these estimators for the submodels. This sort of study is outside the scope of this paper and will be pursued in future papers.

7. The log compound inverse Lindley regression model with censored data

Let X be a random variable that follows the CILPS(λ, θ) distribution. Then, the random variable $Y = \log(X)$ delineates the *log-compound inverse Lindley power series* (LCIPS). Putting $\theta = e^{\mu}$, the PDF of Y is given by

$$f(y) = \frac{\lambda[\exp(-\mu) + \exp(y-\mu)]}{1+\exp(-\mu)} \exp[-2(y-\mu) - \exp(-(y-\mu))] \times \frac{C'\left[\lambda\left\{1 - \left\{1 + \frac{\exp(-(y-\mu))}{\exp(\mu)+1}\right\}\exp[-\exp(-(y-\mu))]\right\}\right]}{C[\lambda]}, \tag{16}$$

where $y \in \mathbb{R}$, $\lambda > 0$, and $\mu \in \mathbb{R}$ is a location parameter. From now and on, we use the notation $Y \sim$ LCILPS(λ, μ) for a random variable Y with PDF given via Equation (16). The corresponding SF is given by

$$S(y) = \frac{C\left[\lambda\left\{1 - \left\{1 + \frac{\exp(-(y-\mu))}{\exp(\mu)+1}\right\}\exp[-\exp(-(y-\mu))]\right\}\right]}{C[\lambda]}. \tag{17}$$

Table 4. AEs and the corresponding RMSEs (in parenthesis) for the sub-model CILP distribution.

n	Parameters	ML	OLS	WLS	MPS
20	$\lambda = 0.5$	0.4300 (0.1884)	0.3676 (0.1896)	0.3755 (0.1909)	0.3433 (0.1847)
	$\theta = 0.5$	0.5034 (0.0683)	0.4903 (0.0713)	0.4912 (0.0700)	0.4776 (0.0708)
	$\lambda = 1$	0.9803 (0.3221)	0.8852 (0.3206)	0.8991 (0.3234)	0.8436 (0.3121)
	$\theta = 0.5$	0.5059 (0.0675)	0.4930 (0.0718)	0.4939 (0.0705)	0.4782 (0.0707)
	$\lambda = 0.5$	0.4306 (0.1888)	0.3892 (0.1887)	0.3949 (0.1913)	0.3636 (0.1868)
	$\theta = 2$	1.9983 (0.2451)	1.9501 (0.2532)	1.9536 (0.2501)	1.9042 (0.2493)
	$\lambda = 3$	2.9934 (0.3192)	2.9270 (0.3162)	2.9299 (0.3186)	2.8679 (0.3097)
	$\theta = 3$	2.9909 (0.2644)	2.9469 (0.2688)	2.9500 (0.2667)	2.8903 (0.2619)
50	$\lambda = 0.5$	0.4348 (0.1850)	0.3985 (0.1921)	0.4050 (0.1911)	0.3604 (0.1868)
	$\theta = 0.5$	0.4975 (0.0492)	0.4896 (0.0490)	0.4903 (0.0478)	0.4804 (0.0482)
	$\lambda = 1$	0.9841 (0.3114)	0.9098 (0.3204)	0.9298 (0.3208)	0.8589 (0.3084)
	$\theta = 0.5$	0.5039 (0.0507)	0.4944 (0.0517)	0.4965 (0.0506)	0.4841(0.0506)
	$\lambda = 0.5$	0.4294 (0.1839)	0.3827 (0.1865)	0.3974 (0.1856)	0.3532 (0.1810)
	$\theta = 2$	1.9860 (0.1888)	1.9511 (0.1990)	1.9585 (0.1919)	1.9104 (0.1865)
	$\lambda = 3$	2.9859 (0.3076)	2.9246 (0.3085)	2.9424 (0.3111)	2.8830 (0.3076)
	$\theta = 3$	2.9913 (0.2142)	2.9545 (0.2233)	2.9635 (0.2192)	2.9158 (0.2163)
80	$\lambda = 0.5$	0.4378 (0.1809)	0.4030 (0.1903)	0.4121 (0.1875)	0.3731 (0.1873)
	$\theta = 0.5$	0.4991 (0.0424)	0.4921 (0.0438)	0.4937 (0.0422)	0.4860 (0.0424)
	$\lambda = 1$	1.0072 (0.3035)	0.9510 (0.3105)	0.9740 (0.3061)	0.9043 (0.3041)
	$\theta = 0.5$	0.5033 (0.0433)	0.4956 (0.0439)	0.4981 (0.0425)	0.4883 (0.0429)
	$\lambda = 0.5$	0.4430 (0.1792)	0.4043 (0.1868)	0.4174 (0.1843)	0.3794 (0.1831)
	$\theta = 2$	1.9859 (0.1706)	1.9528 (0.1741)	1.9610 (0.1707)	1.9280 (0.1712)
	$\lambda = 3$	2.9777 (0.3019)	2.9233 (0.3071)	2.9429(0.3034)	2.8827(0.2955)
	$\theta = 3$	3.0037(0.1851)	2.9745 (0.1904)	2.9849 (0.1855)	2.9400 (0.1883)
100	$\lambda = 0.5$	0.4335 (0.1824)	0.4039 (0.1890)	0.4136 (0.1856)	0.3723 (0.1848)
	$\theta = 0.5$	0.4996 (0.0384)	0.4939 (0.0383)	0.4953 (0.0375)	0.4878 (0.0382)
	$\lambda = 1$	0.9780 (0.2978)	0.9382 (0.3084)	0.9509 (0.3039)	0.8774 (0.2959)
	$\theta = 0.5$	0.5017 (0.0405)	0.4966 (0.0417)	0.4981 (0.0401)	0.4878 (0.0400)
	$\lambda = 0.5$	0.4186 (0.1826)	0.3946 (0.1851)	0.4049 (0.1857)	0.3616 (0.1824)
	$\theta = 2$	1.9841 (0.1537)	1.9644 (0.1561)	1.9696 (0.1494)	1.9328 (0.1526)
	$\lambda = 3$	2.9896 (0.2930)	2.9438 (0.2972)	2.9599 (0.2936)	2.8997 (0.2903)
	$\theta = 3$	2.9971 (0.1846)	2.9735 (0.1839)	2.9808 (0.1807)	2.9403 (0.1841)
150	$\lambda = 0.5$	0.4320 (0.1778)	0.4048 (0.1844)	0.4158 (0.1843)	0.3752 (0.1808)
	$\theta = 0.5$	0.4961 (0.0317)	0.4911 (0.0323)	0.4925 (0.0314)	0.4865 (0.0313)
	$\lambda = 1$	0.9981 (0.2904)	0.9661 (0.2963)	0.9828 (0.2927)	0.9159 (0.292)
	$\theta = 0.5$	0.5015 (0.0350)	0.4972 (0.0356)	0.4990 (0.0344)	0.4907 (0.035)
	$\lambda = 0.5$	0.4413 (0.1746)	0.4135 (0.1811)	0.4236 (0.1778)	0.3868 (0.1789)
	$\theta = 2$	1.9889 (0.1428)	1.9661 (0.1391)	1.9725 (0.1359)	1.9467 (0.1426)
	$\lambda = 3$	2.9881 (0.2829)	2.9469 (0.2894)	2.9670 (0.2858)	2.9110 (0.2854)
	$\theta = 3$	3.0038 (0.1624)	2.9813 (0.1643)	2.9911 (0.1600)	2.9577 (0.1639)
200	$\lambda = 0.5$	0.4355 (0.1772)	0.4117 (0.1847)	0.4208 (0.1819)	0.3883 (0.1791)
	$\theta = 0.5$	0.4980 (0.0281)	0.4939 (0.0287)	0.4952 (0.0275)	0.4902 (0.0282)
	$\lambda = 1$	0.9950 (0.2743)	0.9673 (0.2853)	0.9771 (0.2804)	0.9167 (0.2783)
	$\theta = 0.5$	0.5005 (0.0317)	0.4967 (0.0320)	0.4978 (0.0309)	0.4910 (0.0318)
	$\lambda = 0.5$	0.4363 (0.1730)	0.4134 (0.1784)	0.4217 (0.1769)	0.3871 (0.1766)
	$\theta = 2$	1.9805 (0.1267)	1.9611 (0.1294)	1.9672 (0.1260)	1.9447 (0.1266)
	$\lambda = 3$	3.0104 (0.2686)	2.9755 (0.2769)	2.9935 (0.2718)	2.9426 (0.2731)
	$\theta = 3$	3.0148 (0.1468)	2.9952 (0.1523)	3.0038 (0.1462)	2.9760 (0.1478)

