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Bayesian inference: Weibull Poisson model for censored data using the expectation–maximization algorithm and its application to bladder cancer data

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ABSTRACT

This article focuses on the parameter estimation of experimental items/units from Weibull Poisson Model under progressive type-II censoring with binomial removals (PT-II CBRs). The expectation–maximization algorithm has been used for maximum likelihood estimators (MLEs). The MLEs and Bayes estimators have been obtained under symmetric and asymmetric loss functions. Performance of competitive estimators have been studied through their simulated risks. One sample Bayes prediction and expected experiment time have also been studied. Furthermore, through real bladder cancer data set, suitability of considered model and proposed methodology have been illustrated.

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1. Introduction

The Weibull Poisson Model (*WPM*) is one of the recent compounding of two most greeted probability distributions, i.e. Weibull and zero-truncated Poisson distribution. This distribution was pioneered by Lu and Shi [19]. The cumulative distribution function (CDF) of *WPM* with (α, β, λ) is

$$F(x) = \frac{e^{\lambda e^{-\beta x^{\alpha}}} - e^{\lambda}}{1 - e^{\lambda}}; \quad \alpha > 0,\ \lambda > 0,\ \beta > 0,\ x > 0. \tag{1}$$

The probability density function (pdf) is given by

$$f(x) = \frac{\alpha\beta\lambda e^{-\lambda}}{1 - e^{-\lambda}}\, e^{-\beta x^{\alpha}}\, x^{\alpha-1}\, e^{\lambda e^{-\beta x^{\alpha}}}; \quad \alpha > 0,\ \lambda > 0,\ \beta > 0,\ x > 0, \tag{2}$$

where α and β are the shape and scale parameters of *WPM*, while λ is the rate parameter of zero-truncated Poisson distribution.

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This model has an edge over other Poisson-based distributions like Poisson-gamma, Poisson-log normal, etc. in the sense that it covers all types of failure rates encountered in life-testing experiments, see Gonzales-Barron and Butler [11].

We may note here, a typical feature of life-testing experiments is censoring because situations do arise when items/units are lost or removed from the experiment while they are alive; i.e. quite often, it is very difficult to get failure times of all the items/units put on test owing to various constraints related to time, cost and other resources. Type-I censoring takes place when experimental time is fixed and hence number of failures become random. While type-II censoring occurs when the number of failures is fixed, but experimental time remain random. Even under these conditions, some items/units may drop out of the experiment randomly due to some unknown causes, which are beyond the control of the experimenter. For example, consider that a medical experiment starts with n patients but after the death of first patient, some patients who are alive leave the experiment and go for treatment elsewhere. Similarly, after the death of second patient, a few more are leave and the process continues till predetermined number of failure (say $m < n$) are recorded. It may be assumed here that at each stage participating patient may independently decide to leave the experiment with probability p. Thus the number of patients who leave the experiment at a specified stage will follow binomial distribution with probability p. It may be argued at this stage that probability p may vary at each stage. But for the sake of simplicity, we shall assume that p is same at each stages. Collecting information in this way results to a censored sample and the sampling technique used is called as PT-II CBRs. The mathematical formulation of PT-II CBRs is presented in the next section. For details, one can see Balakrishnan and Sandhu [3] and Balakrishnan and Aggarwala [2].

In the past few decades, estimation of parameters of the Weibull lifetime models based on progressive type-II, PT-II CBRs and optimal progressive censoring samples have been studied by several authors such as Balasooriya *et al.* [4], Tse *et al.* [30], Tang *et al.* [28] and Ng *et al.* [21], etc. Estimation of parameters for Inverse Weibull distribution have been discussed by Sultan *et al.* [27]. And for other lifetime models by Soliman *et al.* [26], Singh *et al.* [23], Kumar *et al.* [14–17], etc. But, it seems as if no attempt has been made to develop estimators for the parameters of *WPM* under PT-II CBRs; although estimation of parameters under classical set up has been attempted by Lu and Shi [19].

Therefore, we propose to develop an estimation procedure to obtain the MLEs (using Expectation–Maximization (EM) algorithm) and Bayes estimators of the parameters of *WPM* under symmetric and asymmetric loss functions when sample is obtained by the use of PT-II CBRs. An important feature of this article is to develop the required mathematics for PT-II CBRs, EM algorithm along with its application to the bladder cancer patients data (remission time in months).

Rest of the paper consists of seven more sections. Section 2 provides the classical and Bayesian estimation procedure under PT-II CBRs for the parameters of *WPM* and likelihood ratio (LR) test is applied for checking the goodness-of-fit. The EM algorithm is proposed to be used to obtain the MLEs. Bayes estimators under symmetric and asymmetric loss functions have been obtained. The Bayes prediction for one sample has been discussed in Section 3. Expression for expected experiment time has been obtained in Section 4. Comparisons of risks of the estimators with corresponding Bayes estimators have been made through Monte Carlo simulation studies and the related discussions are presented in Sections 5 and 6. The proposed methodology is illustrated in Section 7 through

a real data of remission time (in months) of bladder cancer patients, after checking the suitability of *WPM* for it. In the last section, we have provided the conclusion and remarks.

2. Classical and Bayesian estimation under PT-II CBRs

Let us assume that an experimenter conducts a life-testing experiment with n items/units and decides to terminate the experiment as soon as m failure times are recorded. At first failure observed at X_1, R_1 out of the $n-1$ surviving items/units are randomly removed from the experiment and the experiment continues. Similarly, at second failure observed X_2, R_2 of the remaining $n - R_1 - 2$ surviving items/units are again randomly removed from the experiment and in a similar way the experiment continues till the mth failure is recorded and at this stage all the remaining $(n - m - \sum_{i=1}^{m-1} R_i = (R_m))$ surviving items units are removed resulting to termination of the experiment. Since, R_i at ith stage is the total removal out of surviving units, each experiencing the risk of removal with probability p; it is a random variable following the binomial distribution $B(n - m - \sum_{i=1}^{m-1} R_i, p)$. For details see. Viveros and Balakrishnan [31] and Ng *et al.* [21]. Following Cohen [8] for fixed removals, say $R_1 = r_1, R_2 = r_2, R_3 = r_3, \ldots, R_m = r_m$, the conditional likelihood function can be written as,

$$L(\alpha, \beta, \lambda; x|R = r) = c \prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{r_i}; \quad -\infty < x_1 < \cdots < x_m < \infty, \tag{3}$$

$n, m \in N$, $1 \le i \le m$ and $c = \prod_{i=1}^{m} \gamma_i$ where $\gamma_i = \sum_{j=1}^{m}(r_j + 1)$. Substituting $f(x_i)$ and $F(x_i)$ from (1) and (2) into (3), we have

$$L(\alpha, \beta, \lambda; x|R = r) = c \prod_{i=1}^{m} \frac{\alpha\beta\lambda x_i^{\alpha-1}}{1 - e^{-\lambda}} e^{-\lambda - \beta x_i^{\alpha} + \lambda e^{-\beta x_i^{\alpha}}} \left\{ \frac{1 - e^{\lambda e^{-\beta x_i^{\alpha}}}}{1 - e^{\lambda}} \right\}^{r_i}. \tag{4}$$

As mentioned earlier, the number of items/units removed from the experiment is random and independent of each other; therefore

$$p(R_1 = r_1; p) = \binom{n - m}{r_1} p^{r_1} (1 - p)^{n-m-r_1} \tag{5}$$

and for $i = 2, 3, \ldots, m - 1$

$$\begin{aligned} p(R_i; p) &= p(R_i = r_i | R_{i-1} = r_{i-1}, \ldots R_1 = r_1) \\ &= \binom{n - m - \sum_{l=0}^{i-1} r_l}{r_i} p^{r_i} (1 - p)^{n-m-\sum_{l=0}^{i-1} r_l}. \end{aligned} \tag{6}$$

Hence, likelihood function can be written as

$$L(\alpha, \beta, \lambda, p; x) = L(\alpha, \beta, \lambda; x|R = r) p(R = r; p) \tag{7}$$

where,

$$\begin{aligned} p(R = r; p) = p(R_1 = r_1) p(R_2 = r_2 | R_1 = r_1) p(R_3 = r_3 | R_2 = r_2, R_1 = r_1) \cdots \\ p(R_{m-1} = r_{m-1} | R_{m-2} = r_{m-2}, \ldots R_1 = r_1). \end{aligned} \tag{8}$$

Substituting from (5) and (6) into (8), we have

$$p(R=r;p)=\frac{(n-m)!p^{\sum_{i=1}^{m-1}r_i}(1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}}{(n-m-\sum_{l=1}^{i-1}r_l)!\prod_{i=1}^{m-1}r_i!}, \tag{9}$$

now using (4), (7) and (9), the complete likelihood can be expressed in the following form,

$$L(\alpha,\beta,\lambda,p;x)=\Phi L_1(\alpha,\beta,\lambda)L_2(p)$$

where,

$$\begin{aligned}\Phi&=\frac{c(n-m)!}{(n-m-\sum_{l=1}^{i-1}r_l)!\prod_{i=1}^{m-1}r_i!},\\ L_1(\alpha,\beta,\lambda)&=c\prod_{i=1}^{m}\frac{\alpha\beta\lambda x_i^{\alpha-1}}{1-e^{-\lambda}}e^{-\lambda-\beta x_i^{\alpha}+\lambda e^{-\beta x_i^{\alpha}}}\left\{\frac{1-e^{\lambda e^{-\beta x_i^{\alpha}}}}{1-e^{\lambda}}\right\}^{r_i},\\ L_2(p)&=p^{\sum_{i=1}^{m-1}r_i}(1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}.\end{aligned} \tag{10}$$

Now, MLEs of α,β and λ are computed by maximizing L_1 and MLE of p by maximizing L_2. Taking log of both sides to (10), we get

$$\begin{aligned}l_1(\alpha,\beta,\lambda)=\ln(L_1(\alpha,\beta,\lambda))&=m\ln\alpha+m\ln\beta+m\ln\lambda\\&+(\alpha-1)\sum_{i=1}^{m}\ln x_i-m\lambda-\beta\sum_{i=1}^{m}x_i^{\alpha}-m\ln(1-e^{-\lambda})\\&+\lambda\sum_{i=1}^{m}e^{-\beta x_i^{\alpha}}+\sum_{i=1}^{m}r_i(\ln(e^{\lambda e^{-\beta x_i^{\alpha}}}-1)-\ln(e^{\lambda}-1)).\end{aligned} \tag{11}$$

Differentiating (11) with respect to α,β and λ and equating them to zero, we obtain following three normal equations. A simultaneous solution of these provide MLEs of the parameters.

$$\begin{aligned}\frac{\partial l_1(\alpha,\beta,\lambda)}{\partial\alpha}&=\frac{m}{\alpha}+\sum_{i=1}^{m}\ln x_i-\beta\sum_{i=1}^{m}x_i^{\alpha}\ln x_i-\lambda\beta\sum_{i=1}^{m}e^{-\beta x_i^{\alpha}}(x_i^{\alpha}\ln x_i)\\&+\sum_{i=1}^{m}r_i\left[\frac{\lambda e^{-\beta x_i^{\alpha}}e^{\lambda e^{-\beta x_i^{\alpha}}}}{1-e^{\lambda e^{-\beta x_i^{\alpha}}}}\beta x_i^{\alpha}\ln x_i\right]=0,\end{aligned} \tag{12}$$

$$\frac{\partial l_1(\alpha,\beta,\lambda)}{\partial\beta}=\frac{m}{\beta}-\sum_{i=1}^{m}x_i^{\alpha}-\lambda x_i^{\alpha}\sum_{i=1}^{m}e^{-\beta x_i^{\alpha}}+\sum_{i=1}^{m}r_i\left[\frac{\lambda e^{-\beta x_i^{\alpha}}e^{\lambda e^{-\beta x_i^{\alpha}}}}{1-e^{\lambda e^{-\beta x_i^{\alpha}}}}x_i^{\alpha}\right]=0, \tag{13}$$

$$\frac{\partial l_1(\alpha,\beta,\lambda)}{\partial\lambda}=\frac{m}{\lambda}-m-\sum_{i=1}^{m}e^{-\beta x_i^{\alpha}}-\frac{me^{-\lambda}}{1-e^{-\lambda}}-\sum_{i=1}^{m}r_i\left[\frac{e^{\lambda e^{-\beta x_i^{\alpha}}-\beta x_i^{\alpha}}}{1-e^{\lambda e^{-\beta x_i^{\alpha}}}}-\frac{me^{\lambda}}{1-e^{\lambda}}\right]=0. \tag{14}$$

Unfortunately, (12)–(14) cannot be analytically solved simultaneously. Hence we propose the use of numerical iterative procedure, namely, Newton–Rapson method for solving these. The numerical procedure used here for obtaining the iteration function and the choice of initial guesses is based on maximum absolute row sum norms, which has been discussed by Jain *et al.* [12]. The expectation maximization algorithm has been proposed in this article to get the MLEs of α, β and λ. Because, EM algorithm is deterministic optimization technique and can be used for missing and incomplete data. Therefore, the EM algorithm is more suitable alternative approach to existing other numerical methods. This algorithm was introduced by Dempster *et al.* [9] and for its detail see Krishnan and McLachlan [13]. Let Z_{ik} be the unobserved observation for the kth items/units moved out of the experiment at the time of observing ith removal at time X_i; $i = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, r_i$. Thus, the observed X_i's and Z_{ik}'s form the complete data. Hence the complete likelihood is

$$L(\alpha, \beta, \lambda) = \prod_{i=1}^{m} \left[\frac{\alpha\beta\lambda x_i^{\alpha-1}}{1 - e^{-\lambda}} e^{-\lambda - \beta x_i^{\alpha} + \lambda e^{-\beta x_i^{\alpha}}} \prod_{k=1}^{r_i} \frac{\alpha\beta\lambda z_{ik}^{\alpha-1}}{1 - e^{-\lambda}} e^{-\lambda - \beta z_{ik}^{\alpha} + \lambda e^{-\beta z_{ik}^{\alpha}}} \right].$$

The log-likelihood function is as follows

$$\begin{aligned} \ln L(\alpha, \beta, \lambda) &= n \ln(\alpha) + n \ln(\beta) + n \ln(\lambda) - n\lambda - n \ln(1 - e^{-\lambda}) \\ &\quad + (\alpha - 1) \sum_{i=1}^{m} \ln x_i - \beta \sum_{i=1}^{m} x_i^{\alpha} + \lambda \sum_{i=1}^{m} e^{-\beta x_i^{\alpha}} \\ &\quad + (\alpha - 1) \sum_{i=1}^{m} \sum_{k=1}^{r_i} \ln z_{ik} - \beta \sum_{i=1}^{m} \sum_{k=1}^{r_i} z_{ik}^{\alpha} + \lambda \sum_{i=1}^{m} \sum_{k=1}^{r_i} e^{-\beta z_{ik}^{\alpha}}. \end{aligned} \tag{15}$$

Hence, MLEs of the parameters are obtained the simultaneous solution of the following three nonlinear equations

$$\begin{aligned} \frac{\partial \ln L(\alpha, \beta, \lambda)}{\partial \alpha} &= \frac{n}{\alpha} - \alpha\beta \sum_{i=1}^{m} x_i^{\alpha-1} - \alpha\beta\lambda \sum_{i=1}^{m} x_i^{\alpha-1} e^{-\beta x_i^{\alpha}} + \sum_{i=1}^{m} \ln x_i \\ &\quad - \alpha\beta \sum_{i=1}^{m} \sum_{k=1}^{r_i} z_{ik}^{\alpha-1} - \alpha\beta\lambda \sum_{i=1}^{m} \sum_{k=1}^{r_i} z_{ik}^{\alpha-1} e^{-\beta z_{ik}^{\alpha}} \\ &\quad + \sum_{i=1}^{m} \sum_{k=1}^{r_i} \ln z_{ik} = 0, \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{\partial \ln L(\alpha, \beta, \lambda)}{\partial \beta} &= \frac{n}{\beta} - \sum_{i=1}^{m} x_i^{\alpha} - \lambda \sum_{i=1}^{m} x_i^{\alpha} e^{-\beta x_i^{\alpha}} - \sum_{i=1}^{m} \sum_{k=1}^{r_i} z_{ik}^{\alpha} \\ &\quad - \lambda \sum_{i=1}^{m} \sum_{k=1}^{r_i} z_{ik}^{\alpha} e^{-\beta z_{ik}^{\alpha}} = 0, \end{aligned} \tag{17}$$

and

$$\frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial \lambda} = \frac{n}{\lambda} - n + \frac{ne^{-\lambda}}{(1-e^{-\lambda})} + \sum_{i=1}^{m} e^{-\beta x_i^{\alpha}} + \sum_{i=1}^{m}\sum_{k=1}^{r_i} e^{-\beta z_{ik}^{\alpha}} = 0. \tag{18}$$

Now, to perform the EM algorithm, joint distribution of x and z can be written as follows:

$$f(x,z;\alpha,\beta,\lambda) = P(z;\lambda) f(x|z;\alpha,\beta),$$

where,

$$P(z;\lambda) = \frac{e^{-\lambda}\lambda^{z}}{z![1-e^{-\lambda}]}; \quad \lambda > 0,\ z = 1,2,3,\ldots.$$

Since, the conditional pdf is

$$\begin{aligned} P(z|x;\alpha,\beta,\lambda) &= \frac{f(x,z;\alpha,\beta,\lambda)}{f(x;\lambda)} \\ &= \alpha\beta z x^{\alpha-1} e^{-\beta z x^{\alpha}} \lambda^{z} \Gamma^{-1}(z+1)(e^{\lambda}-1)^{-1}; \quad z = 1,2,3,\ldots, \end{aligned} \tag{19}$$

where, $\alpha > 0, \beta > 0$ and $\lambda > 0$. The E-step of EM algorithm needs the computation of the conditional expectation $(Z|X,\alpha^t,\beta^t,\lambda^t)$, where, $(\alpha^t,\beta^t,\lambda^t)$ is the current estimates of (α,β,λ). Hence from (19), we get

$$E(z|x;\alpha^t,\beta^t,\lambda^t) = (1+\lambda^t e^{-\beta^t x^{\alpha^t}}).$$

The EM algorithm is completed with M-step, with complete data, where missing Z's are replaced by their conditional expectations $(Z|X,\alpha^t,\beta^t,\lambda^t)$. Thus, an EM iteration, takes $(\alpha^t,\beta^t,\lambda^t)$ into $(\alpha^{t+1},\beta^{t+1},\lambda^{t+1})$ obtained from the following

$$\begin{aligned} \frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial \alpha} &= \frac{n}{\alpha} - \alpha\beta\sum_{i=1}^{m} x_i^{\alpha-1} - \alpha\beta\lambda\sum_{i=1}^{m} x_i^{\alpha-1} e^{-\beta x_i^{\alpha}} \\ &\quad + \sum_{i=1}^{m}\ln x_i - \alpha\beta\sum_{i=1}^{m}\sum_{k=1}^{r_i}(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}})^{\alpha-1} \\ &\quad - \alpha\beta\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}})^{\alpha-1} e^{-\beta(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}})^{\alpha}} \\ &\quad + \sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}) = 0, \\ \frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial \beta} &= \frac{n}{\beta} - \sum_{i=1}^{m} x_i^{\alpha} - \lambda\sum_{i=1}^{m} x_i^{\alpha} e^{-\beta x_i^{\alpha}} - \sum_{i=1}^{m}\sum_{k=1}^{r_i}(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}})^{\alpha} \\ &\quad - \lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}})^{\alpha} e^{-\beta(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}})^{\alpha}} = 0, \end{aligned}$$

and

$$\frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial \lambda} = \frac{n}{\lambda} - n + \frac{ne^{-\lambda}}{(1-e^{-\lambda})} + \sum_{i=1}^{m} e^{-\beta x_i^{\alpha}} + \sum_{i=1}^{m}\sum_{k=1}^{r_i} e^{-\beta(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}})^{\alpha}} = 0.$$