We define the standardized rv $Z = (Y - \mu)$ having density function

$$\phi(z; \lambda, \mu) = \frac{\lambda[\exp(-\mu) + \exp(z)]}{1 + \exp(-\mu)} \exp[-2z - \exp(-z)] \times \frac{C'\left[\lambda\left\{1 - \left\{1 + \frac{\exp(-z)}{\exp(\mu)+1}\right\}\exp[-\exp(-z)]\right\}\right]}{C[\lambda]}. \tag{18}$$

Table 5. AEs and the corresponding RMSEs (in parenthesis) for the sub-model CILG distribution.

n	Parameters	ML	OLS	WLS	MPS
20	$\lambda = 0.5$	0.4533 (0.1682)	0.3926 (0.1794)	0.4017 (0.1789)	0.3695 (0.1745)
	$\theta = 0.5$	0.5004 (0.0716)	0.4740 (0.0731)	0.4767 (0.0715)	0.4606 (0.0752)
	$\lambda = 0.5$	0.4910 (0.1289)	0.4439 (0.1383)	0.4510 (0.1386)	0.4259 (0.1352)
	$\theta = 1$	1.0252(0.1512)	0.9767 (0.1501)	0.9812 (0.1476)	0.9462 (0.1528)
	$\lambda = 0.7$	0.4662 (0.1033)	0.4432 (0.0863)	0.4483 (0.0904)	0.4374 (0.0822)
	$\theta = 2$	2.1622 (0.2008)	2.1487 (0.2111)	2.1517 (0.2066)	2.0600 (0.2448)
	$\lambda = 0.9$	0.6407 (0.0658)	0.6358 (0.0636)	0.6375 (0.0648)	0.6353 (0.0632)
	$\theta = 1.5$	1.4852 (0.0894)	1.5170 (0.0891)	1.5129 (0.0892)	1.4582 (0.0813)
50	$\lambda = 0.5$	0.4645 (0.1544)	0.4247 (0.1684)	0.4373 (0.1640)	0.3955 (0.1659)
	$\theta = 0.5$	0.4998 (0.0584)	0.4830 (0.0606)	0.4874 (0.0587)	0.4713 (0.0606)
	$\lambda = 0.5$	0.4814 (0.1245)	0.4531 (0.1326)	0.4604 (0.1306)	0.4289 (0.1292)
	$\theta = 1$	1.0159 (0.1194)	0.9880 (0.1197)	0.9934 (0.1169)	0.9623 (0.1181)
	$\lambda = 0.7$	0.4498 (0.0763)	0.4401 (0.0726)	0.4422 (0.0724)	0.4306 (0.0636)
	$\theta = 2$	2.1679 (0.1626)	2.1795 (0.1599)	2.1788 (0.1567)	2.1038 (0.1822)
	$\lambda = 0.9$	0.6186 (0.0393)	0.6180 (0.0383)	0.6189 (0.0394)	0.6165 (0.0367)
	$\theta = 1.5$	1.4600 (0.0765)	1.5133 (0.0829)	1.5047 (0.0837)	1.4435 (0.0677)
80	$\lambda = 0.5$	0.4689 (0.1476)	0.4340 (0.1625)	0.4471 (0.1582)	0.4140 (0.1583)
	$\theta = 0.5$	0.4987 (0.0532)	0.4846 (0.0555)	0.4893 (0.0541)	0.4765 (0.0542)
	$\lambda = 0.5$	0.4801 (0.1185)	0.4550 (0.1250)	0.4650 (0.1215)	0.4334 (0.1213)
	$\theta = 1$	1.0040 (0.1068)	0.9823 (0.1097)	0.9891 (0.1061)	0.9606 (0.1058)
	$\lambda = 0.7$	0.4361 (0.0617)	0.4292 (0.0578)	0.4305 (0.0581)	0.4211 (0.0488)
	$\theta = 2$	2.1497 (0.1447)	2.1728 (0.1445)	2.1686 (0.1412)	2.0991 (0.1545)
	$\lambda = 0.9$	0.6110 (0.0262)	0.6116 (0.0278)	0.6123 (0.0279)	0.6097 (0.0241)
	$\theta = 1.5$	1.4433 (0.0655)	1.5081 (0.0801)	1.4975 (0.0804)	1.4319 (0.0577)
100	$\lambda = 0.5$	0.4696(0.1375)	0.4388 (0.1570)	0.4525 (0.1489)	0.4157 (0.1507)
	$\theta = 0.5$	0.5001 (0.0501)	0.4887 (0.0554)	0.4929 (0.0521)	0.4792 (0.0519)
	$\lambda = 0.5$	0.4880 (0.1092)	0.4600 (0.1215)	0.4718 (0.1175)	0.4474 (0.1176)
	$\theta = 1$	1.0034(0.0862)	0.9813 (0.1005)	0.9907 (0.0960)	0.9691 (0.0966)
	$\lambda = 0.7$	0.4314 (0.0568)	0.4271 (0.0538)	0.4291 (0.0548)	0.4189 (0.0460)
	$\theta = 2$	2.1560 (0.1354)	2.1795 (0.1317)	2.1768 (0.1289)	2.1126 (0.1416)
	$\lambda = 0.9$	0.6080 (0.0215)	0.6079 (0.0219)	0.6085 (0.0224)	0.6074 (0.0208)
	$\theta = 1.5$	1.4399 (0.0591)	1.5104 (0.0767)	1.4986 (0.0767)	1.4301 (0.0527)
150	$\lambda = 0.5$	0.4816 (0.1222)	0.4520 (0.1427)	0.4654 (0.1329)	0.4394 (0.1352)
	$\theta = 0.5$	0.5009 (0.0439)	0.4896 (0.0487)	0.4941 (0.0453)	0.4848 (0.0459)
	$\lambda = 0.5$	0.4882 (0.1087)	0.4688 (0.1167)	0.4781 (0.1142)	0.4543 (0.1149)
	$\theta = 1$	1.0008 (0.0872)	0.9866 (0.0902)	0.9936 (0.0863)	0.9738 (0.0868)
	$\lambda = 0.7$	0.4270 (0.0487)	0.4241 (0.0471)	0.4247 (0.0469)	0.4182 (0.0408)
	$\theta = 2$	2.1561 (0.1195)	2.1872 (0.1128)	2.1812 (0.1119)	2.1243 (0.1229)
	$\lambda = 0.9$	0.6047 (0.0142)	0.6049 (0.0157)	0.6054 (0.0159)	0.6045 (0.0137)
	$\theta = 1.5$	1.4319 (0.0521)	1.5086 (0.0724)	1.4950 (0.0723)	1.4253 (0.0462)
200	$\lambda = 0.5$	0.4782 (0.1164)	0.4561 (0.1355)	0.4665 (0.1264)	0.4421 (0.1285)
	$\theta = 0.5$	0.4996 (0.0402)	0.4915 (0.0462)	0.4949 (0.0424)	0.4860 (0.0420)
	$\lambda = 0.5$	0.4916 (0.1001)	0.4761 (0.1121)	0.4832 (0.1066)	0.4619 (0.1057)
	$\theta = 1$	1.0041 (0.0776)	0.9915 (0.0846)	0.9965 (0.0795)	0.9786 (0.0786)
	$\lambda = 0.7$	0.4210 (0.0424)	0.4211 (0.0436)	0.4211 (0.0427)	0.4137 (0.0351)
	$\theta = 2$	2.1381 (0.1146)	2.1722 (0.1100)	2.1650 (0.1090)	2.1115 (0.1146)
	$\lambda = 0.9$	0.6030 (0.0109)	0.6032 (0.0130)	0.6033 (0.0126)	0.6028 (0.0106)
	$\theta = 1.5$	1.4235 (0.0409)	1.5101 (0.0672)	1.4945 (0.0658)	1.4183 (0.0366)