The iterative procedure obtained for EM algorithm is given below

$$\alpha^{t+1} = \frac{n}{\left\{\begin{array}{c} \alpha\beta\sum_{i=1}^{m} x_i^{\alpha-1} + \alpha\beta\lambda\sum_{i=1}^{m} x_i^{\alpha-1}e^{-\beta x_i^{\alpha}} - \sum_{i=1}^{m}\ln x_i \\ +\alpha\beta\sum_{i=1}^{m}\sum_{k=1}^{r_i}(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}})^{\alpha-1} \\ +\alpha\beta\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}})^{\alpha-1}e^{-\beta(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}})^{\alpha}} \\ -\sum_{i=1}^{m}\sum_{k=1}^{r_i}\ln(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}}) \end{array}\right\}}$$

$$\beta^{t+1} = \frac{n}{\left\{\begin{array}{c} \sum_{i=1}^{m} x_i^{\alpha} + \lambda\sum_{i=1}^{m} x_i^{\alpha}e^{-\beta x_i^{\alpha}} + \sum_{i=1}^{m}\sum_{k=1}^{r_i}(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}})^{\alpha} \\ +\lambda\sum_{i=1}^{m}\sum_{k=1}^{r_i}(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}})^{\alpha}e^{-\beta(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}})^{\alpha}} \end{array}\right\}}$$

and

$$\lambda^{t+1} = \frac{n}{\left\{n - \frac{ne^{-\lambda}}{(1-e^{-\lambda})} - \sum_{i=1}^{m} e^{-\beta x_i^{\alpha}} - \sum_{i=1}^{m}\sum_{k=1}^{r_i} e^{-\beta(1+\lambda^t e^{-\beta^t x_i^{\alpha^t}})^{\alpha}}\right\}}.$$

Then $(\alpha^{t+1}, \beta^{t+1}, \lambda^{t+1})$ is used as the current estimates of (α, β, λ) in the next iteration. The MLEs of (α, β, λ) can be obtained by repeating the E-step and M-step until convergence is achieved.

2.1. Large sample test procedure

Now, we shall discuss LR method for comparing the suitability of competitive models. Note that if we take $r_i = 0$ and $n = m$ in (16)–(18), these reduce to complete sample normal equations. The observed Fisher's Information matrix is

$$J_n(\alpha,\beta,\lambda) = \begin{pmatrix} -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\alpha^2} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\alpha\partial\beta} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\alpha\partial\lambda} \\ -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\beta\partial\alpha} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\beta^2} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\beta\partial\lambda} \\ -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\lambda\partial\alpha} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\lambda\partial\beta} & -\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\lambda^2} \end{pmatrix}_{(\hat{\alpha},\hat{\beta},\hat{\lambda})}$$

where,

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\alpha^2} = \frac{n}{\alpha^2} + \sum_{i=1}^{n}\beta x_i^{\alpha}(\log(x_i))^2(1+\lambda e^{-\beta x_i^{\alpha}} - \beta\lambda x_i^{\alpha}e^{-\beta x_i^{\alpha}}),$$

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\alpha\partial\beta} = \frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\beta\partial\alpha} = \sum_{i=1}^{n}\beta x_i^{\alpha}\log(x_i)(1+\lambda e^{-\beta x_i^{\alpha}} - \beta\lambda x_i^{\alpha}e^{-\beta x_i^{\alpha}}),$$

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\alpha\partial\lambda} = \frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\lambda\partial\alpha} = \sum_{i=1}^{n} \beta x_i^{\alpha} \log(x_i) e^{-\beta x_i^{\alpha}},$$

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\beta^2} = \frac{n}{\beta^2} - \lambda \sum_{i=1}^{n} (x_i^{\alpha})^2 e^{-\beta x_i^{\alpha}},$$

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\beta\partial\lambda} = \frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\lambda\partial\beta} = \sum_{i=1}^{n} x_i^{\alpha} e^{-\beta x_i^{\alpha}},$$

$$\frac{\partial^2 \ln L(\alpha,\beta,\lambda)}{\partial\lambda^2} = \frac{n}{\lambda^2} - n\frac{e^{\lambda}}{(1-e^{\lambda})^2}.$$

Let $T_n(\alpha,\beta,\lambda)$ be the expectation of Fisher Information matrix, i.e.

$$T_n(\alpha,\beta,\lambda) = E(J_n(\alpha,\beta,\lambda)) = n\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

where,

$$T_{11} = \frac{1}{\alpha^2} + \beta E[x_z^{\alpha}(\log(x_z))^2(1+\lambda e^{-\beta x_z^{\alpha}} - \beta\lambda x_z^{\alpha} e^{-\beta x_z^{\alpha}})],$$

$$T_{12} = T_{21} = E[x_z^{\alpha}\log(x_z)(1+\lambda e^{-\beta x_z^{\alpha}} - \beta\lambda x_z^{\alpha} e^{-\beta x_z^{\alpha}})],$$

$$T_{13} = T_{31} = \beta E[x_z^{\alpha}\log(x_z) e^{-\beta x_z^{\alpha}}],$$

$$T_{22} = \frac{1}{\beta^2} - \lambda E[(x_z^{\alpha})^2 e^{-\beta x_z^{\alpha}}],$$

$$T_{23} = T_{32} = E[x_z^{\alpha} e^{-\beta x_z^{\alpha}}],$$

$$T_{33} = \frac{1}{\lambda^2} - \frac{e^{\lambda}}{(1-e^{\lambda})^2}.$$

For large n, under the usual regularity condition, we found that $(\hat{\alpha},\hat{\beta},\hat{\lambda})$ has a multivariate normal distribution with mean (α,β,λ) and covariance matrix $T_n^{-1}(\alpha,\beta,\lambda)$. The asymptotic property of normality is useful for performing a goodness-of-fit test. Here, we can test the significance of the model parameters by comparing this full model with specified nested models based on the LR test. By considering null hypothesis $H_{01}: \alpha = 1$ against $H_{11}: \alpha \neq 1$ and $H_{02}: \lambda = 0$ against $H_{12}: \lambda \neq 0$, one can compare the suitability of Exponential Poisson (EP) and Weibull versus *WPM*, respectively. The test statistic under $H_{0i},\ i = 1, 2$, are

$$R_1 = -2\ln\left(\frac{L(\alpha_0,\hat{\beta},\hat{\lambda})}{L(\hat{\alpha},\hat{\beta},\hat{\lambda})}\right) \quad \text{and} \quad R_2 = -2\ln\left(\frac{L(\hat{\alpha},\hat{\beta},\lambda_0)}{L(\hat{\alpha},\hat{\beta},\hat{\lambda})}\right),$$

respectively, which are asymptotically distributed as chi-square with degrees of freedom equal to the respective dimension of the parameter space under the null hypothesis.

2.2. Bayesian estimation under PT-II CBRs

To obtain the Bayes estimator of α, β and λ, we assume that these are independently distributed prior pdfs for α and λ are chosen by using Jeffery's method i.e. log of the parameters are uniformly distributed; resulting to the following distributions:

$$g_1(\alpha) \propto \frac{1}{\alpha}; \quad \alpha > 0. \tag{20}$$

$$g_2(\lambda) \propto \frac{1}{\lambda}; \quad \lambda > 0. \tag{21}$$

Keeping in mind the wide coverage of variety of prior beliefs, we have chosen gamma distribution given below as prior distribution; see for details, Nassar and Eissa [20] and Box and Tiao [5].

$$g_3(\beta) \propto e^{-a\beta}\beta^{b-1}; \quad a > 0,\ b > 0, \tag{22}$$

where a and b are scale and shape parameters of the gamma distribution. Thus the posterior distribution of the parameters can easily be obtained as

$$\pi(\alpha, \beta, \lambda|x, r) \propto \frac{\alpha^{m-1}\lambda^{m-1}\beta^{m+b-1}e^{-m\lambda-\beta\sum_{i=1}^{m}x_i^{\alpha}-a\beta+\lambda\sum_{i=1}^{m}e^{-\beta x_i^{\alpha}}}}{(1-e^{\lambda})^m} \times \prod_{i=1}^{m} x_i^{\alpha-1}\left[\frac{1-e^{\lambda e^{-\beta x_i^{\alpha}}}}{1-e^{-\lambda}}\right]^{r_i},$$

and the respective marginal posterior pdfs of α, β and λ can be computed from the following

$$\pi_1(\alpha|x, r) = \int_0^{\infty}\int_0^{\infty} \pi(\alpha, \beta, \lambda|x, r)\, \mathrm{d}\beta\, \mathrm{d}\lambda,$$

$$\pi_2(\beta|x, r) = \int_0^{\infty}\int_0^{\infty} \pi(\alpha, \beta, \lambda|x, r)\, \mathrm{d}\alpha\, \mathrm{d}\lambda,$$

and

$$\pi_3(\lambda|x, r) = \int_0^{\infty}\int_0^{\infty} \pi(\alpha, \beta, \lambda|x, r)\, \mathrm{d}\alpha\, \mathrm{d}\beta.$$

Now, let us consider the most popular symmetric loss function (i.e. squared error loss function (SELF)) has equal weight to the over and under estimation of the same magnitude. The SELF is defined as follows:

$$L(\hat{\tau}, \tau) \propto (\tau - \hat{\tau})^2,$$

where, $\hat{\tau}$ is the estimate of the parameter τ. The posterior risk is

$$E_{\text{post}}L(\hat{\tau}, \tau) \propto E_{\text{post}}(\tau - \hat{\tau})^2.$$

The Bayes estimator $\hat{\tau}_S$ of τ comes out to be $E_{\text{post}}(\tau)$, where E_{post} denotes the posterior expectation which minimizes posterior risk. It may also be noted that this symmetric loss

function can only be justified if over and under estimation of equal magnitude are of equal seriousness. But in practical situations, this may not be true. Keeping this point in mind several asymmetric loss functions are available in the statistical literature, and one of the most widely used asymmetric loss function is the general entropy loss function (GELF) which is proposed by Calabria and Pulcini [6]. It is defined as follows:

$$L(\tau, \hat{\tau}) \propto \left(\frac{\hat{\tau}}{\tau}\right)^{\delta} - \delta \ln\left(\frac{\hat{\tau}}{\tau}\right) - 1. \tag{23}$$

The constant δ, involved in (23) is shape parameter. It also reflects departure from symmetry. When $\delta > 0$, over estimation (i.e. positive error) causes more serious consequences than under estimation (i.e. negative error) and converse for $\delta < 0$. The posterior risk is

$$E_{\text{post}} L(\tau, \hat{\tau}) \propto E_{\text{post}} \left(\frac{\hat{\tau}}{\tau}\right)^{\delta} - \delta \ln\left(\frac{\hat{\tau}}{\tau}\right) - 1, \quad \delta \neq 0.$$

The value of τ that minimizes posterior risk i.e. the Bayes estimator $\hat{\tau}_G$ of τ under GELF is given by

$$\hat{\tau}_G = (E_{\text{post}}(\tau^{-\delta}))^{(-1/\delta)}, \tag{24}$$

provided the posterior expectation exits. It may be noted that for $\delta = -1$, the Bayes estimator (24), coincides with the Bayes estimator under SELF. The expressions for the Bayes estimators of the parameters α, β and λ, denoted by $\hat{\alpha}_G, \hat{\beta}_G$ and $\hat{\lambda}_G$, respectively, are given below

$$\hat{\alpha}_G = \left[\int_0^{\infty} \alpha^{-\delta} \pi_1(\alpha|x, r)\, \mathrm{d}\alpha\right]^{-1/\delta}, \tag{25}$$

$$\hat{\beta}_G = \left[\int_0^{\infty} \beta^{-\delta} \pi_2(\beta|x, r) \mathrm{d}\beta\right]^{-1/\delta}, \tag{26}$$

and

$$\hat{\lambda}_G = \left[\int_0^{\infty} \lambda^{-\delta} \pi_3(\lambda|x, r) \mathrm{d}\lambda\right]^{-1/\delta}. \tag{27}$$

It may be noted that the integrals in (25)–(27) cannot be reduced to closed forms. Hence, numerical computational techniques are suggested for their calculations following Tierney [29], who has suggested the use of well-known technique, namely, Markov Chain Monte Carlo (MCMC) technique in which the samples are generated from posterior distribution by Gibbs sampler via Metropolis–Hastings algorithms. The samples thus obtained are then used to evaluate the Bayes estimates under SELF and GELF. It may be noted that Gibbs sampler uses to generate samples from full conditionals to generate samples from posterior distribution and for details [10]. Full conditional posterior distributions of the parameters α, β, and λ can be written in the following form:

$$\pi_1^*(\alpha|\beta, \lambda, x, r) \propto \alpha^{m-1} e^{-\beta \sum_{i=1}^{m} x_i^{\alpha} + \lambda \sum_{i=1}^{m} e^{-\beta x_i^{\alpha}}} \prod_{i=1}^{m} x_i^{\alpha-1} \{1 - e^{\lambda e^{-\beta x_i^{\alpha}}}\}^{r_i}, \tag{28}$$

$$\pi_2^*(\beta|\alpha,\lambda,x,r) \propto \beta^{m+b-1}e^{-\beta\sum_{i=1}^{m}x_i^{\alpha}-a\beta+\lambda\sum_{i=1}^{m}e^{-\beta x_i^{\alpha}}}\prod_{i=1}^{m}\{1-e^{\lambda e^{-\beta x_i^{\alpha}}}\}^{r_i}, \tag{29}$$

and

$$\pi_3^*(\lambda|\alpha,\beta,x,r) \propto \frac{\lambda^{m-1}e^{-m\lambda+\lambda\sum_{i=1}^{m}e^{-\beta x_i^{\alpha}}}}{(1-e^{-\lambda})^m}\prod_{i=1}^{m}\left\{\frac{1-e^{\lambda e^{-\beta x_i^{\alpha}}}}{1-e^{\lambda}}\right\}^{r_i}. \tag{30}$$

The Bayes estimators of α, β and λ are evaluated from the required sample of (28)–(30), generated by using MCMC procedure. The algorithm used for obtaining Bayes estimates and highest posterior density (HPD) credible intervals is given below:

(I) Set α_0, β_0 and λ_0 be the initial guess of α, β and λ.
(II) Set $i = 1$.
(III) Generate α_i from $\pi_1^*(\alpha|\beta_{i-1},\lambda_{i-1},x,r)$, β_i from $\pi_2^*(\beta|\lambda_{i-1},\alpha_{i-1},x,r)$ and λ_i from $\pi_3^*(\lambda|\alpha_{i-1},\beta_{i-1},x,r)$ respectively.
(IV) Repeat steps 2–3, N times.
(V) Obtain the Bayes estimates of α, β and λ under GELF as $[E(\alpha^{-\delta}|x,r)]^{-1/\delta} = [\frac{1}{N-N_0}\sum_{i=1}^{N-N_0}\alpha_i^{-\delta}]^{-1/\delta}$, $[E(\beta^{-\delta}|x,r)]^{-1/\delta} = [\frac{1}{N-N_0}\sum_{i=1}^{N-N_0}\beta_i^{-\delta}]^{-1/\delta}$ and $[E(\lambda^{-\delta}|x,r)]^{-1/\delta} = [\frac{1}{N-N_0}\sum_{i=1}^{N-N_0}\lambda_i^{-\delta}]^{-1/\delta}$, where N_0 is the burn in period. Substituting $\delta = -1$ in step V, we get Bayes estimates of α, β and λ under SELF.
(VI) For computing the HPD credible interval of α, β and λ. We order the MCMC sample values α, β and λ (say $\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_N$ as $\alpha_{(1)},\alpha_{(2)},\alpha_{(3)},\ldots,\alpha_{(N)}$, $\beta_1,\beta_2,\beta_3,\ldots,\beta_N$ as $\beta_{(1)},\beta_{(2)},\beta_{(3)},\ldots,\beta_{(N)}$ and $\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_N$ as $\lambda_{(1)},\lambda_{(2)},\lambda_{(3)},\ldots,\lambda_{(N)}$). Then construct all the 100(1-Ψ)% credible intervals of α β and λ, say $((\alpha_{(1)},\alpha_{N[(1-\Psi)]+1}),\ldots,(\alpha_{[N\Psi]},\alpha_N),(\beta_{(1)},\beta_{N[(1-\Psi)]+1}),\ldots,(\beta_{[N\Psi]},\beta_N)\&$ $(\lambda_{(1)},\lambda_{[N(1-\Psi)]+1}),\ldots,(\lambda_{[N\Psi]},\lambda_N))$ respectively. Where $[\eta]$ denotes the largest integer less than or equal to η. Then the HPD credible interval of α, β and λ is that interval which has the shortest length.

3. Bayes prediction

In this section, we have derived an expression for one sample Bayes prediction, if the experimenter is interested to know the lifetimes of the $(n-m)$ removed surviving units on the basis of observed sample. Let $Y_s = X_{m+s}$, $m < s \le n$, represents the failure lifetime of the remaining units, then conditional distribution of $Y_{(s)}^{\text{th}}$ order statistics given PT-II CBRs sample $\mathbf{x}$ is given by, see Singh *et al.* [25]

$$f(y_{(s)}|x_{(m)},\alpha,\beta,\lambda) = \frac{(n-m)![1-F(y_{(s)})]^{n-m-s}}{(s-1)!(n-m-s)![1-F(x_{(m)})]^{n-m}}[F(y_{(s)})-F(x_{(m)})]^{s-1}f(y_{(s)}). \tag{31}$$

Substituting (1) and (2) in (31), we have

$$f(y_{(s)}|x_{(m)},\alpha,\beta,\lambda) = \alpha\beta y_{(s)}^{\alpha-1}\zeta(y_{(s)})\log(\zeta(y_{(s)}))\frac{(n-m)!}{(s-1)!(n-m-s)!} \times\left[\frac{1-\zeta(y_{(s)})}{1-\zeta(x_{(m)})}\right]^{n-m}[1-\zeta(y_{(s)})]^{-s}[\zeta(x_{(m)})-\zeta(y_{(s)})]^{s-1},$$

where, $\zeta(z) = e^{\lambda e^{-\beta z^{\alpha}}}$. One sample Bayes predictive density of $y_{(s)}^{\text{th}}$ ordered future sample can be obtained as follows:

$$f(y_{(s)}|\mathbf{x}) = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \mathbf{f}(\mathbf{y_{(s)}}|\mathbf{x},\alpha,\beta,\lambda)\pi(\alpha,\beta,\lambda|\mathbf{x})\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\lambda$$

The above equation for $f(y_{(s)}|\mathbf{x})$ cannot be expressed in closed form and hence it cannot be evaluated analytically. Therefore, MCMC techniques is proposed to be used for obtaining the approximate solution of the above predictive density. $\{(\alpha_i,\beta_i,\lambda_i); i = 1,2,\ldots,N-N_0\}$ obtained from $\pi(\alpha,\beta,\lambda|\mathbf{x})$ using Gibbs sampling can be utilized to obtain the consistent estimate of $f(y_{(s)}|\mathbf{x})$. It can be obtained by

$$f(y_{(s)}|\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{N}-\mathbf{N_0}}\sum_{\mathbf{i=1}}^{\mathbf{N-N_0}} \mathbf{f}(\mathbf{y_{(s)}}|\boldsymbol{\alpha_i},\boldsymbol{\beta_i},\boldsymbol{\lambda_i}). \tag{32}$$

Thus, we can obtain the two-sided $100(1-\psi)\%$ prediction interval (l,u) for future sample by solving the following two equations:

$$P(Y_{(s)} > u|\mathbf{x}) = \frac{\psi}{2} \quad \text{and} \quad P(Y_{(s)} > l|\mathbf{x}) = 1-\frac{\psi}{2}.$$

It is not possible to obtain the solutions analytically. We need to apply suitable numerical techniques for solving these non-linear equations. Alternatively, we can also use the MCMC approach discussed by Chen and Shao [7], in the following way: Let $\{(y_{(i:s)});\ i = 1,2,\ldots,N-N_0\}$ be the corresponding ordered MCMC sample of $\{(y_{i:s});\ i = 1,2,\ldots,N-N_0\}$ from (32). Then, the $100(1-\psi)\%$ HPD intervals for $y_{(s)}$ is $(y_{(j*:s)}, y_{j*+[(1-\psi)M]:s})$, where $j*$ is chosen so that

$$y_{j*+[(1-\psi)N-N_0]:s} - y_{(j*:s)} = \min_{1\le j\le N-N_0-[(1-\psi)N-N_0]} [y_{j*+[(1-\psi)N-N_0]:s} - y_{(j*:s)}].$$

For considered real data set, we calculated the mean and 95% credible intervals (predictive bounds) for future samples using one sample prediction technique. The results are summarized in Table 1.