Figure 3 displays the density function for some selected parameter values using the log-compound inverse Lindley Poisson (LCILP), log-compound inverse Lindley geometric (LCILG), and log-compound inverse Lindley logarithm (LCILL) distributions. Based on the CILPS density, we propose a linear location regression model connecting the dependent variable y_i and a set of covariate variables $\mathbf{v}_i^T = (v_{i1}, \ldots, v_{ip})$ as given below:

$$y_i = \mathbf{v}_i^T \boldsymbol{\beta} + z_i, \quad i = 1, \ldots, n, \tag{19}$$

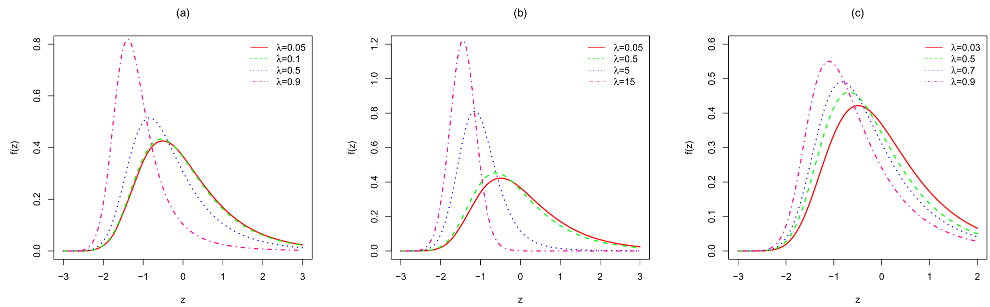

Figure 3. Plots of the LCILPS PDF for some parameter values of λ with $\mu = 0$. (a) LCILG; (b) LCILP; (c) LCILL.

where z_i is a random error with PDF given via Equation (18), $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^T$, λ and μ are unknown parameters. The position of the ith observation, response (y_i) is defined as $\xi_i = \mathbf{v}_i^T \boldsymbol{\beta}$, $i = 1, \ldots, n$. In matrix notation, the vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)^T$ is represented by a linear model $\boldsymbol{\xi} = \mathbf{V}\boldsymbol{\beta}$, where $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)^T$ is a known model matrix.

7.1. The LCILPS regression model for censored data

Let $(y_1, \mathbf{v}_1), \ldots, (y_n, \mathbf{v}_n)$ be a random sample of size n, where $y_i = \min\{\log(X_i), \log(D_i)\}$. In such situations, the most realistic assumption is that the lifetimes and censoring times are independent. Denote by F to be the set of observations for which y_i is the log-lifetime, and the set D to be the set of observations for which y_i is the log-censoring. The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (\lambda, \mu, \boldsymbol{\beta}^T)^T$ from the model given via Equation (19) is given by

$$\begin{aligned}\ell(\boldsymbol{\theta}) &= r[\log(\lambda) - \log(1 + \exp(-\mu))] - n\log(C[\lambda]) \\ &\quad + \sum_{i \in F} \log[\exp(-\mu) + z_i] - \sum_{i \in F} [2z_i - \exp(-z_i)] \\ &\quad + \sum_{i \in F} \log\left(C'\left[\lambda\left\{1 - \left\{1 + \frac{\exp(-z_i)}{\exp(\mu) + 1}\right\}\exp[-\exp(-z_i)]\right\}\right]\right) \\ &\quad + \sum_{i \in D} \log\left(C\left[\lambda\left\{1 - \left\{1 + \frac{\exp(-z_i)}{\exp(\mu) + 1}\right\}\exp[-\exp(-z_i)]\right\}\right]\right), \end{aligned} \tag{20}$$

where $z_i = (y_i - \mathbf{v}_i^T \boldsymbol{\beta})$, and r be the number of uncensored observations (failures). The MLE $\hat{\boldsymbol{\theta}}$ of the vector of unknown parameters can be determined by maximizing the log-likelihood by Equation (20). The fitted CILPS model yields the estimated survival function for y_i ($\hat{z}_i = y_i - \mathbf{v}_i^T \hat{\boldsymbol{\beta}}$) which is given by

$$S(y_i; \hat{\lambda}, \hat{\mu}, \hat{\boldsymbol{\beta}}) = \frac{C\left[\hat{\lambda}\left\{1 - \left\{1 + \frac{\exp(-(y_i - \mathbf{v}_i^T\hat{\boldsymbol{\beta}}))}{\exp(\mu)+1}\right\}\exp[-\exp(-(y_i - \mathbf{v}_i^T\hat{\boldsymbol{\beta}}))]\right\}\right]}{C[\hat{\lambda}]}. \tag{21}$$

For interval estimation of the parameters, we can evaluate numerically the elements of the $(p+2)\times(p+2)$ observed information matrix $-\ddot{\mathbf{L}}(\hat{\boldsymbol{\theta}})$. The multivariate normal distribution $N_{p+2}(0,-\ddot{\mathbf{L}}(\hat{\boldsymbol{\theta}})^{-1})$ for $\hat{\boldsymbol{\theta}}$ can be used to provide approximate confidence intervals for the unknown parameters.

8. Empirical applications

In this section, we shall demonstrate the potentiality of the CILPS model by means of three real data sets, first one is uncensored, the second one is censored, and the third one is a censored data with regression. The fits of the sub-models of CILPS distribution will be compared with some competitive models namely the Weibull (WE), inverse gamma (IGM), generalized inverse exponential (GIE), inverse Gaussian (IG), log-logistic (LL), and log-normal (LN) distributions, whose CDFs (for $x>0$) are given by $F_{\mathrm{WE}}(x,\lambda,\theta)=\exp[-(\frac{\lambda}{x})^{\theta}]$; $F_{\mathrm{IGM}}(x,\lambda,\theta)=TG(\frac{\lambda}{x};\theta)$, where $TG(x;\theta)=\frac{1}{\Gamma(\theta)}\int_x^{\infty}t^{\theta-1}\mathrm{e}^{-t}\,\mathrm{d}t$; $F_{\mathrm{GIE}}(x,\lambda,\theta)=[1-\exp(-\frac{\lambda}{x})]^{\theta}$; $F_{\mathrm{IG}}(x,\lambda,\theta)=\Phi(z_1)+\exp(\frac{2\lambda}{\theta})\Phi(z_2)$, where $z_1=\sqrt{\frac{\beta}{x}}(\frac{x}{\alpha}-1)$ and $z_2=-\sqrt{\frac{\beta}{x}}(\frac{x}{\alpha}+1)$; $F_{\mathrm{LL}}(x,\lambda,\theta)=1-\frac{1}{1+(x/\lambda)^{\theta}}$; $F_{\mathrm{LN}}(x,\lambda,\theta)=\Phi(\frac{\log(x)-\lambda}{\theta})$. The competitive models are compared by using goodness-of-fit criteria and information theoretic criteria including the Kolmogorov–Smirnov (KS) statistic with its p-value, Akaike information criterion (AIC) and Bayesian information criterion (BIC). All statistical computation and modeling are carried out using the statistical software R [30].

8.1. Uncensored data: failure of mechanical components

For the first example, we consider a data set from Murthy *et al.* [23] consisting of the failure times of 20 mechanical components. The data are 0.067, 0.068, 0.076, 0.081, 0.084, 0.085, 0.085, 0.086, 0.089, 0.098, 0.098, 0.114, 0.114, 0.115, 0.121, 0.125, 0.131, 0.149, 0.160 and 0.485. The current data are positively skewed and the data exhibit an upside-down hazard rate as explained by the TTT plot given in Figure 4(a). So the proposed model can be used to model these data. Numerical values of the ML, $\boldsymbol{\ell}$, AIC and BIC are reported in Table 6. Additionally, the table includes the most commonly used non-parametric goodness-of-fit test statistic; K–S test statistic along with its p-value. Based on the K–S test statistics, we may say that all considered models can be used to model the current data except the sub-model CILL and IGG. Table 4 also reveals that the sub-model CILG distribution possesses the lowest values of $\boldsymbol{\ell}$, AIC and BIC in comparison to all other fitted models. Furthermore, Figure 4 (c) (estimated CDFs) shows that the fitted CILG CDF is so near to the empirical distribution. Also, Figure 4(d) (estimated HRFs) demonstrates that the estimated HRF is an upside-down bathtub shape, which reflects the actual behavior of the data. Finally, Figure 5 shows the P–P plot for the proposed model along with some competing models.

8.2. Censored data: cancer data

The second data set refers to the survival times (in years) for 45 patients who were randomized to Chemotherapy plus Radiotherapy for about 8 years. Survival times for the current data as reported in [17] are given as 17, 42, 44, 48, 60, 72, 74, 95, 103, 108, 122,

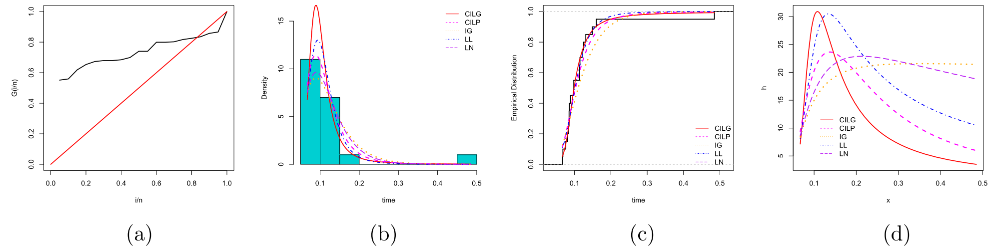

Figure 4. (a) TTT-plot; (b) estimated PDFs along with histogram; (c) estimated CDFs; and (d) estimated HRFs for the failure of mechanical components data.

Table 6. MLEs, standard deviation (in parenthesis) and goodness-of-fit measures for failure data.

Distribution	Estimates		$K-S$	p-value	$-\ell$	AIC	BIC
CILG(λ, θ)	0.9936 (0.0065)	0.6617 (0.1043)	0.13295	0.8714	−38.93	−73.86	−71.87
CILP(λ, θ)	8.5271 (3.1235)	0.3945 (0.0509)	0.1161	0.9503	−36.57	−69.14	−67.15
CILL(λ, θ)	0.0348 (0.1066)	0.1872 (0.0814)	0.3296	< 0.05	−28.38	−52.76	−50.77
WE(λ, θ)	0.1376 (0.0200)	1.6421 (0.2312)	0.2641	0.1227	−26.42	−48.84	−46.85
IGM(λ, θ)	0.1444 (0.0407)	0.6790 (0.0897)	0.5569	< 0.05	−29.28	−54.56	−52.569
GIE(λ, θ)	0.2716 (0.0485)	8.5799 (3.5195)	0.1655	0.6438	−34.40	−64.8	−62.81
IG(λ, θ)	0.5804 (0.1835)	0.1216 (0.0125)	0.1991	0.4061	−33.08	−62.16	−60.17
LL(λ, θ)	0.1016 (0.0075)	5.0860 (0.9630)	0.1124	0.9623	−36.18	−68.36	−66.37
LN(λ, θ)	−2.2298 (0.0939)	0.4201 (0.066)	0.1693	0.6151	−33.56	−63.12	−61.128

144, 167, 170, 183, 185, 193,195, 197, 208, 234, 235, 254, 307, 315, 401, 445, 464, 484, 528, 542, 547, 577,580, 795, 855,1366, 1577, 2060, 2412+, 2486+, 2796+, 2802+, 2934+, 2988+, where + sign indicates a censored observation. The TTT plot for the current data given in Figure 6(a) reveals an upside-down shaped HRF. Specifically, the graph indicates that HRF has a spiked upside down. Interestingly, the estimated HRF shows a spiked upside down, see Figure 6(c) which reflects the actual behavior of the empirical data. Table 7 reports the ML estimates of the model parameters along with the ℓ, AIC, and BIC measures for some fitted models to these data. It is evident that the smallest values of these criteria correspond to the CILG distribution. In order to assess if the proposed model is appropriate, the plots of the fitted CDFs of these distributions and the empirical ones are displayed in Figure 6(b). They indicate that the CILG distribution provides a good fit for these data.