4. Expected experiment time

Cost of an experiment is directly related to the experiment time. Therefore, for a proper planning of the experimentation one is always interested in knowing the expected

Table 1. Mean and 95% predictive bounds for future ordered observations from the bladder cancer data set.

One sample prediction			
		Bounds	
s	Mean	l	u
1	79.04829	77.18001	80.46525
2	79.42236	78.31463	80.52509
3	79.59276	78.47721	80.69601
4	79.89351	78.78346	81.01349

experiment time, which can be defined PT-II CBRs

$$E[X_m] = E_R[E[X_m|R = r]]$$
$$= \sum_{r_1=0}^{g(r_1)} \sum_{r_2=0}^{g(r_2)} \cdots \sum_{r_{m-1}=0}^{g(r_{m-1})} p(R,p)E[X_{m:m:n}|R = r]. \quad (33)$$

Where $g(r_i) = n - m - r_1 - \cdots - r_{i-1}$ and $p(R = r; p)$ is given in (9). Conditioning on R the expected experiment time is

$$E[X_m|R] = \int_0^\infty x f_{X_m}(x)\, \mathrm{d}x,$$

where, $f_{X_{(m)}} = C_{m-1} f(x) \sum_{j=1}^{m} a_{j,m}(1 - F(x))^{\gamma_j}$, $1 \le m \le n$ and $c_{m-1} = \prod_{i=1}^{m} \gamma_i$, $1 \le m \le n$ and $a_{j,m} = \prod_{i=1}^{m} \frac{1}{\gamma_i - \gamma_j}$; $i \ne j$, $1 \le j \le m \le n$. For more details about the procedure of evaluation of conditional expectation of X_m for given R, see Balakrishnan and Aggarwala [2], Singh *et al.* [25], Tse *et al.* [30]. Using the suggested procedure, expected experiment times under PT-II CBRs are computed for different combinations of m and n listed in Table 2. The values of p, considered here are 0.1, 0.3, 0.5, 0.7 and 0.9 while model parameters α, β and λ are arbitrarily taken as 1, 2 and 2 respectively. The results obtained are summarized below in table 2.

Now we can obtain ratio of the expected experiment time (REET) between PT-II CBRs and the complete sampling as

$$\text{REET} = \frac{E[X_m]\text{under PT-II CBRs}}{E[X_n]\text{under complete sampling}}. \quad (34)$$

It may be noted that REET indicates the reduction in experiment time. Figure 1 shows REET for various values of n for $m = 10$ and different removal probability $p = 0.1, 0.3, 0.5, 0.7$ and 0.9. It can be seen from the figure that for each values of p, the REET decreases as n increases. It may be, noted that for larger value of (> 0.5) and larger $n(>25)$; the values of REET do not change for change in the value of p. For $p \le 0.5$ and moderate sample size (25) larger valuers of REET is noted for smaller valuers of p.

Table 2. Expected experiment time $E[X_m]$ under PT-II CBRs.

n	m	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$
30	10	0.15660	0.57392	0.86262	0.93033	0.96077
	15	0.37201	0.99414	1.09666	1.13149	1.13086
	20	0.76695	1.23138	1.27059	1.30419	1.25337
	25	0.93638	1.35836	1.37069	1.34487	1.35232
	30	1.47650	1.45380	1.49828	1.45718	1.47108
20	10	0.28018	0.73055	0.91508	0.95055	0.95818
	15	0.71918	1.10303	1.13051	1.15929	1.14585
	20	1.27924	1.28742	1.27601	1.28292	1.28714
10	3	0.08832	0.12157	0.19727	0.31295	0.43709
	4	0.13669	0.22246	0.36903	0.50305	0.58098
	6	0.29297	0.51585	0.67842	0.75186	0.76813
	10	0.99404	0.98307	0.97925	0.98829	0.99048

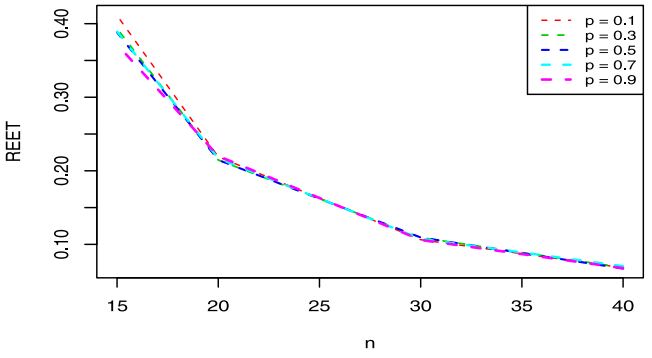

Figure 1. REET under PT-II CBRs to under complete sample.

5. Monte Carlo simulation and comparison

We have seen above that proposed estimators are not obtained in the closed form; therefore, an analytical study of behavior of the estimators is not possible and we propose to study it numerically. For this purpose, we suggest the use of MCMC technique as suggested by Tierney [29] also, for the calculation of risk (average loss over sample space) of estimators of the parameters α, β and λ. Hence, samples are generated from specified *WPM* and PT-II CBRs samples are obtained from these. MLE along with Bayes estimators under SELF and GELF are calculated. The ML estimators are denoted as; $\hat{\alpha}_M$, $\hat{\beta}_M$, $\hat{\lambda}_M$ where as $\hat{\alpha}_S$, $\hat{\beta}_S$, $\hat{\lambda}_S$ and $\hat{\alpha}_G$, $\hat{\beta}_G$, $\hat{\lambda}_G$ denote SELF and GELF estimates of the parameters α, β and λ, respectively. Similarly, (α_L^c, α_U^c), (β_L^c, β_U^c), $(\lambda_L^c, \lambda_U^c)$ and (α_L^h, α_U^h), (β_L^h, β_U^h), $(\lambda_L^h, \lambda_U^h)$ indicate $100(1-\Psi)\%$ confidence interval and HPD credible intervals. Risk are estimated on the basis of 8000 samples. Since risk of the estimators under PI-II CBRs will be function of n, m, p, α, β, λ, δ, a and b. The choice of hyper parameter are made by assuming that the prior information about the parameter is available in the form of its expected value μ and its variance σ^2 reflecting the confidence in expected value. Thus a and b are calculated from equations, which can be taken in such a way that if we consider any two independent pieces of information as prior mean and variance of β are $\mu = \frac{b}{a}$ and $\sigma^2 = \frac{b}{a^2}$, where μ is taken as true values of the parameter β and smaller, moderate and large values of variances, namely 0.5, 1 and 5 which gave $(a = 4,\ b = 8)$, $(a = 2,\ b = 4)$ and $(a = 0.4,\ b = 0.8)$, respectively. We vary the effective samples size $m = 10[5]30$. The value of α, β and λ are arbitrarily taken as 1, 2 and 2, respectively. The value of loss parameter δ is taken as 1.5 for over estimation to be more serious than under estimation and see Singh *et al.* [24]. After an extensive study of results thus obtained, conclusions are drawn regarding the behavior of the estimators. It may be mention here that because of space restriction, results for all the variation in the parameters are not shown here. Only selected figures are included.

6. Discussion of results

In this section, we shall discuss the impact of variation of effective sample size m under PT-II CBRs and compare the risks of all the estimators of α, β and λ, obtained under GELF with the corresponding Bayes estimators under SELF and MLE. It is observed that

the risks of all the estimators of α, β and λ decrease as effective sample observations m increases. The risks of $(\hat{\alpha}_G, \hat{\beta}_G)$ and $(\hat{\alpha}_S, \hat{\beta}_S)$ are found to be close respectively to each other for all the considered situations. A similar trend is observed for $\hat{\lambda}_G$ and $\hat{\lambda}_S$ also. It is further observed that, in general, the risks of the estimators under SELF and GELF decreases, as for $\delta = +1.5$ and $\delta = -1.5$ with each prior belief of the parameter β (see Figures 2–4).

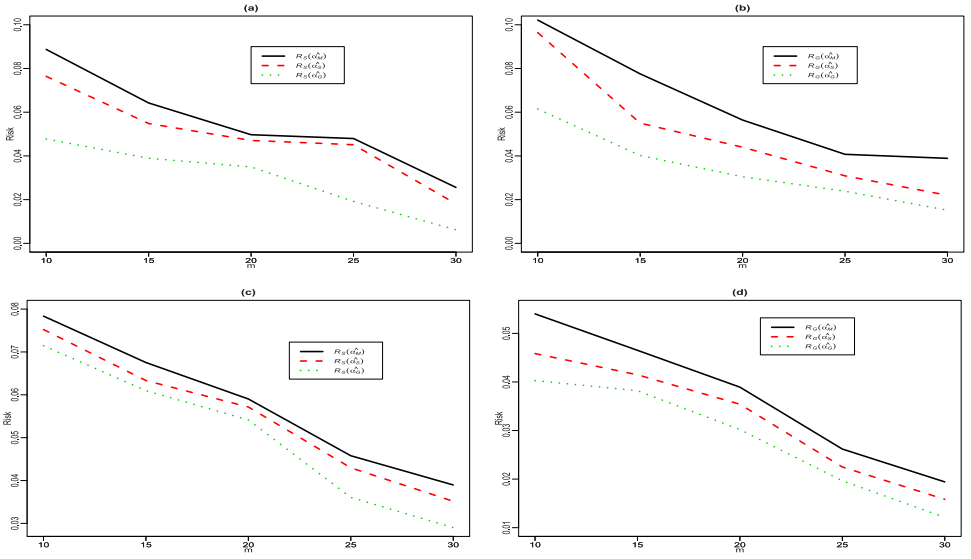

Figure 2. Risks for the estimators of parameter α for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with small prior variance, $\beta = 0.5$; for panels (a) and (b) $\delta = 1.5$; for panels (c) and (d) $\delta = -1.5$.