8.3. Censored LCILPS regression model: cancer tongue data

We demonstrate the proposed regression model using censored cancer data, specifically cancer tongue data. The following analysis refers to data from [17], in which a study was conducted on the effects of ploidy on the prognosis of patients with cancers of the mouth. Patients were selected who had a paraffin-embedded sample of the cancerous. The total sample size is $n = 80$. The response variable in the experiment is the lifetime of the adult flies in days after exposure to the treatments. So, we have the variables used in this study are y_i: log-lifetime of a patient in weeks; δ_i: censoring indicator; v_{i1}: group ($0 =$ Aneuploid Tumor, $1 =$ Diploid Tumor), $i = 1, \ldots, 80$. This analysis aims to compare between survival curves for the two groups. There are a variety of models that can be used such as

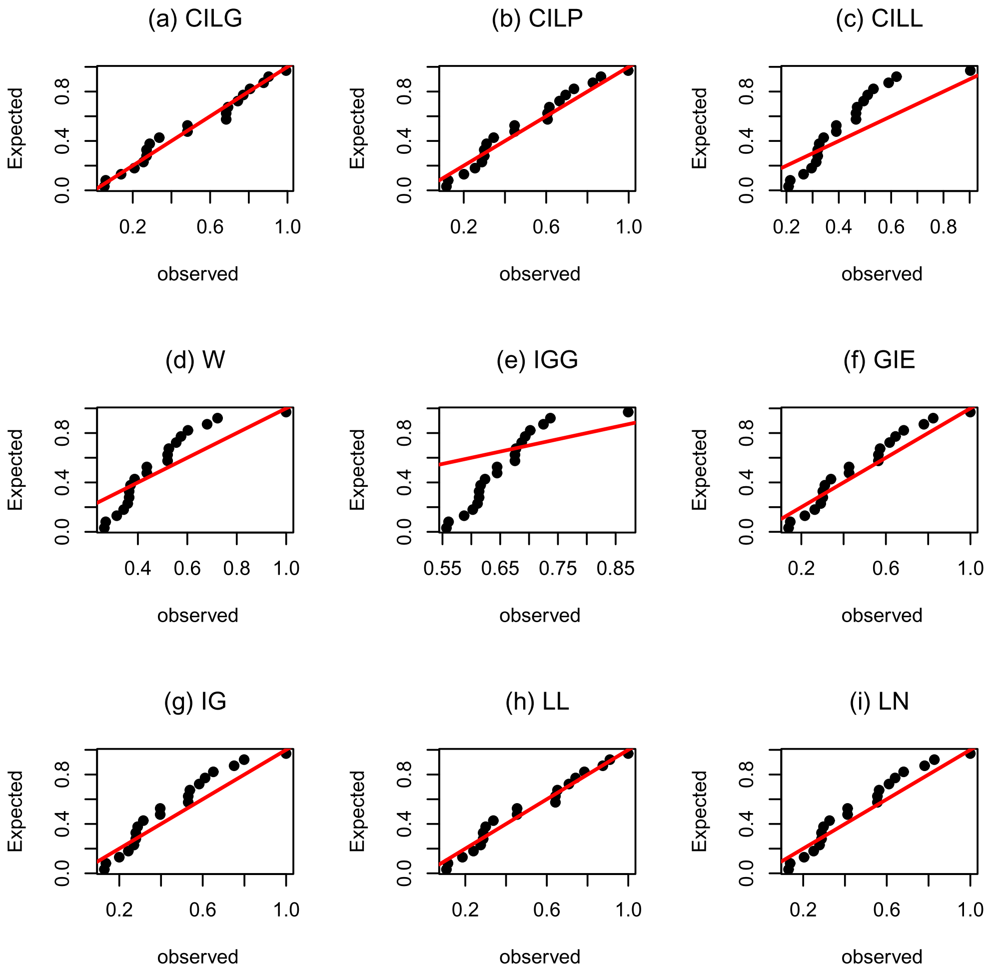

Figure 5. P–P plots for some candidate distributions. (a) CILG, (b) CILP, (c) CILL, (d) W, (e) IGG, (f) GIE, (g) IG, (h) LL and (i) LN.

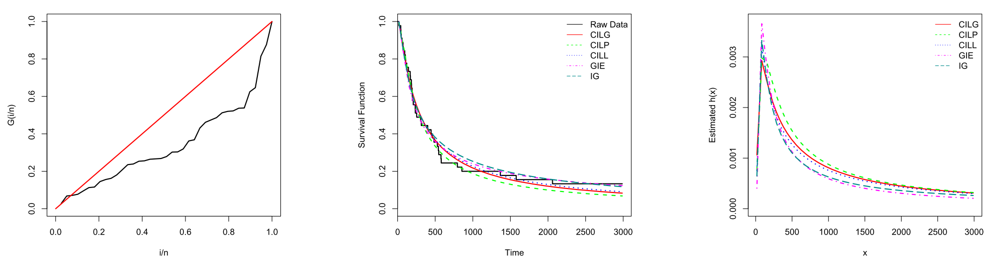

Figure 6. Left: TTT-plot; middle: estimated SFs; and right: estimated HRFs for the cancer data.

the Log-Weibull (LW), Log-loglogistic (LLL) and Log-lognormal (LLN) regression models. We adopt these classical regression models as an example to illustrate that the LCILPS regression model can provide better fits. Therefore, we present the results by fitting the

Table 7. MLEs, standard deviations (in parenthesis) and goodness-of-fit measures for cancer data.

Distribution	Estimates		$-\ell$	AIC	BIC
CILG(λ, θ)	−2.8473 (2.5208)	70.8268 (35.5127)	287.81	579.62	583.23
CILP(λ, θ)	−3.5589 (2.5040)	59.2642 (39.9365)	287.83	579.66	583.273
CILL(λ, θ)	−7.5228 (7.5153)	87.8241 (27.5671)	288.00	580	583.61
WE(λ, θ)	692.7380 (160.999)	0.6998 (0.0875)	295.06	594.12	597.733
IGM(λ, θ)	0.5491 (0.314187)	0.1697 (0.04131)	335.58	675.16	678.77
GIE(λ, θ)	108.1483 (25.4413)	0.6265 (0.12804)	288.08	580.16	583.77
IG(λ, θ)	163.8594 (0.0285)	1771.1575 (0.0006)	288.05	580.1	583.71
LL(λ, θ)	317.6856 (69.3983)	1.1819 (0.1581)	289.36	582.72	586.33
LN(λ, θ)	5.8536 (0.2211)	1.464 (0.1726)	289.49	582.98	586.59

following model

$$y_i = \beta v_i + \epsilon_i, \quad i = 1, 2 \ldots, 80, \tag{22}$$

where ϵ_i is a random error with PDF given via Equation (18), and hence Y_i is a random variable that can follow LCILG, LCILP, or LCILL distributions using the appropriate $C(\cdot)$. Notice that if the regression model given in Equation (22) represents the LW regression, then the errors (ϵ_i) should follow the standard extreme distribution; if it represents the LLL regression model, then the errors (ϵ_i) should follow the standard logistic distribution; and if it represents the LLN regression model, then the errors (ϵ_i) should follow the standard normal distribution. Before we start fitting the proposed log-linear models along with the other models to the current data, it is desirable to provide preliminary analysis that does not depend on a certain functional form or by assuming that the data follow a specific distribution. So, we perform a popular non-parametric test called the log-rank test for testing no difference in survival between Aneuploid and Diploid groups. We find that the value of this test statistic is $\chi^2 = 2.8$ with 1 degree of freedom and a p-value of.09, which implies that the null hypothesis (equality of survival rates) is not rejected at level $\alpha = 0.05$. Table 8 lists the ML estimates for the fitted regression models for these data along with the ℓ, AIC and BIC statistics. It can be seen clearly that the LCILG model for the current data is the best one in terms of possessing the lowest ℓ, AIC and BIC statistics values among those

Table 8. MLEs of the parameters from the fitted LCILG, LCILP, LCILL, LW, LLL, and LLN regression models along with the standard errors (between parentheses) and p-values [between brackets] to tongue cancer data.

Model	λ	θ	β	$-\ell$	AIC	BIC
LCILG	−58.4743 (85.5835)	2.1236 (2.2461)	−0.7766 (0.4099) [0.0582]	123.28	252.56	259.706
LCILP	−4.86373 (0.97318)	9.0662 (1.8958)	−0.7334 (0.2938) [0.0124]	136.24	278.48	285.63
LCILL	−126.1720 (89.0655)	6.6338 (1.7933)	−0.6308 (0.4192) [0.1336]	127.57	261.14	268.29
LW	144.2510 (32.7167)	0.8059 (0.0932)	−0.6690 (0.3510) [0.0574]	123.8	253.6	260.75
LLL	87.3141 (21.1712)	1.0422 (0.1210)	−0.7905 (0.4042) [0.0505]	123.6	253.2	260.35
LLN	4.4504 (0.2586)	1.6802 (0.1726)	−0.80566 (0.4146) [0.0524]	123.61	253.22	260.37

values of the fitted regression models followed by the LLL model as a second close. Additionally, all the considered models except the LCILP model are agreed that the survival rate curves are in difference since the covariate v is insignificant at the 5% level.

In order to provide further assessment whether the proposed model is appropriate, Figure 7 displays the plots of the empirical SF and the estimated SFs from the fitted LCILPS

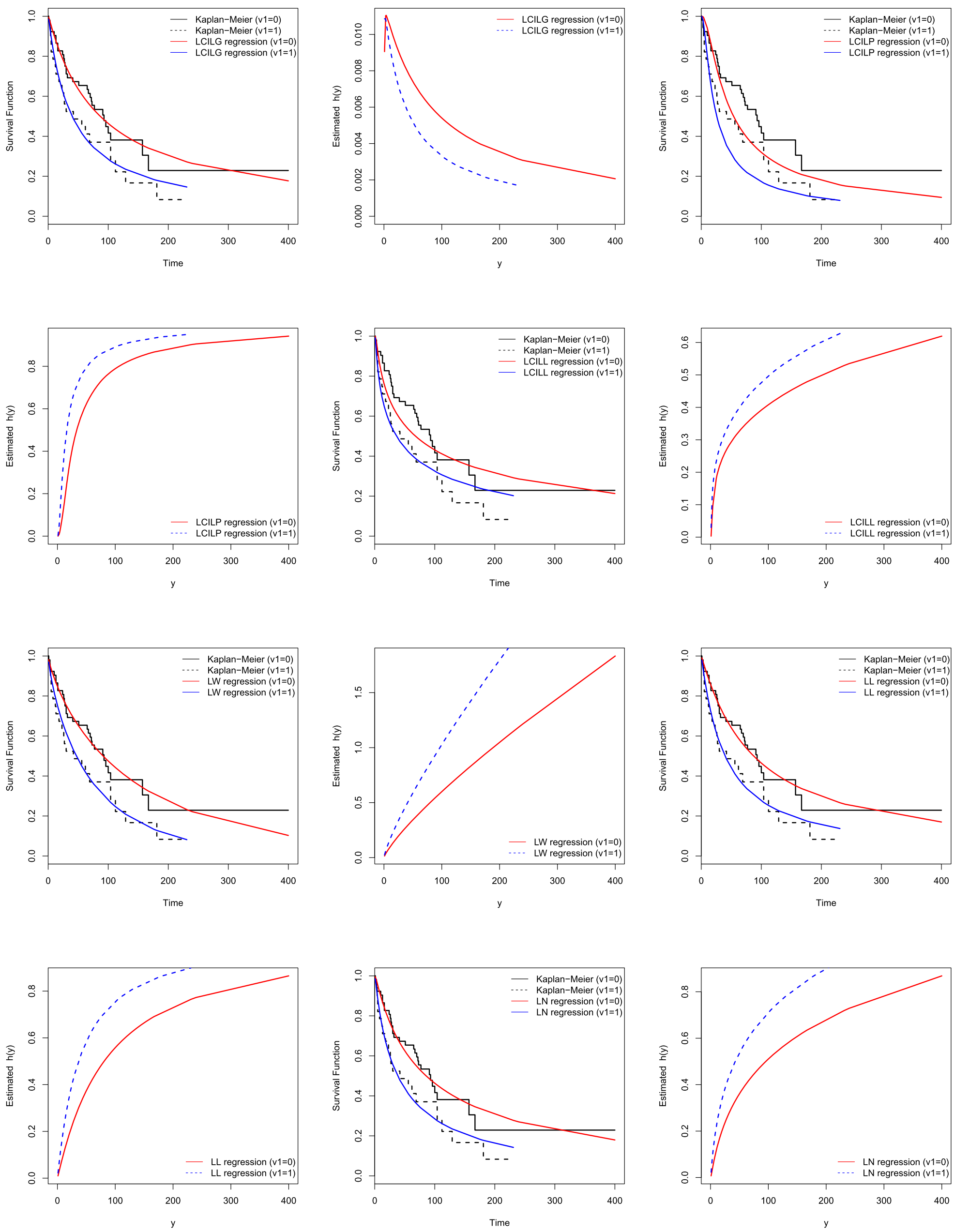

Figure 7. Plots for estimated LCILG, LCILP, LCILL, LW, LLL, and LLN survival functions along with their corresponding HR functions for the tongue data.