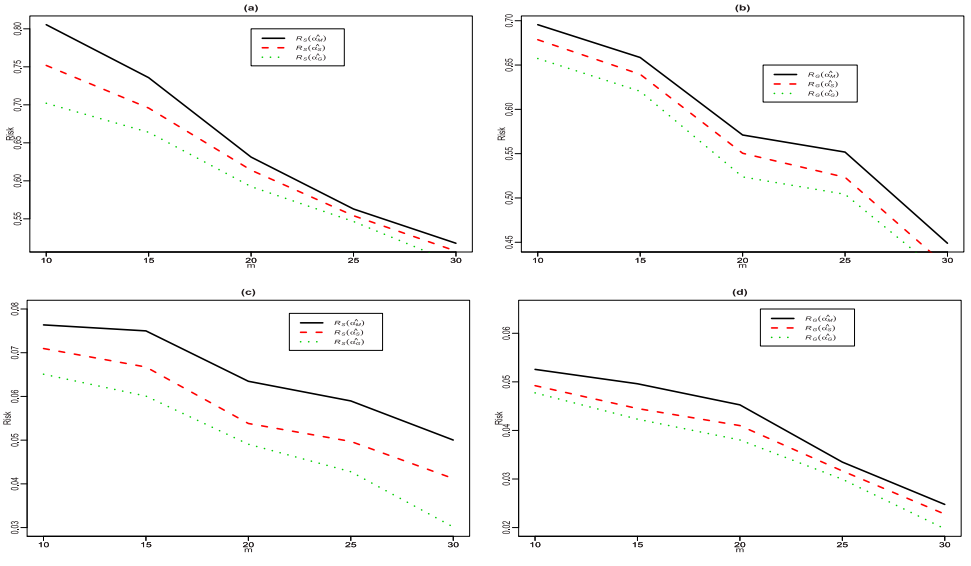

Figure 3. Risks for the estimators of parameter α for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with moderate prior variance, $\beta = 1$; for panels (a) and (b) $\delta = 1.5$; for panels (c) and (d) $\delta = -1.5$.

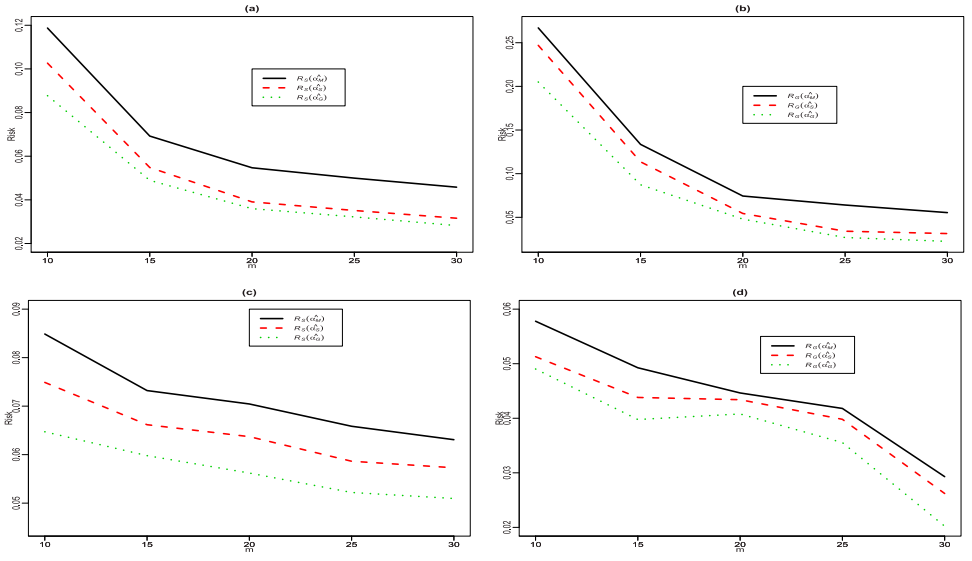

Figure 4. Risks for the estimators of parameter α for fixed $n = 30, p = 0.5, \alpha = 1, \beta = 2, \lambda = 2$ with high prior variance, $\beta = 5$; for panels (a) and (b) $\delta = 1.5$; for panels (c) and (d) $\delta = -1.5$.

For large number of effective sample sizes, the difference between the risks of the estimators are less. The decrease in the risks is more for $\hat{\alpha}_M$ as compared to the other estimators. For almost all values of prior belief of the parameter β and δ, the risk of $\hat{\alpha}_G$ under GELF is found to be least among the considered estimators. It is also interesting to remark here that $\hat{\alpha}_G$ has the least risk under SELF. For positive values of δ, the behavior of risks of estimators under GELF is more or less similar to the one obtained for negative δ (see Figures 2–4).

Similarly, we have studied the risks of Bayes estimators β and λ respectively under SELF and GELF based on PT-II CBRs. The trend remains more or less the same as stated above under both loss functions see results in graphs, which is shown in supplementary material. Further we observed that the risk of $\hat{\beta}_G$ and $\hat{\lambda}_G$ under GELF and SELF are found to be least among the considered estimators, respectively.

Figure 5 shows the CI/HPD credible intervals for α. It may also be noted that average length of CI/HPD credible intervals narrow down as m increases. The HPD credible intervals are better than CIs in respect of average length. While studying the effect of large effective sample sizes m, the difference of average lengths between the CIs and HPD credible intervals are negligibly small. For β and λ also, the trend of CI/HPD credible intervals is similar to that of α. Due to space restriction, results for variations in m of CI/HPD credible intervals of β and λ are not shown here. The CI/HPD credible intervals of β and λ are given in the supplementary material. Thus, we cannot deny from the fact that estimates under Bayesian are more precise and accurate than ML estimates.

We also discussed the expected time to test shown in Table 2. It is meaningful to comment that as the value p and m increase the expected time to test also increases. It is also observed that for fixed m, if increases the value of the sample size, i.e. n, the expected time to test decreases.

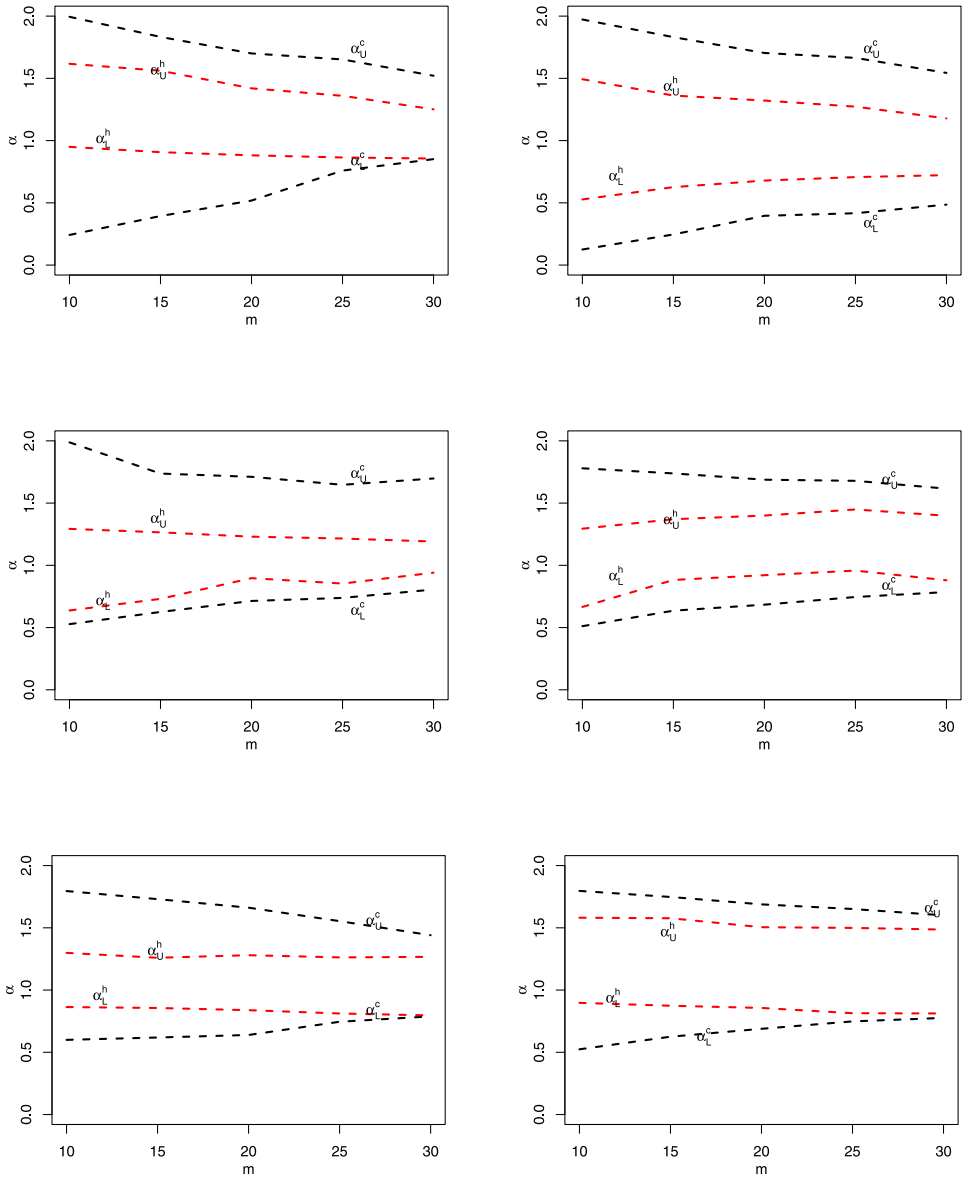

Figure 5. The confidence interval and HPD intervals for α when prior variance is 0.5, 1 and 5 with left panel: $\delta = 1.5$; right panel: $\delta = -1.5$, respectively.

7. An application example

In this section, we have a real data set given by Lee and Wang [18]. It contains a set of remission times (in months) related to 137 cancer patients, and some patients are lost to follow-up. These remission times (in months) are a subset of the data from a bladder cancer study. We have considered here a random set of 128 observations from the study, which are given as follows:4.50, 32.15, 3.88, 13.80, 19.13, 4.87, 5.85, 14.24, 5.71, 7.09, 7.87, 7.59, 20.28, 5.32, 5.49, 3.02, 46.12, 2.02, 4.51, 5.17, 2.83, 9.22, 1.05, 0.20, 8.37, 3.82, 9.47, 36.66, 14.77,

26.31, 79.05, 10.06, 8.53, 2.02, 4.98, 11.98, 2.62, 4.26, 5.06, 1.76, 0.90, 11.25, 16.62, 4.40, 21.73, 10.34, 12.07, 34.26, 10.66, 6.97, 2.07, 0.51, 12.03, 0.08, 17.12, 3.36, 2.64, 1.40, 12.63, 43.01, 14.76, 2.75, 7.66, 0.81, 1.19, 7.32, 4.18, 3.36, 8.66, 1.26, 13.29, 1.46, 14.83, 6.76, 23.63, 5.62, 3.25, 18.10, 7.62, 7.63, 17.14, 25.74, 3.52, 2.87, 15.96, 17.36, 9.74, 3.31, 7.28, 1.35, 0.40, 2.26, 4.33, 9.02, 5.41, 2.69, 22.69, 6.94, 2.54, 11.79, 2.46, 7.26, 2.69, 5.34, 3.48, 8.26, 6.93, 4.23, 3.70, 0.50, 10.75, 6.54, 3.64, 5.32, 13.11, 8.65, 3.57, 5.09, 7.39, 5.41, 11.64, 2.09, 2.23, 6.25, 7.93, 4.34, 25.82, 12.02.

First of all, we checked the suitability of *WPM* to the considered data set and compared with some well-established lifetime models, namely, EP and Weibull distribution. For testing the goodness of fit, we used the method based on maximum likelihood function, the Kolmogorov–Smirnov (KS) distance, the Akaike information criterion (AIC), proposed by Akaike [1], Bayesian information criterion (BIC) proposed by Schwarz *et al.* [22]. The best distribution is that which corresponds to the lowest $-\ln L$, AIC, BIC and KS statistic value and corresponding highest p values. Further, we have used a graphical method also goodness of fit of distributions. We draw quantile–quantile (Q–Q) plots for above-mentioned three lifetime models and are presented in Figure 6. A Q–Q plot depicts the points $\{F^{-1}(\frac{i-0.5}{n}; \hat{\Theta}_M, x_{(i)})\}$, $i = 1, 2, 3, \ldots, n$, where $\hat{\Theta}_M$ is the MLE of the set parameters of lifetime model. The values of MLEs of the parameters of the considered lifetime models, $-\ln L$, AIC, BIC, KS statistic and associated p values are reported in Table 3.

This table shows that *WPM* provides better fit than EP and Weibull distribution. Further, we tested the hypothesis: $H_{01} : \alpha = 1$ (Data follow Exponential Poisson) vs $H_{11} : \alpha \neq 1$ (Data follow Weibull Poisson) and $H_{02} : \lambda = 0$ (Data follow Weibull) vs $H_{12} : \lambda \neq 0$ (Data follow Weibull Poisson), using the large sample test described in Section 2.1. The value of the test statistic R_1 and R_2 are obtained as 8.30737 and 7.79551, respectively. Which reject H_{01} and H_{02}.

Now for the purpose of illustrating the method discussed in this article, PT-II CBR samples are generated from this data set under different schemes. The number of removals are shown in Table 4 under different schemes. The EM algorithm procedure is used to compute the MLEs of α, β and λ. The initial value in this procedure is chosen using contour plots of parameters, and their corresponding log-likelihoods are plotted using R software (Figure 7).

As we have no prior information about the parameter β, and we use non-informative prior for which the hyper parameter of β is taken to be $(a = 0 : 000001; b = 0 : 000001)$. When implementing MCMC algorithm, the values of MLEs are used as initial guess and CUSUM plots are plotted and verified the convergence of Markov chain. Then, we evaluate Bayes estimates and HPD intervals using the formulae given in previous Section 3 under different censoring schemes based on Table 4, the Bayes estimate of α, β and λ under SELF

Table 3. The -Log likelihood ($-\ln L$), KS, p-value and the AIC and BIC values for the Weibull Poisson (WP), Exponential Poisson (EP) and Weibull fitted Models.

	Estimates	$-\ln L$	KS	p-value	AIC	BIC
WP(α, β, λ)	(1.26853,0.01629,4.26518)	−410.189	0.046875	0.99896	826.3782	834.9343
EP(β, λ)	(0.106371,0.0000047)	−414.343	0.078125	0.82955	834.6856	843.2417
Weibull(α, β)	(1.04784,0.09389)	−414.087	0.0703125	0.90972	834.1738	842.7298

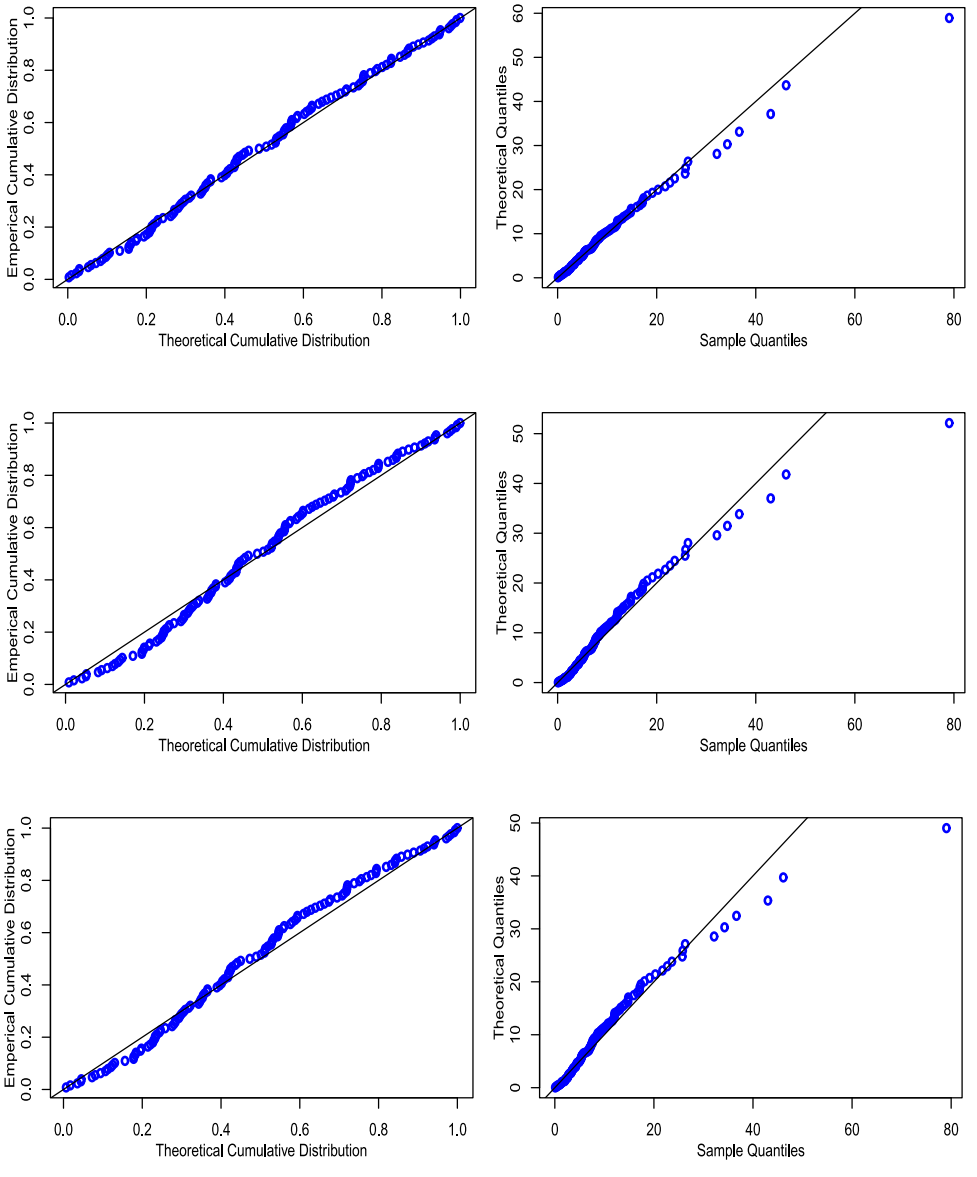

Figure 6. Top row: WP, Middle row: EP, Last row: Weibull distribution shows the PP and QQ plot for bladder cancer data set.

and GELF for $\delta = \pm 1.5$ are presented in Table 5. It may also be seen from Table 5 that various estimates, obtained using PT-II CBRs, are quite close to those obtained under complete samples.