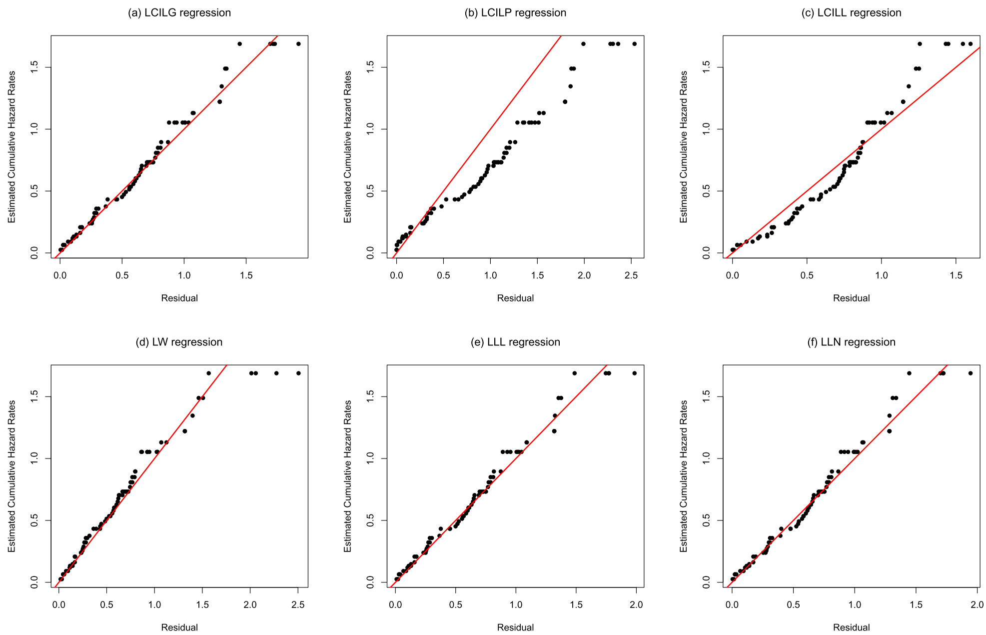

Figure 8. Cox–Snell residuals to assess the fit of LCILG, LCILP, LCILL, LLW, LLL, and LLN regression models for the tongue data.

regression models along with the other competing regression models. The plots reveal that the LCILG regression provides a good fit compared to the considered regression models. An interesting point is that HRFs for patients suffering from diploid and aneuploid tumor using the LCILG regression model reveal a spiked upside-down shape. In contrast, the other competing regression models reveal a fast increasing shape at the early events then start decreasing slowly except for the LLW regression model.

Additionally, we provide further assessment for the adequacy of the fitted log-linear regression model by examining the Cox–Snell residuals to these fitted models. The Cox–Snell residual is defined as the estimated cumulative hazard rate function. Note that the cumulative hazard rate function H is defined as $H_Y(y) = -\ln S_Y(y)$, where $S_Y(\cdot)$ is the survival function of the random variable Y. Therefore, the Cox–Snell residuals of the LCILPS regression model is $r_i = -\ln[S(y_i; \hat{\lambda}, \hat{\mu}, \hat{\boldsymbol{\beta}})]$, where $S(y_i; \hat{\lambda}, \hat{\mu}, \hat{\boldsymbol{\beta}})$ is given via Equation (21). The plot of the Cox–Snell residuals for the fitted LCLIPS regression models can be constructed by graphing r_i against the Nelson–Aalen estimator of r_i. Now, if the LCILPS regression models fit the data, then the plot should produce a straight line. It is worth mentioning that the Cox–Snell residuals are useful for checking the overall fit of the model (for further details regarding these residuals, see [17]). The graphs in Figure 8 reveal good performance for the LCILG, LLL, and LLN regression models but are suspect for the LCILP, LCILL and LW models.

9. Conclusion

In this work, we introduce a new inverse Lindley power series class of distributions, which is obtained by compounding inverse Lindley and power series distributions. The new distribution contains several important lifetime models. We obtain some of its mathematical/statistical properties including moments, moment generating function, mean residual life and order statistics. The model parameters are obtained by the methods of maximum likelihood, least square, weighted least square and maximum product of spacing and compared using Monte Carlo simulation study. Besides, two special models of the new family are investigated. Further, to cater to censored data, we introduce the log compound inverse Lindley regression model. One important aspect of this new model is that it provides better fits than some well-known models using three real data sets.

As for future research, this paper can be extended in several ways. For instance, the new compound family of distribution introduced in this paper can be studied under Type-II progressive censoring sampling schemes to solve real problems related to engineering reliability. Moreover, additional different methods of estimation can be considered particularly, the Bayesian analysis using subjective and objective priors, since this method is effective for small sample sizes. Furthermore, the proposed LCILPS regression model for survival data in presence of a survival fraction can be considered in a future study, as well as by considering the shape parameter depending on covariates. In particular, influential diagnostics and outliers can be investigated further in this context. Additionally, the Bayesian analysis of the censored LCILPS regression model can also be considered including model selection using the conditional predictive ordinate (CPO) statistic, and influential observations using q-divergence measure between two densities which include the Kullback–Leibler divergence and ℓ_1-distance divergence measures as special cases.

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Appendix

Proof of Proposition 3.1: Using the CDF of the CILPS distribution along with the definition of the $C[\cdot]$ function and then taking the limit of the CDF as λ approaches to 0, we have

$$\begin{aligned}
&\lim_{\lambda\to 0} F_{\mathrm{CILPS}}(x;\theta,\lambda)\\
&= 1 - \lim_{\lambda\to 0} \frac{\sum_{n=1}^{\infty} b_n \left[1 - \left\{\left(1 - \left(\left(1 + \frac{\theta}{(\theta+1)x}\right) \mathrm{e}^{-\frac{\theta}{x}}\right)\right)\lambda\right\}^n\right]}{\sum_{n=1}^{\infty} b_n \lambda^n}\\
&= 1 - \frac{\left(1 - \left(1 + \frac{\theta}{(\theta+1)x}\right)\mathrm{e}^{-\frac{\theta}{x}}\right) + b_1^{-1}\lim_{\lambda\to 0}\sum_{n=2}^{\infty} b_n n\lambda^n \left(1 - \left(\left(1 + \frac{\theta}{(\theta+1)x}\right)\mathrm{e}^{-\frac{\theta}{x}}\right)\right)^n}{1 + b_1^{-1}\lim_{\lambda\to 0}\sum_{n=2}^{\infty} b_n n\lambda^n}.
\end{aligned}$$

Since the limit of the second term in the denominator and numerator vanishes as λ approaches to 0, it follows that $F_{\mathrm{CILPS}}(x;\theta,\lambda) = (1 + \frac{\theta}{(\theta+1)x})\,\mathrm{e}^{-\frac{\theta}{x}}$ which is the CDF of ILD as desired. ■