8. Conclusion and remark

On the basis of the discussion of results given in the previous section, we may conclude that the proposed estimators $\hat{\alpha}_G$, $\hat{\beta}_G$ and $\hat{\lambda}_G$ perform better than all other considered

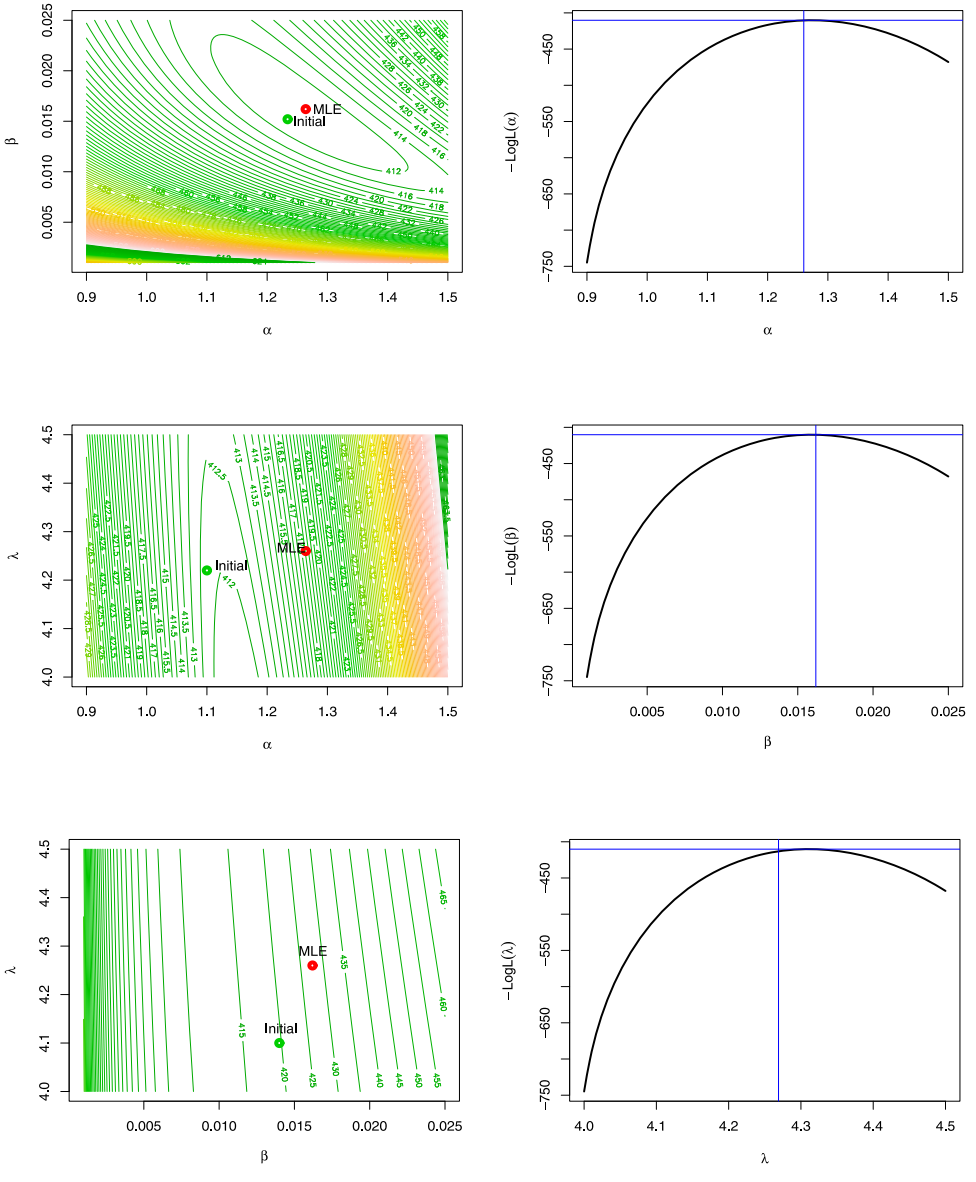

Figure 7. The Contour and -Log likelihood plot of α, β and λ for bladder cancer data set.

competitive estimators, for ($\delta > 0$), i.e. when over estimation is more serious than underestimation and for ($\delta < 0$), i.e. when under estimation is more serious than over estimation, under both the loss functions. Thus, the use of the proposed estimator $\hat{\alpha}_G$, $\hat{\beta}_G$ and $\hat{\lambda}_G$ are recommended under SELF and GELF. Moreover, a brief study has done on the expected experiment time by taking the various combinations of effective parameters n, p and m and it observed that on increases the value of p and m, the expected time to test increases. While, for fixed m, on increases the value of n, the expected time to test decreases. The likelihood ratio test has performed the goodness of fit. The one sample Bayes prediction has also presented. Furthermore, a real data set is fitted to show the practical applicability of *WPM*.

Table 4. PT-II CBR samples under different censoring scheme ($S_{n:m}$) for fixed $n = 128, p = 0.5$.

	$S_{128:64}$		$S_{128:77}$				$S_{128:102}$			
i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i
1	0.08	23	0.08	18	20.28	0	0.08	7	10.34	0
2	2.69	17	2.26	9	21.73	0	1.19	4	10.66	0
3	4.23	7	3.02	8	22.69	0	1.76	3	10.75	0
4	4.98	2	3.7	6	23.63	0	2.09	4	11.25	0
5	5.17	3	4.34	2	25.74	0	2.62	0	11.64	0
6	5.41	1	4.51	3	25.82	0	2.64	2	11.79	0
7	5.49	5	5.09	2	26.31	0	2.75	2	11.98	0
8	6.76	4	5.32	1	32.15	0	3.02	0	12.02	0
9	7.26	0	5.41	1	34.26	0	3.25	2	12.03	0
10	7.28	1	5.49	0	36.66	0	3.36	0	12.07	0
11	7.39	1	5.62	0	43.01	0	3.48	1	12.63	0
12	7.62	0	5.71	0	46.12	0	3.57	0	13.11	0
13	7.63	0	5.85	0	79.05	0	3.64	0	13.29	0
14	7.66	0	6.25	1			3.7	1	13.8	0
15	7.87	0	6.76	0			3.88	0	14.24	0
16	7.93	0	6.93	0			4.18	0	14.76	0
17	8.26	0	6.94	0			4.23	0	14.77	0
18	8.37	0	6.97	0			4.26	0	14.83	0
19	8.53	0	7.09	0			4.33	0	15.96	0
20	8.65	0	7.26	0			4.34	0	16.62	0
21	8.66	0	7.28	0			4.4	0	17.12	0
22	9.02	0	7.32	0			4.5	0	17.14	0
23	9.22	0	7.39	0			4.51	0	17.36	0
24	9.47	0	7.59	0			4.87	0	18.1	0
25	9.74	0	7.62	0			4.98	0	19.13	0
26	10.06	0	7.63	0			5.06	0	20.28	0
27	10.34	0	7.66	0			5.09	0	21.73	0
28	10.66	0	7.87	0			5.17	0	22.69	0
29	10.75	0	7.93	0			5.32	0	23.63	0
30	11.25	0	8.26	0			5.32	0	25.74	0
31	11.64	0	8.37	0			5.34	0	25.82	0
32	11.79	0	8.53	0			5.41	0	26.31	0
33	11.98	0	8.65	0			5.41	0	32.15	0
34	12.02	0	8.66	0			5.49	0	34.26	0
35	12.03	0	9.02	0			5.62	0	36.66	0
36	12.07	0	9.22	0			5.71	0	43.01	0
37	12.63	0	9.47	0			5.85	0	46.12	0
38	13.11	0	9.74	0			6.25	0	79.05	0
39	13.29	0	10.06	0			6.54	0		
40	13.8	0	10.34	0			6.76	0		
41	14.24	0	10.66	0			6.93	0		
42	14.76	0	10.75	0			6.94	0		
43	14.77	0	11.25	0			6.97	0		
44	14.83	0	11.64	0			7.09	0		
45	15.96	0	11.79	0			7.26	0		
46	16.62	0	11.98	0			7.28	0		
47	17.12	0	12.02	0			7.32	0		
48	17.14	0	12.03	0			7.39	0		
49	17.36	0	12.07	0			7.59	0		
50	18.1	0	12.63	0			7.62	0		
51	19.13	0	13.11	0			7.63	0		
52	20.28	0	13.29	0			7.66	0		
53	21.73	0	13.8	0			7.87	0		
54	22.69	0	14.24	0			7.93	0		
55	23.63	0	14.76	0			8.26	0		
56	25.74	0	14.77	0			8.37	0		
57	25.82	0	14.83	0			8.53	0		

(*continued*)

Table 4. Continued.

	$S_{128:64}$		$S_{128:77}$				$S_{128:102}$			
i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i	X_i	R_i
58	26.31	0	15.96	0			8.65	0		
59	32.15	0	16.62	0			8.66	0		
60	34.26	0	17.12	0			9.02	0		
61	36.66	0	17.14	0			9.22	0		
62	43.01	0	17.36	0			9.47	0		
63	46.12	0	18.1	0			9.74	0		
64	79.05	0	19.13	0			10.06	0		

Table 5. Bayes and ML estimates, CI/HPD interval for *WPM* parameters α, β and λ with pre-defined censoring schemes for the bladder cancer data set.

Scheme	Parameter	MLE	SELF	GELF		CI		HPD	
$S_{n:m}$				$\delta = -1.5$	$\delta = 1.5$	θ_L^C	θ_U^C	θ_L^h	θ_U^h
$S_{128:64}$	α	1.752554	1.753276	1.753308	1.753111	1.722654	1.782455	1.723427	1.78316
	β	0.001915	0.015308	0.016975	0.00012	3.52E−07	0.031815	4.53E−07	0.034268
	λ	4.664404	4.664876	4.665002	4.664246	4.569793	4.759014	4.571824	4.760873
$S_{128:77}$	α	1.690516	1.708283	1.709049	1.704437	1.550406	1.830627	1.569133	1.853984
	β	0.002777	0.080686	0.087632	0.000115	1.26E−07	0.469512	1.37E−07	0.168236
	λ	4.652349	4.671151	4.674187	4.655874	4.185615	5.119084	4.209607	5.1385
$S_{128:102}$	α	1.523263	1.511028	1.513516	1.498465	1.279468	1.767057	1.267033	1.742682
	β	0.006193	0.006015	0.006602	0.000115	4.43E−07	0.013385	8.43E−07	0.012561
	λ	4.499809	4.739747	4.982471	0.314773	1.033094	7.966523	3.946512	6.012466

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No potential conflict of interest was reported by the author(s).

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