Proof of Proposition 3.2: We need the following facts about the PDF and CDF of the ILD. It is not difficult to show that $\lim_{x\to 0} g_{\mathrm{ILD}}(x;\theta) = \lim_{x\to 0} h_{\mathrm{ILD}}(x;\theta) = 0$ and $\lim_{x\to\infty} g_{\mathrm{ILD}}(x;\theta) = \lim_{x\to\infty} h_{\mathrm{ILD}}(x;\theta) = 0$. Additionally, the first derivative of the $g_{\mathrm{ILD}}(x;\theta)$ is $g'_{\mathrm{ILD}}(x;\theta) = -\frac{\theta^2}{(1+\theta)x^5}\exp(-\theta/x)[2x^2 - (\theta-3)x - \theta]$, and $\lim_{x\to 0} g'_{\mathrm{ILD}}(x;\theta) = +\infty$, $\lim_{x\to\infty} g'_{\mathrm{ILD}}(x;\theta) = -\infty$ and $\lim_{x\to 0}\frac{C'[x]}{C[x]} = \lambda^{-1}$. Therefore, we have that

$$\begin{aligned}
\lim_{x\to 0} h_{\mathrm{CILPS}}(x;\theta,\lambda) &= \lim_{x\to 0}\left\{\frac{\frac{\lambda\theta^2}{\theta+1}\left(\frac{1+x}{x^3}\right)\mathrm{e}^{-\frac{\theta}{x}} C'\left[\left(1 - \left(1 + \frac{\theta}{(\theta+1)x}\right)\mathrm{e}^{-\frac{\theta}{x}}\right)\lambda\right]}{C\left[\left(1 - \left(1 + \frac{\theta}{(\theta+1)x}\right)\mathrm{e}^{-\frac{\theta}{x}}\right)\lambda\right]}\right\}\\
&= \frac{\lambda C'(\lambda)}{C(\lambda)}\lim_{x\to 0} g_{\mathrm{ILD}}(x;\theta) = 0.
\end{aligned}$$

Similarly, as $x \to \infty$, we have $\lim_{x\to\infty} h(x;\theta,\lambda) = \lim_{x\to\infty} \frac{\lambda C'(0)}{C(0)} g_{\rm ILD}(x;\theta) = \lim_{x\to\infty} g_{\rm ILD}(x;\theta) = 0$. To show that the HRF of the CILDPS distribution is unimodal, we differentiate Equation (4) and examine the behavior of the resulting equation. Note that the HRF of the CILDPS distribution given via Equation (4) can be written as $h_{\rm CILPS}(x;\theta,\lambda) = \lambda g_{\rm ILD}(x;\theta)\frac{C'[\lambda(1-G_{\rm ILD}(x;\theta))]}{C[\lambda(1-G_{\rm ILD}(x;\theta))]}$. So, the derivative of $h_{\rm CILPS}(x;\theta,\lambda)$ and after some algebra reduces to

$$h'_{\rm CILPS}(x;\theta,\lambda) = -\lambda[\lambda h_{\rm IL}(x;\theta) g_{\rm ILD}(x;\theta)\{C''[\lambda(1-G_{\rm ILD}(x;\theta))] - (C'[\lambda(1-G_{\rm IL}(x;\theta))])^2\} - \boldsymbol{\rho}(C'[x],C[x])g'_{\rm ILD}(x;\theta)],$$

where $\boldsymbol{\rho}(C'[x],C[x]) = \frac{C'[\lambda(1-G_{\rm ILD}(x;\theta))]}{C[\lambda(1-G_{\rm ILD}(x;\theta))]}$. Observe that $\lim_{x\to\infty}\boldsymbol{\rho}(C'[x],C[x]) = \frac{C'[0]}{C[0]} = \lambda^{-1}$. The modal, say x_0, is the value such that $h'_{\rm CILPS}(x_0;\theta,\lambda) = 0$. To verify that x_0 is the value that maximizes $h_{\rm CILPS}(x;\theta,\lambda)$, we examine the behavior of $h'_{\rm CILPS}(x;\theta,\lambda)$. Note that $x_0 \in (0,\infty)$. Therefore, as $x \to \infty$, we have

$$\lim_{x\to\infty} h'_{\rm CILPS}(x;\theta,\lambda) = -\lambda\left[0 - \frac{1}{\lambda}\lim_{x\to\infty} g'_{\rm ILD}(x;\theta)\right] = \lim_{x\to\infty} g'_{\rm ILD}(x;\theta) = -\infty < 0.$$

Similarly, it can be readily seen that $\lim_{x\to 0} h'_{\rm CILPS}(x;\theta,\lambda) = \frac{C'[\lambda]}{C[\lambda]}\lim_{x\to 0} g'_{\rm ILD}(x;\theta) = +\infty > 0$, and hence, x_0 is a mode for $h_{\rm CILPS}(x;\theta,\lambda)$ as required. ■

Proof of Proposition 3.3: We have from Equation (2) that

$$F_{\rm CILPS}(x;\theta,\lambda) = 1 - \frac{1}{C(\lambda)}\sum_{n=0}^{\infty} b_n\lambda^n\left[1-\left(1+\frac{\theta}{(1+\theta)x}\right)e^{-\frac{\theta}{x}}\right]^n$$
$$= 1 - \frac{1}{C(\lambda)}\sum_{n=0}^{\infty} b_n\lambda^n[\bar{G}_{\rm ILD}(x,\theta)]^n, \qquad \text{(A1)}$$

where $\bar{G}_{\rm ILD}(x,\theta) = 1 - G_{\rm ILD}(x;)$. Consider the Lehmann type II (LTII) CDF; $\Pi_c(x) = 1 - \{1 - G(x)\}^c$ [19] with power parameter $c > 0$ defined from the baseline $G(x)$. Thus, the LTII density is given by $\pi_c(x) = c\bar{G}(x)^{c-1}g(x)$, where $g(x) = dG(x)/dx$. By differentiating the last equation of $F(x)$, the pdf of X follows as given in Equation (5). ■

Proof of Proposition 3.4: We have that $m(t) = \frac{1}{S_{\rm CILPS}(t)C(\lambda)}\int_t^\infty C((1-(1+\frac{\theta}{(\theta+1)x})e^{-\frac{\theta}{x}})\lambda)\,dx$. On using Equation (A1), the above equation reduces to

$$m(t) = \frac{1}{S_{\rm CILPS}(t)C(\lambda)}\sum_{n=1}^{\infty} b_n\int_t^\infty\left[\left\{1-\left(1+\frac{\theta}{(\theta+1)x}\right)e^{-\frac{\theta}{x}}\right\}\lambda\right]^n dx.$$

Now applying the binomial expansion to the integrand term in the above equation, we have that

$$m(t) = \frac{1}{S_{\rm CILPS}(t)C(\lambda)}\sum_{n=1}^{\infty} b_n\sum_{k=0}^{n}(-1)^k\lambda^k\binom{n}{k}\int_t^\infty\left(1+\frac{\theta}{(\theta+1)x}\right)^k e^{-\frac{\theta k}{x}}\,dx.$$

On using the binomial and exponential expansions followed by rearranging the terms, the following expression is obtained:

$$m(t) = \frac{1}{S_{\rm CILPS}(t)C(\lambda)}\sum_{n=1}^{\infty} b_n\sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{l=0}^{\infty}(-1)^{k+l}\frac{\lambda^k(\theta k)^l}{l!}\binom{n}{k}\binom{k}{m}k\left(\frac{\theta}{\theta+1}\right)^m \frac{1}{(m+l-1)t^{m+l-1}}\,dx,$$

and hence, the result. ■